ODÜ SOBİAD
ORDU ÜNİVERSİTESİ SOSYAL BİLİMLER ENSTİTÜSÜ

2025, VOL 15, ISRIS 2025 SPECIAL ISSUE, 826-845
DOI: 10.48146/odusobiad.1729711
Received Date: 29.06.2025
Acceptance Date: 10.07.2025

Review Article *Open Access*

Systematic review of studies on mathematical thinking[1]

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ABSTRACT

Mathematical thinking plays a central role in mathematics education. Equipping students with mathematical thinking skills is essential to move beyond rote memorization and foster critical inquiry, interpretation, and reflective thinking, thereby recognizing mathematics as a product of thought. In this context, this study aimed to map the articles on mathematical thinking published between 1990-2021 in the Web of Science database using the bibliometric analysis method and to investigate the research trends. As a result of the search made with the phrase "mathematical thinking" in the Web of Science (WoS) database, 668 articles published in English between January 1990 and December 2021 in the field of mathematics education were included in this research. This research revealed a consistent increase in the number of studies on mathematical thinking over the years examined. The most frequently used keywords in the studies examined were *mathematics, noticing, mathematics education, student mathematical thinking, problem solving, advanced mathematical thinking, teacher education, teacher knowledge, reasoning, evaluation, teacher candidate education, technology, mathematics teaching, fractions, task design, cognitive demand*, and *reflection*. Country-based analysis revealed that the United States, Australia, and Turkey produced the highest number of articles. The bibliometric analysis of the articles on mathematical thinking revealed a four-cluster structure formed according to the relationships of the keywords reflecting the articles. These sets are: recognition, development, advanced mathematical thinking and technology. It is thought that this study on the dynamics of mathematical thinking will help researchers, teachers, and students have an idea about the areas that need research and make plans accordingly.

KEYWORDS

Bibliometric analysis, mathematical thinking, mathematics education.

Matematiksel düşünme üzerine yapılan çalışmaların sistematik olarak incelenmesi

ÖZET

Matematiksel düşünme, matematik eğitiminde önemli bir yere sahiptir. Öğrencilerin sadece formül ezberleyen bireyler olmayıp, sorgulayan, yorumlayan, düşünen bireyler olarak yetiştirilebilmesi, matematiğin bir düşünce ürünü olduğunun anlaşılabilmesi için öğrencilere matematiksel düşünme becerisi kazandırmak önem taşımaktadır. Bu kapsamda çalışmanın amacı Web of Science (WoS) veri tabanında 1990-2021 yılları arasında matematiksel düşünme üzerine yayınlanan makalelerin bibliyometrik analiz yöntemi kullanılarak haritalandırılması ve araştırma eğilimlerinin izlenmesi olarak belirlenmiştir. Web of Science veri tabanında "mathematical thinking" ifadesi ile yapılan tarama sonucunda; matematik eğitimi alanında Ocak 1990-Aralık 2021 tarihleri arasında İngilizce dilinde yayınlanan 668 adet makale araştırmaya dâhil edilmiştir. Araştırmanın sonucunda belirlenen yıllar arasında matematiksel düşünme üzerine yapılan çalışma sayısının düzenli olarak artış gösterdiği belirlenmiştir. Ayrıca incelenen çalışmalarda en sık kullanılan anahtar kelimelerin matematik, fark etme, matematik eğitimi, öğrenci matematiksel düşünmesi, problem çözme, ileri matematiksel düşünme, öğretmen eğitimi, öğretmen bilgisi, akıl yürütme, değerlendirme, aday öğretmen eğitimi, teknoloji, matematik öğretimi, kesirler, görev tasarımı, bilişsel talep ve yansıtma olduğu görülmüştür. Ülkelere göre inceleme yapıldığında ABD, Avustralya ve Türkiye'nin en fazla makale üreten ülkeler olduğu fark edilmiştir. Matematiksel düşünme ile ilgili makalelerin bibliyometrik

[1] This article is derived from the first author's master's thesis and was presented as an oral presentation at ISRIS-II.

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analizi makaleleri yansıtan anahtar kelimelerin birbirleriyle ilişkilerine göre oluşan dört kümeli bir yapıyı ortaya çıkarmıştır. Bu kümeler; fark etme, geliştirme, ileri matematiksel düşünme ve teknoloji olarak isimlendirilmiştir. Matematiksel düşünmenin dinamiklerine yönelik olarak yapılan bu çalışmanın araştırmacıların, öğretmenlerin, öğrencilerin araştırmaya ihtiyaç duyulan alanlar hakkında fikir sahibi olmalarına ve bu doğrultuda planlamalar yapmalarına yardımcı olacağı düşünülmektedir.

ANAHTAR KELİMELER

Bibliyometrik analiz, matematiksel düşünme, matematik eğitimi.

Introduction

Individuals utilize diverse cognitive structures to address everyday problems. Mathematics is a domain in which such cognitive processes and logical relationships are utilized effectively (Duran, 2005). Mathematics, as a discipline, fosters essential cognitive habits—reasoning, recognizing patterns, making predictions, and problem solving—that are transferable to real-life contexts (Umay, 2003).

The National Council of Teachers of Mathematics (2000) emphasized that in an increasingly dynamic and evolving world, the need for individuals to interpret and apply mathematics in everyday life is growing steadily, and that mathematical thinking and problem-solving skills are becoming more essential across various domains. The mathematics curriculum aims to support students in developing mathematical thinking skills, interpreting mathematical concepts, fostering positive attitudes toward mathematics, solving real-life problems, and making interdisciplinary connections. Furthermore, the program highlights the importance of understanding mathematical concepts and the relationships between them, with the overarching goal of cultivating individuals who possess strong mathematical reasoning and are effective problem-solvers (Ministry of Education [MoNE], 2018).

Mathematics is an intellectual activity that has developed out of humanity's curiosity to "know and understand the truth" (Altun, 2018). Despite the many definitions of mathematics put forward by researchers over time, the fundamental question of what mathematics truly is remains unanswered. For some, it is a type of intellectual game governed by specific rules; for others, it is a computational technique applied in science and daily life. Some define it as a branch of science that deals with abstract entities, such as numbers. However, from the perspective of mathematicians, mathematics is regarded as a unique mode of thinking that leads us to certain knowledge and truth (Yıldırım, 1988).

Baykul (2009) proposed four perspectives on the nature of mathematics, shaped by individuals' purposes for engaging with mathematics, their experiences within the field, and their attitudes and interests toward it:

1. Mathematics is the measurement, counting, calculation, and drawing used to solve problems encountered in daily life.
2. Mathematics is a language whose alphabet consists of symbols.
3. Mathematics is a logical system that contributes to human rational thinking.
4. Mathematics is an auxiliary tool we rely on to understand and contribute to the development of the world we live in.

Despite extensive discussions and attempts to define it, mathematics remains a subject often approached with prejudice in schools due to its abstract nature and is commonly perceived as difficult and tedious (Şengül et al., 2014). Dreyfus (1991) observed that some students merely memorize formulas in mathematics classes, failing to engage with the underlying principles that define mathematical thinking. Cultivating students as critical thinkers, beyond mere memorization, is essential for helping them recognize mathematics as a product of thought and to develop their mathematical thinking skills. Mathematics is a discipline that contributes to the formation of mathematical thinking in individuals (Samo & Kartasasmita, 2017). Mathematical thinking is not limited to knowing how to reach a solution for a specific type of problem; it is a mode of thinking that also aids in generating solutions to new and complex problems encountered in daily life (Fisher et al., 2012).

Like mathematics itself, there is no consensus among researchers on a single definition of mathematical thinking. According to Alkan and Bukova-Güzel (2005), mathematical thinking is a type of thought that proves beneficial due to its usefulness and productivity in addressing needs and solving problems. Liu Po-Hung (2003) defined mathematical thinking as a set of complex processes including prediction, induction, deduction, description, generalization, exemplification, formal and informal reasoning, verification, and similar cognitive activities. In this context, it can be stated that mathematical thinking begins with an individual's effort to perceive objects in their environment and to make meaningful connections between them (Tall, 1995). Considering these definitions, it is understood that to concretize mathematical thinking, which is understood to be abstract, researchers have tried to examine its characteristics, components and the issues that distinguish mathematical thinking from other thinking (Arslan & Yıldız, 2010). In this context, Alkan and Bukova-Güzel (2005) point out that the most prominent feature that distinguishes mathematical thinking from other forms of thinking is the ability to reach a new concept or knowledge through abstraction, estimation, generalization, reasoning, hypothesizing, testing hypotheses, description and proving by using previously learned mathematical concepts and knowledge (Figure 1).

