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RESEARCH ARTICLE

Curvature Relations for Lagrangian Submersions From Globally Conformal Kaehler Manifolds

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ABSTRACT
In this paper, we derive curvature identities for Lagrangian submersions from globally conformal Kaehler manifolds onto Riemannian manifolds. Then, we give a relation between the horizontal lift of the curvature tensor of the base manifold and the curvature tensor of a fiber. We examine the necessary and sufficient conditions for the total manifolds of Lagrangian submersions to be Einstein. We also obtain Ricci, scalar, sectional, holomorphic bisectional and holomorphic sectional curvatures for these submersions. Finally, we give some inequalities involving the scalar and Ricci curvatures, and we also provide Chen-Ricci inequality for Lagrangian submersions from globally conformal Kaehler space forms.

Mathematics Subject Classification (2020): 53C18, 53C25, 53C55

Keywords: Riemannian submersion, Lagrangian submersion, globally conformal Kaehler manifold, Chen-Ricci inequality.

1. INTRODUCTION

Curvature invariants play the most important role in Riemannian geometry. They determine the intrinsic and extrinsic properties of Riemannian manifolds. Chen (1993) established a relationship between the intrinsic and extrinsic invariants. He also obtained an inequality between Ricci curvature and the squared mean curvature of a submanifold of a real space form (Chen (1999)). After then, he obtained a generalization of this inequality which is known as Chen-Ricci inequality (Chen (2005)).

On the other hand, the notion of Riemannian submersion is a generalization of an isometry between two Riemannian manifolds which was introduced by O'Neill (1966) and Gray (1967), independently. This notion was extended to almost complex and almost contact manifolds (Watson (1976), Chinea (1985)). After that, Riemannian submersions are studied widely in various kinds of structures for both almost complex and almost contact manifolds such as almost Hermitian (Şahin (2017)), almost contact (Taştan (2017)), cosymplectic (Taştan and Gerdan Aydın (2019)) and Sasakian (Taştan and Gerdan (2016)). These structures have also examined in different types of Riemannian submersions such as anti-invariant submersions (Şahin (2010)), Lagrangian submersions (Taştan (2014)) etc. Riemannian submersions have also been studied in globally conformal Kaehler manifolds which are a special class of Kaehler manifolds. The globally and locally conformal Kaehler manifolds were studied widely by Vaisman (1980). Then, locally conformal Kaehler submersions were introduced by Marrero and Rocha (1994) and studied by many researchers (Çimen et al. (2024), Pirinççi et al. (2023)).

In this paper, we study the curvature relations for Lagrangian submersions which are defined from globally conformal Kaehler manifolds onto Riemannian manifolds. First, we obtain curvature identities for Lagrangian submersions whose total manifolds are globally conformal Kaehler manifolds. Then, we give a relation between the horizontal lift of the curvature tensor of the base manifold and the curvature tensor of a fiber. We obtain Ricci curvatures and scalar curvatures for these submersions. Then, we give the necessary and sufficient conditions for the total manifolds of such submersions to be Einstein. We also obtain sectional, holomorphic bisectional and holomorphic sectional curvatures. Finally, we derive some inequalities involving the scalar curvature and Ricci curvature of Lagrangian submersions from globally conformal Kaehler space forms and give Chen-Ricci inequality for such submersions as well.

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2. GLOBALLY CONFORMAL KAEHLER MANIFOLDS

A Hermitian manifold (M^{2n}, J, g) with an almost complex structure J and a Hermitian metric g is called a *locally conformal Kaehler (l.c.K.) manifold*, if there exists an open cover $\{O_i\}_{i \in I}$ of M^{2n} with a family $\{\sigma_i\}_{i \in I}$ of smooth functions $\sigma_i : O_i \to \mathbb{R}$ such that

$$g_i = e^{-\sigma_i} g|_{O_i}$$

are Kaehler metrics for every $i \in I$ (Dragomir and Ornea (1998)). If $\tilde{g} = e^{-\sigma} g$ is Kaehlerian for a smooth function $\sigma : M^{2n} \to \mathbb{R}$, then (M^{2n}, J, g) is called a *globally conformal Kaehler (g.c.K.) manifold*. Dragomir and Ornea (1998) gave the following theorem for locally conformal Kaehler manifolds.

Theorem 2.1. *Let Φ be a 2-form defined by $\Phi(X, Y) = g(X, JY)$ on a Hermitian manifold (M^{2n}, J, g), where X, Y are vector fields on M^{2n}. Then (M^{2n}, J, g) is a locally conformal Kaehler manifold if and only if there exists a closed 1- form ω defined on M^{2n} globally such that $d\Phi = \omega \wedge \Phi$.*

If ω is exact, then (M^{2n}, J, g) is a g.c.K. manifold. In the case $\omega \equiv 0$, a g.c.K. manifold reduces a Kaehler manifold. The 1-form ω is called Lee form of (M^{2n}, J, g) and a g.c.K. manifold (M^{2n}, J, g) with Lee form ω is denoted by (M^{2n}, J, g, ω).

For the Riemannian connections ∇ of (M^{2n}, J, g, ω) and $\tilde{\nabla}$ of Kaehler metric $\tilde{g}$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\Big\{\omega(X)Y + \omega(Y)X - g(X, Y)B\Big\}, \tag{1}$$

where X, Y are vector fields on M^{2n} and B is the g- dual vector field of ω which is called *Lee vector field* of (M^{2n}, J, g, ω). $\tilde{\nabla}$ is a torsion-free connection and also satisfies $\tilde{\nabla} J = 0$. Hence, using (1), we have

$$(\nabla_X J)Y = \frac{1}{2}\Big\{\omega(JY)X - \omega(Y)JX - \Phi(X, Y)B + g(X, Y)JB\Big\}.$$

Now, from (1), Vaisman (1980) gave curvature identity between Riemannian curvature tensors of ∇ and $\tilde{\nabla}$ as follows:

$$\begin{aligned} \tilde{R}(X, Y)Z = {} & R(X, Y)Z \\ & - \frac{1}{2}\Big\{L(X, Z)Y - L(Y, Z)X - g(Y, Z)\Big[\nabla_X B + \frac{1}{2}\omega(X)B\Big] \\ & + g(X, Z)\Big[\nabla_Y B + \frac{1}{2}\omega(Y)B\Big]\Big\} \\ & - \frac{||\omega||^2}{4}\Big\{g(Y, Z)X - g(X, Z)Y\Big\}, \end{aligned} \tag{2}$$

where $||\omega||^2 = g(B, B)$,

$$L(X, Y) = (\nabla_X \omega)(Y) + \frac{1}{2}\omega(X)\omega(Y) = g(\nabla_X B, Y) + \frac{1}{2}\omega(X)\omega(Y),$$

and X, Y, Z are vector fields on M^{2n}.

