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A new class of Hardy spaces in the plane

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Abstract

We introduce new spaces that are extensions of the Hardy spaces and prove a removable singularity result for holomorphic functions within these spaces. Additionally we provide non-trivial examples.

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1. Introduction

This paper deals with a construction of a holomorphic function space on an arbitrary open connected subset of the complex plane $\mathbb{C}$. In this paper we suggest a method of constructing a function space W^p in any arbitrary domain. The definition of the norm on W^p makes use of growth information of the function locally in the domain. We show that W^p is Banach when $p \geq 1$ and prove a removable singularity theorem. This generalizes the result of M. Parreau in [8]. In the definition of W^p we make use of the recently studied Poletsky-Stessin-Hardy (PSH) spaces. These spaces were introduced in several complex variables context in [9] and recently studied in planar domains in [1] and for the disk in the papers [10] and [11].

In general, PSH norm depends on the choice of the subharmonic exhaustion function which exists only when the domain is regular with respect to the classical Dirichlet problem. Our motivation for such a construction comes from the question that which subspaces of the classical Hardy space H^p can be obtained as a Poletsky-Stessin-Hardy space. For example the subspace zH^p of H^p is not a Poletsky-Stessin-Hardy space because if the function z belongs to this space, then so does the constant function 1. However we show in section 4 that $B(z)H^p$ can be viewed as a W^p space when B is a finite Blaschke product.

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2. Poletsky-Stessin-Hardy spaces

A function $u \leq 0$ on a bounded open set $G \subset \mathbb{C}$ is called an exhaustion on G if the set

$$B_{c,u} := \{z \in G : u(z) < c\}$$

is relatively compact in G for any $c < 0$. We denote the class of harmonic functions and subharmonic functions on a domain G by $har(G)$ and $sh(G)$, respectively. It is known that there is a subharmonic exhaustion function on G if and only if G is regular with respect to the classical Dirichlet problem. Let us denote the class of continuous subharmonic exhaustion functions on a domain G by $\mathcal{E}(G)$. If u is an exhaustion and $c < 0$ is a number, we set

$$u_c := \max\{u, c\}, \quad S_{c,u} := \{z \in G : u(z) = c\}.$$

Since u_c is a continuous subharmonic function the measure Δu_c is well-defined. Following Demailly [2] we define

$$\mu_{c,u} := \Delta u_c - \chi_{G \setminus B_{c,u}} \Delta u,$$

where χ_ω is the characteristic function of a set $\omega \subset G$. We shall call these measures as Demailly measures.

If u is a negative subharmonic exhaustion function on G, then the Demailly-Lelong-Jensen formula takes the form

$$\int_{S_{c,u}} v \, d\mu_{c,u} = \int_{B_{c,u}} (v\Delta u - u\Delta v) + c \int_{B_{c,u}} \Delta v, \tag{2.1}$$

where $\mu_{c,u}$ is the Demailly measure which is supported in the level sets $S_{c,u}$ of u and $v \in sh(G)$. This formula is the one variable version of the result which was proved by Demailly [2]. Let us recall that by [2] if $\int_G \Delta u < \infty$, then the measures $\mu_{c,u}$ converge as $c \to 0$ weak-$*$ to a measure μ_u supported in the boundary ∂G.

Let $u \in sh(G)$ be an exhaustion function which is continuous with values in $\mathbb{R} \cup \{-\infty\}$. Following [9] we set

$$sh_u(G) := sh_u := \left\{ v \in sh(G) : v \geq 0, \sup_{c<0} \int_{S_{c,u}} v \, d\mu_{c,u} < \infty \right\},$$

and

$$H_u^p(G) := H_u^p := \{f \in hol(G) : |f|^p \in sh_u\}$$

for every $p > 0$. We write

$$\|v\|_u := \sup_{c<0} \int_{S_{c,u}} v \, d\mu_{c,u} = \int_G (v\Delta u - u\Delta v) \tag{2.2}$$

for the norm of a nonnegative function $v \in sh(G)$ and set

$$\|f\|_{u,p} := \sup_{c<0} \left(\int_{S_{c,u}} |f|^p \, d\mu_{c,u} \right)^{1/p} \tag{2.3}$$

for the norm of a holomorphic function f on G. Let us write $\|f\|_u$ when $p = 1$. It is known in view of [9, Theorem 4.1] that H_u^p is a Banach space when $p \geq 1$. It is clear that the function 1 belongs to H_u^p if and only if the Demailly measure μ_u has finite mass. If G is a regular bounded domain in $\mathbb{C}$ and $w \in G$, then we have the Green function $v(z) = g_G(z, w)$ which is a subharmonic exhaustion function for G.

