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ON THE RATE OF CONVERGENCE OF THE g-NAVIER-STOKES EQUATIONS

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Abstract. In this paper we consider 2D g-Navier-Stokes equations in a bounded domain by Ω. We give an error estimate between the solutions of Galerkin approximation of the g-Navier-Stokes equations and the exact solutions of them.

1. Introduction

We essentially focus on studying the rate of convergence for the g-Navier-Stokes equations (g-NSE). The error estimates for the differences between the solutions of the 2D α-models and solutions of their corresponding Galerkin approximation systems are given by Cao and Titi in [3]. Inspired from this article we get an estimate on g-NSE. There exist extensive analytical studies on the global regularity of solutions and the existence of global attractor of the g-NSE in [1], [7]-[10]. Using their result about the weak and strong solutions of g-NSE under the periodic conditions in [10] we give our estimate. The $L_2(\Omega, g)$-norm of the difference $|u - u_m|$ is the order $O\left(\frac{1}{\lambda_{m+1}}(\log \lambda_{m+1})^{\frac{1}{2}}\right)$. Here, u and u_m are the solution of the g-NSE and the solution of finite-dimensional Galerkin system of them respectively. We use Brezis-Gallouet inequality [2] in our proof. Using the equivalent norms related to the Stokes operator and g-Stokes operator, we rewrite the Brezis-Gallouet inequality stated under periodic boundary conditions.

The g-NSE are given in the following form

$$\frac{\partial u}{\partial t} + \nu\Delta u + (u.\nabla)\, u + \nabla p = f, \tag{1.1}$$

$$\nabla.(gu) = 0. \tag{1.2}$$

with the initial condition

$$u(x,0) = u_0(x), \tag{1.3}$$

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in $\Omega = (0,1) \times (0,1) \subset R^2$. This system is equipped with the periodic boundary conditions where ν and f are given, the velocity u and pressure p are the unknowns functions. We assume u, p and the first derivative of u to be spatially periodic, i.e.,

$$u(x_1+1, x_2) = u(x_1, x_2) = u(x_1, x_2+1) \qquad (x_1, x_2) \in R^2.$$

These equations are derived from 3D Navier stokes equations by Roh [8]. Here g is a suitable smooth real valued function on Ω.

Throughout this paper, we assume that

(i)

$$g(x_1, x_2) \in C^{\infty}_{per}(\Omega). \tag{1.4}$$

(ii) There exist two constants m_0, M_0 such that

$$0 < m_0 \le g(x_1, x_2) \le M_0 \quad \text{for every} \quad (x_1, x_2) \in \Omega. \tag{1.5}$$

Throughout this paper c will denote a generic positive constant. It can be different from line to line. This paper is organized as follows. In section 2 we give some notations and present the mathematical spaces. We also give some preliminary results given by Roh for the 2D g-NSE. In section 3, we obtain the error estimates between the solution of the g-NSE and the solution of Galerkin system of them.

2. Preliminaries and Functional Setting

In this section we introduce the usual notations used in the context, which are adopted by the works of [9, 10].

Let Ω be bounded domain in R^2. We define the Hilbert space $L^2(\Omega, g)$ which is the space $L^2(\Omega)$ with the scalar product and the norm given by

$$(u,v)_g = \int_{\Omega} (u,v)\, g dx \text{ and } |u|_g^2 = (u,u)_g\,,$$

and we also define the space $H^1(\Omega, g)$ which is the space $H^1(\Omega)$ with the norm by

$$\|u\|_{H^1(\Omega,g)} = \left[|u|_g^2 + \sum_{i=1}^{2} |D_i u|_g^2 \right]^{\frac{1}{2}}$$

where $D_i u = \frac{\partial u}{\partial x_i}$. The two spaces $L^2(\Omega)$ and $L^2(\Omega, g)$ have equivalent norms in the following inequalities

$$\sqrt{m_0}\,|u| \le |u|_g \le \sqrt{M_0}\,|u|\,, \tag{2.1}$$

where m_0 and M_0 are positive constants.

