On converses of some comparison inequalities for homogeneous means

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Abstract

In this paper, the necessary and sufficient conditions for the converses of comparison inequalities for Stolarsky means and for Gini means to hold are proved, and the necessary and sufficient conditions for some companion inequalities for bivariate means to hold are given, which unify, generalize and improve known results.

1. Introduction

Let $p, q \in \mathbb{R}$ and $a, b \in \mathbb{R}_+ := (0, \infty)$ with $a \neq b$. The Stolarsky means $S_{p,q}(a, b)$ were defined by Stolarsky [30] as

$$
(1.1) \quad S_{p,q}(a, b) = \begin{cases}
\left(\dfrac{q(a^p - b^p)}{p(a^q - b^q)} \right)^{1/(p-q)} & \text{if } p \neq q, pq \neq 0, \\[2mm]
\left(\dfrac{a^p - b^p}{p(\ln a - \ln b)} \right)^{1/p} & \text{if } p \neq 0, q = 0, \\[2mm]
\left(\dfrac{a^q - b^q}{q(\ln a - \ln b)} \right)^{1/q} & \text{if } p = 0, q \neq 0, \\[2mm]
\exp\left(\dfrac{a^p \ln a - b^p \ln b}{a^p - b^p} - \dfrac{1}{p} \right) & \text{if } p = q \neq 0, \\[2mm]
\sqrt{ab} & \text{if } p = q = 0.
\end{cases}
$$

Also, $S_{p,q}(a, a) = a$. The Stolarsky means contain many famous means, for example, $S_{1,0}(a, b) = L(a, b)$ –the logarithmic mean, $S_{1,1}(a, b) = I(a, b)$ –the identric (exponential) mean, $S_{2,1}(a, b) = A(a, b)$ –arithmetic mean, $S_{3/2,1/2}(a, b) = He(a, b)$ –Heronian mean, $S_{p,0}(a, b) = L^{1/p}(a^p, b^p) = L_p$ –the p-order logarithmic mean, $S_{p,p}(a, b) = I^{1/p}(a^p, b^p) =$

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I_p –the p-order identic (exponential) mean, $S_{2p,p}(a,b) = A^{1/p}(a^p, b^p) = A_p$ –the p-order power mean, etc.

Another well-known two-parameter family of bivariate means was introduced by C. Gini in [8]. That is defined as

$$(1.2) \qquad G_{p,q}(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{a^q + b^q} \right)^{1/(p-q)} & \text{if } p \neq q, \\[2ex] \exp\left(\dfrac{a^p \ln a + b^p \ln b}{a^p + b^p} \right) & \text{if } p = q. \end{cases}$$

The Gini means also contain many famous means, for example, $G_{1,0}(a,b) = A(a,b)$ – arithmetic mean, $G_{1,1}(a,b) = Z(a,b)$ –the power-exponential mean, $G_{p,0}(a,b) = A^{1/p}(a^p, b^p) = A_p$ –the p-order power mean, $G_{p,p}(a,b) = Z^{1/p}(a^p, b^p) = Z_p$ –the p-order power-exponential mean, etc.

The comparison problem for Stolarsky means $S_{p,q}(a,b) \leq S_{r,s}(a,b)$ $(a,b \in \mathbb{R}_+)$ was first solved by Leach and Sholander [12]. Páles presented a new proof for this result in [21] and dealt with the same comparison problem for Gini means $G_{p,q}(a,b)$ in [20]. For later use, we record the two comparison theorems as follows.

1.1. Theorem ([12], [21]). *The comparison inequality*

$$(1.3) \qquad S_{p,q}(a,b) \leq S_{r,s}(a,b)$$

holds for all $a,b \in \mathbb{R}_+$ if and only if

$$(1.4) \qquad p + q \leq r + s$$

and

$$(1.5) \qquad \begin{cases} (i)\ l(p,q) \leq l(r,s) & \text{if } \min(p,q,r,s) \geq 0 \text{ or } \max(p,q,r,s) \leq 0, \\ (ii)\ \mu(p,q) \leq \mu(r,s) & \text{if } \min(p,q,r,s) < 0 < \max(p,q,r,s), \end{cases}$$

where

$$(1.6) \qquad l(u,v) = \begin{cases} \dfrac{u-v}{\ln(u/v)} & \text{if } uv > 0, \\ 0 & \text{if } uv = 0; \end{cases} \qquad \mu(u,v) = \begin{cases} \dfrac{|u|-|v|}{u-v} & \text{if } u \neq v, \\ \operatorname{sgn}(u) & \text{if } u = v. \end{cases}$$