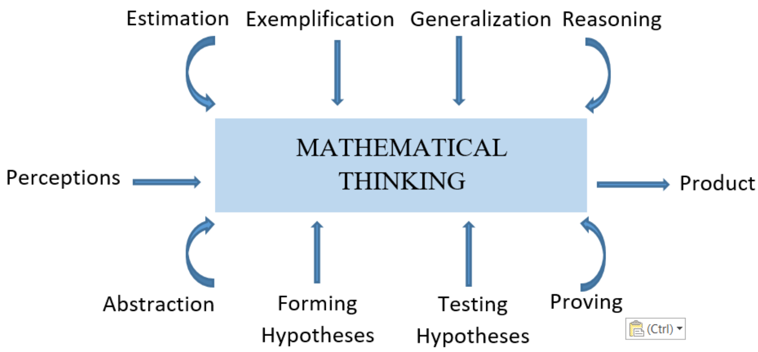

Figure 1 The operational structure of mathematical thinking (Alkan & Bukova-Güzel, 2005)

Examining studies on mathematical thinking, which enrich cognitive development and are applied both in daily life and mathematics education, is essential to identify key areas of focus, associated concepts, and their interrelationships. Accordingly, this study aims to map the publications on mathematical thinking between 1990 and 2021 in the Web of Science (WoS) database using bibliometric analysis and to track research trends. In line with this purpose, the following research questions were addressed:

- How are studies on mathematical thinking distributed across publication years and countries?
- Which keywords are most frequently used in studies on mathematical thinking, and how do they evolve over time?
- What is the country-wise distribution of citation counts in studies on mathematical thinking?
- What structure emerges from the co-word analysis conducted on studies on mathematical thinking?

Method

This study employed bibliometric analysis to map articles on mathematical thinking published in the WoS database between 1990 and 2021. VOSviewer software (van Eck & Waltman, 2020) was used to analyze the articles, offering statistical insights into items, their interrelations, and cluster formations. The results obtained indicated relevant terms, their occurrences, and their clustering patterns. This study aims to associate theoretical knowledge with observable elements and offer empirical evidence for representing structures associated with mathematical thinking.

Research design

The research design was based on the framework developed by Gökçe and Güner (2021), Gökçe and Güner (2022), and Güner and Gökçe (2021). Accordingly, as shown in Figure 2, this study consisted of four stages: exploration, visualization, identification, and verification.

Figure 2 Research design

The first stage, the exploration phase, involved a search in the WoS database using the term *mathematical thinking*. The visualization phase encompassed the results of clustering terms related to mathematical thinking using the VOSviewer software. The identification phase consisted of naming the networks that emerged because of clustering. In the verification phase, evidence from the publications included in this study was provided to support the emerging cognitive structure. In this study, to determine the overall trends of articles published on mathematical thinking between 1990 and 2021, the results obtained through bibliometric analysis were interpreted in conjunction with an in-depth examination of the selected publications.

As a result of a search conducted in the WoS database using the term *"mathematical thinking"* within the topic field (title, abstract, and keywords), 1,423 studies related to mathematical thinking were identified. When the criteria outlined in Table 1 were applied, the number of articles included in this study was reduced to 668, and the analysis continued with these selected articles.

It was found that the earliest accessible article on mathematical thinking in the WoS database was published in 1980. However, since the keywords of the five articles published between 1980 and 1990 could not be accessed, the starting year of the study was determined as 1990.

Table 1 Exploration details

Criteria	
Database	Web of Science
Search term	"mathematical thinking"
Publication year	January 1990 - December 2021
Document type	Article
Language	English

The data analysis began with descriptive analyses conducted through the WoS system. Subsequently, bibliometric analyses were performed using the VOSviewer software. Data from the articles obtained from the WoS database were transferred to the VOSviewer software, where bibliometric analyses were conducted and mappings were generated. The analyses carried out included co-word and citation analyses. Prior to the co-word analysis, necessary adjustments and consolidations were applied to terms with the same or similar meanings. For instance, in cases where the keywords *student mathematical thinking*, *student thinking*, and *students' mathematical thinking* appeared, the dataset was standardized by assigning the label *student mathematical thinking*. Keywords were presented in italics throughout the article. During the visualization process in VOSviewer, network visualization, overlay visualization, and density visualization techniques were utilized. This section constitutes the visualization phase of the study.

During the identification phase, the networks obtained in the previous visualization stage were named by considering their general characteristics and trends. In this process, studies containing the high-frequency keywords emerging in each cluster were first identified, and their purpose statements and abstracts were examined. The analysis revealed that studies sharing keywords within the same cluster also exhibited a common research trend. Based on the focal areas of these studies, the clusters were accordingly named. In the verification phase, evidence supporting these labels was provided through the examined articles, and both were interpreted together.

Validity and reliability

Validity refers to the extent to which the intended measurement accurately reflects the concept without being confounded by other variables, whereas reliability pertains to the degree to which the measurement is free from errors (Büyüköztürk et al., 2017). It can be stated that validity in a study is ensured through the researchers' effort to observe the work as objectively as possible. In addition to the data obtained during the detailed examination of the subject matter or data, certain methods, such as participant confirmation, peer review, or expert evaluation may be necessary during the process of creating a conceptual map (Yıldırım & Şimşek, 2008). Miles and Huberman (1994) emphasize that internal validity, which concerns the accuracy of research findings, requires that the findings be consistent and meaningful within themselves. They highlight the need for clear procedures to confirm findings, identify transparency gaps, and ensure that inferences and generalizations align with the collected data. In this context, to ensure the internal validity of the study, detailed information was provided regarding the criteria for selecting the articles examined, the total number of articles included, and the data collection method. Evidence was presented during the validation phase to ensure the consistency of the labeling process.

External validity, which is related to the generalizability of research findings, requires that the results can be generalized to similar settings and situations (Guba & Lincoln, 1982; Şencan, 2005). Miles and Huberman (1994) state that to ensure external validity, the sample should be described in detail so that comparisons with other samples can be made, the sample should be diversified enough to allow generalization, necessary explanations should be provided to enable testing of findings in other studies, and attention should be paid to ensuring that findings can be easily tested in similar contexts. In this regard, the data, design, and the research process have been clearly described to allow other researchers to review the study.

On the other hand, reliability refers to the replicability of research findings, which ensures the external reliability of the study. For internal reliability, the analysis of the data obtained can be confirmed by another researcher. Data analysis conducted based on a pre-established and clearly defined conceptual framework also enhances internal reliability (Yıldırım & Şimşek, 2008). To ensure reliability in this study, expert opinions were sought after the networks formed during the labeling phase were named. Additionally, the data analysis process was transparently documented.

Results

The results obtained from this research are presented in accordance with the phases outlined in the research design.

Exploration

Descriptive analyses of the articles retrieved through a search using the term *"mathematical thinking"* in the WoS database are presented in this section. First, the change in the number of articles accessed over the years was examined. Figure 3 displays the number of articles published between 1992 and 2021, presented in five-year intervals. Since no accessible publications prior to 1992 were identified in the WoS database, the periods were defined starting from 1992.

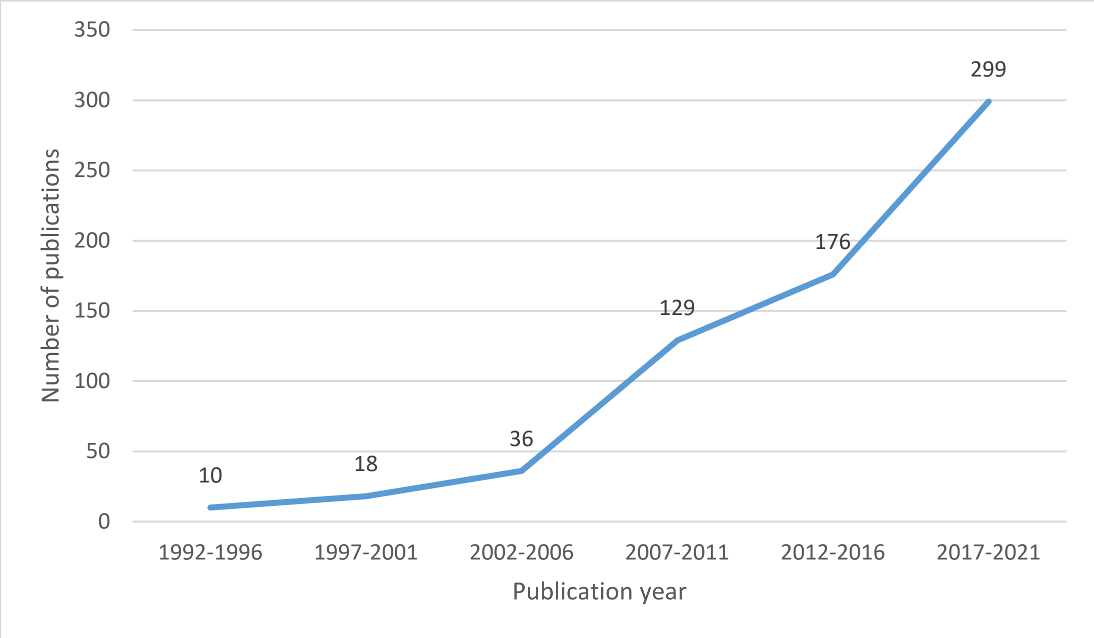

Figure 3 The distribution of articles published on mathematical thinking by year

As shown in Figure 3, the number of studies on mathematical thinking accessible through the WoS database have increased in each five-year period than the previous one. In the second and third periods, the number of articles approximately doubled compared to the preceding period; in the fourth period, it increased by about 3.5 times, and in the last two periods, it rose by approximately 1.5 times compared to their respective previous periods. Although the rate of increase varies, an overall upward trend in the number of studies throughout the examined timeframe is evident.

