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REFINEMENT OF SOME INEQUALITIES FOR OPERATORS

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Abstract. In this paper, we will use a refinement of the classical Young inequality to improve some inequalities of operators.

1. Introduction

Let $\boldsymbol{H}$ be a complex Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let $\mathcal{B}(\boldsymbol{H})$ denote the algebra of all bounded linear operators on $\boldsymbol{H}$, $\|.\|$ will also denote the operator norm on $\mathcal{B}(\boldsymbol{H})$.
For $A \in \mathcal{B}(\boldsymbol{H})$ the numerical radius is defined as follows,

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \boldsymbol{H}, \|x\| = 1\}.$$

We recall the following results that were proved in [2, 5].

Lemma 1.1. *Let $A \in \mathcal{B}(\boldsymbol{H})$ and let $\omega(.)$ be the numerical radius. Then*
(i) $\omega(.)$ is a norm on $\mathcal{B}(\boldsymbol{H})$,
(ii) $\omega(UAU^*) = \omega(A)$, for all unitary operators U,
(iii) $\omega(A^k) \leq \omega(A)^k, k = 1, 2, 3, \ldots$ (power inequality)
(iv) $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$.

Moreover, $\omega(.)$ is not a unitarily invariant norm and is not submultiplicative. For positive real numbers a, b, the classical Young inequality says that if $p, q > 1$ such that $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.1}$$

Replacing a, b by their squares, we could write (1.1) in the form

$$(ab)^2 \leq \frac{a^{2p}}{p} + \frac{b^{2q}}{q}. \tag{1.2}$$

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A refinement of the scalar Young inequality is as follows [9],

$$ab + r_0(a^{p/2} - b^{q/2})^2 \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1.3}$$

where $r_0 = \min\{1/p, 1/q\}$.

Some authors considered replacing the numbers a, b by positive operators A, B. But there are some difficulties, for example if A and B are positive operators, the operator AB is not positive in general. Hence the authors studied the singular values and the norms of the operators instead of operators in some inequalities. Let us denote by $\mathbb{M}_n$ the algebra of all $n \times n$ complex matrices. Bhatia and Kittaneh in 1990 [3] established a matrix mean inequality as follows:

$$|||A^*B||| \leq \frac{1}{2}\,|||A^*A + B^*B|||\,, \tag{1.4}$$

for matrices $A, B \in \mathbb{M}_n$.

In [2] a generalization of (1.4) was proved, for all $X \in \mathbb{M}_n$,

$$|||A^*XB||| \leq \frac{1}{2}\,|||AA^*X + XBB^*|||\,. \tag{1.5}$$

Ando in 1995 [1] established a matrix Young inequality:

$$|||AB||| \leq \left|\left|\left| \frac{A^p}{p} + \frac{B^q}{q} \right|\right|\right| \tag{1.6}$$

for $p, q > 1$ with $1/p + 1/q = 1$ and positive matrices A, B. Also, in [11], we showed that $|||AXB||| \leq \left|\left|\left| \frac{1}{p}A^pX + \frac{1}{q}XB^q \right|\right|\right|$ does not hold in general. In [10] we considered the inequalities (1.4) and (1.6) with the numerical radius norm as follows:

Proposition 1.1. [10, Proposition 1] *If A, B are $n \times n$ matrices, then*

$$\omega(A^*B) \leq \frac{1}{2}\omega(A^*A + B^*B). \tag{1.7}$$

Also if A and B are positive matrices and $p, q > 1$ with $1/p + 1/q = 1$, then

$$\omega(AB) \leq \omega(\frac{A^p}{p} + \frac{B^q}{q}).$$

In this paper we obtain some generalized matrix versions of the inequalities (1.2) and (1.7).

2. MAIN RESULTS

Let $A \in \mathcal{B}(\boldsymbol{H})$. We know that $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$(see Lemma 1.1(iv)). These inequalities were improved in [6, 8] as follows:

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}(\|A\| + \|A^2\|^{1/2}), \tag{2.1}$$

$$\frac{1}{4}\|A^*A + AA^*\| \leq \omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|, \tag{2.2}$$

where $|A| := (A^*A)^{\frac{1}{2}}$ is the absolute value of A.