We note that L is a symmetric (0,2)-tensor on M^{2n}. From (2), he also obtained the well-known formula:

$$\begin{aligned} e^{\sigma}\tilde{R}(X, Y, Z, W) = {} & R(X, Y, Z, W) \\ & - \frac{1}{2}\Big\{L(X, Z)g(Y, W) - L(Y, Z)g(X, W) \\ & - L(X, W)g(Y, Z) + L(Y, W)g(X, Z)\Big\} \\ & - \frac{||\omega||^2}{4}\Big\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\Big\}, \end{aligned} \tag{3}$$

where $\tilde{R}$ denotes the Riemannian curvature tensor of the Kaehler metric $\tilde{g}$. Now, since $\tilde{g}$ is a Kaehler metric, the Riemannian curvature tensor $\tilde{R}$ satisfies $\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W)$. If we use the last equation in (3), then we get the following result as in the l.c.K submersion case in Pirinççi et al. (2023).

Theorem 2.2. *Let (M, J, g, ω) be a g.c.K. manifold. Then we have*

$$\begin{aligned} R(X,Y,Z,W) = \ & R(JX,JY,JZ,JW) \\ & + \frac{1}{2}\Big\{\delta(X,Z)g(Y,W) - \delta(Y,Z)g(X,W) \\ & - \delta(X,W)g(Y,Z) + \delta(Y,W)g(X,Z)\Big\}, \end{aligned} \tag{4}$$

where

$$\delta(X,Y) = L(X,Y) - L(JX,JY),$$

and X, Y, Z, W are vector fields on M^{2n}.

3. LAGRANGIAN SUBMERSIONS

In this section, we will give the notion of Riemannian submersion and its special type Lagrangian submersion. We deduce the curvature relations for Lagrangian submersions.

Let (M_1^n, g_1) and (M_2^m, g_2) be Riemannian manifolds with dimensions n and m, respectively. O'Neill (1966) called a mapping ψ of (M_1^n, g_1) onto (M_2^m, g_2) that satisfies the following two conditions a *Riemannian submersion*:

(i) The rank of ψ is maximal;

which means that the derivative map ψ_* is surjective. Hence for each $y \in M_2^m$, $\psi^{-1}(y)$ is an $(n-m)$-dimensional closed submanifold of M_1^n. A submanifold $\psi^{-1}(y)$ is called a *fiber*. The vector fields on M_1^n which are tangent to a fiber is called *vertical*, and the vector fields on M_1^n orthogonal to a fiber is called *horizontal*. Vertical and horizontal distributions of the tangent space of M_1^n are denoted by $ker\psi_*$ and $(ker\psi_*)^\perp$, respectively. A horizontal vector field X on M_1^n is called *basic* if $\psi_*(X) = X_*$, for a vector field X_* on M_2^m.

(ii) ψ_* is a linear isometry on $(ker\psi_*)^\perp$.

Let E^v and E^h be the vertical and horizontal part of a vector field on M_1^n, respectively. Then, the covariant derivatives of vertical and horizontal vector fields are defined by O'Neill (1966) as follows:

$$\mathcal{T}_E F = (\nabla_{E^v} F^v)^h + (\nabla_{E^v} F^h)^v, \tag{5}$$

$$\mathcal{A}_E F = (\nabla_{E^h} F^v)^h + (\nabla_{E^h} F^h)^v, \tag{6}$$

where E and F are vector fields on M_1^n and ∇ is the Riemannian connection of g_1. The tensors $\mathcal{T}$ and $\mathcal{A}$ defined above are called *O'Neill's tensors*. $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators and each one reverses the vertical distribution to the horizontal distribution, and vice versa.

Lemma 3.1. *Let $\psi : (M_1^n, g_1) \to (M_2^m, g_2)$ be a Riemannian submersion, and X, Y be basic vector fields on M_1^n. Then,*

(i) $g_1(X,Y) = g_2(X_*, Y_*) \circ \psi$,

(ii) $\psi_*([X,Y]^h) = [X_*, Y_*]$,

(iii) $\psi_*((\nabla_X Y)^h) = \nabla^*_{X_*} Y_*$.

Using (5), (6) and Lemma 3.1 we obtain the following equations:

$$\mathcal{T}_U V = \mathcal{T}_V U, \tag{7}$$

$$\mathcal{A}_X Y = - \mathcal{A}_Y X = \frac{1}{2}[X,Y]^v, \tag{8}$$

$$\nabla_U V = \mathcal{T}_U V + (\nabla_U V)^v, \tag{9}$$

$$\nabla_U X = (\nabla_U X)^h + \mathcal{T}_U X, \tag{10}$$

$$\nabla_X U = \mathcal{A}_X U + (\nabla_X U)^v, \tag{11}$$

$$\nabla_X Y = (\nabla_X Y)^h + \mathcal{A}_X Y, \tag{12}$$

$$(\nabla_U X)^h = \mathcal{A}_X U, \qquad \textit{for a basic vector field } X, \tag{13}$$

where $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*)^\perp$. $\mathcal{T}$ is the second fundamental form of all the fibers. We say that the fibers are totally geodesic when $\mathcal{T} = 0$. If $\{U_1, ..., U_n\}$ is an ortonormal frame of $ker\psi_*$, then $H = \frac{1}{n}\sum_{i=1}^n \mathcal{T}_{U_i} U_i$ is called the *mean curvature vector field* of the fibers. For more information about Riemannian submersions, we refer to O'Neill (1966) and Falcitelli et al. (2004).

Now, using Lemma 3.1 and the equations (7)~(13) we have the following curvature relations for every $U, V, W, W' \in ker\psi_*$

and $X, Y, Z, Z' \in (ker\psi_*)^\perp$:

$$R_1(U,V,W,W') = \hat{R}(U,V,W,W') + g_1(\mathcal{T}_U W', \mathcal{T}_V W) - g_1(\mathcal{T}_U W, \mathcal{T}_V W'), \tag{14}$$
$$R_1(U,V,W,X) = g_1((\nabla_V \mathcal{T})_U W, X) - g_1((\nabla_U \mathcal{T})_V W, X), \tag{15}$$
$$R_1(X,Y,Z,Z') = R^*(X,Y,Z,Z') - 2g_1(\mathcal{A}_X Y, \mathcal{A}_Z Z') - g_1(\mathcal{A}_X Z, \mathcal{A}_Y Z') + g_1(\mathcal{A}_X Z', \mathcal{A}_Y Z), \tag{16}$$
$$R_1(X,Y,Z,U) = g_1((\nabla_Z \mathcal{A})_X Y, U) + g_1(\mathcal{A}_X Y, \mathcal{T}_U Z) + g_1(\mathcal{A}_X Z, \mathcal{T}_U Y) - g_1(\mathcal{A}_Y Z, \mathcal{T}_U X), \tag{17}$$
$$R_1(X,Y,U,V) = g_1((\nabla_U \mathcal{A})_X Y, V) - g_1((\nabla_V \mathcal{A})_X Y, U) + g_1(\mathcal{A}_X U, \mathcal{A}_Y V) - g_1(\mathcal{A}_X V, \mathcal{A}_Y U) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y) + g_1(\mathcal{T}_V X, \mathcal{T}_U Y), \tag{18}$$
$$R_1(X,U,Y,V) = g_1((\nabla_X \mathcal{T})_U V, Y) + g_1((\nabla_U \mathcal{A})_X Y, V) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y) + g_1(\mathcal{A}_X U, \mathcal{A}_Y V), \tag{19}$$

where R_1 and R_2 are Riemannian curvature tensors of M_1^n and M_2^m, respectively, R^* is the horizontal lift of the curvature tensor of R_2, i.e., $R^*(X,Y,Z,Z') = g_1(R^*(Z,Z')Y,X) = R_2(\psi_* X, \psi_* Y, \psi_* Z, \psi_* Z') \circ \psi$ and $\hat{R}$ is the curvature tensor of $\psi^{-1}(y)$ (see O'Neill (1966)).