The following result is obtained in [1].

2.1. Theorem. Let G be a bounded domain and $u \in \mathcal{E}(G)$. Let $p > 0$. The following statements are equivalent:

i. $f \in H^p_u(G)$.
ii. There exists a least harmonic function h in G which belongs to the class sh_u so that $|f|^p \leq h$ on G. Furthermore,

$$\|f\|^p_{u,p} = \int_G h\Delta u = \|h\|_u.$$

Now let G be a bounded domain with C^1 boundary or a bounded simply connected domain with rectifiable boundary. Let $u \in \mathcal{E}(G)$ and $p \geq 1$ ($p > 0$ if G is simply connected). Then the space $H^p_u(G)$ (thinking of boundary values) is a closed subspace of the weighted space $L^p(V_u d\sigma)$ on the boundary ∂G, where (see [1]) $d\sigma$ is the usual Lebesgue measure on ∂G and

$$V_u(\zeta) = \int_G P_G(z,\zeta)\Delta u(z), \quad \zeta \in \partial G$$

is the balayage of the positive measure Δu to the boundary ∂G. Then $V_u(\zeta) = \frac{\partial u}{\partial \mathbf{n}}(\zeta)$ is the directional derivative of u in the normal direction at a point $\zeta \in \partial G$ (see [4] and [11]). The next results are restatements from [9] and they establish basic observations on the classes of Hardy spaces.

2.2. Proposition. [9, Corollary 3.2] Let v be a continuous subharmonic exhaustion function on a bounded regular domain G and let $v(z) = g(z,w)$ be the Green function. Then $sh^p_u(G) \subset sh^p_v(G)$ and there is a constant c such that $\|\varphi\|_v \leq c\|\varphi\|_u$ for every nonnegative subharmonic function φ on G.

2.3. Proposition. [9, Corollary 3.2] Let u and v be continuous subharmonic exhaustion functions on G and let K be a compact set in G such that $bv(z) \leq u(z)$ for some constant $b > 0$ and all $z \in G\backslash K$. Then $sh_v \subset sh_u$ and $\|\varphi\|_u \leq b\|\varphi\|_v$ for every $\varphi \in sh_v$.

The following result is basically contained in the proof of [9, Theorem 3.6] taking $n = 1$.

2.4. Proposition. Let v be a continuous subharmonic exhaustion function on G, $K \subset G$ be compact and $V \subset\subset G$ be an open set containing K. Suppose that there exists a constant $s > 0$ so that $v(z) \leq s g_G(z,w)$ for every $w \in K$ and $z \in G\backslash\overline{V}$. Then

$$\varphi(w) \leq \frac{s}{2\pi}\|\varphi\|_v, \quad w \in K$$

for every nonnegative $\varphi \in sh(G)$.

3. Hardy spaces in arbitrary open sets

In this section we propose a way to define weighted Hardy spaces in arbitrary planar domains. For Hardy spaces in multiply connected domains we refer to [3]. Let us set the notation first. Let Ω be a domain, $E \subset \Omega$ be a compact polar subset and let Ω_j be a sequence of regular domains so that $\Omega_j \subset \Omega$ and the union of all Ω_j is the open set $\Omega\backslash E$. Also for each j let $u_j \in \mathcal{E}(\Omega_j)$, that is, u_j is a subharmonic exhaustion function for Ω_j. We define the class W^p of holomorphic functions on Ω as follows:

$$W^p := \{f \in hol(\Omega) : \sup_j \|f\|_{u_j,p} < \infty\}.$$

Let us define

$$\|f\|_{W^p} := \sup_j \|f\|_{u_j,p}$$

for any $f \in W^p$. We will write $W^p[u_j, \Omega, E]$ if we wish to emphasize the sequence of functions u_j used in the definition or the underlying domain Ω and the polar set E. Before showing that $(W^p, \|\|_{W^p})$ is a Banach space we need the following removable singularity theorem for bounded holomorphic functions due to Lelong.

3.1. Theorem. [7, p.35], [5, p. 107] Let E be a relatively closed pluri-polar set and let f be holomorphic in $\Omega \backslash E$. Suppose that f is bounded on $\Omega \backslash E$. Then f has a unique holomorphic extension to the whole of Ω.

We prove an auxiliary result.

