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Research Article

Euler–Maclaurin-type Inequalities for $h-$convex Functions via Riemann-Liouville Fractional Integrals

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Abstract
In this paper, some Euler-Maclaurin-type inequalities are established by using $h-$convex functions involving Riemann-Liouville fractional integrals. In precisely, using the properties of h-convex functions, we prove new Euler-Maclaurin-type inequalities. In addition, we present some Euler-Maclaurin-type inequalities for Riemann-Liouville fractional integrals by using Hölder inequality. Moreover, some Euler-Maclaurin-type inequalities are established by using power-mean inequality. Finally, by using the special choices of the obtained results, we obtain some Euler-Maclaurin-type inequalities.

Keywords: Convex functions, Fractional calculus, Maclaurin's formula, Quadrature formula
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1. Introduction

Inequality theory is a well-established and still fascinating field of research, with a wide range of applications across various areas of mathematics. In mathematical analysis, convex functions play a crucial role in the study of inequalities due to their distinct geometric and analytical properties.

The author of [1] introduces a novel class of functions called h-convex functions.

Definition 1.1. *Let $h:(0,1)\to\mathbb{R}$ be a non-negative function, $h\neq 0$. We say that $f: I\subseteq\mathbb{R}\to\mathbb{R}$ is an h-convex function, if f is non-negative and for all $x,y\in I$, $t\in(0,1)$ we have*

$$f(tx+(1-t)y)\leq h(t)f(x)+h(1-t)f(y). \tag{1.1}$$

If the inequality (1.1) is reversed, then f is said to be h-concave.

By setting

- $h(t)=t$, Definition 1.1 becomes to convex function [2].
- $h(t)=t^s$, Definition 1.1 reduces to s-convex functions [3].
- $h(t)=1$, Definition 1.1 equals to P-functions [4].

Theorem 1.2. *(Hölder inequality). Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |f(t)g(t)|\,dt \le \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,dt\right)^{\frac{1}{q}}.$$

The power-mean integral inequality, derived from the Hölder inequality, can be expressed as follows:

Theorem 1.3. *(Power mean integral inequality). Let $p\ge 1$ and f, g be two real functions defined on $[a,b]$. If $|f|$, $|f||g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(t)g(t)|\,dt \le \left(\int_a^b |f(t)|\,dt\right)^{1-\frac{1}{p}} \left(\int_a^b |f(t)|\,|g(t)|^p\,dt\right)^{\frac{1}{p}}.$$

For further information and clarification of the power-mean integral inequality, go to references [5].

Subsequently, mathematicians have become increasingly interested in fractional calculus due to its fundamental properties and wide-ranging applications. The *Riemann–Liouville integrals* $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\ge 0$ are given by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ \ x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ \ x<b,$$

respectively [6,7]. Here, f belongs to $L_1[a,b]$ and $\Gamma(\alpha)$ denotes the Gamma function defining as

$$\Gamma(\alpha):=\int_0^{\infty} e^{-u}u^{\alpha-1}du.$$

The fractional integral coincides with the classical integral for the case of $\alpha=1$.

The formula for Simpson's quadrature, commonly referred as Simpson's $1/3$ rule, is as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a four times differentiable and continuous function on (a,b), and let $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

In the paper [8], Dragomir provided an estimate for the remainder in Simpson's formula for functions of bounded variation, with applications in the theory of special means. For further details on Simpson-type inequalities and other related topics involving Riemann–Liouville fractional integrals, readers are referred to [9, 10] and its references.

The Newton-Cotes quadrature formula, frequently referred as Simpson's second formula (also known as Simpson's $3/8$ rule; see [11]), is defined as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right].$$

Theorem 1.5. *If $f:[a,b]\to\mathbb{R}$ is a four times differentiable and continuous function on (a,b), and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$, then one has the inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{6480}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

In the literature, evaluations for three-step quadrature kernels are frequently referred to as Newton-type results because the three-point Newton-Cotes quadrature is a rule of Simpson's second rule. Newton-type inequalities have been extensively studied by a number of mathematicians. For instance, in paper [12], Erden et al. investigated several Newton-type integral inequalities for functions whose first derivative is arithmetically-harmonically convex in absolute value at a given power. Please refer to [13–15] and its references for more details on Newton-type inequality, which includes convex differentiable functions.