Secondly, the highly occurring keywords in articles published on mathematical thinking were identified and are presented in Table 2.

Table 2 Highly occurring keywords

Keyword	Frequency	Keyword	Frequency
Mathematics	48	reasoning	14
Noticing	41	assessment	13
mathematics education	39	preservice teacher education	13
student mathematical thinking	37	technology	12
problem solving	30	mathematics teaching	11
advanced mathematical thinking	26	fractions	10
teacher education	18	task design	9
teacher learning	16	cognitive demand	8
teacher knowledge	15	reflection	8

When examining the highly occurring keywords, it was found that the keyword appearing in the highest number of articles was *mathematical thinking*. However, considering its limited discriminative effect, this keyword was excluded from the table.

As seen in Table 2, the most frequently used keywords in articles written on mathematical thinking include *mathematics*, *noticing*, *mathematics education*, and *student mathematical thinking*. Additionally, keywords, such as *problem solving*, *advanced mathematical thinking*, *teacher education*, *teacher learning*, *teacher knowledge*, *reasoning*, *assessment*, *preservice teacher education*, *technology*, *mathematics teaching*, *fractions*, *task design*, *cognitive demand*, and *reflection* are also among the commonly used keywords.

Although the most frequently used keywords in articles on mathematical thinking have been mentioned above, the emergence dates of these keywords over time are also believed to provide more detailed information about the trends in studies on mathematical thinking. Therefore, the top 10 keywords that appeared in each five-year period are presented through a timeline.

Figure 4 illustrates the change over time in the keywords used in articles on mathematical thinking. It is observed that some keywords, such as *mathematics education*, *problem solving*, *assessment*, and *teacher education*, appeared in all periods, even if they were not always among the top ten.

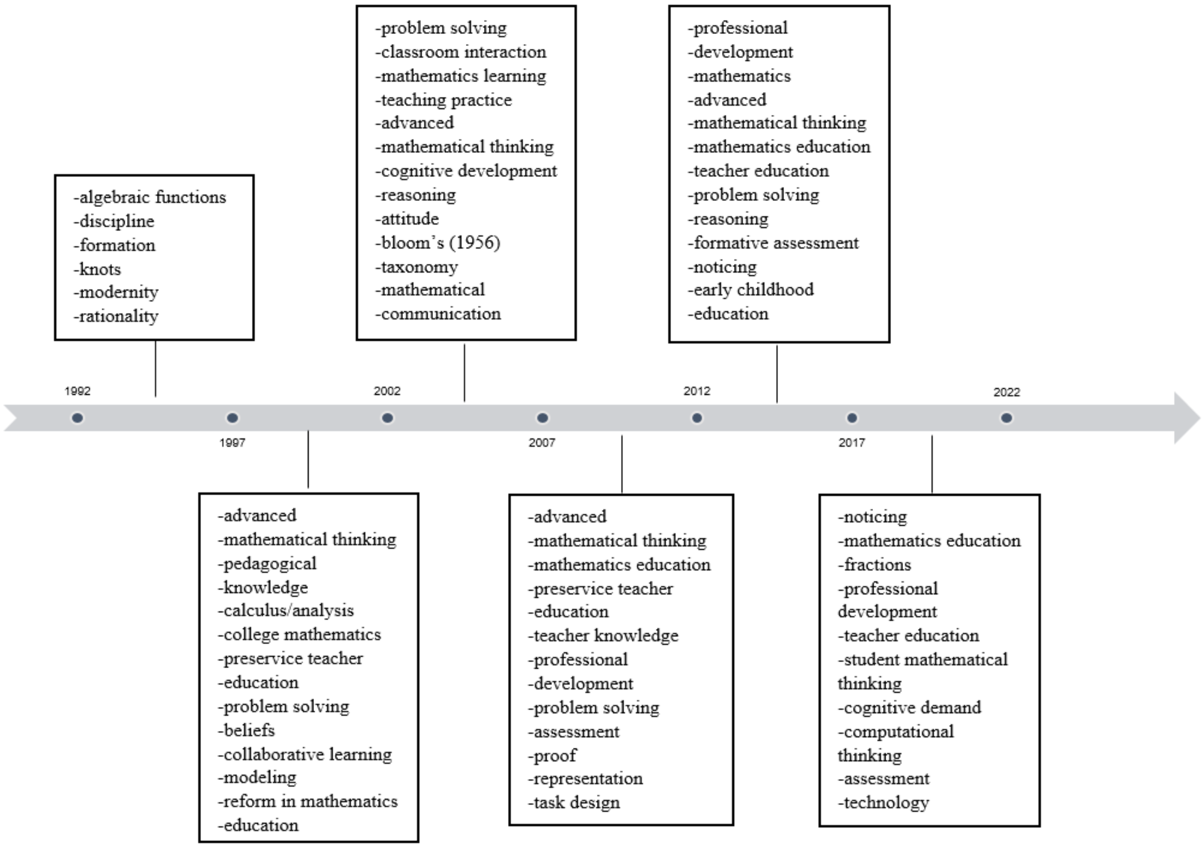

Figure 4 Timeline of frequently used keywords

For the period between 1992 and 1997, keyword data could be accessed from only one study. During the 1997-2001 period, the top-ranking keywords included *advanced mathematical thinking*, *college mathematics*, and *preservice teacher education*. This suggests that the studies in this timeframe were more focused on older student groups. In addition, *problem solving*, *beliefs*, *collaborative learning*, *modeling*, and *reform in mathematics education* are among the notable keywords. During the 2002-2006 period, prominent keywords included *classroom interaction*, *teaching practice*, *cognitive development*, *reasoning*, *attitude*, *mathematical communication*, and *Bloom's (1956) taxonomy*. The data suggest that, during this period, studies focusing on classroom interaction and the use of mathematical language came to the forefront. During the 2007-2011 period, unlike previous periods, keywords, such as *teacher knowledge*, *professional development*, *assessment*, *proof*, *representation*, and *task design* appeared among the top 10 most frequently used keywords. In the 2012-2016 period, keywords like *formative assessment*, *noticing*, and *early childhood education* were observed among the most frequently used for the first time. In the five-year period between 2017 and 2021, the keyword *noticing*, which had already emerged in the previous period, became the most frequently used keyword, drawing particular attention. Additionally, *cognitive demand*, *computational thinking*, and *technology* appeared among the top 10 keywords for the first time.

Subsequently, the top 10 countries were identified based on the number of articles on mathematical thinking retrieved from the WoS database. The findings are presented in Table 3.

Table 3 Top 10 Countries and citation counts in articles on mathematical thinking

Rank	Country	Frequency	Number of Citations	Average Citations per Publication
1	USA	315	8623	27.37
2	Australia	42	375	8.92
3	Turkey	39	134	3.43
4	United Kingdom	34	430	12.64
5	Germany	26	175	6.73
6	Canada	23	165	7.17
7	Israel	23	303	13.17
8	New Zealand	22	228	10.36
9	Netherlands	18	204	11.33
10	China	18	90	5

In the ranking based on the number of publications on mathematical thinking, the United States ranks first with 315 articles. Australia and Turkey follow in second and third place, respectively. China ranks tenth with 18 publications. However, when examining the average number of citations per publication, the ranking shifts. While the United States maintains its lead, Israel and the United Kingdom follow the second and third places, respectively. Although Turkey ranks third in total citation count, it ranks tenth in average citations per publication.

Visualization

In the visualization phase, the aim is to reveal the network of relationships among the keywords used in articles on mathematical thinking through the VOSviewer software. For this purpose, an overlay visualization was first generated for the studies published between 2002 and 2021. Studies published before 2002 were excluded from the visualization due to their low number.

Figure 5 presents a visualization of the keywords used in studies on mathematical thinking between 2002 and 2021, with colors indicating the year of emergence. Keywords appearing around 2002 are represented in dark blue, transitioning to green and then yellow as they approach 2021. Upon examining the map, it is observed that keywords closest to the dark blue color include *pedagogical knowledge*, *constructivism*, *proof*, *college mathematics*, *preservice teacher education*, *advanced mathematical thinking*, and *discourse*.