We note that $(\nabla_E \mathcal{A})_F$ and $(\nabla_E \mathcal{T})_F$ are skew-symmetric and linear operators defined by

$$(\nabla_E \mathcal{A})_F G = \nabla_E(\mathcal{A}_F G) - \mathcal{A}_{(\nabla_E F)} G - \mathcal{A}_F(\nabla_E G),$$
$$(\nabla_E \mathcal{T})_F G = \nabla_E(\mathcal{T}_F G) - \mathcal{T}_{(\nabla_E F)} G - \mathcal{T}_F(\nabla_E G),$$

respectively, where E, F and G are vector fields on M_1^n. Moreover, $g_1((\nabla_E \mathcal{A})_X Y, U)$ is alternate in X and Y, $g_1((\nabla_E \mathcal{T})_U V, X)$ is symmetric in U and V, where $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*)^\perp$. Furthermore, (14) and (15) are the corresponding Gauss and Codazzi equations, (16) and (17) are their dual equations.

Definition 3.2. Let ψ be a Riemannian submersion from a Hermitian manifold (M_1^{2n}, J, g_1) onto a Riemannian manifold (M_2^m, g_2). ψ is called an *anti-invariant Riemannian submersion*, if its vertical distribution is anti-invariant with respect to J, i.e. $J(ker\psi_*) \subseteq (ker\psi_*)^\perp$. Especially, ψ is called a *Lagrangian submersion* when $J(ker\psi_*) = (ker\psi_*)^\perp$. In this case J reverses the vertical (horizontal) distributions to the horizontal (vertical) distributions, and $m = n$.

4. CURVATURE IDENTITIES FOR LAGRANGIAN SUBMERSIONS

In this section, we obtain curvature relations using the following result due to Pirinççi (2025). From now on, $(M_1^{2n}, J, g_1, \omega)$ represents a g.c.K. manifold and (M_2^n, g_2) represents a Riemannian manifold.

Lemma 4.1. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion. Then we have*

$$\mathcal{T}_U JV = J\mathcal{T}_U V + \frac{1}{2}\{\omega(JV)U + g_1(U,V)JB^h\},$$
$$\mathcal{T}_U JX = J\mathcal{T}_U X - \frac{1}{2}\{\omega(X)JU - g_1(JU,X)B^h\},$$
$$\mathcal{A}_X JU = J\mathcal{A}_X U - \frac{1}{2}\{\omega(U)JX + g_1(X,JU)B^v\},$$
$$\mathcal{A}_X JY = J\mathcal{A}_X Y + \frac{1}{2}\{\omega(JY)X + g_1(X,Y)JB^v\},$$

where $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*^\perp)$.

Pirinççi (2025) showed that for a Lagrangian submersion ψ from a l.c.K. manifold onto a Riemannian manifold, if JU is a basic vector field for any $U \in ker\psi_*$, then the Lee vector field B cannot be vertical. Therefore, we will examine the curvature relations in the special case where the Lee vector field B is horizontal. In this case he showed that the horizontal distribution is integrable and totally geodesic, i.e., $\mathcal{A} \equiv 0$. Then we get the following result from (13)~(18):

Corollary 4.2. Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Then, we have the following curvature relations for every $U, V, W, W' \in ker\psi_*$ and $X, Y, Z, Z' \in (ker\psi_*)^{\perp}$:

$$
\begin{aligned}
R_1(U, V, W, W') &= \hat{R}(U, V, W, W') + g_1(\mathcal{T}_U W', \mathcal{T}_V W) - g_1(\mathcal{T}_U W, \mathcal{T}_V W'), && (20)\\
R_1(U, V, W, X) &= g_1((\nabla_V \mathcal{T})_U W, X) - g_1((\nabla_U \mathcal{T})_V W, X), && (21)\\
R_1(X, Y, Z, Z') &= R^*(X, Y, Z, Z'), && (22)\\
R_1(X, Y, Z, U) &= 0, && (23)\\
R_1(X, Y, U, V) &= g_1(\mathcal{T}_V X, \mathcal{T}_U Y) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y), && (24)\\
R_1(X, U, Y, V) &= g_1((\nabla_X \mathcal{T})_U V, Y) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y). && (25)
\end{aligned}
$$

Now, if we use $\mathcal{A} \equiv 0$, (20) and (22) in (4), then we get the following relation between the horizontal lift of the curvature tensor of R_2 and the curvature tensor of a fiber.

Theorem 4.3. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Then, we have the following curvature relation for every $U, V, W, W' \in ker\psi_*$:*

$$
\begin{aligned}
\hat{R}(U, V, W, W') = {} & R^*(JU, JV, JW, JW') - g_1(\mathcal{T}_U W', \mathcal{T}_V W) + g_1(\mathcal{T}_U W, \mathcal{T}_V W') \\
& + \frac{1}{2}\Big\{\delta(U, W) g_1(V, W') - \delta(V, W) g_1(U, W') \\
& - \delta(U, W') g_1(V, W) + \delta(V, W') g_1(U, W)\Big\}. \qquad (26)
\end{aligned}
$$

In a similar way, if we use (21) and (23) in (4), then we have

$$
g_1((\nabla_V \mathcal{T})_U W, X) - g_1((\nabla_U \mathcal{T})_V W, X) = \frac{1}{2}\Big\{\delta(V, X) g_1(U, W) - \delta(U, X) g_1(V, W)\Big\}.
$$

Using the skew-symmetry property of the operator $(\nabla_V \mathcal{T})_U$, we get

$$
(\nabla_U \mathcal{T})_V X - (\nabla_V \mathcal{T})_U X = \frac{1}{2}\Big\{\delta(V, X) U - \delta(U, X) V\Big\}. \qquad (27)
$$

We will examine the conditions for M_1^{2n} to be an Einstein manifold. To obtain these conditions we will first find the Ricci and scalar curvatures of M_1^{2n} using the following notation:

$$
\begin{aligned}
\mathcal{T}_{ij}^k &= g_1(\mathcal{T}_{U_i} U_j, JU_k), && (28)\\
||\mathcal{T}||^2 &= \sum_{i,j=1}^{n} g_1(\mathcal{T}_{U_i} U_j, \mathcal{T}_{U_i} U_j) = \sum_{k=1}^{n} \sum_{i,j=1}^{n} (\mathcal{T}_{ij}^k)^2, && (29)\\
\delta(\mathcal{T}) &= \sum_{i,j=1}^{n} g_1((\nabla_{JU_i} \mathcal{T})_{U_j} U_j, JU_i), && (30)
\end{aligned}
$$

where $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$.