The Maclaurin rule, which is derived from the Maclaurin formula (see to [11]), is equivalent to the corresponding dual Simpson's $3/8$ formula:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right].$$

The Maclaurin rule, which is derived from the Maclaurin inequality, is equivalent to the corresponding dual Simpson's $3/8$ formula:

Theorem 1.6. *If $f:[a,b]\to\mathbb{R}$ is a four times differentiable and continuous function on (a,b), and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$, then the following inequality holds:*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{7}{51840}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Dedić et al. [16] are constructed a set of inequalities using Euler-Maclaurin-type inequalities, and the results were utilized to derive specific error estimates in the case of the Maclaurin quadrature rules. In the paper [17], these results are applied to provide error estimates for the Simpson 3/8 quadrature rules. In [18], several Euler-Maclaurin-type inequalities are considered for differentiable convex functions. Additionally, in [19], several corrected Euler-Maclaurin-type inequalities are established using Riemann-Liouville fractional integrals. For further information on such types of inequalities, the reader is referred to [20–22] and the references therein.

2. A Crucial Equality

In this section, we express integral equality in order to demonstrate the main results of the study.

Lemma 2.1. *[23] If $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function (a,b) such that $f'\in L_1[a,b]$, then the equality*

$$\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f(a)+J^\alpha_{\frac{a+b}{2}+}f(b)\right]$$
$$=\frac{b-a}{4}\left[I_1+I_2\right].$$

is valid. Here,

$$\begin{cases} I_1=\int\limits_0^{\frac{1}{3}} t^\alpha\left[f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt, \\ I_2=\int\limits_{\frac{1}{3}}^{1}\left(t^\alpha-\frac{3}{4}\right)\left[f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt. \end{cases}$$

3. Euler-Maclaurin-type Inequalities for h-Convex Functions

In this section, we obtain several Euler-Maclaurin-type inequalities for differentiable h-convex functions by using the Riemann-Liouville fractional integrals.

Theorem 3.1. *Suppose that Lemma 2.1 holds and the function $|f'|$ is h-convex on the interval $[a,b]$. Then, one can prove fractional Euler-Maclaurin-type inequality*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f(a)+J^\alpha_{\frac{a+b}{2}+}f(b)\right]\right|$$
$$\le\frac{b-a}{4}\left(\Omega_1(\alpha;h)+\Omega_2(\alpha;h)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Here,

$$\Omega_1(\alpha;h)=\int_0^{\frac{1}{3}} t^{\alpha}\left[h\left(\frac{t}{2}\right)+h\left(\frac{2-t}{2}\right)\right]dt,$$

and

$$\Omega_2(\alpha;h)=\int_{\frac{1}{3}}^{1} \left|t^{\alpha}-\frac{3}{4}\right|\left[h\left(\frac{t}{2}\right)+h\left(\frac{2-t}{2}\right)\right]dt.$$

Proof. By taking into account the absolute value of Lemma 2.1, one may directly obtain

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \\ &\leq\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}|t^{\alpha}|\left[\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|+\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|\right]dt\right. \\ &\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|+\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|\right]dt\right\}.\end{aligned}\tag{3.1}$$

Since $|f'|$ is h-convex, it yields

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \\ &\leq\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}t^{\alpha}\left[h\left(\frac{t}{2}\right)|f'(b)|+h\left(\frac{2-t}{2}\right)|f'(a)|+h\left(\frac{t}{2}\right)|f'(a)|+h\left(\frac{2-t}{2}\right)|f'(b)|\right]dt\right. \\ &\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[h\left(\frac{t}{2}\right)|f'(b)|+h\left(\frac{2-t}{2}\right)|f'(a)|+h\left(\frac{t}{2}\right)|f'(a)|+h\left(\frac{2-t}{2}\right)|f'(b)|\right]dt\right\} \\ &=\frac{b-a}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

which complete the proof of Theorem 3.1. □

Remark 3.2. *If we choose $h(t)=t$ in Theorem 3.1, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \\ &\leq\frac{b-a}{4}\left(\psi_1(\alpha)+\psi_2(\alpha)\right)\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

$$\psi_1(\alpha)=\int_0^{\frac{1}{3}}t^{\alpha}dt=\frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1},$$

and

$$\psi_2(\alpha)=\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{\alpha+1}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)-\frac{1}{2}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}, \\ \frac{2\alpha}{\alpha+1}\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}+\frac{1}{\alpha+1}-1, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha.\end{cases}$$

which is established by Gumus et al. in paper [23, Theorem 4].

