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An Extension of the τ-Gauss Hypergeometric Functions and its Properties

Dinesh Kumar*

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Abstract
Recently, an extension of the τ-hypergeometric functions ${}_2R_1^{\tau}(z)$ was given by Parmar [6]. The main object of this paper is to introduce an extension of the τ-hypergeometric function ${}_3R_2^{\tau}(z)$ and investigate its various properties such as integral representations, derivative formula, Mellin transform and fractional calculus operators. Some published results are the special cases of our main results.

Keywords: *Generalized Gamma functions; Gauss's hypergeometric function; Generalized Gauss hypergeometric function; Extended τ-hypergeometric function; Mellin transform; Fractional calculus operators.*

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****Corresponding author***

1. Introduction and Preliminaries

Chaudhry and Zubair [1, 2] introduced extended gamma function, as follows:

$$\Gamma_p(z) = \begin{cases} \int_0^\infty t^{z-1}\exp\left(-t-\frac{p}{t}\right)dt, & (\Re(p)>0; z\in\mathbb{C}) \\ \Gamma(z), & (\Re(z)>0;\ p=0) \end{cases}.$$

Srivastava et al. [9, p.487, eq.(15)] introduced the following family of generalized hypergeometric functions:

$${}_rF_s\left[\begin{matrix}(\alpha_1,p),\alpha_2,\cdots,\alpha_r; \\ \beta_1,\beta_2,\cdots,\beta_s;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1;p)_n\,(\alpha_2)_n\cdots(\alpha_r)_n}{(\beta_1)_n\,(\beta_2)_n\cdots(\beta_s)_n}\,\frac{z^n}{n!},$$

where, $(\alpha_1;p)_n$ is the generalized Pochhammer symbol [9, p.485, eq.(8)], defined as

$$(\lambda;p)_\mu = \begin{cases} \frac{\Gamma_p(\lambda+\mu)}{\Gamma(\lambda)}, & (\Re(p)>0;\ \lambda,\mu\in\mathbb{C}) \\ (\lambda)_\mu, & (\lambda,\mu\in\mathbb{C};\ p=0) \end{cases}, \tag{1.1}$$

here, $(\lambda)_\mu$ $(\lambda,\mu\in\mathbb{C})$ denotes the Pochhammer symbol [10].
The integral representation of (1.1) is given by

$$(\lambda;p)_\mu = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+\mu-1}\ \exp\left(-t-\frac{p}{t}\right)dt, \quad (\Re(p)>0,\ \Re(\lambda+\mu)>0 \text{ when } p=0).$$

The τ-hypergeometric function was investigated and studied by Virchenko et al. [11], defined by the following manner:

$${}_2R_1^{\tau}(z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\,\frac{z^n}{n!}, \tag{1.2}$$

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with $\tau > 0; |z| < 1; \Re(c) > \Re(b) > 0$.
Integral representation of (1.2) given by [11, p.91, eq.(6)]:

$$ {}_2R_1(a, b; c; \tau; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1 - zt^{\tau})^{-a} \, dt, \tag{1.3} $$

with $\tau > 0; |\arg(1-z)| < \pi; \Re(c) > \Re(b) > 0$.
When $\tau = 1$ in (1.2) and (1.3) yields the representations of Gauss's hypergeometric function [7].

Very recently, Parmar [6, p.423, eq.(2.1)] introduced an extended τ-hypergeometric function ${}_2R_1^{\tau}(z)$, as follows:

$$ {}_2R_1^{\tau}(z) = {}_2R_1^{\tau}((a,p), b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a;p)_n \, \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!}, \tag{1.4} $$

where, $a, b \in \mathbb{C}$, $c \in \mathbb{C} \backslash \mathbb{Z}_0^-$, $p \geq 0, \tau > 0$, $|z| < 1$, and $\Re(c) > \Re(b) > 0$ when $p = 0$,
and its integral representation [6, p.423, eq.(3.1)]:

$$ {}_2R_1^{\tau}((a,p), b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \, {}_1F_0\left[(a,p); -; zt^{\tau}\right] dt, $$

with $\tau > 0; \Re(p) > 0$; $\Re(c) > \Re(b) > 0$ when $p = 0$.