Generalizations of the first inequality in (2.1) and the second inequality in (2.2) have been established in [4]. It has been shown that if $A, B \in \mathcal{B}(\boldsymbol{H})$, for $0 < \alpha < 1$ and $r \geq 1$, then

$$\omega^r(A+B) \leq 2^{r-2} \left\| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} + |B|^{2r\alpha} + |B^*|^{2r(1-\alpha)} \right\|, \tag{2.3}$$

$$\omega^r(A) \leq \frac{1}{2} \left\| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} \right\|. \tag{2.4}$$

In 2005, Kittaneh extended the above inequalities as follows:

Theorem 2.1. [8, Theorem 2] *If $A, B, C, D, S, T \in \mathcal{B}(\boldsymbol{H})$, then for all $\alpha \in (0,1)$,*

$$\omega(ATB+CSD) \leq \frac{1}{2} \left(\left\| A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2(\alpha)}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2(\alpha)}D \right\| \right). \tag{2.5}$$

In 2009, Shebrawi and Albadawi extended the inequality (2.5), in the following form:

Theorem 2.2. [12, Theorem 2.5] *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, \ldots, n)$, and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $r \geq 1$,*

$$\omega^r(\sum_{i=1}^{n} A_i^* X_i B_i) \leq \frac{n^{r-1}}{2} \left(\left\| \sum_{i=1}^{n} ([A_i^* g^2(|X_i^*|)A_i]^r + [B_i^* f^2(|X_i|)B_i]^r) \right\| \right). \tag{2.6}$$

In [10] we established a numerical radius inequality that generalizes (2.6) and consequently, generalize (2.3), (2.4), (2.5).

Theorem 2.3. [10, Theorem 5] *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, \ldots, n)$, and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. If $p \geq q > 1$ with $1/p + 1/q = 1$, then for all $r \geq \frac{2}{q}$,*

$$\omega^r(\sum_{i=1}^{n} A_i^* X_i B_i) \leq n^{r-1} \left\| \sum_{i=1}^{n} \frac{1}{p}(B_i^* f^2(|X_i|)B_i)^{rp/2} + \frac{1}{q}(A_i^* g^2(|X_i^*|)A_i)^{rq/2} \right\|. \tag{2.7}$$

In this section, we refine this inequality by using the inequality (1.3) to improve our results, we need the following basic lemmas.

Lemma 2.1. [7, Theorem 1] *Let A be an operator in $\mathcal{B}(\boldsymbol{H})$, and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all x and y in $\boldsymbol{H}$,*

$$|\langle Ax, y \rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|. \tag{2.8}$$

The following lemma is a consequence of the spectral theorem for positive operators and Jensen's inequality (see, e.g., [7]).

Lemma 2.2. *Let A be a positive operator in $\mathcal{B}(\boldsymbol{H})$ and let $x \in \boldsymbol{H}$ be any unit vector. Then for all $r \geq 1$,*

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle. \tag{2.9}$$

Now, we state the following theorem which is a refinement of (2.7).

Theorem 2.4. *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, \ldots, n)$, and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. If $p \geq q > 1$ with $1/p + 1/q = 1$, then for all $r \geq \frac{2}{q}$,*

(2.10)

$$\omega^r(\sum_{i=1}^{n} A_i^* X_i B_i) \leq n^{r-1} \left(\left\| \sum_{i=1}^{n} \frac{1}{p}(B_i^* f^2(|X_i|)B_i)^{rp/2} + \frac{1}{q}(A_i^* g^2(|X_i^*|)A_i)^{rq/2} \right\| - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x) \right),$$

where $\eta(x) := \sum_{i=1}^{n} \left(\left\langle B_i^* f^2(|X_i|)B_i x, x \right\rangle^{rp/4} - \left\langle A_i^* g^2(|X_i^*|)A_i x, x \right\rangle^{rq/4} \right)^2.$