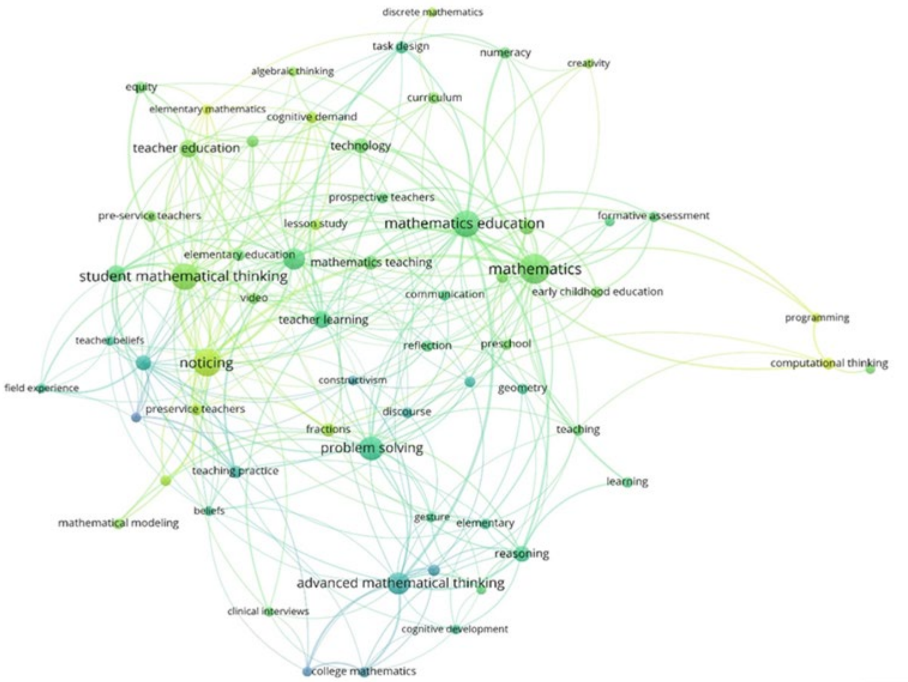

Figure 5 Overlay visualization of keywords related to mathematical thinking

These keywords appear to have been used more intensively around 2005. Keywords, such as *creativity*, *technology*, *problem solving*, *communication*, *professional development*, *early childhood education*, *video*, *abstraction*, and *reasoning,* are represented in green. It can be said that these keywords were more frequently encountered around the years 2010-2015. Keywords closest to yellow include *computational thinking*, *programming*, *noticing*, *lesson study*, *cognitive demand*, *fractions*, *pre-service teachers*, and *elementary mathematics*. Since these keywords have usage both before and after 2020, the terms represented in yellow are considered to reflect more recent research topics. Figure 6 presents a density visualization that examines heat maps based on the frequency intensity of the items.

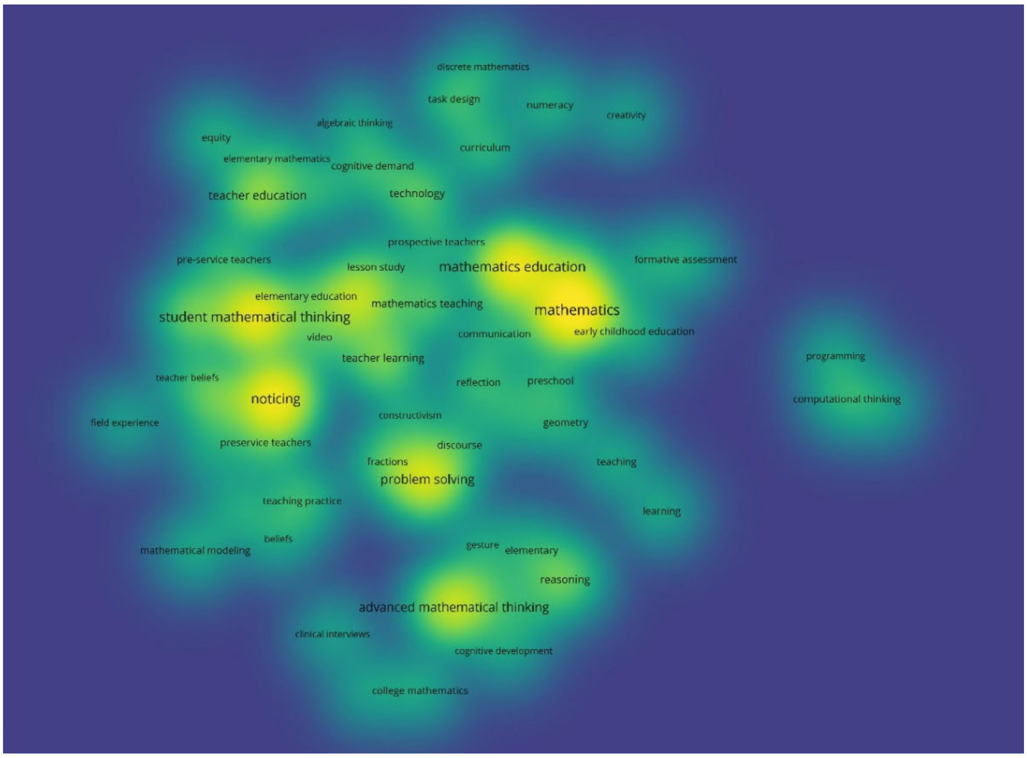

Figure 6 Density visualization of keywords related to mathematical thinking

In the density visualization, the most frequently used keywords are shown in colors closer to yellow. Accordingly, upon examining the map, the most used keywords are observed to be *mathematics*, *mathematics education*, *problem solving*, *advanced mathematical thinking*, *noticing*, *student mathematical thinking*, and *professional development*. Thirdly, the network visualization was used to observe the connections and relationship networks among the keywords (Figure 7).

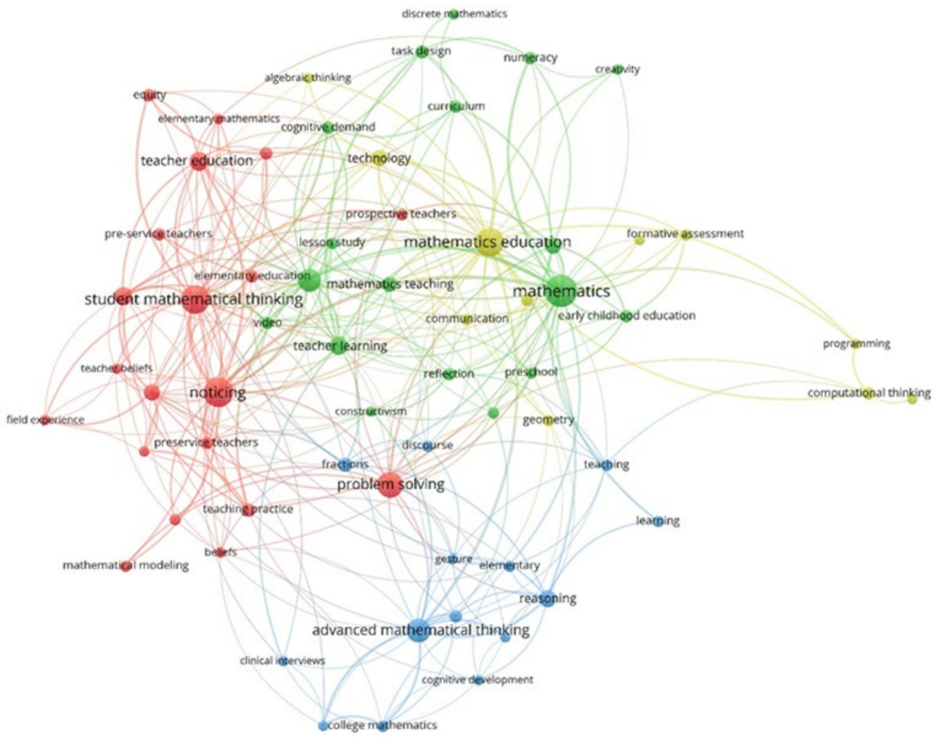

Figure 7 Network visualization of keywords related to mathematical thinking

In the network visualization map, each circle represents a keyword. Keywords that are used more frequently are depicted with larger circles, while less frequently used keywords are represented by smaller circles. The lines connecting the circles indicate the strength of the relationship between two keywords. A thicker line represents a stronger connection between two keywords, whereas a thinner line signifies a weaker connection. Additionally, keywords more closely related to each other are positioned closer together. Examining Figure 7 reveals that the keywords are grouped into four clusters. These clusters, shown in yellow, red, blue, and green, are formed based on the frequency of co-occurrence of the keywords. Like the relationships between the circles, clusters that are more closely related to each other are located nearer to one another.

Identification

At this stage, the clusters identified in the previous visualization phase were named, and descriptive information for each cluster is presented in Table 4.

Table 4 Identification of clusters

Color	Cluster Name	
Red	Noticing	noticing (41), student mathematical thinking (37), problem solving (30), teacher education (18), teacher knowledge (15), preservice teacher education (13), teaching practice (9), preservice teachers (8), equity (7), mathematical modeling (6)
green	Development	professional development (25), teacher learning (16), assessment (13), mathematics teaching (11), task design (9), video (8), curriculum (8), cognitive demand (8), reflection (8), numeracy (7)
blue	Advanced mathematical thinking	advanced mathematical thinking (26), reasoning (14), fractions (10), proof (7), teaching (6), college mathematics (6), discourse (6), calculus (5), clinical interviews (5), cognitive development (5)
yellow	Technology	mathematics education (39), technology (12), algebra (7), computational thinking (7), geometry (6), formative assessment (6), communication (6), programming (5), classroom discourse (5), abstraction (5), algebraic thinking (5)

As shown in Table 4, the red cluster has been named “Noticing” because it includes studies related to the recognition and identification of students’ mathematical thinking. The frequently used keywords in this cluster are *noticing, student mathematical thinking, problem solving, teacher education, teacher knowledge,* and *preservice teacher education*. The green cluster has been named “Development” as it encompasses studies to enhance students’ mathematical thinking. In this cluster, frequently used keywords include *professional development, assessment, mathematics teaching,* and *curriculum*. The blue cluster has been named “Advanced Mathematical Thinking” because it contains studies related to advanced mathematical thinking. Frequently used keywords in the blue cluster are *advanced mathematical thinking, reasoning, fractions, discourse, clinical interviews,* and *cognitive development*. The yellow cluster has been named “Technology” as it includes studies aimed at designing instructional environments that enhance mathematical thinking through the use of technological tools. Frequently used keywords in this cluster include *mathematics education, technology, computational thinking, programming,* and *communication*.