2. Extended τ-Hypergeometric Function ${}_3R_2^{\tau}(z)$

Motivated mainly by investigations of the extended τ-hypergeometric function ${}_2R_1^{\tau}(z)$ defined by (1.2), we introduced the extended τ-hypergeometric function ${}_3R_2^{\tau}(z)$ as follows:
For $\lambda, a, b \in \mathbb{C}$ and $c, d \in \mathbb{C} \backslash \mathbb{Z}_0^-$, we have

$$ {}_3R_2^{\tau}(z) = {}_3R_2^{\tau}((\lambda,p), a, b; c; d; z) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda;p)_n \, \Gamma(a+\tau n)\, \Gamma(b+\tau n)}{\Gamma(c+\tau n)\, \Gamma(d+\tau n)} \frac{z^n}{n!}, \tag{2.1} $$

where, $p \geq 0, \tau > 0$, $|z| < 1$, and $\Re(d) > \Re(a) > 0$, $\Re(c) > \Re(b) > 0$ when $p = 0$.

Special Cases:

1. If we take $b = d$, then (2.1) reduces to the extended τ-hypergeometric function ${}_2R_1^{\tau}(z)$ given by Parmar [6, p.422, eq.(2.1)] as defined in (1.4).

2. If we put $b = d$ and set $\tau = 1$, then (2.1) reduces to the extended Gauss hypergeometric function [9, p.487, eq.(17)] given by

$$ {}_2F_1((\lambda,p), a; c; z) = \sum_{n=0}^{\infty} \frac{(\lambda;p)_n \, (a)_n}{(c)_n} \frac{z^n}{n!}. $$

3. If we take $\tau = 1$ and $p = 0$ in (2.1), then it reduces to the classical Gauss's hypergeometric function as

$$ {}_3F_2(\lambda, a, b; c; d; z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n \, (a)_n \, (b)_n}{(c)_n \, (d)_n} \frac{z^n}{n!}. $$

4. If we put $b = d$ and set $\tau = 1$, $p = 0$ in (2.1), then it reduces to the classical Gauss's hypergeometric function as

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n \, (b)_n}{(c)_n} \frac{z^n}{n!}. $$

3. Integral Representation and Derivative Formula

In this section, we obtain integral representation and differential formula for ${}_3R_2^\tau(z)$ as given in (2.1).

Theorem 1. *The following integral representation for ${}_3R_2^\tau(z)$ in (2.1) holds true:*

$$\begin{aligned}&{}_3R_2^\tau\left((\lambda,p),a,b;c;d;z\right)\\&=\frac{1}{B\left(a,c-a\right)}\int_0^1 t^{a-1}\left(1-t\right)^{c-a-1}\ {}_2R_1^\tau\left[(\lambda,p),b,d;zt^\tau\right]\,dt,\end{aligned}\tag{3.1}$$

where ${}_2R_1^\tau(z)$ is given in (1.4), and $\tau>0; \Re\left(p\right)>0; \Re\left(d\right)>\Re\left(a\right)>0, \Re\left(c\right)>\Re\left(b\right)>0$ when $p=0$.

Proof. Using (2.1) and considering the following elementary identity for the Beta function:

$$B\left(m,n\right)=\frac{\Gamma\left(m\right)\Gamma\left(n\right)}{\Gamma\left(m+n\right)}=\int_0^1 t^{m-1}\left(1-t\right)^{n-1}\,dt,\tag{3.2}$$

then we arrive at

$${}_3R_2^\tau\left((\lambda,p),a,b;c;d;z\right)=\frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(c-a\right)}\sum_{n=0}^\infty\frac{\left(\lambda;p\right)_n\Gamma\left(b+\tau n\right)}{\Gamma\left(d+\tau n\right)}B\left(a+\tau n,c-a\right)\frac{z^n}{n!},$$

now using (3.2), then we have

$$=\frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(c-a\right)}\sum_{n=0}^\infty\frac{\left(\lambda;p\right)_n\Gamma\left(b+\tau n\right)}{\Gamma\left(d+\tau n\right)}\int_0^1 t^{a+\tau n-1}\left(1-t\right)^{c-a-1}\frac{z^n}{n!}\,dt.$$

Next, interchanging the order of integration and summation which is permissible

$$=\frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(c-a\right)}\int_0^1 t^{a-1}\left(1-t\right)^{c-a-1}\left\{\frac{\Gamma\left(d\right)}{\Gamma\left(b\right)}\sum_{n=0}^\infty\frac{\left(\lambda;p\right)_n\Gamma\left(b+\tau n\right)\left(zt^\tau\right)^n}{\Gamma\left(c+\tau n\right)n!}\right\}dt,$$

by using (1.4), then we easily get the desired result in (3.1). This complete the proof of the Theorem 1. □