Lemma 4.4. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Then the Ricci tensor Ric_1 and the scalar curvature ρ_1 of M_1^{2n} satisfy the following relations:*

$$
\begin{aligned}
Ric_1(U, V) = {} & \hat{Ric}(U, V) - \frac{1}{2}\omega(JU)\omega(JV) + g_1(U, V)||\omega||^2 \\
& - n g_1(H, \mathcal{T}_U V) + \sum_{i=1}^{n} g_1((\nabla_{JU_i} \mathcal{T})_U V, JU_i), \qquad (31)\\
Ric_1(U, X) = {} & \frac{n-1}{2}(\nabla_U \omega) X, \qquad (32)\\
Ric_1(X, Y) = {} & Ric^*(X, Y) + \sum_{i=1}^{n} \Big\{g_1((\nabla_X \mathcal{T})_{U_i} U_i, Y) - g_1(\mathcal{T}_{U_i} X, \mathcal{T}_{U_i} Y)\Big\}, \qquad (33)\\
\rho_1 = {} & \hat{\rho} + \rho^* + (2n-1)||\omega||^2 - n^2||H||^2 - ||\mathcal{T}||^2 + 2\delta(\mathcal{T}), \qquad (34)
\end{aligned}
$$

where $U, V \in ker\psi_*$, $X, Y \in (ker\psi_*)^{\perp}$, $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$, $\hat{Ric}$ is Ricci tensor of any fiber, Ric^* is the horizontal lift of Ricci tensor of M_2^n, $\hat{\rho}$ is scalar curvature of any fiber and ρ^* is the lift of scalar curvature of M_2^n.

Proof. Let $\{U_1,...,U_n\}$ be an ortonormal frame of $ker\psi_*$. From the definition of Ricci tensor, (20) and (25) we have

$$
\begin{aligned}
Ric_1(U,V) &= \sum_{i=1}^{n} R_1(U_i,U,U_i,V) + \sum_{i=1}^{n} R_1(JU_i,U,JU_i,V)\\
&= \sum_{i=1}^{n}\Big\{\hat{R}(U_i,U,U_i,V) + g_1(\mathcal{T}_{U_i}V,\mathcal{T}_U U_i) - g_1(\mathcal{T}_{U_i}U_i,\mathcal{T}_U V)\Big\}\\
&\quad + \sum_{i=1}^{n}\Big\{g_1((\nabla_{JU_i}\mathcal{T})_U V, JU_i) - g_1(\mathcal{T}_U JU_i,\mathcal{T}_V JU_i)\Big\}\\
&= \hat{Ric}(U,V) - n g_1(H,\mathcal{T}_U V)\\
&\quad + \sum_{i=1}^{n}\Big\{g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) + g_1((\nabla_{JU_i}\mathcal{T})_U V, JU_i) - g_1(\mathcal{T}_U JU_i,\mathcal{T}_V JU_i)\Big\}.
\end{aligned}
\tag{35}
$$

Now, from Lemma 4.1 we have

$$
\begin{aligned}
\sum_{i=1}^{n} g_1(\mathcal{T}_U JU_i,\mathcal{T}_V JU_i) &= \sum_{i=1}^{n} g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) + \frac{1}{2}\sum_{i=1}^{n}\Big\{\omega(JU_i)g_1(J\mathcal{T}_U U_i,V)\\
&\quad + g_1(V,U_i)g_1(\mathcal{T}_U U_i,B^h) + \omega(JU_i)g_1(U,J\mathcal{T}_V U_i)\\
&\quad + g_1(U,U_i)g_1(\mathcal{T}_V U_i,B^h)\Big\}\\
&= \sum_{i=1}^{n} g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) - \frac{1}{2}\sum_{i=1}^{n}\Big\{g_1(U_i,JB)g_1(U_i,\mathcal{T}_U JV)\\
&\quad + g_1(V,U_i)g_1(U_i,\mathcal{T}_U B) + g_1(U_i,JB)g_1(\mathcal{T}_V JU,U_i)\\
&\quad + g_1(U,U_i)g_1(U_i,\mathcal{T}_V B)\Big\}\\
&= \sum_{i=1}^{n} g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i)\\
&\quad + \frac{1}{2}\Big\{\omega(J\mathcal{T}_U JV) + 2\omega(\mathcal{T}_U V) + \omega(J\mathcal{T}_V JU)\Big\}.
\end{aligned}
\tag{36}
$$

Moreover, from Lemma 4.1, since

$$
\begin{aligned}
\omega(J\mathcal{T}_U JV) &= -g_1(\mathcal{T}_U JV, JB)\\
&= -g_1(J\mathcal{T}_U V + \frac{1}{2}\{\omega(JV)U + g_1(U,V)JB^h\}, JB)\\
&= -\omega(\mathcal{T}_U V) + \frac{1}{2}\omega(JV)\omega(JU) - g_1(U,V)||\omega||^2\\
&= \omega(J\mathcal{T}_V JU),
\end{aligned}
$$

equation (36) becomes

$$
\sum_{i=1}^{n} g_1(\mathcal{T}_U JU_i,\mathcal{T}_V JU_i) = \sum_{i=1}^{n} g_1(\mathcal{T}_U U_i,\mathcal{T}_V U_i) + \frac{1}{2}\omega(JU)\omega(JV) - g_1(U,V)||\omega||^2. \tag{37}
$$

Using (37) in (35), we obtain (31).
Similarly, using (21), (22), (23), (25) and (27) we obtain (32) and (33). Now, from the definition of scalar curvature, (31) and (33)

we have

$$\begin{aligned}
\rho_1 &= \sum_{j=1}^{n} Ric_1(U_j, U_j) + \sum_{j=1}^{n} Ric_1(JU_j, JU_j) \\
&= \sum_{j=1}^{n} \Big\{ \hat{Ric}(U_j, U_j) - \frac{1}{2}\omega(JU_j)\omega(JU_j) + g_1(U_j, U_j)||\omega||^2 \\
&\quad + \sum_{i=1}^{n} \Big[g_1((\nabla_{JU_i}\mathcal{T})_{U_j}U_j, JU_i) - g_1(\mathcal{T}_{U_i}U_i, \mathcal{T}_{U_j}U_j) \Big] \Big\} \\
&\quad + \sum_{j=1}^{n} \Big\{ Ric^*(JU_j, JU_j) + \sum_{i=1}^{n} \Big[g_1((\nabla_{JU_j}\mathcal{T})_{U_i}U_i, JU_j) - g_1(\mathcal{T}_{U_i}JU_j, \mathcal{T}_{U_i}JU_j) \Big] \Big\} \\
&= \hat{\rho} + \rho^* + \frac{2n-1}{2}||\omega||^2 - n^2||H||^2 \\
&\quad + 2\sum_{i,j=1}^{n} g_1((\nabla_{JU_i}\mathcal{T})_{U_j}U_j, JU_i) - \sum_{i,j=1}^{n} g_1(\mathcal{T}_{U_i}JU_j, \mathcal{T}_{U_i}JU_j)
\end{aligned}$$

Finally, if we use (29), (30) and (37) in the last equation, we get (34).