1.2. Theorem ([20]). *The comparison inequality*

$$(1.7) \qquad G_{p,q}(a,b) \leq G_{r,s}(a,b)$$

holds for all $a,b \in \mathbb{R}_+$ if and only if (1.4) holds and

$$(1.8) \qquad \begin{cases} (i)\ \min(p,q) \leq \min(r,s) & \text{if } \min(p,q,r,s) \geq 0, \\ (ii)\ \max(p,q) \leq \max(r,s) & \text{if } \max(p,q,r,s) \leq 0, \\ (iii)\ \mu(p,q) \leq \mu(r,s) & \text{if } \min(p,q,r,s) < 0 < \max(p,q,r,s), \end{cases}$$

where $\mu(u,v)$ is defined by (1.6).

Other references involving these comparison problems can be found in [6], [4], [2], [3], [22], [14], [17], [5], [33].

There are many inequalities for bivariate means (see [13], [26], [31], [24], [29] [16], [10], [18], [36], [37], [38], [39], [41], [42]). However, most of comparison inequalities for bivariate means are derived from Theorem 1.1 and 1.2.

On the other hand, the converses of certain inequalities for bivariate means have also attracted the attention of scholars.

Let $M_1(a,b)$ and $M_2(a,b)$ be two means of positive numbers a and b and inequality

$$(1.9) \qquad M_1(a,b) \leq M_2(a,b)$$

holds for all $a, b \in \mathbb{R}_+$. If there exits a constant $k > 1$ such that inequality

$$(1.10) \quad kM_1(a,b) \geq M_2(a,b) \quad \text{or} \quad M_1(a,b) \geq k^{-1}M_2(a,b)$$

holds for all $a, b \in \mathbb{R}_+$, then (1.10) is called a converse of (1.9). And inequalities (1.10) and (1.9) are called a pair of companion inequalities for means $M_1(a,b)$ and $M_2(a,b)$ by Neuman and Sándor [19].

For $a, b \in \mathbb{R}_+$ with $a \neq b$, it is known that

$$(1.11) \quad I(a,b) < A(a,b).$$

In 1988, Alzer [1] gave a converse of inequality (1.11), that is,

$$(1.12) \quad I(a,b) > 2e^{-1}A(a,b)$$

(also see [27]). In 2002, Hästö derived two pairs of companion inequalities involving Stolarsky means and Gini means from the *strong inequalities*. For convenience, we read them as follows.

1.3. Theorem([9, Corollary 1.2]). *Let $p, q, r, s \in \mathbb{R}_+$, $r > s$ and $p > q$. If $p + q \geq r + s$ and $s \geq q$ (note: there is a misprinted and it should be "$q \geq s$" instead of "$s \geq q$") then*

$$(1.13) \quad S_{r,s}(a,b) \leq S_{p,q}(a,b) \leq (q/p)^{1/(p-q)}(r/s)^{1/(r-s)}S_{r,s}(a,b).$$

Both inequalities are sharp.

1.4. Theorem([9, Corollary 1.2]). *Let $p, q, r, s \in \mathbb{R}_+$, $r > s$ and $r + s \leq 3(p + q)$. Assume also that $1/3 \leq p/q \leq 3$ or $s \leq p + q$. Then*

$$(1.14) \quad S_{r,s}(a,b) \leq G_{p,q}(a,b) \leq (r/s)^{1/(r-s)}S_{r,s}(a,b).$$

Both inequalities are sharp.

Using different methods and ideals, Neuman and Sándor [15, 19], Yang [35, 36, 40], Du [7] also obtained some companion inequalities for bivariate means, which are all special cases of Hästö's results above. For example, for $a, b \in \mathbb{R}_+$ with $a \neq b$, putting $(p, q) = (1, 1)$, $(r, s) = (1, 1/2)$ in Theorem 1.3 yields

$$(1.15) \quad A_{1/2}(a,b) < I(a,b) < 4e^{-1}A_{1/2}(a,b),$$

where the second inequality of (1.15) was rediscovered by Neuman and Sándor [15]; putting $(p, q) = (1, 1)$, $(r, s) = (3/4, 2/3)$, $(3/2, 1/2)$ in Theorem 1.3 lead to

$$(1.16) \quad 1 < I(a,b)/He(a,b) < 3e^{-1} \quad \text{and} \quad 1 < I(a,b)/A_{2/3}(a,b) < 2\sqrt{2}e^{-1}$$

reobtained by Yang [36]; putting $(p, q) = (1, 1)$, $(r, s) = (2, 2)$ in Theorem 1.4 leads to

$$(1.17) \quad A_2(a,b) < Z(a,b) < \sqrt{2}A_2(a,b)$$

given in [19]. It is clear that, however, the following companion inequalities

$$(1.18) \quad \sqrt{I_p(a,b)I_q(a,b)} < S_{p,q}(a,b) < e^{1/A_{-1}(p,q)-1/L(p,q)}\sqrt{I_p(a,b)I_q(a,b)}$$

for $p, q > 0$ with $p \neq q$ proved by Yang [36] do not follow from the results of Hästö.