Proof. For every unit vector $x \in \boldsymbol{H}$, we have

$$\begin{aligned}
\left| \left\langle \left(\sum_{i=1}^{n} A_i^* X_i B_i \right) x, x \right\rangle \right|^r &\leq \left(\sum_{i=1}^{n} |\langle X_i B_i x, A_i x \rangle| \right)^r \\
&\underset{(2.8)}{\leq} \left(\sum_{i=1}^{n} \left\langle f^2(|X_i|)B_i x, B_i x \right\rangle^{1/2} \left\langle g^2(|X_i^*|)A_i x, A_i x \right\rangle^{1/2} \right)^r \\
&\leq n^{r-1} \sum_{i=1}^{n} \left\langle f^2(|X_i|)B_i x, B_i x \right\rangle^{r/2} \left\langle g^2(|X_i^*|)A_i x, A_i x \right\rangle^{r/2} \\
&= n^{r-1} \sum_{i=1}^{n} \left\langle B_i^* f^2(|X_i|)B_i x, x \right\rangle^{r/2} \left\langle A_i^* g^2(|X_i^*|)A_i x, x \right\rangle^{r/2} \\
&\leq n^{r-1} \sum_{i=1}^{n} (\frac{1}{p} \left\langle (B_i^* f^2(|X_i|)B_i)^{rp/2} x, x \right\rangle + \frac{1}{q} \left\langle (A_i^* g^2(|X_i^*|)A_i)^{rq/2} x, x \right\rangle \\
&- \frac{1}{p} \left(\left\langle (B_i^* f^2(|X_i|)B_i) x, x \right\rangle^{rp/4} - \left\langle (A_i^* g^2(|X_i^*|)A_i) x, x \right\rangle^{rq/4} \right)^2) \\
&\underset{(1.3),(2.9)}{=} n^{r-1} (\langle \sum_{i=1}^{n} (\frac{1}{p}(B_i^* f^2(|X_i|)B_i)^{rp/2} \\
&+ \frac{1}{q}(A_i^* g^2(|X_i^*|)A_i)^{rq/2}) x, x \rangle - (\frac{1}{p}) \eta(x)).
\end{aligned}$$

Now, the result follows by taking the supremum over all unit vectors in $\boldsymbol{H}$. □

Remark 2.1. Let $p = q = r = 2$. Then $\eta(x) \equiv 0$ if and only if $\omega(B_i^* f^2(|X_i|)B_i - A_i^* g^2(|X_i^*|)A_i) = 0$, for all $i = 1, \ldots, n$. In general, $\eta(x) = 0$ if and only if $\left\langle B_i^* f^2(|X_i|)B_i x, x \right\rangle^{rp/4} = \left\langle A_i^* g^2(|X_i^*|)A_i x, x \right\rangle^{rq/4}$, for all $i = 1, \ldots, n$. Moreover, in the refinement of the Kittaneh's inequalitity, $\inf \eta(x) = 0$.
Because $0 \in \sigma_{app}(|A| - |A^*|)$ (approximate point spectrum) and the approximate point spectrum is a subset of the closure of the numerical range. Then $\inf \langle |A| - |A^*| x, x \rangle = 0$, where $\langle x, x \rangle = 1$ and hence $\inf \eta(x) = 0$.

In the following example we show that (2.10) is a refinement of the inequality (2.7) and $\inf_{\|x\|=1} \eta(x) > 0$.

Example 2.1. Let $X = I, n = 1, f(t) = g(t) = t^{1/2}, r = p = q = 2$ and $|A|^2 = diag(5, 1), |B|^2 = diag(2, 0)$ in the inequality (2.10). Then $\inf_{\|x\|=1} \eta(x) > 0$ and (2.10) is a refinement of the inequality (2.7).

The inequality (2.10) includes several numerical radius inequalities as special cases. Examples of inequalities are shown in the following.

Corollary 2.1. *Let $A_i, B_i, \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, \dots, n)$. If $p \geq q > 1$ with $1/p + 1/q = 1$ and $r \geq \frac{2}{q}$, then*

$$\omega^r(\sum_{i=1}^{n} A_i^* B_i) \leq n^{r-1} \left(\left\| \sum_{i=1}^{n} (\frac{1}{p}|B_i|^{rp} + \frac{1}{q}|A_i|^{rq}) \right\| - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x) \right),$$

where $\eta(x) := \sum_{i=1}^{n} \left(\langle |B_i|^2 x, x\rangle^{rp/4} - \langle |A_i|^2 x, x\rangle^{rq/4} \right)^2$.
In particular, if $n = 1$, then

$$\omega^r(A^* B) \leq \left\| \frac{1}{p}|B|^{rp} + \frac{1}{q}|A|^{rq}) \right\| - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x),$$

where $\eta(x) := \left(\langle |B| x, x\rangle^{rp/4} - \langle |A| x, x\rangle^{rq/4} \right)^2$.