In our problem, we consider the following closed subspace of $L^2(\Omega, g)$:

$$
\begin{aligned}
H_g &= CL_{L^2(\Omega,g)} \left\{ u \in C_{per}^{\infty}(\Omega) : \nabla .gu = 0, \int_{\Omega} u dx = 0 \right\}, \\
Q &= CL_{L^2(\Omega)} \left\{ \nabla \phi : \phi \in C_{per}^{1}(\overline{\Omega}, R) \right\},
\end{aligned}
$$

where H_g is equipped with the scalar product and the norm in $L^2(\Omega, g)$. And we use the following space

$$
V_g = \left\{ u \in H_{per}^{1}(\Omega, g) : \nabla .gu = 0, \int_{\Omega} u dx = 0 \right\},
$$

with the scalar product and the norm given by

$$
(u, v)_{V_g} = \int_{\Omega} (D_i u, D_i v)\, g dx \text{ and } \quad \|u\|_{V_g}^2 = (u, u)_{V_g}.
$$

Then, we can define the orthogonal projection $P_g : L_{per}^2(\Omega, g) \longrightarrow H_g$ which is similar to the Leray Projection, as $P_g v = u$ and we obtain $Q \subset H_g^{\perp}$ where Q doesn't depend on the function g. Now we consider the $g-$Laplacian Δ_g defined by

$$
-\Delta_g u = -\frac{1}{g}(\nabla .g\nabla)u = -\Delta u - \frac{1}{g}(\nabla g.\nabla)u. \tag{2.2}
$$

So, using (2.2), (1.1) can be written as

$$
\frac{\partial u}{\partial t} - \nu \Delta_g u + \frac{1}{g}(\nabla g.\nabla)u + (u.\nabla)u + \nabla p = f \quad \text{in } \Omega. \tag{2.3}
$$

Now we rewrite the equation (2.3) as abstract evolution equations;

$$
\frac{du}{dt} + \nu A_g u + B_g(u, u) + Ru = f, \tag{2.4}
$$

where

$$
u(0) = u_0,
$$

$$
A_g u = P_g(-\Delta_g u),\ B_g(u, u) = P_g(u.\nabla)u,\ Ru = P_g \left[\frac{1}{g}(\nabla g.\nabla)u \right].
$$

For the linear operator A_g, the following proposition holds (see [10]).

Proposition 1. [10] *For the $g-$Stokes operator A_g, the followings hold;*

(i) A_g is a positive, self adjoint operator with compact inverse, where the domain of A_g, $D(A_g) = V_g \cap H^2(\Omega, g)$.

(ii) There exist countable eigenvalues of A_g satisfying

$$
0 < \lambda(g) \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...
$$

where $\lambda(g) = \frac{4\pi^2 m_0}{M_0}$ and λ_1 is the smallest eigenvalue of A_g. Moreover, there exist the corresponding collection of eigenfunctions $\{e_1, e_2, e_3 ...\}$ forms an orthonormal basis for H_g.

Now we recall the following inequalities. Let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis of H_g consisting of eigenfunctions of the operator A_g. Denote by $H_m^g = Span\{w_1, w_2, ..., w_m\}$, for $m \geq 1$ and let P_m^g be the orthogonal projection from H_g onto H_m^g, then we give the following inequalities [5].

$$|(I - P_m^g)u|_g^2 \leq \frac{1}{\lambda_{m+1}} \|u\|_g^2, \quad \text{for all } u \in V_g, \tag{2.5}$$

$$\|(I - P_m^g)u\|_g^2 \leq \frac{1}{\lambda_{m+1}} |A_g u|_g^2, \quad \text{for all } u \in D(A_g), \tag{2.6}$$

$$|AP_m^g u|_g^2 \leq \lambda_m \|P_m^g u\|_g^2. \tag{2.7}$$

Since the operator A_g is self-adjoint, the fractional power of the g-Stokes operator is defined as

$$\left(A_g^{\frac{1}{2}} u, A_g^{\frac{1}{2}} u\right)_g = (A_g u, u)_g, \quad \text{for } u \in D(A_g) = V_g \cap H^2(\Omega).$$

And since the orthogonal projection P_g is self-adjoint operator, by using integration by parts we write

$$\left(A_g^{\frac{1}{2}} u, A_g^{\frac{1}{2}} u\right)_g = \left(P_g\left[-\frac{1}{g}(\nabla.g\nabla)u\right], u\right)_g = \int_\Omega (\nabla u, \nabla u)\, g dx.$$

Thus we get

$$\left|A_g^{\frac{1}{2}} u\right|_g^2 = |\nabla u|_g^2 = \|u\|_g^2, \quad \text{for} \quad u \in V_g.$$

Theorem 1. (g-*Poincare inequality on* V_g) [8] *Assume that g satisfies (1.4). Then we have*

$$\frac{2\pi\sqrt{m_0}}{\sqrt{M_0}} |u|_g \leq \|u\|_g \quad \textit{for } u \in V_g.$$

where $m_0 \leq g(x) \leq M_0$ for all $x \in \Omega$.