3.2. Theorem. Let f_n be a holomorphic function on a domain Ω and E be a compact polar set in Ω. Suppose that f_n converges uniformly to a function f on compact subsets of $\Omega \backslash E$. Then the function f can be extended to a holomorphic function on Ω.

Proof. Let Γ be a bounded open region in Ω with piecewise smooth boundary γ so that $E \subset \Gamma \subset \overline{\Gamma} \subset \Omega$. Since $|f_n|$ converges uniformly to $|f|$ on γ, we see that $\sup_n |f_n|$ is uniformly bounded on γ, that is, there exists a number M so that $|f_n| \leq M$ on γ for every n. We write $P_\Gamma \varphi$ for the Poisson integral of a continuous function φ on γ. Then

$$|f_n(z)| \leq P_\Gamma |f_n|(z) \leq M$$

for every n for every $z \in \Gamma$. Therefore $|f(z)| \leq M$ for every $z \in \Gamma \backslash E$. By Theorem 3.1 f has a holomorphic extension to Γ. Since f is already holomorphic outside of Γ we conclude that f can be extended to a holomorphic function on Ω. □

We can now prove that $(W^p, \|\|_{W^p})$ is Banach.

3.3. Theorem. $(W^p, \|\|_{W^p})$ is a Banach space for $p \geq 1$.

Proof. If $\|f\|_{W^p} = 0$, then $\|f\|_{u_j,p} = 0$, that is why $f = 0$ in Ω_j for every j. Hence $f = 0$ on $\Omega \backslash E$, and since E is polar, $f = 0$ on Ω. The other properties of norm can be easily checked for $\|f\|_{W^p}$. So let us prove that it is complete. Take a Cauchy sequence $\{f_n\}$ from W^p. This implies first that the sequence of holomorphic functions $\{f_n\}$ is Cauchy in $H^p_{u_j}$ for every j. We conclude that f_n converges uniformly to a function f on every compact subset of Ω_j for each j, hence on every compact subset of $\Omega \backslash E$. By Theorem 3.2 f extends to a holomorphic function to the whole of Ω.

To prove that $f \in W^p$ we will now show that $\|f_n - f\|_{W^p}$ converges to zero. Given $\varepsilon > 0$ there exists an integer $N \geq 1$ so that

$$\sup_j \|f_n - f_m\|_{u_j,p} < \varepsilon$$

whenever $n, m \geq N$. This gives that

$$\|f_n - f\|_{W^p} = \sup_j \|f_n - f\|_{u_j,p} \leq \varepsilon$$

for every $n \geq N$. Therefore $\|f_n - f\|_{W^p}$ converges to zero and $f \in W^p$. □

It is known that a polar set is a removable singularity for the classical Hardy spaces in the plane (see [6] and [8]). The next result can be considered as a removable singularity theorem for the W^p spaces. There by $W^p[u_j, \Omega, E]|_{\Omega \backslash E}$ we denote the class of restrictions of the functions from $W^p[u_j, \Omega, E]$ to $\Omega \backslash E$.

3.4. Theorem. Let $\overline{\Omega}_j \subset \Omega_{j+1}$, $E \subset \Omega$ be a compact polar set for every j and let $p > 0$. If there exists an open set $U \subset \Omega \backslash E$ so that $\sup_j u_j(z) \leq \ell < 0$ for every $z \in U$, then

$$W^p[u_j, \Omega, E]|_{\Omega \backslash E} = W^p[u_j, \Omega \backslash E, \emptyset].$$

Proof. The inclusion $W^p[u_j,\Omega,E]|_{\Omega\setminus E}\subset W^p[u_j,\Omega\setminus E,\emptyset]$ is immediate from the definitions. To prove the reverse inclusion we claim that if f belongs to the space $W^p[u_j,\Omega\setminus E,\emptyset]$, then f extends to a holomorphic function to the whole set Ω. Since $f\in H^p_{u_j}(\Omega_j)$, according to Theorem 2.1 we see that the function $h_j:=P_{\Omega_j}(|f|^p)$ has the properties that $h_j\in har\cap sh_{u_j}(\Omega_j)$ and that $|f|^p\leq h_j$ on Ω_j. Then $h_j\leq h_{j+1}$, and thanks to the Harnack theorem the limit $h=\lim h_j$ is a harmonic function on $\Omega\setminus E$ unless $h=\infty$ identically everywhere. Let $z_0\in U$ and $r>0$ so that $\{z:|z-z_0|\leq r\}\subset U$. We claim that $h(z_0)<\infty$. There exists a constant $s>0$ so that $\ell<sg_\Omega(z,z_0)\leq sg_{\Omega_j}(z,z_0)$ for every $z\in\Omega_j$ with $|z-z_0|=r$, $j\geq 1$. Harmonicity of the Green's function $g_{\Omega_j}(z,z_0)$ on $\Omega_j\setminus\{z:|z-z_0|\leq r\}$ implies that $u_j(z)\leq\ell<sg_{\Omega_j}(z,z_0)$ for every $z\in\Omega_j\setminus\{z:|z-z_0|\leq r\}$ for every $j\geq 1$. By Theorem 2.1 and Proposition 2.4 we see that