Verification

The verification stage refers to the process of seeking evidence, including the examination of articles, to determine the consistency of the naming. Below is the review of the articles related to each cluster:

Cluster 1: Noticing

Upon examining the first cluster, represented by red, the prominent keywords identified include: *noticing*, *student mathematical thinking*, *problem solving*, *teacher education*, *teacher knowledge*, *preservice teacher education*, and *teaching practice*. Since these keywords are interrelated in the context of teachers and preservice teachers working with students at various educational levels to notice and elicit students' mathematical thinking, the largest cluster has been labeled “Noticing.”

For instance, studies that examine the extent to which preservice teachers can notice children's mathematical thinking (Superfine et al., 2018; McDuffie et al., 2014; Coskun et al., 2021), research investigating the effects of a curriculum module designed to enhance preservice teachers' ability to notice students’ mathematical thinking (Krupa et al., 2017), and studies focusing on the impact of an intervention aimed at professionally noticing students’ conceptual development in integers and arithmetic reasoning on elementary preservice teachers' professional noticing skills, attitudes toward mathematics, and mathematical knowledge for teaching (Fisher et al., 2018) are included in this cluster. Similarly, studies involving preservice teachers include research by Fernandez et al. (2013), which analyzed how elementary preservice teachers notice students’ mathematical thinking and proposed an initial framework consisting of four developmental levels, and Fernandez et al. (2012), which aimed to characterize the development of preservice mathematics teachers’ professional noticing of students’ mathematical thinking in online contexts. Wager (2014) examined how teachers participating in a professional development course responded to what they noticed about children’s engagement in elementary mathematics classrooms and how their noticing was linked to their stance toward equitable mathematics pedagogy. On the other hand, Huang and Li (2012) compared how teachers with varying levels of professional experience noticed events in mathematics classrooms. Nickerson et al. (2017) focused on the challenges of assessing middle school mathematics teachers’ professional noticing of students’ mathematical thinking, while Lee and Francis (2018) investigated the relationships among elementary teachers’ perceptions of using student thinking in instructional decision-making, their professional noticing skills, and their actual use of student thinking during instruction.

Cluster 2: Development

An examination of the second cluster, represented by green, reveals frequently occurring keywords, such as *professional development*, *teacher learning*, *assessment*, *mathematics teaching*, *task design*, *video*, *curriculum*, *cognitive demand*, *reflection*, and *numeracy*. Since this cluster encompasses studies to enhance students' mathematical thinking, it has been labeled "Development." For instance, studies included in this cluster comprise Leatham et al. (2015), which aimed to enhance student thinking by conceptualizing mathematically significant pedagogical opportunities; Fraivillig et al. (1999), which proposed a pedagogical framework to support children's conceptual understanding of mathematics and the development of their mathematical thinking; and Hudson et al. (2015), which examined participants in a newly developed master's course designed to promote the development of mathematical thinking in elementary school, while also aiming to encourage university researchers to engage in curriculum development research. On the other hand, Wager and Parks (2016) examined how play in the early childhood period can serve as a context for assessing and fostering children's mathematical thinking. Van Oers (2010) conducted a study aimed at promoting mathematical thinking by supporting young children in engaging with schematic representations and notations within play contexts. Similarly, Fouze and Amit (2018) carried out a study that aimed to develop mathematical thinking by integrating ethnomathematical folklore games -a learning process in mathematics that is interesting, enjoyable, and effective -into mathematics education. Efforts have been made to develop teachers' expertise in understanding, assessing, and fostering children's mathematical thinking using task-based one-on-one assessment interviews (Clarke et al., 2011). Akcay and Boston (2018) examined preservice teachers' ability to integrate technology into instructional practices in ways that support students' mathematical thinking and reasoning, employing the Instructional Quality Assessment. Additionally, Schoenfeld (2017) explored how video can enhance both the understanding and promotion of mathematical thinking and teaching.

Cluster 3: Advanced Mathematical Thinking

Upon examining the third cluster, represented by blue, the prominent keywords identified include *advanced mathematical thinking*, *reasoning*, *fractions*, *proof*, *teaching*, *college mathematics*, *discourse*, *calculus*, *clinical interviews*, and *cognitive development*. Based on these themes, the cluster has been labeled "Advanced Mathematical Thinking." For example, Yoon et al. (2011) argued that gestures play a role beyond simply conveying thought and supporting understanding; they demonstrated how gestures can help construct a virtual mathematical structure. Engelbrecht (2010), on the other hand, proposed certain frameworks to facilitate the transition from school mathematics -often perceived as traumatic by many students- to advanced mathematics. Dickerson and Pitman (2016) conducted a qualitative study involving ten undergraduate mathematics departments to examine students' definition-writing abilities. Inglis and Alcock (2012) presented a comparison of proof-validation behaviors between novice undergraduate students and practicing research mathematicians. Meanwhile, Radu and Weber (2011) explored improvements in undergraduate mathematics students' reasoning about completed infinite iterative processes, and Weber and Mejia-Ramos (2011) investigated the goals that guided nine research mathematicians as they read published proofs, along with the types of reasoning they employed to achieve these goals. Among the studies identified in this cluster is Tsamir and Tirosh (1999), which demonstrated how research-based insights into high school students' conflicting solutions to various representations of the same problem can be used to raise students' awareness of inconsistencies in their reasoning. Also included are Szydlik (2000), who investigated the mathematical beliefs of 27 university mathematics students and the connections between those beliefs and their understanding of limits; Dickerson and Doerr (2014), who examined high school mathematics teachers' perspectives on the purposes of mathematical proof in school mathematics; and Gavin et al. (2013), who explored the impact of challenging geometry and measurement units on the achievement of second-grade elementary students. These studies contribute to the third cluster.

Cluster 4: Technology

An analysis of the fourth cluster, represented by yellow, reveals key recurring terms, such as *mathematics education*, *technology*, *algebra*, *computational thinking*, *geometry*, *formative assessment*, *communication*, and *programming*. Since this cluster includes studies focused on designing instructional environments that aim to enhance mathematical thinking through technological support, it has been labeled "Technology." For example, Huscroft-D'Angelo et al. (2014) examined the effects of an intervention designed to promote the communication of mathematical thinking among students with learning difficulties through a digital writing platform. Lee (2005) provided insights into how three preservice teachers interpreted and developed their roles in supporting students' mathematical problem solving using a technology tool. Similarly, Cui and Ng (2021) investigated the interaction between mathematical and computational thinking in elementary students' mathematical problem solving within a block-based programming environment. Benton et al. (2018) implemented the ScratchMaths program curriculum designed to foster mathematical and computational thinking skills through learning to program. Rowlett (2015) conducted a study on promoting strategic and mathematical thinking through the game *Quarto*, played in *Maths Arcade*, an extracurricular university club. Meanwhile, Sherman et al. (2020) conducted a systematic analysis of how technology-based tasks are integrated into secondary mathematics curricula based on an analysis of 20 textbook samples. Studies included in this cluster also feature El-Demerdash et al. (2016) focused on designing and evaluating digital resources to foster creative mathematical thinking in the context of biomathematics, and Kaur (2020), who investigated the impact of using dynamic geometry environments on young children's thinking about angles. Additionally, Henning et al. (2012) examined the relationship between instructional design and classroom discourse as implemented by a mathematics teacher, while Yilmaz and Argun (2018) conducted a case study involving five secondary mathematics preservice teachers to explore the role of visualizations in mathematical abstraction—a key process in mathematical thinking. These studies contribute to the fourth cluster.

Discussion, conclusion and recommendations

This study conducted a bibliometric analysis of articles on mathematical thinking published between 1990 and 2021 in the WoS database. Articles were categorized by publication year, keywords, and country of origin. Bibliometric networks were developed to map the relationships among keywords associated with mathematical thinking.