Corollary 3.3. *Let us consider $h(t)=t^s$ in Theorem 3.1. Then, the following Euler-Maclaurin-type inequality for s-convex functions by using the Riemann-Liouville fractional integrals*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$
$$\leq\frac{b-a}{4}\left(\phi_1(\alpha,s)+\phi_2(\alpha,s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Here,

$$\phi_1(\alpha,s)=\int_0^{\frac{1}{3}}t^{\alpha}\left[\left(\frac{t}{2}\right)^s+\left(\frac{2-t}{2}\right)^s\right]dt$$

and

$$\phi_2(\alpha,s)=\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[\left(\frac{t}{2}\right)^s+\left(\frac{2-t}{2}\right)^s\right]dt.$$

Corollary 3.4. *If we assign $h(t)=1$ in Theorem 3.1, then we get the following Euler-Maclaurin-type inequality for $P-$functions by using the Riemann-Liouville fractional integrals*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$
$$\leq\frac{b-a}{2}\left(\psi_1(\alpha)+\psi_2(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Corollary 3.5. *If we assign $\alpha=1$ in Theorem 3.1, then we can obtain Euler-Maclaurin-type inequality for h-convex functions*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq\frac{b-a}{4}\left(\Omega_1(1;h)+\Omega_2(1;h)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

where

$$\Omega_1(1;h)=\int_0^{\frac{1}{3}}t\left[h\left(\frac{t}{2}\right)+h\left(\frac{2-t}{2}\right)\right]dt,$$

and

$$\Omega_2(1;h)=\int_{\frac{1}{3}}^{1}\left|t-\frac{3}{4}\right|\left[h\left(\frac{t}{2}\right)+h\left(\frac{2-t}{2}\right)\right]dt.$$

Remark 3.6. *If we choose $h(t)=t$ in Corollary 3.5, then we have the following Euler-Maclaurin-type inequality for h-convex functions functions*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq\frac{25(b-a)}{576}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

This is established by Hezenci and Budak in paper [18, Corollary 1].

Corollary 3.7. *Let us consider $h(t)=t^s$ in Corollary 3.5. Then, we obtain the following Euler-Maclaurin-type inequality for s-convex functions*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{b-a}{4}\left(\varphi_1(s)+\varphi_2(s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Here,

$$\varphi_1(s)=\int_0^{\frac{1}{3}} t\left[\left(\frac{t}{2}\right)^s+\left(\frac{2-t}{2}\right)^s\right]dt=\frac{1}{9\cdot 6^s(s+2)}\left[1+\frac{6^{s+2}-5^{s+1}(s+7)}{(s+1)}\right],$$

and

$$\varphi_2(s)=\int_{\frac{1}{3}}^1 \left|t-\frac{3}{4}\right|\left[\left(\frac{t}{2}\right)^s+\left(\frac{2-t}{2}\right)^s\right]dt.$$

Corollary 3.8. *If we assign $h(t)=1$ in Corollary 3.5, then we get the following Euler-Maclaurin-type inequality for $P-$functions*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{25(b-a)}{288}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Theorem 3.9. *Let us consider the assumptions in Lemma 2.1 and the function $|f'|^q$, $q>1$ is $h-$convex on $[a,b]$. Then, the following Euler-Maclaurin-type inequality holds:*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$
$$\leq \frac{b-a}{4}\left\{\left(\frac{1}{(\alpha p+1)}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\int_0^{\frac{1}{3}}\left(h\left(\frac{t}{2}\right)\left|f'(b)\right|^q+h\left(\frac{2-t}{2}\right)\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^{\frac{1}{3}}\left(h\left(\frac{t}{2}\right)\left|f'(a)\right|^q+h\left(\frac{2-t}{2}\right)\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{3}}^1\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{3}}^1\left(h\left(\frac{t}{2}\right)\left|f'(b)\right|^q+h\left(\frac{2-t}{2}\right)\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\int_{\frac{1}{3}}^1\left(h\left(\frac{t}{2}\right)\left|f'(a)\right|^q+h\left(\frac{2-t}{2}\right)\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\right\}.$$

Here, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If we apply Hölder's inequality to (3.1), then we get

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \\
&\leq \frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad +\left(\int_0^{\frac{1}{3}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad +\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad \left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Taking advantage of the $h-$convexity $|f'|^q$, we can easily get