It should be noted that Theorem 1.3 only gives a sufficient condition for the converse of comparison inequalities for Stolarsky means (1.3) to hold. From the published relative literatures, however, it is the best result. For this reason, in this paper we devote to further investigate the necessary and sufficient condition such that the converse of (1.3) is true. One of our main results is the following statement.

1.5. Theorem. *Let* $p, q, r, s \in \mathbb{R}_+$. *Then the converse of comparison inequality for Stolarsky means*

$$(1.19) \quad e^{1/L(p,q)} S_{p,q}(a,b) \leq e^{1/L(r,s)} S_{r,s}(a,b)$$

holds for all $a, b \in \mathbb{R}_+$ *with* $a \neq b$ *if and only if*

$$(1.20) \quad L(p,q) \geq L(r,s) \text{ and } \min(p,q) \geq \min(r,s).$$

We also look for the condition so that the converse of comparison inequalities for Gini means (1.7) holds, and obtain the other one of our main results as follows.

1.6. Theorem. *Let* $p, q, r, s \in \mathbb{R}_+$. *Then the converse of comparison inequality for Gini means*

$$(1.21) \quad e^{1/L(p,q)} G_{p,q}(a,b) \leq e^{1/L(r,s)} G_{r,s}(a,b)$$

holds for all $a, b \in \mathbb{R}_+$ *with* $a \neq b$ *if*

$$(1.22) \quad p + q \geq r + s \text{ and } \min(p,q) \geq \min(r,s).$$

Based on our main results, in section 4 we shall give a necessary and sufficient condition for (1.13) to hold, and prove this result is also true for Gini means. Additionally, we also derive a necessary and sufficient condition for the companion inequalities for Gini mean and power mean. Lastly, a simple proof of a part of Theorem 1.4 is presented.

2. PRELIMINARY

In 1992 Páles [23] offered a unified treatment for comparison problems on bounded intervals of the positive real. For convenience, we recall it as follows.

2.1. Theorem([23, Theorem 2]). *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a four times differentiable even function such that* $\phi''(0) > 0$ *and*

$$\phi''(x) > 0, \phi'''(x) < 0 \quad \text{and} \quad (x\phi'''(x)/\phi''(x))' < 0$$

holds for all $x > 0$. *Define* $\Phi_{p,q}(t)$ *by*

$$(2.1) \quad \Phi_{p,q}(t) := \begin{cases} \dfrac{\phi(pt) - \phi(qt)}{p - q} & \text{if } p \neq q, \\ t\phi'(pt) & \text{if } p = q. \end{cases}$$

Let p, q, r, s *and* $c > 0$ *be fixed real values. Then the inequality*

$$\Phi_{p,q}(t) \leq \Phi_{r,s}(t)$$

holds for all $t \in [-c, c]$ *if and only if*

$$p + q \leq r + s \quad \text{and} \quad \Phi_{p,q}(c) \leq \Phi_{r,s}(c).$$

To prove our main results in this paper, we also need another comparison theorem, which is similar to Páles comparison theorem above.

2.2. Theorem. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a three times differentiable function satisfying*

(2.2) $\phi''(x) > (<)0$ and $\kappa(x) = x\phi'''(x)/\phi''(x)$ *is strictly monotone for all $x > 0$.*

Then for fixed real values $p, q, r, s \in \mathbb{R}_+$ and $c_2 > c_1 > 0$ the comparison inequality

(2.3) $\Phi_{p,q}(t) \leq \Phi_{r,s}(t)$

holds for all $t \in [c_1, c_2]$ if and only if

(2.4) $\Phi_{p,q}(c_1) \leq \Phi_{r,s}(c_1)$ *and* $\Phi_{p,q}(c_2) \leq \Phi_{r,s}(c_2)$.

The train of thoughts of proving Theorem 2.2 is to decompose one function into the product of one positive function and another monotone one.

To prove Theorem 2.2, we need some lemmas.