Next, we denote the bilinear operator $B_g : V_g \times V_g \to V_g'$

$$B_g(u, v) = P_g(u.\nabla)v, \tag{2.8}$$

and the trilinear form

$$b_g(u, v, w) = \sum_{i,j=1}^{n} \int_\Omega u_i (D_i v_j) w_j g dx = (P_g(u.\nabla)v, w)_g, \tag{2.9}$$

where u, v, w lie in appropriate subspaces of $L_{per}^2(\Omega, g)$ and V_g' is the dual space of V_g.

b_g trilinear form have the following properties

$$i)\ b_g(u,v,w) = -b_g(u,w,v), \quad (2.10)$$
$$ii)\ b_g(u,v,v) = 0, \quad (2.11)$$

for sufficient smooth functions $u \in H_g$, $v, w \in V_g$ [4, 6, 9, 11].

Now we will give the following lemma see [3, 4, 11].

Lemma 1. *The bilinear operator B_g defined in (2.8) satisfies the following inequalities;*

$$\left|\langle B_g(u,v), w\rangle_{V_g'}\right| \le c\,|u|_g^{\frac{1}{2}}\,\|u\|_g^{\frac{1}{2}}\,\|v\|_g\,|w|_g^{\frac{1}{2}}\,\|w\|_g^{\frac{1}{2}} \quad \text{for all } u,v,w \in V_g, \quad (2.12)$$

$$|(B_g(u,v),w)| \le c\,\|u\|_{L^\infty}\,\|v\|_g\,|w|_g\,, \text{ for all } u \in D(A_g), v \in V_g, w \in H_g, \quad (2.13)$$

$$\left|\langle B_g(u,v), w\rangle_{(D(A_g))'}\right| \le c\,|u|_g\,\|v\|_g\,\|w\|_{L^\infty}\,, \text{ for all } u \in H_g, v \in V_g, w \in D(A_g). \quad (2.14)$$

Proposition 2. [10] *We assume that $\|g\|_\infty^2 < \frac{m_0^3\pi^2}{M_0}$ and $f \in L^2(\Omega,g)$. Then the followings hold;*

(i) For $u_0 \in H_g$, one has

$$|u(t)|_g^2 \le e^{-\alpha_1 t}\,|u_0|_g^2 + \alpha_2\,|f|_g^2 = K_0^2, \quad (2.15)$$

and

$$\left(1 - \frac{m_0}{2M_0}\right)\int_{t_1}^{t}\left|A_g^{\frac{1}{2}}u(s)\right|_g^2 ds \le |u(t_1)|_g^2 + \frac{2\,(t-t_1)}{\lambda_1}\,|f|_g^2\,,$$

for $0 \le t_1 \le t < \infty$.

(ii) For $u_0 \in V_g$ then there exist constants, $r_1 = r_1(m_0, M_0, f)$, $r_2 = r_2(m_0, M_0, f)$ and $L_1 = L_1(m_0, M_0, f)$ such that $0 \le t$,

$$\left|A_g^{\frac{1}{2}}u(t)\right|_g^2 \le r_1\left(1 + \left|A_g^{\frac{1}{2}}u_0\right|_g^2\right)e^{-\alpha_1 t} + L_1 = K_1^2(u_0, f, m_0, M_0). \quad (2.16)$$

In addition, if $u_0 \in D(A_g)$ and the forcing term $f \in V_g$ then there exist constants $r_3 = r_3(m_0, M_0, f)$ and $L_2 = L_2(m_0, M_0, f)$ such that

$$|A_g u(t)|_g^2 \le r_3\left(1 + |A_g u(0)|_g^2\right)e^{-\alpha_1 t} + L_2 = K_2^2(u_0, f, m_0, M_0), \text{ for } t \ge 0, \quad (2.17)$$

where $\alpha_1 = \lambda_1 - \frac{2}{m_0^2}\,\|\nabla g\|_\infty^2 > \frac{2m_0\pi^2}{M_0}$, $\alpha_2 = \frac{2}{\lambda_1\alpha_1} < \frac{M_0^2}{4m_0^2\pi^4}$.