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \\
&\leq \frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(h\left(\frac{t}{2}\right)|f'(b)|^q+h\left(\frac{2-t}{2}\right)|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\quad +\left(\int_0^{\frac{1}{3}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(h\left(\frac{t}{2}\right)|f'(a)|^q+h\left(\frac{2-t}{2}\right)|f'(b)|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad +\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left(h\left(\frac{t}{2}\right)|f'(b)|^q+h\left(\frac{2-t}{2}\right)|f'(a)|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad \left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left(h\left(\frac{t}{2}\right)|f'(a)|^q+h\left(\frac{2-t}{2}\right)|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right\} \\
&= \frac{b-a}{4}\left\{\left(\frac{1}{(\alpha p+1)}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\right. \\
&\quad \times\left[\left(\int_0^{\frac{1}{3}}\left(h\left(\frac{t}{2}\right)|f'(b)|^q+h\left(\frac{2-t}{2}\right)|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^{\frac{1}{3}}\left(h\left(\frac{t}{2}\right)|f'(a)|^q+h\left(\frac{2-t}{2}\right)|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{3}}^{1}\left(h\left(\frac{t}{2}\right)\left|f'(b)\right|^{q}+h\left(\frac{2-t}{2}\right)\left|f'(a)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\int_{\frac{1}{3}}^{1}\left(h\left(\frac{t}{2}\right)\left|f'(a)\right|^{q}+h\left(\frac{2-t}{2}\right)\left|f'(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right]\right\}.$$

This ends the proof of Theorem 3.9. □

Remark 3.10. *If we choose $h(t)=t$ in Theorem 3.9, then the Theorem 3.9 reduces to the result in paper [23, Theorem 5].*

Corollary 3.11. *Let us consider $h(t)=t^{s}$ in Theorem 3.9. Then, the following Euler-Maclaurin-type inequality for s-convex functions by using the Riemann-Liouville fractional integrals*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)\right]\right|$$
$$\leq\frac{b-a}{4}\left\{\left(\frac{1}{(\alpha p+1)}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\frac{|f'(b)|^{q}+\left(6^{s+1}-5^{s+1}\right)|f'(a)|^{q}}{3\cdot 6^{s}(s+1)}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+\left(6^{s+1}-5^{s+1}\right)|f'(b)|^{q}}{3\cdot 6^{s}(s+1)}\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\frac{\left(3^{s+1}-1\right)|f'(b)|^{q}+\left(5^{s+1}-3^{s+1}\right)|f'(a)|^{q}}{3\cdot 6^{s}(s+1)}\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{\left(3^{s+1}-1\right)|f'(a)|^{q}+\left(5^{s+1}-3^{s+1}\right)|f'(b)|^{q}}{3\cdot 6^{s}(s+1)}\right)^{\frac{1}{q}}\right]\right\}.$$

Corollary 3.12. *If we assign $h(t)=1$ in Theorem 3.9, then we get the following Euler-Maclaurin-type inequality for $P-$functions by using the Riemann-Liouville fractional integrals*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)\right]\right|$$
$$\leq\frac{b-a}{2}\left\{\left(\frac{1}{(\alpha p+1)}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^{q}+|f'(a)|^{q}}{3}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^{p}dt\right)^{\frac{1}{p}}\left(\frac{2|f'(b)|^{q}+2|f'(a)|^{q}}{3}\right)^{\frac{1}{q}}\right\}.$$

Corollary 3.13. *If we assign $\alpha=1$ in Theorem 3.9, then we can obtain Euler-Maclaurin-type inequality*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|$$
$$\leq\frac{b-a}{4}\left\{\left(\frac{1}{(p+1)}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\int_0^{\frac{1}{3}}\left(h\left(\frac{t}{2}\right)\left|f'(b)\right|^q+h\left(\frac{2-t}{2}\right)\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{\frac{1}{3}}\left(h\left(\frac{t}{2}\right)\left|f'(a)\right|^q+h\left(\frac{2-t}{2}\right)\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{3}}^{1}\left(h\left(\frac{t}{2}\right)\left|f'(b)\right|^q+h\left(\frac{2-t}{2}\right)\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{\frac{1}{3}}^{1}\left(h\left(\frac{t}{2}\right)\left|f'(a)\right|^q+h\left(\frac{2-t}{2}\right)\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\right\}.$$

Remark 3.14. *If we choose $h(t)=t$ in Corollary 3.13, then the following inequality holds:*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq\frac{b-a}{4}\left\{\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\frac{4\left|f'(b)\right|^q+2\left|f'(a)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{4\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{9}\right)^{\frac{1}{q}}\right]$$

$$\left.+\left(\frac{1}{p+1}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{11\left|f'(b)\right|^q+\left|f'(a)\right|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{11\left|f'(a)\right|^q+\left|f'(b)\right|^q}{36}\right)^{\frac{1}{q}}\right]\right\},$$

which is established by Gumus et al. in paper [23, Corollary 1].