The findings of the study are presented in the stages of exploration, visualization, labeling, and verification, respectively. Bibliometric analyses were conducted using VOSviewer software to identify interrelated keywords by examining articles on mathematical thinking. A review of studies on mathematical thinking in the WoS database since 1990 reveals that the number of publications was relatively low in the first decade. Although there was no consistent growth rate in the following years, the number of articles continued to increase in each subsequent five-year period. An examination of the frequently used keywords revealed various terms associated with mathematical thinking in the reviewed studies. Keywords, such as *noticing* (Coşkun et al., 2021; Fernandez et al., 2012; Fernandez et al., 2013; Fisher et al., 2018; Huang & Li, 2012; Krupa et al., 2017; Lee & Francis, 2018; McDuffie et al., 2014; Nickerson et al., 2017; Superfine et al., 2019; Wager, 2014), *problem solving* (Bloom, 2007; Hashemi et al., 2015; Hino, 2007), *advanced mathematical thinking* (Inglis & Alcock, 2012; Dickerson & Doerr, 2014; Dickerson & Pitman, 2016; Engelbrecht, 2010; Weber, 2011), *teacher education* (Jacobs et al., 2010; Sleep & Boerst, 2012; Stockero et al., 2017), *reasoning* (Bayazit & Osmanoglu, 2017; Tsamir & Tirosh, 1999; Woods et al., 2006), *assessment* (Drijvers et al., 2019; Rowlett et al., 2019; Wager & Parks, 2016), *technology* (Hitt et al., 2016; Kaur, 2020; Nickels & Cullen, 2017), *task design* (Norton & Kastberg, 2012; Paterson & Sneddon, 2011), and *cognitive demand* (Akcay & Boston, 2018; Estrella et al., 2020; Hallman-Thrasher, 2017) were among the most commonly co-occurring with mathematical thinking.

The timeline analysis of keywords provides insights into the trends of studies conducted on mathematical thinking. During the initial period from 1992 to 1996, keywords could be retrieved from only one article. The available keywords from this period constitute the basis of the data collected. Between 1997 and 2001, the studies focused on areas, such as *advanced mathematical thinking* (Szydlik, 2000; Tsamir & Tirosh, 1999; Williams, 2001), *pedagogical knowledge* (Fraivillig et al., 1999; Vacc & Bright, 1999), *university-level mathematics*, *preservice teacher education* (Mewborn, 1999), and *young children's perceptions of mathematics in problem-solving contexts* (Franke & Carey, 1997). In the early 2000s, the research began to concentrate on themes, such as *classroom interaction and teaching practices* (Hufferd-Ackles et al., 2004; Woods et al., 2006), *cognitive development* (Yuzawa et al., 2005), *reasoning* (Selden & Selden, 2003), *attitudes* (Oers, 2002), *mathematical communication* (Cooke & Buchholz, 2005), *problem solving*, and *mathematical discourse* (Kieran, 2001). Between 2007 and 2011, *advanced mathematical thinking* continued to be among the most frequently used keywords. Notably, the studies on *advanced mathematical thinking* conducted during these years primarily focused on university-level mathematics (Weber, 2009; Inglis & Simpson, 2009; Oehrtman, 2009). Unlike earlier periods, keywords, such as *teacher knowledge* (Philipp et al., 2007; Wilson, 2011), *professional development* (Ryken, 2009; Van Es & Sherin, 2008), *assessment* (Hino, 2007; Ryken, 2009; Young-Loveridge, 2011), *proof* (Koichu, 2010), *representation* (Ryken, 2009; Stewart & Thomas, 2009), and *task design* (Paterson & Sneddon, 2011) also emerged among the top ten most frequently used keywords. Between 2012 and 2016, keywords, such as *formative assessment* (Ginsburg, 2016; Henning et al., 2012; Sleep & Boerst, 2012), *noticing* (Carter & Amador, 2015; Roth McDuffie et al., 2014; Wager, 2014), and *early childhood education* (Ginsburg, 2016) emerged for the first time among the most frequently used keywords. During the 2017–2021 period, the number of studies related to *noticing* (Lee, 2019; Nickerson et al., 2017; Superfine et al., 2017) increased significantly, and keywords such as *cognitive demand* (Estrella et al., 2020; Hallman-Thrasher, 2017; Otten et al., 2017), *computational thinking* (Kallia et al., 2021; Pérez, 2018), and *technology* (Amador, 2017; Nickels & Cullen, 2017) appeared among the top ten for the first time. These findings suggest that in the most recent period, there was a greater integration of technology into education, with increased use of technological tools compared to earlier periods. The consistent recurrence of keywords such as *preservice teacher education*, *teacher education*, *teaching practice*, and *professional development* across all periods underscores a sustained emphasis on enhancing teacher competencies and the quality of teacher preparation programs in relation to mathematical thinking. Additionally, the timeline analysis reveals that keyword *problem solving* appears in nearly every period. This suggests that problem solving lies at the core of mathematical thinking and aligns with studies that argue problem solving is a fundamental component of mathematical thinking, and that mathematical thinking skills can be developed through problem-solving activities (Piggott, 2004; Tall, 2002; Yıldız, 2016).

An analysis of the number of publications and the average number of citations per publication in articles on mathematical thinking reveals a notable discrepancy between these two rankings. While the top three countries in terms of publication frequency are the United States, Australia, and Turkey, the ranking is based on average citations per publication lists, such as the United States, Israel, and the United Kingdom. In this latter ranking, Australia ranks sixth and Turkey tenth. This variation in average citation counts is believed to be influenced by the publication language. Publications in languages more widely spoken globally, particularly in English -a language commonly learned and used in academic contexts- tend to receive higher citation rates.

The overlay visualization indicates that keywords, such as *pedagogical knowledge*, *constructivism*, *proof*, *university mathematics*, *preservice teacher education*, *advanced mathematical thinking*, and *discourse* were more intensively used around 2005. Between 2010 and 2015, terms including *creativity*, *technology*, *communication*, *problem solving*, *professional development*, *early childhood education*, *video*, *abstraction*, and *reasoning* were frequently emphasized. In the period following 2015, keywords, such as *computational thinking*, *programming*, *awareness*, *lesson study*, *cognitive demand*, *fractions*, and *preservice teachers* gained prominence. These findings are consistent with the results of the timeline analysis.

The network visualization map illustrates that the keywords in studies on mathematical thinking are grouped into four clusters based on the degree of their interrelationships. In forming these clusters, keywords that are closely related and frequently co-occurring were placed in the same cluster by the VOSviewer software. The first cluster includes studies related to students' noticing and elicitation of mathematical thinking; the second cluster encompasses research focused on the development of mathematical thinking; the third cluster comprises studies on advanced mathematical thinking; and the fourth cluster involves research on designing instructional environments related to mathematical thinking using technological tools. On the other hand, it should not be overlooked that none of the clusters are entirely independent from one another; each cluster supports and contributes to others, collectively enriching the concept of mathematical thinking. It is noteworthy that within the third cluster, views asserting that studies on advanced mathematical thinking are both dependent on classroom and content level and can emerge at every stage of mathematics education are represented together.

A key finding of this study is the limited identification of mathematical thinking components through co-word analysis. Since the co-word analysis was based on keywords, it appears that researchers rarely include these components explicitly as keyword terms. It is therefore recommended that future researchers conduct co-word analyses using abstract terms to explore the nature of emerging clusters. Examining the extent to which components of mathematical thinking act as clustering determinants could yield fresh perspectives for further research.

This study is limited to articles published in English between 1990 and 2021 and accessed through the WoS database. Future research could extend the scope by broadening criteria, such as publication year, language, database, and publication type, allowing the study to be replicated with a wider dataset. Additionally, repeating this study in five-year intervals is expected to contribute to the literature by enabling the monitoring and comparison of changes, developments, and trends in research on mathematical thinking.

Although Turkey ranks third in the country-based publication ranking in articles on mathematical thinking, it accounts for only 5.8% of the total publications. In this context, it is important to contribute to the field with qualified and original studies. Additionally, considering the recent increase in studies linking mathematical thinking with technology, greater emphasis can be placed on designing instructional environments that enhance mathematical thinking through technological support.

Author contribution rates

The first and second authors each contributed 50% to the study.

Conflict of ınterest declaration

Our article titled "Systematic review of studies on mathematical thinking" has no financial conflict of interest with any institution, organization, or person. There is also no conflict of interest between the authors.

References

Akcay, A. O., & Boston, M. D. (2018). Preservice mathematics teachers' effective use of technology: Analyzing the cognitive demands of technology-based instructional activities. In *Educating prospective secondary mathematics teachers* (pp. 143–166). Springer. https://doi.org/10.1007/978-3-319-90179-4_7

Alkan, H., & Bukova Güzel, E. (2005). Öğretmen adaylarında matematiksel düşünmenin gelişimi. *Gazi Üniversitesi Gazi Eğitim Fakültesi Dergisi, 25*(3), 221-236.

Altun, M. (2018). *Ortaokullarda (5, 6, 7, 8. sınıflarda) matematik öğretimi* (12. baskı). Aktüel Yayınları.

Amador, J. M. (2017). Preservice teachers' video simulations and subsequent noticing: A practice-based method to prepare mathematics teachers. *Research in Mathematics Education, 19*(3), 217-235.

Arslan, S., & Yıldız, M. (2010). 11. sınıf öğrencilerinin matematiksel düşünme becerileri. *Eğitim ve Bilim, 35*(156), 143-154.

Bayazıt, Y., & Osmanoğlu, A. (2017). The effect of problem solving instruction on mathematical thinking. *Pedagogika-Pedagogy, 89*(1), 80-89.

Baykul, Y. (2009). *İlköğretimde matematik öğretimi (6-8. sınıflar)*. Pegem Akademi Yayıncılık.