2.3. Lemma. *Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and monotone. Then $f(x) \geq (\leq)0$ for all $x \in [a, b]$ if and only if $f(a) \geq (\leq)0$ and $f(b) \geq (\leq)0$.*

2.4. Lemma. *Let $x_1, x_2, x_3 \in [a, b]$ and ϕ be two times differentiable on $[a, b]$. Define that*

(2.5) $[x_1, x_2; \phi] \quad := \quad \begin{cases} \dfrac{\phi(x_1) - \phi(x_2)}{x_1 - x_2} & \text{if } x_1 \neq x_2, \\ \phi'(x_2) & \text{if } x_1 = x_2. \end{cases}$

(2.6) $[x_1, x_2, x_3; \phi] \quad := \quad \begin{cases} \dfrac{[x_1, x_2; \phi] - [x_2, x_3; \phi]}{x_1 - x_3} & \text{if } x_1 \neq x_3, \\ \left.\dfrac{\partial [x_1, x_2; \phi]}{\partial x_1}\right|_{x_1 = x_3} & \text{if } x_1 = x_3. \end{cases}$

Then we have

1) $[x_1, x_2, x_3; \phi]$ are symmetric with respect to x_1, x_2 and x_3, that is,

$$[x_1, x_2, x_3; \phi] \quad = \quad [x_1, x_3, x_2; \phi] = [x_2, x_3, x_1; \phi]$$
$$= \quad [x_2, x_1, x_3; \phi] = [x_3, x_1, x_2; \phi] = [x_3, x_2, x_1; \phi].$$

2) $[x_1, x_2, x_3; \phi] \geq (\leq)0$ if and only if ϕ is convex (concave) on $[a, b]$.

3) (Mean Value Theorem) If ϕ is two times differentiable on $[a, b]$ and $x_1, x_2, x_3 \in [a, b]$, then there is a ξ between the smallest and the largest x_i such that

(2.7) $[x_1, x_2, x_3; \phi] = \dfrac{\phi''(\xi)}{2!}$.

The following lemma will play an important role in the proof of Theorem 2.2.

2.5. Lemma. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a three times differentiable function which satisfies (2.2). Then for fixed $p, q, r, s \in \mathbb{R}_+$ with $p \geq q, r \geq s$ and $(s-p)(q-r)(s-q)(r-p) \neq 0$, the function $V : \mathbb{R}_+ \to \mathbb{R}$ defined by*

(2.8) $V(t) := \dfrac{\Phi_{r,s}(t) - \Phi_{r,p}(t)}{s - p} \Big/ \dfrac{\Phi_{p,q}(t) - \Phi_{p,r}(t)}{q - r}$

is monotone on $\mathbb{R}_+$, where $\Phi_{p,q}(t)$ is defined by (2.1).

Proof. To prove this lemma, it suffice to show that $\operatorname{sgn}(V'(t))$ is a constant. Without loss of generality, we assume that $\kappa(x) = x\phi'''(x)/\phi''(x)$ is strictly monotone increasing on $\mathbb{R}_+$. We stepwise prove this lemma.

Step 1. Finding an appreciate expression of $V(t)$.

By (2.6) $V(t)$ can be written as

(2.9) $V(t) = \dfrac{[rt, st, pt; \phi]}{[pt, qt, rt; \phi]}$,

which implies that $V(t)$ is symmetric with respect to p and r, and so we assume that $p > r$. This together with assumption $p \geq q, r \geq s$ implies that $p = \max(p, q, r, s)$.

On the other hand,

$$
\begin{aligned}
V(t) &= \frac{[rt, st, pt; \phi]}{[pt, qt, rt; \phi]} = \frac{\dfrac{1}{st - rt} \left([st, pt; \phi] - [rt, pt; \phi]\right)}{\dfrac{1}{qt - rt} \left([qt, pt; \phi] - [rt, pt; \phi]\right)} \\
&= \frac{\Phi_{s,p}(t) - \Phi_{r,p}(t)}{s - r} \bigg/ \frac{\Phi_{q,p}(t) - \Phi_{r,p}(t)}{q - r} \text{ if } s \neq r, \\
V(t) &= \lim_{s \to r} \frac{\Phi_{s,p}(t) - \Phi_{r,p}(t)}{s - r} \bigg/ \frac{\Phi_{q,p}(t) - \Phi_{r,p}(t)}{q - r} \text{ if } s = r.
\end{aligned}
$$

Step 2. Calculating $V'(t)$ and treating the $\operatorname{sgn}(V'(t))$.