Corollary 3.15. *Let us consider $h(t)=t^s$ in Corollary 3.13. Then, the following inequality*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq\frac{b-a}{4}\left\{\left(\frac{1}{(p+1)}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\frac{\left|f'(b)\right|^q+\left(6^{s+1}-5^{s+1}\right)\left|f'(a)\right|^q}{3\cdot 6^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+\left(6^{s+1}-5^{s+1}\right)\left|f'(b)\right|^q}{3\cdot 6^s(s+1)}\right)^{\frac{1}{q}}\right]$$

$$+\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\left[\left(\frac{\left(3^{s+1}-1\right)\left|f'(b)\right|^q+\left(5^{s+1}-3^{s+1}\right)\left|f'(a)\right|^q}{3\cdot 6^s(s+1)}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{\left(3^{s+1}-1\right)\left|f'(a)\right|^q+\left(5^{s+1}-3^{s+1}\right)\left|f'(b)\right|^q}{3\cdot 6^s(s+1)}\right)^{\frac{1}{q}}\right]\right\}.$$

Corollary 3.16. *If we assign $h(t)=1$ in Corollary 3.13, then we get the following inequality*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{b-a}{2}\left\{\left(\frac{1}{(p+1)}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(a)|^q}{3}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\left(\frac{2|f'(b)|^q+2|f'(a)|^q}{3}\right)^{\frac{1}{q}}\right\}.$$

Theorem 3.17. *Assume that the assumptions of Lemma 2.1 satisfy and the function $|f'|^q$, $q\geq 1$ is $h-$convex on $[a,b]$. Then, we obtain the following Euler-Maclaurin-type inequality*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$
$$\leq \frac{b-a}{4}\left\{(\varphi_1(\alpha))^{1-\frac{1}{q}}\left[\left[\varphi_3(\alpha;h)\left|f'(b)\right|^q+\varphi_4(\alpha;h)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.\right.$$
$$\left.+\left[\varphi_3(\alpha;h)\left|f'(a)\right|^q+\varphi_4(\alpha;h)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right]$$
$$+(\varphi_2(\alpha))^{1-\frac{1}{q}}\left[\left[\varphi_5(\alpha;h)\left|f'(b)\right|^q+\varphi_6(\alpha;h)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.\left.+\left[\varphi_5(\alpha;h)\left|f'(a)\right|^q+\varphi_6(\alpha;h)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right]\right\}.$$

Here,

$$\varphi_1(\alpha)=\int_0^{\frac{1}{3}}t^{\alpha}dt=\frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1},$$

$$\varphi_2(\alpha)=\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{\alpha+1}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)-\frac{1}{2}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\[2mm] \frac{2\alpha}{\alpha+1}\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}+\frac{1}{\alpha+1}-1, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha.\end{cases}$$

and

$$\begin{cases}\varphi_3(\alpha;h)=\int_0^{\frac{1}{3}}t^{\alpha}h\left(\frac{t}{2}\right)dt, & \varphi_5(\alpha;h)=\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|h\left(\frac{t}{2}\right)dt,\\ \varphi_4(\alpha;h)=\int_0^{\frac{1}{3}}t^{\alpha}h\left(\frac{2-t}{2}\right)dt, & \varphi_6(\alpha;h)=\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|h\left(\frac{2-t}{2}\right)dt.\end{cases}$$

Proof. When we apply (3.1) to the power-mean inequality, we have

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$

$$\leq \frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}|t^{\alpha}|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}|t^{\alpha}|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{3}}|t^{\alpha}|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}|t^{\alpha}|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

By using the $h-$convexity of $|f'|^q$, it follows

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$
$$\leq \frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}t^{\alpha}\left[h\left(\frac{t}{2}\right)|f'(b)|^q+h\left(\frac{2-t}{2}\right)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{3}}t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}t^{\alpha}\left[h\left(\frac{t}{2}\right)|f'(a)|^q+h\left(\frac{2-t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[h\left(\frac{t}{2}\right)|f'(b)|^q+h\left(\frac{2-t}{2}\right)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[h\left(\frac{t}{2}\right)|f'(a)|^q+h\left(\frac{2-t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}.$$

This finishes the proof of Theorem 3.17. □

4. Summary and Concluding Remarks

In this paper, several Euler-Maclaurin-type inequalities are investigated for differentiable $h-$convex functions by using the Riemann-Liouville fractional integrals. Moreover, by using Hölder inequality, we give some Euler-Maclaurin-type inequalities for Riemann-Liouville fractional integrals. Furthermore, by using the special choices of the obtained results, we obtain the some Euler-Maclaurin-type inequalities.

In future papers, the ideas and strategies behind our results on Euler-Maclaurin-type inequalities using Riemann-Liouville fractional integrals may pave the way for new avenues of research in this field. Improvements or generalizations of our results can be explored by considering different classes of convex functions or other types of fractional integral operators. Additionally, one could derive Euler-Maclaurin-type inequalities for various function classes with the aid of quantum calculus.

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