Benton, L., Kalas, I., Saunders, P., Hoyles, C., & Noss, R. (2018). Beyond jam sandwiches and cups of tea: An exploration of primary pupils' algorithm evaluation strategies. *Journal of Computer Assisted Learning, 34*, 590-601. https://doi.org/10.1111/jcal.12266

Bloom, I. (2007). Extended analyses: Promoting mathematical inquiry with preservice mathematics teachers. *Journal of Mathematics Teacher Education, 10*(4), 399-403.

Büyüköztürk, Ş., Çakmak, E. K., Akgün, Ö. E., Karadeniz, Ş., & Demirel, F. (2017). *Bilimsel araştırma yöntemleri*. Ankara: Pegem Akademi Yayıncılık.

Carter, I. S. W., & Amador, J. (2015). Lexical and indexical conversational components that mediate professional noticing during lesson study. *Eurasia Journal of Mathematics, Science and Technology Education, 11*(6), 1339-1361.

Clarke, D., Clarke, B., & Roche, A. (2011). Building teachers' expertise in understanding, assessing and developing children's mathematical thinking: The power of task-based, one-to-one assessment interviews. *ZDM Mathematics Education, 43*, 901-913. https://doi.org/10.1007/s11858-011-0345-2

Cooke, B. D., & Buchholz, D. (2005). Mathematical communication in the classroom: A teacher makes a difference. *Early Childhood Education Journal, 32*(6), 365-369.

Coşkun, S. D., Sitrava, R. T., & Bostan, M. I. (2021). Pre-service elementary teachers' noticing expertise of students' mathematical thinking: The case of fractions. *International Journal of Mathematical Education in Science and Technology*, 1-18.

Cui, Z., & Ng, O. L. (2021). The interplay between mathematical and computational thinking in primary school students' mathematical problem-solving within a programming environment. *Journal of Educational Computing Research, 59*(5), 988-1012.

Dickerson, D. S., & Damien, J. P. (2016). An examination of college mathematics majors' understandings of their own written definitions. *The Journal of Mathematical Behavior, 41*, 1-9. DOI:10.1016/j.jmathb.2015.11.001

Dickerson, D. S., & Doerr, H. M. (2014). High school mathematics teachers' perspectives on the purposes of mathematical proof in school mathematics. *Mathematics Education Research Journal, 26*, 711-733. https://doi.org/10.1007/s13394-013-0091-6

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking*. Kluwer Academic Publishers.

Drijvers, P., Kodde-Buitenhuis, H., & Doorman, M. (2019). Assessing mathematical thinking as part of curriculum reform in the Netherlands. *Educational Studies in Mathematics, 102*(3), 435-456.

Duran, N. (2005). *Matematiksel düşünme becerilerine ilişkin bir araştırma* [Yüksek lisans tezi, Hacettepe Üniversitesi Sosyal Bilimler Enstitüsü].

El-Demerdash, M., Mercat, C., Trgalova, J., Essonnier, N., & Lealdino Filho, P. (2016). Design and evaluation of digital resources to enhance creative mathematical thinking in a biomathematics context. *Bulletin of the IEEE Technical Committee on Learning Technology, 18*(1), 6-9.

Engelbrecht, J. (2010). Adding structure to the transition process to advanced mathematical activity. *International Journal of Mathematical Education in Science and Technology, 41*(2), 143-154. DOI: 10.1080/00207390903391890

Estrella, S., Zakaryan, D., Olfos, R., & Espinoza, G. (2020). How teachers learn to maintain the cognitive demand of tasks through Lesson Study. *Journal of Mathematics Teacher Education, 23*(3), 293-310.

Fernández, C., Llinares, S., & Valls, J. (2012). Learning to notice students' mathematical thinking through on-line discussions. *ZDM Mathematics Education, 44*, 747-759. https://doi.org/10.1007/s11858-012-0425-y

Fernández, C., Llinares, S., & Valls, J. (2013). Primary school teacher's noticing of students' mathematical thinking in problem solving. *The Mathematics Enthusiast, 10*(1), 441-468.

Fisher, K., Hirsh-Pasek, K., & Golinkoff, R. M. (2012). Fostering mathematical thinking through playful learning. In E. Reese & S. P. Segate (Eds.), *Contemporary debates on child development and education* (pp. 81-92). Routledge.

Fisher, M. H., Thomas, J., Schack, E. O., Jong, C., & Tassell, J. (2018). Noticing numeracy now! Examining changes in preservice teachers' noticing, knowledge, and attitudes. *Mathematics Education Research Journal, 30*(2), 209-232.

Fouze, A., & Amit, M. (2017). Development of mathematical thinking through integration of ethnomathematic folklore game in math instruction. *Eurasia Journal of Mathematics, Science and Technology Education, 14*(2), 617-630. 10.12973/ejmste/80626

Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (1999). Advancing children's mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education, 30*(2), 148-170. https://doi.org/10.2307/749608

Franke, M. L., & Carey, D. A. (1997). Young children's perceptions of mathematics in problem-solving environments. *Journal for Research in Mathematics Education, 28*(1), 8-25.

Gavin, M. K., Casa, T. M., Adelson, J. L., & Firmender, J. M. (2013). The impact of challenging geometry and measurement units on the achievement of grade 2 students. *Journal for Research in Mathematics Education, 44*(3), 478-509.

Ginsburg, H. P. (2016). Helping early childhood educators to understand and assess young children's mathematical minds. *ZDM, 48*(7), 941-946.

Gökçe, S., & Güner, P. (2021). Forty years of mathematics education: 1980-2019. *International Journal of Education in Mathematics, Science and Technology, 9*(3), 514-539.

Gökçe, S., & Güner, P. (2022). Dynamics of GeoGebra ecosystem in mathematics education. *Education and Information Technologies, 27*(4), 5301-5323.

Guba, E. G., & Lincoln, Y. S. (1982). Epistemological and methodological bases of naturalistic inquiry. *ECTJ, 30*(4), 233-252.

Güner, P., & Gökçe, S. (2021). Monitoring the nomological network of number sense studies. *International Journal of Mathematical Education in Science and Technology, 52*(4), 580-608.

Hallman-Thrasher, A. (2017). Prospective elementary teachers' responses to unanticipated incorrect solutions to problem-solving tasks. *Journal of Mathematics Teacher Education, 20*(6), 519-555.

Hashemi, N., Abu, M. S., Kashefi, H., Mokhtar, M., & Rahimi, K. (2015). Designing learning strategy to improve undergraduate students' problem solving in derivatives and integrals: A conceptual framework. *Eurasia Journal of Mathematics, Science and Technology Education, 11*(2), 227-238.

Henning, J. E., McKeny, T., Foley, G. D., & Balong, M. (2012). Mathematics discussions by design: Creating opportunities for purposeful participation. *Journal of Mathematics Teacher Education, 15*, 453-479.

Hino, K. (2007). Toward the problem-centered classroom: Trends in mathematical problem solving in Japan. *ZDM, 39*(5), 503-514.

Hitt, F., Saboya, M., & Cortés Zavala, C. (2016). An arithmetic-algebraic work space for the promotion of arithmetic and algebraic thinking: Triangular numbers. *ZDM, 48*(6), 775-791.

Huang, R., & Li, Y. (2012). What matters most: A comparison of expert and novice teachers' noticing of mathematics classroom events. *School Science and Mathematics, 112*, 420-432.

Hudson, B., Henderson, S., & Hudson, A. (2015). Developing mathematical thinking in the primary classroom: Liberating students and teachers as learners of mathematics. *Journal of Curriculum Studies, 47*(3), 374-398.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 35*(2), 81-116.

Huscroft-Dâ, J. N., Higgins, K. N., & Crawford, L. L. (2014). A descriptive study examining the impact of digital writing environments on communication and mathematical reasoning for students with learning disabilities. *Learning Disabilities: A Multidisciplinary Journal, 20*(4), 177-188.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*(4), 358-390.

Inglis, M., & Simpson, A. (2009). Conditional inference and advanced mathematical study: Further evidence. *Educational Studies in Mathematics, 72*(2), 185-198.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

Kallia, M., van Borkulo, S. P., Drijvers, P., Barendsen, E., & Tolboom, J. (2021). Characterising computational thinking in mathematics education: A literature-informed Delphi study. *Research in Mathematics Education, 23*(2), 159-187.

Kaur, H. (2020). Introducing the concept of angle to young children in a dynamic geometry environment. *International Journal of Mathematical Education in Science and Technology, 51*(2), 161-182.

Kieran, C. (2001). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. *Educational Studies in Mathematics, 46*(1), 187-228.

Koichu, B. (2010). On the relationships between (relatively) advanced mathematical knowledge and (relatively) advanced problem-solving behaviours. *International Journal of Mathematical Education in Science and Technology, 41*(2), 257-275.

Krupa, E. E., Huey, M., Lesseig, K., Casey, S., & Monson, D. (2017). Investigating secondary preservice teacher noticing of students' mathematical thinking. In E. Schack, M. Fisher, & J. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks*. Springer. https://doi.org/10.1007/978-3-319-46753-5_4

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*, 88-124.