$$\frac{2\pi}{s}h_j(z_0)\leq\|h_j\|_{p,u_j}=\|f\|_{p,u_j}\leq\|f\|_{W^p[u_j,\Omega\setminus E,\emptyset]}<\infty$$

for every $j\geq 1$. This proves that $h(z_0)<\infty$. Hence $h\in har(\Omega\setminus E)$ and satisfies $|f|^p\leq h$. Now this means f belongs to the Hardy class of functions mentioned in [8]. By [8, Theorem A] f admits a unique holomorphic extension to Ω and therefore $f\in W^p[u_j,\Omega,E]$. This completes the proof. □

4. Examples

In view of [9, Proposition 3.5] $H^p_u\subset H^p$ when u is a subharmonic exhaustion on the disk. It is our purpose to construct examples of subsets of the classical Hardy space H^p on the disk which can be described using the Hardy spaces of the form W^p. The next examples are of this sort. In the next example we construct a family of exhaustion functions inside the unit disk to descripe the space of functions in H^p which are zero at 0.

4.1. *Example.* For any $0<R<1$ let Γ_R denote the annulus

$$\Gamma_R:=\{z\in\mathbb{C}:R<|z|<1\}.$$

If $t>0$, define a subharmonic exhaustion function u_t in Γ_R by

$$u(z):=u_t(z):=u_{t,R}(z):=\max\left\{t\log\left(\frac{R}{|z|}\right),\log|z|\right\}.$$

Some properties of u_t are listed below.

(1) $u_t(z)=0$, if $|z|=1$ or $|z|=R$.

(2) We solve $t\log\left(\frac{R}{|z|}\right)=\log|z|$ to get $|z|=R^{t/t+1}$ and hence

$$u_t(z)=\begin{cases}t\log\left(\frac{R}{|z|}\right) & \text{if } R<|z|\leq R^{t/t+1};\\ \log|z|, & \text{if } R^{t/t+1}<|z|<1.\end{cases}$$

(3) We compute the measure μ_u of u.

$$V_u(e^{i\theta})=\frac{\partial u}{\partial\mathbf{n}}|_{z=e^{i\theta}}=1$$

and

$$V_u(Re^{i\theta})=\frac{\partial u}{\partial\mathbf{n}}|_{z=Re^{i\theta}}=t/R$$

for every $\theta\in[0,2\pi]$. Hence, for any positive measurable function φ on $\partial\Gamma_R$ we have

$$\int_{\partial\Gamma_R}\varphi d\mu_u=\frac{t}{2\pi R}\int_0^{2\pi}\varphi(Re^{i\theta})d\theta+\frac{1}{2\pi}\int_0^{2\pi}\varphi(e^{i\theta})d\theta.$$

Now we are ready to state the main purpose of this example.

4.2. Theorem. Let H^p be the classical Hardy space in the unit disc for $p>0$ and $k\geq 1$ be an integer. Let (R_n) be any sequence of numbers converging to 0 so that $0<R_n<1$. Take α so that $1-kp\leq\alpha<1-kp+p$. Then we have

$$z^kH^p=W^p[u_{R_n^\alpha,R_n},\mathbb{D},\{0\}]$$

and two spaces have equivalent norms.

Proof. Let $W^p=W^p[u_{R_n^\alpha,R_n},\mathbb{D},\{0\}]$. If $h\in z^kH^p$, we will show that $h\in W^p$. Let $h=z^kf$, where $f\in H^p$. Then

$$\begin{aligned}\|z^kf\|_{W^p}^p &= \sup_n\|z^kf\|^p_{u_{R_n^\alpha},p}\\ &= \sup_n\left(\frac{R_n^{\alpha+kp-1}}{2\pi}\int_0^{2\pi}|f(R_ne^{i\theta})|^pd\theta+\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^pd\theta\right)\\ &\leq \sup_n(R_n^{\alpha+kp-1}+1)\|f\|^p_{H^p}\leq 2\|f\|^p_{H^p}<\infty.\end{aligned}$$

Hence $h\in W^p$ and $z^kH^p\subset W^p$.