Theorem 4.5. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Then, $(M_1^{2n}, J, g_1, \omega)$ is an Einstein manifold if and only if the following relations hold:*

$$\begin{aligned}
\hat{Ric}(U, V) &= (\frac{\rho_1}{2n} - ||\omega||^2)g_1(U, V) + ng_1(H, \mathcal{T}_U V) + \frac{1}{2}\omega(JU)\omega(JV) \\
&\quad - \sum_{i=1}^{n} g_1((\nabla_{JU_i}\mathcal{T})_U V, JU_i), \\
Ric^*(X, Y) &= \frac{\rho_1}{2n}g_1(X, Y) - \sum_{i=1}^{n} \Big\{ g_1((\nabla_X\mathcal{T})_{U_i}U_i, Y) - g_1(\mathcal{T}_{U_i}X, \mathcal{T}_{U_i}Y) \Big\},
\end{aligned}$$

and

$$(\nabla_U\omega)X = 0,$$

where $U, V \in ker\psi_*$, $X, Y \in (ker\psi_*)^\perp$ and $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$.

Proof. $(M_1^{2n}, J, g_1, \omega)$ is an Einstein manifold if and only if $Ric_1 = \frac{\rho_1}{2n}g_1$. Using this in (31), (32) and (33), we get the results.

Theorem 4.6. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*)^\perp$, then the sectional curvature K_1 is given by*

$$\begin{aligned}
K_1(U, V) &= \hat{K}(U, V) + \frac{||\mathcal{T}_U V||^2 - g_1(\mathcal{T}_U U, \mathcal{T}_V V)}{||U \wedge V||^2}, \\
K_1(X, Y) &= K^*(X, Y), \\
K_1(X, U) &= \frac{g_1((\nabla_X\mathcal{T})_U U, X) - ||\mathcal{T}_U X||^2}{||X||^2||U||^2},
\end{aligned}$$

where $||U \wedge V||^2 = ||U||^2||V||^2 - (g_1(U, V))^2$.

Proof. If we use the definition of the sectional curvature $K_1(E, F) = \frac{R_1(E, F, E, F)}{||E \wedge F||^2}$ in (20), then we have

$$\begin{aligned}
K_1(U, V) &= \frac{R_1(U, V, U, V)}{||U \wedge V||^2} \\
&= \frac{1}{||U \wedge V||^2}\Big\{ \hat{R}(U, V, U, V) + g_1(\mathcal{T}_U V, \mathcal{T}_V U) - g_1(\mathcal{T}_U U, \mathcal{T}_V V) \Big\} \\
&= \hat{K}(U, V) + \frac{||\mathcal{T}_U V||^2 - g_1(\mathcal{T}_U U, \mathcal{T}_V V)}{||U \wedge V||^2}.
\end{aligned}$$

Similarly, using (22) and (25), we obtain the other two equations.

The *holomorphic bisectional curvature* and the *holomorphic sectional curvature* of an almost Hermitian manifold (M^{2n}, J, g) are defined for any nonzero vector fields E, F on M^{2n} as

$$B(E,F) = \frac{R(E,JE,F,JF)}{||E||^2||F||^2},$$

and

$$H(E) = B(E,E),$$

respectively. Hence, we have the following results.

Theorem 4.7. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $U, V \in ker\psi_*$ and $X, Y \in (ker\psi_*)^\perp$, then the holomorphic bisectional curvature B_1 is given by*

$$B_1(U,V) = \frac{g_1((\nabla_{JU}\mathcal{T})_U V, JV) - g_1(\mathcal{T}_U JU, \mathcal{T}_V JV)}{||U||^2||V||^2},$$
$$B_1(X,Y) = \frac{g_1((\nabla_X \mathcal{T})_{JX} JY, Y) - g_1(\mathcal{T}_{JX} X, \mathcal{T}_{JY} Y)}{||X||^2||Y||^2},$$
$$B_1(X,U) = \frac{g_1(\mathcal{T}_{JX} X, \mathcal{T}_U JU) - g_1((\nabla_X \mathcal{T})_{JX} U, JU)}{||X||^2||U||^2}.$$

Proof. Using the definition of the holomorphic bisectional curvature in (25), we obtain the results immediately.

Theorem 4.8. *Let $\psi : (M_1^{2n}, J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $U \in ker\psi_*$ and $X \in (ker\psi_*)^\perp$, then the holomorphic sectional curvature H_1 is given by*

$$H_1(U) = \frac{g_1((\nabla_{JU}\mathcal{T})_U U, JU) - ||\mathcal{T}_U JU||^2}{||U||^4},$$
$$H_1(X) = \frac{g_1((\nabla_X \mathcal{T})_{JX} JX, X) - ||\mathcal{T}_{JX} X||^2}{||X||^4}.$$

Proof. Using the definition of the holomorphic bisectional curvature in Theorem 4.7, we obtain the above equations.

5. CHEN-RICCI INEQUALITY

A Kaehler manifold (M^{2n}, J, g) with constant holomorphic sectional curvature c is called a *complex space form* and denoted by $(M^{2n}(c), J, g)$. The curvature tensor R of $(M^{2n}(c), J, g)$ satisfies

$$\begin{aligned} R(X,Y,Z,W) = \frac{c}{4}\Big\{ & g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(JX,W)g(JY,Z) \\ & - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,W)\Big\} \end{aligned} \tag{38}$$

for every vector fields X, Y, Z, W on $M^{2n}(c)$.

A g.c.K. manifold $(M_1^{2n}, J, g_1, \omega)$ with constant holomorphic sectional curvature c is called a *globally conformal complex space form* and denoted by $(M_1^{2n}(c), J, g_1, \omega)$. Using (3) and (38), we get

$$\begin{aligned} R_1(X,Y,Z,W) = e^{-\sigma}\frac{c}{4}\Big\{ & g_1(X,W)g_1(Y,Z) - g_1(X,Z)g_1(Y,W) + g_1(JX,W)g_1(JY,Z) \\ & - g_1(JX,Z)g_1(JY,W) - 2g_1(JX,Y)g_1(JZ,W)\Big\} \\ + \frac{1}{2}\Big\{ & L(X,Z)g_1(Y,W) - L(Y,Z)g_1(X,W) \\ & - L(X,W)g_1(Y,Z) + L(Y,W)g_1(X,Z)\Big\} \\ + \frac{||\omega||^2}{4}\Big\{ & g_1(Y,Z)g_1(X,W) - g_1(X,Z)g_1(Y,W)\Big\} \end{aligned} \tag{39}$$

for every vector fields X, Y, Z, W on $M_1^{2n}(c)$.