Lee, H. S. (2005). Facilitating students' problem solving in a technological context: Prospective teachers' learning trajectory. *Journal of Mathematics Teacher Education, 8*, 223–254. https://doi.org/10.1007/s10857-005-2618-6

Lee, M. Y., & Cross Francis, D. (2018). Investigating the relationships among elementary teachers' perceptions of the use of students' thinking, their professional noticing skills, and their teaching practices. *The Journal of Mathematical Behavior, 51*, 118-128.

Lee, M. Y. (2019). The development of elementary pre-service teachers' professional noticing of students' thinking through adapted lesson study. *Asia-Pacific Journal of Teacher Education, 47*(4), 383-398.

Liu, P. H. (2003). *The relationship of a problem-based calculus course and students' views of mathematical thinking* [Unpublished doctoral dissertation]. Oregon State University.

Mewborn, D. S. (1999). Reflective thinking among preservice elementary mathematics teachers. *Journal for Research in Mathematics Education, 30*(3), 316-341.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Sage.

Milli Eğitim Bakanlığı (MEB). (2018). *Matematik dersi öğretim programı*. Milli Eğitim Bakanlığı.

Nickels, M., & Cullen, C. J. (2017). Mathematical thinking and learning through robotics play for children with critical illness: The case of Amelia. *Journal for Research in Mathematics Education, 48*(1), 22-77.

Nickerson, S. D., Lamb, L., & LaRochelle, R. (2017). Challenges in measuring secondary mathematics teachers' professional noticing of students' mathematical thinking. In *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 381-398). Springer.

Norton, A., & Kastberg, S. (2012). Learning to pose cognitively demanding tasks through letter writing. *Journal of Mathematics Teacher Education, 15*(2), 109-130.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426.

Oers, B. V. (2002). Educational forms of initiation in mathematical culture. In *Learning discourse* (pp. 59-85). Springer.

Otten, S., Webel, C., & de Araujo, Z. (2017). Inspecting the foundations of claims about cognitive demand and student learning: A citation analysis of Stein and Lane (1996). *The Journal of Mathematical Behavior, 45*, 111-120.

Paterson, J., & Sneddon, J. (2011). Conversations about curriculum change: Mathematical thinking and team-based learning in a discrete mathematics course. *International Journal of Mathematical Education in Science and Technology, 42*(7), 879-889.

Pérez, A. (2018). A framework for computational thinking dispositions in mathematics education. *Journal for Research in Mathematics Education, 49*(4), 424-461.

Philipp, R. A., Ambrose, R., Lamb, L. L., Sowder, J. T., Schappelle, B. P., Sowder, L., ... & Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education, 38*(5), 438-476.

Piggott, J. (2004, January). *Developing a framework for mathematical enrichment*. Paper presented at the Symposium on Critical Thinking in Teaching and Learning, University of West Indies, St. Augustine, Trinidad.

Radu, I., & Weber, K. (2011). Refinements in mathematics undergraduate students' reasoning on completed infinite iterative processes. *Educational Studies in Mathematics, 78*(2), 165-182. http://www.jstor.org/stable/41486160

Roth McDuffie, A., Foote, M. Q., Bolson, C., Turner, E. E., Aguirre, J. M., Bartell, T. G., Drake, C., & Land, T. (2014). Using video analysis to support prospective K-8 teachers' noticing of students' multiple mathematical knowledge bases. *Journal of Mathematics Teacher Education, 17*(3), 245–270. https://doi.org/10.1007/s10857-013-9257-0

Rowlett, P. (2015). Developing strategic and mathematical thinking via game play: Programming to investigate a risky strategy for Quarto. *The Mathematics Enthusiast, 12*(1), 55-61.

Rowlett, P., Webster, K., Bradshaw, N. A., & Hind, J. (2019). Evaluation of the maths arcade initiative at five UK universities. *The Mathematics Enthusiast, 16*(1), 451-460.

Ryken, A. E. (2009). Multiple representations as sites for teacher reflection about mathematics learning. *Journal of Mathematics Teacher Education, 12*(5), 347-364.

Samo, D. D., Darhim, D., & Kartasasmita, B. (2017). Developing contextual mathematical thinking learning model to enhance higher-order thinking ability for middle school students. *International Education Studies, 10*(2), 17-29.

Schoenfeld, A. H. (2017). Uses of video in understanding and improving mathematical thinking and teaching. *Journal of Mathematics Teacher Education, 20*, 415-432. https://doi.org/10.1007/s10857-017-9381-3

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34*(1), 4-36.

Sherman, M. F., Cayton, C., Walkington, C., & Funsch, A. (2020). An analysis of secondary mathematics textbooks with regard to technology integration. *Journal for Research in Mathematics Education, 51*(3), 361-374.

Sleep, L., & Boerst, T. A. (2012). Preparing beginning teachers to elicit and interpret students' mathematical thinking. *Teaching and Teacher Education, 28*(7), 1038-1048.

Stewart, S., & Thomas, M. O. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951-961.

Stockero, S. L., Rupnow, R. L., & Pascoe, A. E. (2017). Learning to notice important student mathematical thinking in complex classroom interactions. *Teaching and Teacher Education, 63*, 384-395.

Superfine, A. C., Fisher, A., Bragelman, J., & Amador, J. M. (2017). Shifting perspectives on preservice teachers' noticing of children's mathematical thinking. In *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 409-426). Springer.

Superfine, A., Amador, J., & Bragelman, J. (2018). Facilitating video-based discussions to support prospective teacher noticing. *The Journal of Mathematical Behavior, 54*. https://doi.org/10.1016/j.jmathb.2018.11.002

Şencan, H. (2005). *Sosyal ve davranışsal ölçümlerde güvenilirlik ve geçerlilik.* Ankara: Seçkin Yayıncılık.

Şengül, S., Katrancı, Y., & Gerez-Cantimer, G. (2014). Metaphor perceptions of secondary school students about "mathematics teacher". *The Journal of Academic Social Science Studies, 25*(1), 89-111.

Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education, 31*(3), 258-276. https://doi.org/10.2307/749807

Tall, D. (1995). Cognitive growth in elementary and advanced mathematical thinking. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 61-75). Recife, Brasil: Universidade Federal de Pernambuco.

Tall, D. (2002). *Advanced mathematical thinking*. Kluwer Academic Publishers.

Tsamir, P., & Tirosh, D. (1999). Consistency and representations: The case of actual infinity. *Journal for Research in Mathematics Education, 30*(2), 213-219. https://doi.org/10.2307/749611

Umay, A. (2003). Matematiksel muhakeme yeteneği. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi, 24*(3), 234-243.

Vacc, N. N., & Bright, G. W. (1999). Elementary preservice teachers' changing beliefs and instructional use of children's mathematical thinking. *Journal for Research in Mathematics Education, 30*(1), 89-110.

van Eck, N. J., & Waltman, L. (2011). *VOSviewer manual: Manual for VOSviewer version 1.0*.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24*(2), 244-276.

van Oers, B. (2010). Emergent mathematical thinking in the context of play. *Educational Studies in Mathematics, 74*(1), 23-37.

Wager, A. A. (2014). Noticing children's participation: Insights into teacher positionality toward equitable mathematics pedagogy. *Journal for Research in Mathematics Education, 45*(3), 312-350.

Wager, A. A., & Parks, A. N. (2016). Assessing early number learning in play. *ZDM, 48*(7), 991-1002.

Weber, K. (2009). How syntactic reasoners can develop understanding, evaluate conjectures, and generate counterexamples in advanced mathematics. *The Journal of Mathematical Behavior, 28*(2-3), 200-208.

Weber, K., & Mejia-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics, 76*(3), 329-344.

Williams, S. R. (2001). Predications of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education, 32*(4), 341-367.

Wilson, P. H., Lee, H. S., & Hollebrands, K. F. (2011). Understanding prospective mathematics teachers' processes for making sense of students' work with technology. *Journal for Research in Mathematics Education, 42*(1), 39-64.

Woods, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education, 37*(3), 222-255.

Yıldırım, A., & Şimşek, H. (2008). *Sosyal bilimlerde nitel araştırma yöntemleri.* Ankara: Seçkin Yayıncılık.

Yıldırım, C. (1988). *Matematiksel düşünme.* İstanbul: Remzi Kitabevi.

Yıldız, C. (2016). Comparing the mathematical thinking experiences of students at faculty of education and faculty of arts and sciences. *TOJET: The Turkish Online Journal of Educational Technology-Special Issue for INTE 2016*, 480-488.

Yilmaz, R., & Argun, Z. (2017). Role of visualization in mathematical abstraction: The case of congruence concept. *International Journal of Education in Mathematics, Science and Technology, 6*, 1-11.

Yoon, C., Thomas, M. O., & Dreyfus, T. (2011). Gestures and insight in advanced mathematical thinking. *International Journal of Mathematical Education in Science and Technology, 42*(7), 891-901.

Young-Loveridge, J. (2011). Assessing the mathematical thinking of young children in New Zealand: The initial school years. *Early Child Development and Care, 181*(2), 267-276.

Yuzawa, M., Bart, W. M., Yuzawa, M., & Junko, I. (2005). Young children's knowledge and strategies for comparing sizes. *Early Childhood Research Quarterly, 20*(2), 239-253.