3. Error Estimates Of The Galerkin Approximation Of The g-NSE

Before giving the main result, we give the following Lemmas. First we state a 2D periodic boundary condition version of the well known Brezis-Gallouet inequality [3]. Then we state this inequality for the A_g operator.

Lemma 2. [3] *There exists a constant $c > 0$ such that for every $u \in D(A_g)$*

$$\|u\|_{L^\infty} \leq c\,\|u\|_g \left(1 + \log(\frac{1}{\sqrt{\lambda_g}} \frac{|A_g u|_g}{\|u\|_g})\right)^{\frac{1}{2}}. \tag{3.1}$$

We give the following Lemma by using (2.16), (2.17) in (3.1).

Lemma 3. *Let $u_0 \in D(A_g)$ and $T > 0$. Assume that $\frac{1}{\sqrt{\lambda_g}\|\nabla g\|_\infty^2} \geq 1$ and $\frac{K_2}{K_1}\|\nabla g\|_\infty^2 \geq 1$. Let $u(t)$ satisfy (2.16), (2.17) in Lemma 3, then we get*

$$\|u\|_{L^\infty}^2 \leq c(1 + \log \frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}),$$

where we denote c as a generic constant.

Proof. Using the Brezis-Gallouet inequality in Lemma 2 we write

$$\|u\|_{L^\infty}^2 \leq \frac{c^2}{m_0^2}\|u\|_g^2 \left(1 + \log\left(\frac{|A_g u|_g}{\sqrt{\lambda_g}\,\|u\|_g}\right)\right).$$

From (2.16), (2.17) we have $\left|A_g^{\frac{1}{2}} u(t)\right|_g^2 = \|u(t)\| \leq K_1$ and $|A_g u(t)|_g \leq K_2$ for all $t \in [0, T]$ so we get

$$\begin{aligned} \|u\|_{L^\infty}^2 \quad &\leq \quad \frac{c^2}{m_0^2}[K_1^2 + K_1^2 \log\left(\frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}\right) \\ &\qquad + K_1^2 \frac{\|u\|_g}{K_1} \log\left(\frac{K_2\,\|\nabla g\|_\infty^2}{\|u\|_g}\right)]. \end{aligned} \tag{3.2}$$

Now from (2.16) and $\frac{\left\|A_g^{\frac{1}{2}} u\right\|}{K_1} \leq 1$ for all $t \in [0, T]$ we obtain

$$\frac{\|u\|}{K_1} \log\left(\frac{K_1}{\|u\|_g}\right) \leq \frac{1}{e}. \tag{3.3}$$

Using (3.3) and the assumption $\frac{K_2}{K_1}\|\nabla g\|_\infty^2 \geq 1$ in (3.2). Therefore the proof is completed. □

Now we will give the error estimate between the approximation solutions u_m of the finite dimensional Galerkin system and the exact solution u of the $g-$NSE. The error is given in terms of m and $\|\nabla g\|_\infty$.

First of all, we can decompose u as in the case $u = p_m + q_m$, where $p_m = P^g_m u$, $q_m = (I - P^g_m)u$, P^g_m is the orthogonal project from H_g onto H^g_m. H^g_m is defined in Section 2.

Since $u = p_m + q_m$, we can decompose the equation (2.4) into the following coupled system of equations;

$$\frac{dp_m}{dt} + \nu A_g p_m + P^g_m B_g(u,u) + P^g_m R p_m = P^g_m f, \tag{3.4}$$

$$\frac{dq_m}{dt} + \nu A_g q_m + (I - P^g_m) B_g(u,u) + (I - P^g_m) R q_m = (I - P^g_m) f. \tag{3.5}$$

For the Galerkin approximation system of the $g-$NSE, we write the following equation

$$\frac{du_m}{dt} + \nu A_g u_m + P^g_m B_g(u_m, u_m) + R u_m = P^g_m f. \tag{3.6}$$

We will proceed by first estimating the H_g-norm of q_m and then H_g-norm the difference $\delta_m = p_m - u_m$.

$$|u - u_m|_g^2 = |p_m - u_m|_g^2 + |q_m|_g^2 .$$

From (3.4) and (3.6) we observe that $\delta_m = p_m - u_m$ satisfies the following equation;

$$\frac{d\delta_m}{dt} + \nu A_g \delta_m + P^g_m B_g(\delta_m + q_m, u) + P^g_m B_g(u_m, \delta_m + q_m) + P^g_m R \delta_m = 0. \tag{3.7}$$

Now we will give the following main theorem.