Remark 2.2. By replacing $n = 1$ in Theorem 2.4, we obtain the following

$$\omega^r(A^* X B) \leq \left\| \frac{1}{p}(B^*|X|B)^{rp/2} + \frac{1}{q}(A^*|X^*|A)^{rq/2} \right\| - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x), \tag{2.11}$$

where $\eta(x) := \left(\langle (B^*|X|B)x, x\rangle^{rp/4} - \langle (A^*|X^*|)A)x, x\rangle^{rq/4} \right)^2$.
Furthermore, by Lemma 1.1, for all $A, B, X \in \mathcal{B}(\boldsymbol{H})$, we obtain the following inequalities:

$$\omega((A^* X B)^2) \leq \omega(\frac{1}{p}(A^*|X^*|A)^p + \frac{1}{q}(B^*|X|B)^q) - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x), \tag{2.12}$$

where $\eta(x) := \left(\langle (B^*|X|B)x, x\rangle^{p/2} - \langle A^*|X^*|Ax, x\rangle^{q/2} \right)^2$, and

$$\omega(A^* X B) \leq \frac{1}{2}\omega(A^*|X^*|A + B^*|X|B) - (\frac{1}{2}) \inf_{\|x\|=1} \eta(x), \tag{2.13}$$

where $\eta(x) := \left(\langle (B^*|X|B)x, x\rangle^{1/2} - \langle (A^*|X^*|)A)x, x\rangle^{1/2} \right)^2$.
The inequalities (2.12) and (2.13) are generalized matrix versions of the inequalities (1.2) and (1.7), respectively.

Remark 2.3. By the Example 2.1 we can show that $\inf_{\|x\|=1} \eta(x) > 0$, in Corollary 2.1 and the inequalities (2.11), (2.12), (2.13).

3. Additional results

Some of usual operator norm inequalities for summation of operators have been proved. It has been shown in [4] that if A and B are normal and $r \geq 1$, then

$$\|A + B\|^r \leq 2^{r-1} \||A|^r + |B|^r\|. \tag{3.1}$$

In this section, we get a norm inequality for Hilbert space operators, so that new inequalities for operators and generalizations of earlier results will be obtained. By the same method as in the proof of Theorem 2.4 we obtain the following:

Proposition 3.1. *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, ..., n)$, and let f and g be as in (2.1) and $p \geq q > 1$ with $1/p + 1/q = 1$. Then for all $r \geq \frac{2}{q}$,*

$$\begin{aligned}\left\|\sum_{i=1}^{n} A_i^* X_i B_i\right\|^r \leq n^{r-1}(\frac{1}{p}&\left\|\sum_{i=1}^{n}(B_i^* f^2(|X_i|)B_i)^{rp/2}\right\| \\ &+\frac{1}{q}\left\|\sum_{i=1}^{n}(A_i^* g^2(|X_i^*|)A_i)^{rq/2}\right\| - (\frac{1}{p})\inf_{\|x\|=\|y\|=1}\eta(x,y)),\end{aligned} \tag{3.2}$$

where $\eta(x,y) := \sum_{i=1}^{n}\left(\langle(B_i^* f^2(|X_i|)B_i)x, x\rangle^{rp/4} - \langle(A_i^* g^2(|X_i^*|)A_i)y, y\rangle^{rq/4}\right)^2$.

Inequality (3.2) yields several norm inequalities as special cases. Samples of these inequalities are demonstrated below.

Corollary 3.1. *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, ..., n), r \geq \frac{2}{q}$ and $p \geq q > 1$ with $1/p + 1/q = 1$ and $\alpha \in (0, 1)$. Then*

$$\begin{aligned}\left\|\sum_{i=1}^{n} A_i^* X_i B_i\right\|^r \leq n^{r-1}(\frac{1}{p}&\left\|\sum_{i=1}^{n}(B_i^* |X_i|^{2\alpha} B_i)^{rp/2}\right\| \\ &+\frac{1}{q}\left\|\sum_{i=1}^{n}(A_i^* |X_i^*|^{2(1-\alpha)}A_i)^{rq/2}\right\| - (\frac{1}{p})\inf_{\|x\|=\|y\|=1}\eta(x,y)),\end{aligned} \tag{3.3}$$

where $\eta(x,y) := \sum_{i=1}^{n}\left(\langle(B_i^* |X_i|^{2\alpha}B_i)x, x\rangle^{rp/4} - \langle(A_i^* |X_i^*|^{2(1-\alpha)}A_i)y, y\rangle^{rq/4}\right)^2$.