Conversely, let $h\in W^p$. By the definition of the norm of W^p (by item (3) above for instance) it is clear that $\|h\|_{H^p}\leq\|h\|_{W^p}$ for every $h\in W^p$, that is, $W^p\subset H^p$. We will show that $h\in z^kH^p$. Suppose on the contrary that $h(z)=z^mf(z)$, where $0\leq m\leq k-1$, $f\in H^p$ and $|f(0)|>0$. Then

$$\begin{aligned}\|h\|_{W^p} &\geq \sup_n\frac{R_n^{\alpha+(k-1)p-1}}{2\pi}\int_0^{2\pi}|f(R_ne^{i\theta})|^pd\theta\\ &\geq \sup_n R_n^{\alpha+(k-1)p-1}|f(0)|^p=\infty.\end{aligned}$$

The contradiction shows that $h(z)=z^kf(z)$ for some $f\in H^p$. Hence $W^p=z^kH^p$. □

Finally we can do the previous construction for finite Blaschke products.

4.3. Theorem. Let $a_1,\cdots,a_N$ be distinct points in $\mathbb{D}$ and let

$$B(z):=\prod_{j=1}^N\left(\frac{z-a_j}{1-\overline{a}_jz}\right)^{k_j},$$

where $k_j\geq 1$ are integers. Let $p>0$. Then there exists a sequence $\{\Omega_n\}\subset\mathbb{D}$ of $N+1$-connected domains and functions $u_n\in\mathcal{E}(\Omega_n)$ so that

$$B(z)H^p=W^p[u_n,\mathbb{D}]$$

and two spaces have equivalent norms.

Proof. Choose $R>0$ small enough so that the circles

$$C_j=\left\{z:\left|\frac{z-a_j}{1-\overline{a}_jz}\right|^{k_j}=R\right\},$$

$j=1,\cdots,N$, are pairwise disjoint. Let Ω_R be the $N+1$-connected domain with boundary $\partial\mathbb{D}\cup\cup_{j=1}^NC_j$. For each j choose α_j so that $-k_jp\leq\alpha_j<-k_jp+p$. Let ψ_R be the function defined on $\partial\mathbb{D}$ by 1, on C_j by R^α. Then ψ_R is lower semicontinuous on Ω_R, $\psi_R\geq t_R>0$ for some constant $t=t_R$ and by [4, Theorem 2.1] there exists a subharmonic exhaustion $u_R\in\mathcal{E}(\Omega_R)$ so that $\partial u_R/\partial\mathbf{n}=\psi_R$ on $\partial\Omega_R$.

Now let $0 < R_n < R$ be numbers decreasing to 0 and consider the space $W^p = W^p[u_{R_n}, \mathbb{D}]$. If $h = Bf \in B(z)H^p$, then

$$\begin{aligned} \|Bf\|_{W^p}^p &= \sup_n \left(\sum_{j=1}^N R_n^{\alpha_j + k_j p} \int_{C_j} |f(\zeta)|^p d\sigma_j + \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right) \\ &\leq \sup_n (\sum_{j=1}^N R_n^{\alpha_j + k_j p} + 1) \|f\|_{H^p}^p \leq (N+1) \|f\|_{H^p}^p < \infty. \end{aligned}$$

Hence $h \in W^p$ and $BH^p \subset W^p$.

Conversely, let $h \in W^p$. By the definition of the norm of W^p it is clear that $\|h\|_{H^p} \leq \|h\|_{W^p}$ for every $h \in W^p$, that is, $W^p \subset H^p$. We will show that $h \in BH^p$. Suppose on the contrary that the multiplicity m_j of zero of $h(z)$ at a_j is strictly less than k_j, that is $0 \leq m_j \leq k_j - 1$ for some j. Then $h(z) = \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{m_j} f(z)$ for a function $f \in H^p$ with $|f(a_j)| > 0$ and therefore,

$$\begin{aligned} \|h\|_{W^p} &\geq \sup_n R_n^{\alpha_j + (k_j - 1)p} \int_{C_j} |f(\zeta)|^p d\sigma_j \\ &\geq C \sup_n R_n^{\alpha_j + (k_j - 1)p} |f(a_j)|^p = \infty. \end{aligned}$$

The contradiction shows that $h(z) = B(z) f(z)$ for some $f \in H^p$. Hence $W^p = B(z) H^p$. □

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