Theorem 2. *Let $T > 0$ and let u be a solution of the $g-$NSE (2.4) with the initial data $u_0 \in D(A_g)$ and let u_m be the solution of (3.6) with the initial data $u_{0m} = P^g_m u_0$ over the interval $[0, T]$. For a given $m \geq 1$, then*

$$ess \sup_{0 \leq t < T} |u(t) - u_m(t)|_g^2 \leq \epsilon^2$$

where

$$\epsilon^2 := \frac{1}{(\lambda_{m+1})^2} e^{\frac{2}{\nu}} (c + \|\nabla g\|_\infty^2) (L_m + 1) Z_1,$$

where Z_1 is defined in (3.13) and $L_m = 1 + \log \frac{\lambda_{m+1}}{\lambda_1}$ respectively .

Proof. First of all, we estimate the H_g norm of q_m. We take the inner product of equation (3.5) with q_m and obtain

$$\frac{1}{2} \frac{d}{dt} |q_m|_g^2 + \nu \|q_m\|_g^2 \leqslant J_1 + J_2 + J_3. \tag{3.8}$$

where

$$J_1 = |((I - P^g_m) B_g(u,u), q_m)_g| ,$$
$$J_2 = |((I - P^g_m) R q_m, q_m)_g| ,$$
$$J_3 = |((I - P^g_m) f, q_m)_g| .$$

Now for estimating J_1 we use (2.5) and (2.13), we have

$$J_1 \leq \|u\|_{L^\infty} \|u\|_g \frac{\|q_m\|_g}{(\lambda_{m+1})^{\frac{1}{2}}}. \tag{3.9}$$

Using Lemma 3 and (2.16), we write

$$J_1 \leq \frac{cK_1}{(\lambda_{m+1})^{\frac{1}{2}}} \left(1+\log \frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}\right)^{\frac{1}{2}} \|q_m\|_g .$$

Applying Young's inequality, we have

$$J_1 \leq \frac{\nu}{4}\|q_m\|_g^2 + \frac{c}{\nu\lambda_{m+1}}\left(1+\log \frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}\right),$$

where $c = c^2K_1^2$ is a constant. And then, for estimating J_2, we apply Cauchy-Schwarz's inequality and Young's inequality, we yield

$$J_2 \leq \frac{\nu}{4}\|q_m\|^2 + \frac{1}{\nu m_0^2\lambda_{m+1}}\|\nabla g\|_\infty^2 |q_m|_g^2 .$$

Let us use (2.5) and Young inequality, we get

$$J_3 = \left|((I-P_m^g)f, q_m)_g\right| \leq \frac{\nu}{4}\|q_m\|_g^2 + \frac{1}{\nu\lambda_{m+1}}|f|_g^2 . \tag{3.10}$$

Thus we obtain

$$\begin{aligned} \frac{d}{dt}|q_m|_g^2 + \frac{\nu}{2}\|q_m\|_g^2 \quad &\leq \quad \frac{2c}{\nu\lambda_{m+1}}\left(1+\log \frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}\right)\|u\|_g \\ &\quad + \frac{2}{\nu m_0^2\lambda_{m+1}}\|\nabla g\|_\infty^2 \|q_m\|_g^2 + \frac{2}{\nu\lambda_{m+1}}|f|_g^2 , \end{aligned}$$

$$\begin{aligned} &\frac{d}{dt}|q_m|_g^2 + \frac{\nu\lambda_{m+1}}{4}|q_m|_g^2 + \left(\frac{\nu}{4} - \frac{2}{\nu m_0^2\lambda_{m+1}}\|\nabla g\|_\infty^2\right)\|q_m\|_g^2 \\ &\leq \frac{1}{\lambda_{m+1}}\left[\frac{2c}{\nu}\left(1+\log \frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}\right) + \frac{2}{\nu}|f|_g^2\right]. \end{aligned} \tag{3.11}$$