Let $\psi : (M_1^{2n}(c), J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. Now, using (20) and (22) in (39), we have

$$\hat{R}(U,V,W,W') = \left(\frac{ce^{-\sigma}+||\omega||^2}{4}\right)\Big\{g_1(U,W')g_1(V,W) - g_1(U,W)g_1(V,W')\Big\}$$
$$+\frac{1}{2}\Big\{L(U,W)g_1(V,W') - L(V,W)g_1(U,W') \tag{40}$$
$$- L(U,W')g_1(V,W) + L(V,W')g_1(U,W)\Big\}$$
$$+ g_1(\mathcal{T}_U W, \mathcal{T}_V W') - g_1(\mathcal{T}_U W', \mathcal{T}_V W)$$

and

$$R^*(X,Y,Z,Z') = \left(\frac{ce^{-\sigma}+||\omega||^2}{4}\right)\Big\{g_1(X,Z')g_1(Y,Z) - g_1(X,Z)g_1(Y,Z')\Big\}$$
$$+\frac{1}{2}\Big\{L(X,Z)g_1(Y,Z') - L(Y,Z)g_1(X,Z')$$
$$- L(X,Z')g_1(Y,Z) + L(Y,Z')g_1(X,Z)\Big\},$$

for every $U,V,W,W' \in ker\psi_*$ and $X,Y,Z,Z' \in (ker\psi_*)^{\perp}$.

We will use the following remark in the examination of the curvature relations.

Remark 5.1. Pirinççi (2025) showed that the vertical distribution of a Lagrangian submersion from a l.c.K. manifold onto a Riemannian manifold cannot be totally geodesic, i.e., $\mathcal{T} \neq 0$.

Proposition 5.2. *Let $\psi : (M_1^{2n}(c), J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$, then we have*

$$\hat{Ric}(U_1) < (1-n)\left(\frac{ce^{-\sigma}+||\omega||^2}{4}\right) + ng_1(\mathcal{T}_{U_1}U_1, H) \tag{41}$$
$$-\frac{1}{2}\Big\{(n-2)\omega(\mathcal{T}_{U_1}U_1) + n\omega(H)\Big\},$$

where

$$\hat{Ric}(U_1) = \sum_{i=1}^{n} \hat{R}(U_1, U_i, U_1, U_i). \tag{42}$$

Proof. We note that if the Lee vector field B is horizontal, then $L(U,V) = -\omega(\mathcal{T}_U V)$. So for every $U,V,W,W' \in ker\psi_*$, (40) becomes

$$\hat{R}(U,V,W,W') = \left(\frac{ce^{-\sigma}+||\omega||^2}{4}\right)\Big\{g_1(U,W')g_1(V,W) - g_1(U,W)g_1(V,W')\Big\}$$
$$-\frac{1}{2}\Big\{\omega(\mathcal{T}_U W)g_1(V,W') - \omega(\mathcal{T}_V W)g_1(U,W') \tag{43}$$
$$- \omega(\mathcal{T}_U W')g_1(V,W) + \omega(\mathcal{T}_V W')g_1(U,W)\Big\}$$
$$+ g_1(\mathcal{T}_U W, \mathcal{T}_V W') - g_1(\mathcal{T}_U W', \mathcal{T}_V W).$$

Using (43) in (42), we have

$$\hat{Ric}(U_1) = \sum_{i=1}^{n}\left(\frac{ce^{-\sigma}+||\omega||^2}{4}\right)\Big\{g_1(U_1,U_i)g_1(U_i,U_1) - g_1(U_1,U_1)g_1(U_i,U_i)\Big\}$$
$$-\frac{1}{2}\sum_{i=1}^{n}\Big\{\omega(\mathcal{T}_{U_1}U_1)g_1(U_i,U_i) - \omega(\mathcal{T}_{U_i}U_1)g_1(U_1,U_i)$$
$$- \omega(\mathcal{T}_{U_1}U_i)g_1(U_i,U_1) + \omega(\mathcal{T}_{U_i}U_i)g_1(U_1,U_1)\Big\}$$
$$+\sum_{i=1}^{n}\Big\{g_1(\mathcal{T}_{U_1}U_1, \mathcal{T}_{U_i}U_i) - g_1(\mathcal{T}_{U_1}U_i, \mathcal{T}_{U_i}U_1)\Big\}$$
$$= (1-n)\left(\frac{ce^{-\sigma}+||\omega||^2}{4}\right) + ng_1(\mathcal{T}_{U_1}U_1, H) - ||\mathcal{T}_{U_i}U_1||^2$$
$$-\frac{1}{2}\Big\{(n-2)\omega(\mathcal{T}_{U_1}U_1) + n\omega(H)\Big\}.$$

Hence, (41) comes from Remark 5.1.

Proposition 5.3. *Let $\psi : (M_1^{2n}(c), J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$, then the scalar curvature of the vertical distribution holds*

$$\hat{\rho} < n(1-n)\Big(\frac{ce^{-\sigma} + ||\omega||^2}{4} + \omega(H)\Big) + n^2||H||^2, \tag{44}$$

where

$$\hat{\rho} = \sum_{i,j=1}^{n} \hat{R}(U_i, U_j, U_i, U_j). \tag{45}$$

Proof. If we use (43) in (45), then we have

$$\begin{aligned}
\hat{\rho} = &\sum_{i,j=1}^{n} \Big(\frac{ce^{-\sigma} + ||\omega||^2}{4}\Big)\Big\{g_1(U_i, U_j)g_1(U_j, U_i) - g_1(U_i, U_i)g_1(U_j, U_j)\Big\} \\
&- \frac{1}{2}\sum_{i,j=1}^{n} \Big\{\omega(\mathcal{T}_{U_i}U_i)g_1(U_j, U_j) - \omega(\mathcal{T}_{U_j}U_i)g_1(U_i, U_j) \\
&- \omega(\mathcal{T}_{U_i}U_j)g_1(U_j, U_i) + \omega(\mathcal{T}_{U_j}U_j)g_1(U_i, U_i)\Big\} \\
&+ \sum_{i,j=1}^{n} \Big\{g_1(\mathcal{T}_{U_i}U_i, \mathcal{T}_{U_j}U_j) - g_1(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_j}U_i)\Big\} \\
= &\, n(1-n)\Big(\frac{ce^{-\sigma} + ||\omega||^2}{4}\Big) - ||\mathcal{T}||^2 + n^2||H||^2 + n(1-n)\omega(H).
\end{aligned}$$

The result is obtained by using Remark 5.1.