In particular,

$$\|A^* X B\|^r \leq \frac{1}{p}\left\|(B^*|X|B)^{rp/2}\right\| + \frac{1}{q}\left\|(A^*|X^*|A)^{rq/2}\right\| - (\frac{1}{p})\inf_{\|x\|=\|y\|=1}\eta(x,y)$$

where $\eta(x,y) := \left(\langle(B^*|X|B)x, x\rangle^{rp/4} - \langle(A^*|X^*|A)y, y\rangle^{rq/4}\right)^2$.

For $X_i = I(i = 1, 2, ..., n)$ in inequality (3.3), we get norm inequalities for products of operators.

Corollary 3.2. *Let $A_i, B_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, ..., n), r \geq \frac{2}{q}$. Then*

$$\left\|\sum_{i=1}^{n} A_i^* B_i\right\|^r \leq n^{r-1}\left(\frac{1}{p}\left\|\sum_{i=1}^{n}|B_i|^{rp}\right\| + \frac{1}{q}\left\|\sum_{i=1}^{n}|A_i|^{rq}\right\| - (\frac{1}{p})\inf_{\|x\|=\|y\|=1}\eta(x,y)\right),$$

where $\eta(x,y) := \sum_{i=1}^{n}\left(\langle|B_i|^2 x, x\rangle^{rp/4} - \langle|A_i|^2 y, y\rangle^{rq/4}\right)^2$. *In particular,*

$$\begin{aligned}\left\|\sum_{i=1}^{n} A_i^* B_i\right\|^2 \leq n(\frac{1}{p}&\left\|\sum_{i=1}^{n}|B_i|^{2p}\right\| \\ &+\frac{1}{q}\left\|\sum_{i=1}^{n}|A_i|^{2q}\right\| - (\frac{1}{p})\inf_{\|x\|=\|y\|=1}\sum_{i=1}^{n}\left(\langle|B_i|^2 x, x\rangle^{p/2} - \langle|A_i|^2 y, y\rangle^{q/2}\right)^2).\end{aligned}$$

Example 3.1. Let $X = I, n = 1, f(t) = t^{\alpha}, g(t) = t^{1-\alpha}, \alpha = 1/2, r = p = q = 2$ and $|A|^2 = diag(5, 7), |B|^2 = diag(2, 3)$ in the inequalities (3.2) and (3.3) and Corollary 3.2 if needed. Then $\eta(x, y) := \left(\langle|B|^2 x, x\rangle - \langle|A|^2 y, y\rangle\right)^2$ and hence

$$\inf_{\|x\|=\|y\|=1}\eta(x,y) \geq 4 > 0.$$

For $n=2$ in inequality (3.3), we get the interesting norm inequalities that give an
estimate for the operator norm of commutators. Also for $A_i = B_i = I (i=1,2,...,n)$ in the inequality (3.3), we get the following operator inequalities for summation of operators.

Corollary 3.3. *Let $X_i \in \mathcal{B}(\boldsymbol{H}) (i=1,2,...,n), r \geq \frac{2}{q}$ and $\alpha \in (0,1)$. Then*

$$\left\| \sum_{i=1}^{n} X_i \right\|^r \leq n^{r-1} \left(\frac{1}{p} \left\| \sum_{i=1}^{n} |X_i|^{\alpha r p} \right\| + \frac{1}{q} \left\| \sum_{i=1}^{n} |X_i^*|^{(1-\alpha) r q} \right\| \right),$$

In particular, if $X_i (i=1,2,...,n)$ are normal, then

$$\left\| \sum_{i=1}^{n} X_i \right\|^r \leq n^{r-1} \left(\frac{1}{p} \left\| \sum_{i=1}^{n} |X_i|^{\alpha r p} \right\| + \frac{1}{q} \left\| \sum_{i=1}^{n} |X_i|^{(1-\alpha) r q} \right\| \right), \tag{3.4}$$

The inequality (3.4) is a generalized form of (3.1) and this inequality is not true for arbitrary operators.
The following example shows that in the inequality (3.4) normality of X_i is necessary,

Example 3.2. Let $X_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, (non normal) and $X_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $p=q=2, \alpha = 1/2$ and $r=1$. Then $\|X_1 + X_2\| = \sqrt{2}$ as $|X_1| + |X_2| = I$, consequently $\||X_1| + |X_2|\| = 1$, that is a contradiction with $\sqrt{2} \leq 1$.

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