If we put $b=d$, then (3.1) reduces to the known integral representation of the extended τ-hypergeometric function ${}_2R_1^\tau(z)$ given by Parmar [6, p.423, eq.(3.1)], as given in the following corollary:

Corollary 1.1.

$${}_2R_1^\tau\left((\lambda,p),b;c;z\right)=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\ {}_1F_0\left[(\lambda,p);-;zt^\tau\right]\,dt,$$

where $\Re(p)>0;\ \tau>0;\ \Re(c)>\Re(b)>0$ when $p=0$.

Remark 3.1. *If we set $\tau=1$ in above corollary then we obtain the following result in terms of the extended hypergeometric function [9, p.488, eq.(24)]:*

$${}_2F_1\left((\lambda,p),b;c;z\right)=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\ {}_1F_0\left[(\lambda,p);-;zt\right]\,dt.\tag{3.3}$$

Further, if we take $p=0$ in (3.3), then we arrive at the known integral representation of the classical Gauss's hypergeometric function [3, p.19, eq.(1.11.10)], as follows.

$${}_2F_1\left(\lambda,b;c;z\right)=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-\lambda}\,dt.$$

Theorem 2. *The following derivative formula for ${}_3R_2^\tau(z)$ in (2.1) holds true:*

$$\left(\frac{d}{dz}\right)^n\left[z^{c-1}\,{}_3R_2^\tau\left((\lambda,p),a,b;c;d;\nu z^\tau\right)\right]=\frac{z^{c-n-1}\Gamma\left(c\right)}{\Gamma\left(c-n\right)}\,{}_3R_2^\tau\left[(\lambda,p),a,b;c-n;d;\nu z^\tau\right].\tag{3.4}$$

Proof. By using the series representation of ${}_3R_2^{\tau}(z)$ as given in (2.1), and interchanging the order of differentiation and summation, then we arrive at the following:

$$\left(\frac{d}{dz}\right)^n \left[z^{c-1}\, {}_3R_2^{\tau}\left((\lambda,p),a,b;c;d;\nu z^{\tau}\right)\right]$$

$$=\frac{\Gamma(c)\,\Gamma(d)}{\Gamma(a)\,\Gamma(b)}\sum_{m=0}^{\infty}\frac{(\lambda;p)_m\,\Gamma(a+\tau m)\,\Gamma(b+\tau m)}{\Gamma(c+\tau m)\,\Gamma(d+\tau m)}\frac{\nu^m}{m!}\left(\frac{d}{dz}\right)^n\left(z^{c+\tau m-1}\right),$$

□

now, differentiating term by term under the sign of summation, we have

$$=\frac{\Gamma(c)\,\Gamma(d)}{\Gamma(a)\,\Gamma(b)}\sum_{m=0}^{\infty}\frac{(\lambda;p)_m\,\Gamma(a+\tau m)\,\Gamma(b+\tau m)}{\Gamma(c-n+\tau m)\,\Gamma(d+\tau m)}\frac{\nu^m}{m!}z^{c+\tau m-n-1}$$

$$=z^{c-n-1}\frac{\Gamma(c)\,\Gamma(d)}{\Gamma(a)\,\Gamma(b)}\sum_{m=0}^{\infty}\frac{(\lambda;p)_m\,\Gamma(a+\tau m)\,\Gamma(b+\tau m)}{\Gamma(c-n+\tau m)\,\Gamma(d+\tau m)}\frac{(\nu z^{\tau})^m}{m!}.$$

By using (2.1), we obtain the R.H.S. of (3.4) after little simplifications. This complete the proof.

4. Mellin Transform of the Function ${}_3R_2^{\tau}(z)$

The Mellin transform of a function $f(x)$ is defined by

$$\mathcal{M}\left[f(x);s\right]=F(s)=\int_0^{\infty}x^{s-1}f(x)\,dx,\qquad (s\in\mathbb{C}),\tag{4.1}$$

provided that the improper integral in (4.1) exists.

Theorem 3. *The Mellin transform of the extended τ-hypergeometric function ${}_3R_2^{\tau}(z)$ defined in (2.1) is given by*

$$\mathcal{M}\left[{}_3R_2^{\tau}\left((\lambda,p),a,b;c;d;z\right);s\right]=\Gamma(s)\,(\lambda)_s\,{}_3R_2^{\tau}\left(\lambda+s,a,b;c;d;z\right),\tag{4.2}$$

where $\Re(s)>0$ and $\Re(\lambda+s)>0$ when $p=0$.