Dropping the last term of the left hand side of the equation (3.11), providing $\nu^2 m_0^2\lambda_{m+1} - 8\|\nabla g\|_\infty^2 > 0$ and applying Gronwall's inequality, we get

$$\begin{aligned} |q_m(t)|_g^2 \quad &\leqslant \quad e^{-\frac{\nu}{4}\lambda_{m+1}t}|q_m(0)|_g^2 + \frac{c}{\nu\lambda_{m+1}}\left(1+\log \frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}\right)\int_0^t e^{-\frac{\nu}{4}\lambda_{m+1}(t-s)}ds \\ &\quad + \frac{2}{\nu\lambda_{m+1}}|f|_g^2 \int_0^t e^{-\frac{\nu}{4}\lambda_{m+1}(t-s)}ds. \end{aligned}$$

Using the inequality (2.5) we have

$$|q_m(t)|_g^2 \le \frac{1}{(\lambda_{m+1})^2}\,|A_g q_m(0)|_g^2 + \frac{4c}{\upsilon^2(\lambda_{m+1})^2}\left(1+\log\frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}\right) + \frac{8}{\nu^2\lambda_{m+1}}\,|f|_g^2 .$$

Thus

$$|q_m(t)|_g^2 \le \frac{1}{(\lambda_{m+1})^2} Z_1, \tag{3.12}$$

where

$$Z_1 = |A_g q_m(0)|_g^2 + \frac{4}{\upsilon^2}\left(c\left(1+\log\frac{1}{\sqrt{\lambda_g}\,\|\nabla g\|_\infty^2}\right) + |f|_g^2\right). \tag{3.13}$$

Next, we estimate the L^2-norm of δ_m by taking the inner product of equation (3.7) with δ_m and using (2.10), (2.11) and then we get the H_g-norm of δ_m. Taking the inner product of equation (3.7) with δ_m and using (2.10), (2.11) and then we estimate

$$\frac{1}{2}\frac{d}{dt}\,|\delta_m|_g^2 + \nu\,\|\delta_m\|_g^2 \leqslant J_4 + J_5 + J_6 + J_7, \tag{3.14}$$

where

$$\begin{aligned} J_4 &= \left|\langle B_g(\delta_m, u), \delta_m\rangle_g\right|, \\ J_5 &= \left|\langle B_g(q_m, u), \delta_m\rangle_g\right|, \\ J_6 &= \left|\langle B_g(u_m, q_m), \delta_m\rangle_g\right|, \\ J_7 &= \left|\langle R\delta_m, \delta_m\rangle_g\right|. \end{aligned}$$

First of all we estimate J_4. Using (2.14), and applying (2.16) and also using Young's inequality, we obtain

$$\begin{aligned} J_4 &\le \frac{c}{2}\,|\delta_m|_g\,\|u\|_g\,\|\delta_m\|_g, \\ &\le \frac{\nu}{4}\,\|\delta_m\|_g^2 + \frac{c}{\nu}\,|\delta_m|_g^2\,\|u\|_g^2, \\ &\le \frac{\nu}{4}\,\|\delta_m\|_g^2 + \frac{c}{\nu}\,|\delta_m|_g^2. \end{aligned}$$

Now we estimate J_5. Using (2.13) and applying (2.16) and Lemma 2 we write

$$J_5 \le c\,|q_m|_g\,\|u\|_g\,\|\delta_m\|_g\left(1+\log\frac{1}{\sqrt{\lambda_g}}\frac{|A_g\delta_m|_g}{\|\delta_m\|_g}\right)^{\frac{1}{2}}.$$

Now using (2.16), (2.7) and Young's inequality in the above inequality we write

$$J_5 \le \frac{\nu}{4}\,\|\delta_m\|_g^2 + \frac{c}{\nu} L_m\,|q_m|_g^2,$$

where $L_m = 1+\log\frac{\lambda_{m+1}}{\lambda_g}$. Next we estimate J_6. We will proceed by applying (2.10), (2.13) we obtain

$$J_6 \leq c\left|q_m\right|_g \left\|u_m\right\|_\infty \left\|\delta_m\right\|_g .$$

And then, we use Lemma 3, (2.13) and Young's inequality we have

$$J_6 \leq \frac{\nu}{4}\left\|\delta_m\right\|_g^2 + \frac{c}{\nu}\left(1+\log\frac{\lambda_{m+1}}{\lambda_g}\right)\left|q_m\right|_g^2 . \tag{3.15}$$