Proposition 5.4. *Let $\psi : (M_1^{2n}(c), J, g_1, \omega) \to (M_2^n, g_2)$ be a Lagrangian submersion and the Lee vector field B be horizontal. If $\{U_1, ..., U_n\}$ is an orthonormal frame of $ker\psi_*$, then the scalar curvature of the horizontal distribution holds*

$$\rho^* < n(1-n)\Big(\frac{ce^{-\sigma} + ||\omega||^2}{4} - \omega(H)\Big) + ||\mathcal{T}||^2 + (n-1)\text{Trace}(L), \tag{46}$$

where

$$\rho^* = \sum_{i,j=1}^{n} R^*(JU_i, JU_j, JU_i, JU_j). \tag{47}$$

Proof. From (26) we have

$$\begin{aligned}\sum_{i,j=1}^{n}\hat{R}(U_i,U_j,U_i,U_j) &= \sum_{i,j=1}^{n}R^*(JU_i,JU_j,JU_i,JU_j)\\ &\quad-\sum_{i,j=1}^{n}g_1(\mathcal{T}_{U_i}U_j,\mathcal{T}_{U_j}U_i)+\sum_{i,j=1}^{n}g_1(\mathcal{T}_{U_i}U_i,\mathcal{T}_{U_j}U_j)\\ &\quad+\frac{1}{2}\sum_{i,j=1}^{n}\Big\{\delta(U_i,U_i)g_1(U_j,U_j)-\delta(U_j,U_i)g_1(U_i,U_j)\\ &\qquad-\delta(U_i,U_j)g_1(U_j,U_i)+\delta(U_j,U_j)g_1(U_i,U_i)\Big\}\\ &=\sum_{i,j=1}^{n}R^*(JU_i,JU_j,JU_i,JU_j)-||\mathcal{T}||^2+n^2||H||^2\\ &\quad+(n-1)\sum_{i=1}^{n}\delta(U_i,U_i)\\ &=\sum_{i,j=1}^{n}R^*(JU_i,JU_j,JU_i,JU_j)-||\mathcal{T}||^2+n^2||H||^2\\ &\quad+(n-1)\sum_{i=1}^{n}\Big\{L(U_i,U_i)-L(JU_i,JU_i)\Big\}\\ &=\sum_{i,j=1}^{n}R^*(JU_i,JU_j,JU_i,JU_j)-||\mathcal{T}||^2+n^2||H||^2\\ &\quad-n(n-1)\omega(H)-(n-1)\sum_{i=1}^{n}L(JU_i,JU_i).\end{aligned}$$

If we use (45), (47) and

$$\mathrm{Trace}(L)=\sum_{i=1}^{n}\Big\{L(U_i,U_i)+L(JU_i,JU_i)\Big\}=-n\omega(H)+\sum_{i=1}^{n}L(JU_i,JU_i),$$

in the last equation, then we have

$$\hat{\rho}=\rho^*-||\mathcal{T}||^2+n^2||H||^2-2n(n-1)\omega(H)-(n-1)\mathrm{Trace}(L). \tag{48}$$

Finally, using (44) in (48), we get (46).

Now, we give the Chen-Ricci inequality for a Lagrangian submersion from a g.c.K manifold onto a Riemannian manifold by using the following equation which was introduced by Gülbahar et al. (2017):

$$\begin{aligned}||\mathcal{T}||^2 &= \frac{n^2}{2}||H||^2+\frac{1}{2}\sum_{k=1}^{n}(\mathcal{T}_{11}^k-\mathcal{T}_{22}^k-\ldots-\mathcal{T}_{nn}^k)^2+2\sum_{k=1}^{n}\sum_{j=2}^{n}(\mathcal{T}_{1j}^k)^2\\ &\quad-2\sum_{k=1}^{n}\sum_{2\le i<j}^{n}\Big\{\mathcal{T}_{ii}^k\mathcal{T}_{jj}^k-(\mathcal{T}_{ij}^k)^2\Big\}.\end{aligned} \tag{49}$$

Theorem 5.5. *Let* $\psi:(M_1^{2n}(c),J,g_1,\omega)\to(M_2^n,g_2)$ *be a Lagrangian submersion and the Lee vector field* B *be horizontal. If* $\{U_1,\ldots,U_n\}$ *is an orthonormal frame of* $ker\psi_*$, *then we have*

$$\begin{aligned}\hat{Ric}(U_1)+Ric^*(JU_1)+\delta(\mathcal{T})+\frac{(n^2+5n-2)ce^{-\sigma}}{4}+\frac{n^2+6n-4}{4}||\omega||^2\\ <||\mathcal{T}||^2+\frac{n^2}{4}||H||^2+\frac{n+1}{2}\mathrm{Trace}(L)\\ +\frac{n-2}{2}\Big\{L(U_1,U_1)+L(JU_1,JU_1)\Big\},\end{aligned}$$

where

$$Ric^*(JU_1)=\sum_{i=1}^{n}Ric^*(JU_1,JU_i,JU_1,JU_i).$$

Proof. Using (39), we have

$$\sum_{i,j=1}^{n} R_1(U_i, JU_j, U_i, JU_j) = \sum_{i,j=1}^{n} \Big[e^{-\sigma} \frac{c}{4} \Big\{ g_1(U_i, JU_j) g_1(JU_j, U_i) - g_1(U_i, U_i) g_1(JU_j, JU_j) - g_1(JU_i, JU_j) g_1(U_j, U_i) + g_1(JU_i, U_i) g_1(U_j, JU_j) - 2 g_1(JU_i, JU_j) g_1(JU_i, JU_j) \Big\} + \frac{1}{2} \Big\{ L(U_i, U_i) g_1(JU_j, JU_j) - L(JU_j, U_i) g_1(U_i, JU_j) - L(U_i, JU_j) g_1(JU_j, U_i) + L(JU_j, JU_j) g_1(U_i, U_i) \Big\} + \frac{||\omega||^2}{4} \Big\{ g_1(JU_j, U_i) g_1(U_i, JU_j) - g_1(U_i, U_i) g_1(JU_j, JU_j) \Big\} \Big]$$
$$= \frac{n}{2} \mathrm{Trace}(L) - \frac{n(n+3)ce^{-\sigma}}{4} - \frac{n^2}{4} ||\omega||^2.$$

If we substitute the above equation in the definition of the scalar curvature,

$$\rho_1 = \sum_{j=1}^{n} Ric_1(U_j, U_j) + \sum_{j=1}^{n} Ric_1(JU_j, JU_j)$$
$$= \sum_{i,j=1}^{n} R_1(U_i, U_j, U_i, U_j) + 2 \sum_{i,j=1}^{n} R_1(U_i, JU_j, U_i, JU_j) + \sum_{i,j=1}^{n} R_1(JU_i, JU_j, JU_i, JU_j), \tag{50}$$

then we have

$$\rho_1 = 2 \sum_{1 \le i < j}^{n} R_1(U_i, U_j, U_i, U_j) + 2 \sum_{1 \le i < j}^{n} R_1(JU_i, JU_j, JU_i, JU_j) + n \mathrm{Trace}(L) - \frac{n(n+3)ce^{-\sigma}}{2} - \frac{n^2}{2} ||\omega||^2. \tag{51}$$