Proof. Using (2.1) and applying Mellin transform, then we get

$$\mathcal{M}\left[{}_3R_2^{\tau}\left((\lambda,p),a,b;c;d;z\right);s\right]$$

$$=\int_0^{\infty}p^{s-1}\left(\frac{\Gamma(c)\,\Gamma(d)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{(\lambda;p)_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{z^n}{n!}\right)dp.$$

Interchanging the order of integration and summation which is permissible, we have

$$=\frac{\Gamma(c)\,\Gamma(d)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{z^n}{n!}\frac{1}{\Gamma(\lambda)}\int_0^{\infty}p^{s-1}\,\Gamma_p(\lambda+n)\,dp.$$

Now, using the result [2, p.16, eq.(1.110)] given by

$$\int_0^{\infty}p^{s-1}\,\Gamma_p(\lambda+n)\,dp=\Gamma(\lambda+s+n)\,\Gamma(s),\qquad (\Re(s)>0),$$

then we arrive at the following:

$$\mathcal{M}\left[{}_3R_2^{\tau}\left((\lambda,p),a,b;c;d;z\right);s\right]$$

$$=\frac{\Gamma(s)\,\Gamma(c)\,\Gamma(d)}{\Gamma(\lambda)\,\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+s+n)\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{z^n}{n!}$$

$$=\Gamma(s)\,(\lambda)_s\frac{\Gamma(c)\,\Gamma(d)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{(\lambda+s)_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{z^n}{n!}.$$

By using the definition of extended τ-hypergeometric function as given by (2.1), we obtain the desired result in (4.2). This complete the proof of the Theorem 3. □

5. Fractional Calculus Approach

The Riemann-Liouville left-sided fractional calculus operators I_{a+}^{α} and D_{a+}^{α} of order α are defined by [8]

$$I_{a+}^{\alpha} f(x) = {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \qquad (x > a),$$

is called the Riemann-Liouville left-sided fractional integral of order α.

$$D_{a+}^{\alpha} f(x) = {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)\, dt}{(t-x)^{\alpha-n+1}}, \qquad (n = [\alpha]+1),$$

is called the left-sided Riemann-Liouville derivative of order α.

In this section we consider the fractional differintegral operators containing extended τ-hypergeometric function ${}_3R_2^{\tau}(z)$ as in kernel.

Theorem 4. *Let $\mu \in \mathbb{R}_+,\ \lambda, \alpha, a, b, c, d, \nu \in \mathbb{C}$ and $\Re(\alpha) > 0,\ \Re(c) > 0,\ \Re(\tau) > 0$, then for $x > \mu$ the following result holds true:*

$$\begin{aligned} &\left(I_{\mu+}^{\alpha}\left[(t-\mu)^{c-1}\ {}_3R_2^{\tau}\left((\lambda,p), a, b; c; d; \nu(t-\mu)^{\tau}\right)\right]\right)(x) \\ &= \frac{(x-\mu)^{c+\alpha-1}\ \Gamma(c)}{\Gamma(c+\alpha)}\ {}_3R_2^{\tau}\left((\lambda,p), a, b; c+\alpha; d; \nu(x-\mu)^{\tau}\right). \end{aligned} \tag{5.1}$$

Proof. By using the series representation of extended τ-hypergeometric function ${}_3R_2^{\tau}(z)$ as given by (2.1) and interchanging the order of integration and summation, we have

$$\begin{aligned} &\left(I_{\mu+}^{\alpha}\left[(t-\mu)^{c-1}\ {}_3R_2^{\tau}\left((\lambda,p), a, b; c; d; \nu(t-\mu)^{\tau}\right)\right]\right)(x) \\ &= \frac{\Gamma(c)\ \Gamma(d)}{\Gamma(a)\ \Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda;p)_n\ \Gamma(a+\tau n)\ \Gamma(b+\tau n)}{\Gamma(c+\tau n)\ \Gamma(d+\tau n)} \frac{\nu^n}{n!}\ I_{\mu+}^{\alpha}\left((t-\mu)^{c+\tau n-1}\right)(x), \end{aligned}$$

now for $x > \mu$, taking the following power function formula into account:

$$I_{a+}^{\alpha}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \qquad (\alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0),$$

then we have

$$\begin{aligned} &\left(I_{\mu+}^{\alpha}\left[(t-\mu)^{c-1}\ {}_3R_2^{\tau}\left((\lambda,p), a, b; c; d; \nu(t-\mu)^{\tau}\right)\right]\right)(x) \\ &= \frac{(x-\mu)^{c+\alpha-1}\ \Gamma(c)}{\Gamma(c+\alpha)} \frac{\Gamma(c+\alpha)\ \Gamma(d)}{\Gamma(a)\ \Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda;p)_n\ \Gamma(a+\tau n)\ \Gamma(b+\tau n)}{\Gamma(c+\alpha+\tau n)\ \Gamma(d+\tau n)} \frac{\left[\nu(x-\mu)^{\tau}\right]^n}{n!}, \end{aligned}$$

next, by using the definition (2.1), we obtain the desired result in (5.1). □

Theorem 5. *Let $\mu \in \mathbb{R}_+,\ \lambda, a, b, c, d, \nu \in \mathbb{C}$ and $\alpha > 0,\ \Re(c) > 0,\ \Re(\tau) > 0$, then for $x > \mu$ the following relation holds true:*

$$\begin{aligned} &\left(D_{\mu+}^{\alpha}\left[(t-\mu)^{c-1}\ {}_3R_2^{\tau}\left((\lambda,p), a, b; c; d; \nu(t-\mu)^{\tau}\right)\right]\right)(x) \\ &= \frac{(x-\mu)^{c-\alpha-1}\ \Gamma(c)}{\Gamma(c-\alpha)}\ {}_3R_2^{\tau}\left((\lambda,p), a, b; c-\alpha; d; \nu(x-\mu)^{\tau}\right). \end{aligned} \tag{5.2}$$

Proof. By using the series representation given by (2.1) and using the following relation

$$\left(D_{\mu+}^{\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{\mu+}^{n-\alpha} f\right)(x), \quad (\alpha \in \mathbb{C},\ \alpha > 0;\ n = [\alpha]+1),$$

then we have

$$\left(D_{\mu+}^{\alpha}\left[(t-\mu)^{c-1}\ {}_3R_2^{\tau}\left((\lambda,p),a,b;c;d;\nu\,(t-\mu)^{\tau}\right)\right]\right)(x)$$

$$=\left(\frac{d}{dx}\right)^{n}\left(I_{\mu+}^{n-\alpha}\left[(t-\mu)^{c-1}\ {}_3R_2^{\tau}\left((\lambda,p),a,b;c;d;\nu\,(t-\mu)^{\tau}\right)\right]\right)(x)\,,$$

by using (5.1) and setting $\alpha=n-\alpha$, then we arrive at the following:

$$\left(D_{\mu+}^{\alpha}\left[(t-\mu)^{c-1}\ {}_3R_2^{\tau}\left((\lambda,p),a,b;c;d;\nu\,(t-\mu)^{\tau}\right)\right]\right)(x)$$

$$=\left(\frac{d}{dx}\right)^{n}\left[\frac{(x-\mu)^{c+n-\alpha-1}\,\Gamma\left(c\right)}{\Gamma\left(c+n-\alpha\right)}\ {}_3R_2^{\tau}\left((\lambda,p),a,b;c+n-\alpha;d;\nu\,(x-\mu)^{\tau}\right)\right].$$

next, by using the result (3.4), we obtain the desired result in (5.2) after little simplification. This complete the proof of the Theorem 5. □

Remark 5.1. *For the detail of fractional calculus operators the reader can refer the work (see, [4, 5]).*

6. Concluding Remarks

In the present paper we derive a new extended τ-hypergeometric function ${}_3R_2^{\tau}(z)$. Our results motivated mainly by investigations of the τ-hypergeometric function ${}_2R_1^{\tau}(z)$ [11] and its extension [6]. We obtained certain integral representation, a derivative formula, Mellin transform and fractional calculus approach of this new extended τ-hypergeometric function ${}_3R_2^{\tau}(z)$. The provided results are new and have uniqueness identity in the literature. On account of the general nature of the extended τ-hypergeometric function, a number of known results can easily be found as special cases of our main results.

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Affiliations

DINESH KUMAR
ADDRESS: Department of Mathematics & Statistics Jai Narain Vyas University, Jodhpur - 342005, India
E-MAIL: dinesh_dino03@yahoo.com