Finally, we estimate the last term

$$J_7 \leq \frac{\nu}{4}\left\|\delta_m\right\|_g^2 + \frac{1}{\nu m_0^2}\left\|\nabla g\right\|_\infty^2 \left|\delta_m\right|_g^2 .$$

Let us substitute the bounds for J_4, J_5, J_6, J_7 into (3.14), we get

$$\frac{d}{dt}\left|\delta_m\right|_g^2 + \nu\left\|\delta_m\right\|_g^2 \leq \frac{1}{\nu}\left(c+\left\|\nabla g\right\|_\infty^2\right)\left|\delta_m\right|_g^2 + \frac{c}{\nu}L_m\left|q_m\right|_g^2 . \tag{3.16}$$

Neglecting the second term of left hand side of the equation (3.16), using (3.12) and Gronwall's inequality and recalling that $|\delta_m(0)| = 0$, we obtain

$$\left|\delta_m(t)\right|_g^2 \leq \frac{c}{\nu\left(\lambda_{m+1}\right)^2} L_m Z_1 e^{\frac{1}{\nu}\left(c+\|\nabla g\|_\infty^2\right)T}.$$

Therefore we have the following inequality

$$\begin{aligned} ess\sup_{0\leq t\leq T}\left|u(t)-u_m(t)\right|_g^2 &\leq ess\sup_{0\leq t\leq T}\left|\delta_m\right|_g^2 + ess\sup_{0\leq t\leq T}\left|q_m\right|_g^2 \\ &\leq \frac{1}{(\lambda_{m+1})^2}\left(\frac{c}{\nu}L_m e^{\frac{1}{\nu}\left(c+\|\nabla g\|_\infty^2\right)T}+1\right)Z_1. \end{aligned}$$

□

Remark. The result which is given in the above for the $g-$NSE is the same order as that of the error estimates for the usual Galerkin approximation of NSE. Indeed, in [13] the order of error estimate for the 2D NSE is given by $O\left(\frac{1}{\lambda_{m+1}}\left(\log\left(\lambda_{m+1}\right)\right)^{\frac{1}{2}}\right)$.

References

[1] Ahn, C.T., Quyet, D.T. and Tinh, D.T., Existence and finite time approximation of strong solutions to the 2D g-Navier-Stokes equations. *Acta Math. Vietnam.* 38 (2013) 413-428.

[2] Brezis, H. and Gallouet, T., Nonlinear Schrodinger evolution equations. *Nonlinear Analysis, Theory, Methods & Applications*, (1980) 677–681.

[3] Cao, Y. and Titi, E.S.,On the rate of convergence of the two-dimensional $\alpha-$models of turbulence to the Navier-Stokes Equations. *Numer. Funct. Anal. Optim. 30.* (2009) 11-12:1231–1271.

[4] Constantin, P. and Foias C., *Navier-Stokes equations.* University of Chicago Press, Chicago, (1988).

[5] Foias, C. , Manley, O., Rosa, R. and Temam R., *Navier-Stokes equations and turbulence.* Encyclopedia of Mathematics and its Applications, 83. Cambridge University Press, Cambridge, (2001).

[6] Courant, R. and Hilbert, D., (1989). *Methods of Mathematical Physics Vol. II.* John Wiley & Sons, New York.

[7] Kwak, M., Kwean, H. and Roh, J., The dimension of attractor of the 2D g-Navier-Stokes equations. *J. Math. Anal. Appl. 315*. 2 (2006) 436–461.

[8] Roh, J., *g-Navier Stokes equations.* Thesis, University of Minnesota (2001).

[9] Roh, J., Convergence of the g-Navier Stokes equations. *Taiwanese J. Math. 13*. 1 (2009) 189–210.

[10] Roh, J., Dynamics of the g-Navier Stokes equations. *J. Differential Equations 211*. 2 (2005) 452–484.

[11] Temam, R., *Navier-Stokes equations. Theory and numerical analysis.* North-Holland Publishing Co. Amsterdam (1977).

[12] Temam, R., *Navier-Stokes equations and nonlinear functional analysis.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1983).

[13] Titi, E.S. On approximate inertial manifolds to the Navier-Stokes equations. *J. Math. Anal. App.* 149 (1990) 540–557.

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