On the other hand if we use (20), (25), (30) and (37) , then we have

$$\sum_{i,j=1}^{n} R_1(U_i, U_j, U_i, U_j) = \sum_{i,j=1}^{n} \Big\{ \hat{R}(U_i, U_j, U_i, U_j) + g_1(\mathcal{T}_{U_i} U_j, \mathcal{T}_{U_j} U_i) - g_1(\mathcal{T}_{U_i} U_i, \mathcal{T}_{U_j} U_j) \Big\}$$
$$= 2 \sum_{1 \le i < j}^{n} \hat{R}(U_i, U_j, U_i, U_j) + ||\mathcal{T}||^2 - n^2 ||H||^2,$$

and

$$\sum_{i,j=1}^{n} R_1(U_i, JU_j, U_i, JU_j) = \sum_{i,j=1}^{n} R_1(JU_j, U_i, JU_j, U_i)$$
$$= \sum_{i,j=1}^{n} \Big\{ g_1((\nabla_{JU_j} \mathcal{T})_{U_i} U_i, JU_j) - g_1(\mathcal{T}_{U_i} JU_j, \mathcal{T}_{U_i} JU_j) \Big\}$$
$$= \delta(\mathcal{T}) - \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{n} g_1(\mathcal{T}_{U_i} U_j, \mathcal{T}_{U_i} U_j) + \frac{1}{2} \omega(JU_i) \omega(JU_i) - g_1(U_i, U_i) ||\omega||^2 \Big\}$$
$$= \delta(\mathcal{T}) - ||\mathcal{T}||^2 + \Big(\frac{2n-1}{2} \Big) ||\omega||^2.$$

If we write the last two equations in (50) and use (22), then we get

$$\rho_1 = 2 \sum_{1 \le i < j}^{n} \hat{R}(U_i, U_j, U_i, U_j) + 2 \sum_{1 \le i < j}^{n} R^*(JU_i, JU_j, JU_i, JU_j) + 2\delta(\mathcal{T}) + (2n-1) ||\omega||^2 - n^2 ||H||^2 - ||\mathcal{T}||^2. \tag{52}$$

Making use of (51) and (52), we have

$$\begin{aligned}\sum_{1\le i<j}^{n} R_1(U_i,U_j,U_i,U_j) &+ \sum_{1\le i<j}^{n} R_1(JU_i,JU_j,JU_i,JU_j)\\ &+\frac{n}{2}\mathrm{Trace}(L)-\frac{n(n+3)ce^{-\sigma}}{4}-\frac{n^2+4n-2}{4}||\omega||^2\\ &=\sum_{1\le i<j}^{n}\hat{R}(U_i,U_j,U_i,U_j)+\sum_{1\le i<j}^{n}R^*(JU_i,JU_j,JU_i,JU_j)\\ &+\delta(\mathcal{T})-\frac{n^2}{2}||H||^2-\frac{1}{2}||\mathcal{T}||^2.\end{aligned} \tag{53}$$

Now, using (28) and (29) in (20) we obtain

$$\sum_{2\le i<j}^{n} R_1(U_i,U_j,U_i,U_j)=\sum_{2\le i<j}^{n}\hat{R}(U_i,U_j,U_i,U_j)-\sum_{k=1}^{n}\sum_{2\le i<j}^{n}\Big\{\mathcal{T}_{ii}^k\mathcal{T}_{jj}^k-(\mathcal{T}_{ij}^k)^2\Big\}.$$

Substituting the last equation in (53), we get

$$\begin{aligned}\sum_{j=2}^{n} R_1(U_1,U_j,U_1,U_j)&+\sum_{j=2}^{n}R_1(JU_1,JU_j,JU_1,JU_j)-\sum_{k=1}^{n}\sum_{2\le i<j}^{n}\Big\{\mathcal{T}_{ii}^k\mathcal{T}_{jj}^k-(\mathcal{T}_{ij}^k)^2\Big\}\\ &+\frac{n}{2}\mathrm{Trace}(L)-\frac{n(n+3)ce^{-\sigma}}{4}-\frac{n^2+4n-2}{4}||\omega||^2\\ &=\hat{Ric}(U_1)+Ric^*(JU_1)+\delta(\mathcal{T})-\frac{n^2}{2}||H||^2-\frac{1}{2}||\mathcal{T}||^2.\end{aligned}$$

If we use (39) for the first and the second terms on the left hand side of the last equation, namely

$$\sum_{j=2}^{n} R_1(U_1,U_j,U_1,U_j)=(1-n)\Big(\frac{ce^{-\sigma}+||\omega||^2}{4}\Big)+\frac{1}{2}\Big\{(n-2)L(U_1,U_1)-n\omega(H)\Big\},$$

and

$$\begin{aligned}\sum_{j=2}^{n} R_1(JU_1,JU_j,JU_1,JU_j)&=(1-n)\Big(\frac{ce^{-\sigma}+||\omega||^2}{4}\Big)\\ &+\frac{1}{2}\Big\{(n-2)L(JU_1,JU_1)+\mathrm{Trace}(L)+n\omega(H)\Big\},\end{aligned}$$

then we have

$$\begin{aligned}\frac{n+1}{2}\mathrm{Trace}(L)&+\frac{n-2}{2}\Big\{L(U_1,U_1)+L(JU_1,JU_1)\Big\}-\frac{(n^2+5n-2)ce^{-\sigma}}{4}\\ &-\frac{n^2+6n-4}{4}||\omega||^2-\sum_{k=1}^{n}\sum_{2\le i<j}^{n}\Big\{\mathcal{T}_{ii}^k\mathcal{T}_{jj}^k-(\mathcal{T}_{ij}^k)^2\Big\}\\ &=\hat{Ric}(U_1)+Ric^*(JU_1)+\delta(\mathcal{T})-\frac{1}{2}||\mathcal{T}||^2-\frac{n^2}{2}||H||^2.\end{aligned}$$

Now, using (49) in the last equation, we have

$$\begin{aligned}\hat{Ric}(U_1)&+Ric^*(JU_1)+\delta(\mathcal{T})-||\mathcal{T}||^2-\frac{n^2}{4}||H||^2\\ &=\frac{n+1}{2}\mathrm{Trace}(L)+\frac{n-2}{2}\Big\{L(U_1,U_1)+L(JU_1,JU_1)\Big\}\\ &-\frac{(n^2+5n-2)ce^{-\sigma}}{4}-\frac{n^2+6n-4}{4}||\omega||^2\\ &-\frac{1}{4}\sum_{k=1}^{n}(\mathcal{T}_{11}^k-\mathcal{T}_{22}^k-\ldots-\mathcal{T}_{nn}^k)^2-\sum_{k=1}^{n}\sum_{j=2}^{n}(\mathcal{T}_{1j}^k)^2.\end{aligned}$$

The result comes from Remark 5.1.

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