Denote by

$$
(2.10) \qquad h(x) \quad : \quad = \Phi_{x,p}(t) = \begin{cases} \dfrac{\phi(xt) - \phi(pt)}{x - p} & \text{if } x \neq p, \\ t\phi'(pt) & \text{if } x = p. \end{cases}
$$

$$
(2.11) \qquad g(x) \quad : \quad = \frac{\partial \Phi_{x,p}(t)}{\partial t} = \begin{cases} \dfrac{x\phi'(xt) - p\phi'(pt)}{x - p} & \text{if } x \neq p, \\ \phi'(pt) + pt\phi''(pt) & \text{if } x = p. \end{cases}
$$

Some simple calculations lead to

$$
(2.12) \qquad h'(x) \quad = \quad \begin{cases} t^2 \dfrac{\int_x^p (p-u)\phi''(ut)\,du}{(p-x)^2} & \text{if } x \neq p, \\ \dfrac{t^2}{2}\phi''(pt) & \text{if } x = p; \end{cases}
$$

$$
(2.13) \qquad g'(x) \quad = \quad \begin{cases} t \dfrac{\int_x^p (p-u)\left(2\phi''(ut) + ut\phi'''(ut)\right)du}{(x-p)^2} & \text{if } x \neq p, \\ \dfrac{t}{2}\left(2\phi''(pt) + pt\phi'''(pt)\right) & \text{if } x = p. \end{cases}
$$

Since $\phi''(x) > (<)0$ we see that $h'(x) > (<)0$, which implies that $h(x)$ is strictly increasing (decreasing), and so $h(x)$ is reversible.

In the case of $s \neq r$. Applying logarithmic derivative yields

$$
\begin{aligned}
\frac{V'(t)}{V(t)} &= \frac{\frac{\partial \Phi_{s,p}(t)}{\partial t} - \frac{\partial \Phi_{r,p}(t)}{\partial t}}{\Phi_{s,p}(t) - \Phi_{r,p}(t)} - \frac{\frac{\partial \Phi_{q,p}(t)}{\partial t} - \frac{\partial \Phi_{r,p}(t)}{\partial t}}{\Phi_{q,p}(t) - \Phi_{r,p}(t)} \\
&= \frac{g(s) - g(r)}{h(s) - h(r)} - \frac{g(q) - g(r)}{h(q) - h(r)} \\
(2.14) \qquad &= (h(s) - h(q)) \frac{\frac{g(s)-g(r)}{h(s)-h(r)} - \frac{g(q)-g(r)}{h(q)-h(r)}}{h(s) - h(q)}.
\end{aligned}
$$

Using notations given by (2.5) and (2.6) yield

$$
\begin{aligned}
h(s) - h(q) &= \Phi_{s,p}(t) - \Phi_{q,p}(t) = t\left([st, pt; \phi] - [qt, pt; \phi]\right) \\
(2.15) \qquad &= t(st - qt) \frac{[st, pt; \phi] - [qt, pt; \phi]}{st - qt} = t^2(s - q)[st, qt, pt; \phi],
\end{aligned}
$$

$$
(2.16) \qquad \frac{\frac{g(s)-g(r)}{h(s)-h(r)} - \frac{g(q)-g(r)}{h(q)-h(r)}}{h(s) - h(q)} = [h(s), h(r), h(q); g \circ h^{-1}].
$$

Thus, $V'(t)/V(t)$ can be expressed as

$$
\begin{aligned}
(2.17) \qquad \frac{V'(t)}{V(t)} &= t^2(s - q) \cdot [st, qt, pt; \phi] \cdot [h(s), h(r), h(q); g \circ h^{-1}] \\
&: \; = V_1(t) \cdot V_2(t) \cdot V_3(t),
\end{aligned}
$$

consequently,

(2.18) $V'(t) = V(t) \cdot V_1(t) \cdot V_2(t) \cdot V_3(t).$

where,

$$\begin{aligned}
V_1(t) &= t^2(s-q), \\
V_2(t) &= [st, qt, pt; \phi], \\
V_3(t) &= [h(s), h(r), h(q); g \circ h^{-1}].
\end{aligned}$$

It is easy to verify that (2.18) is also true in the case of $s = r$.

From (2.7) and $\phi''(x) > (<)0$ it follows that

(2.19) $V(t) = \dfrac{[rt, st, pt; \phi]}{[pt, qt, rt; \phi]} > 0$ and $V_2(t) = [st, qt, pt; \phi] > (<)0;$

while

(2.20) $\operatorname{sgn} V_1(t) = \operatorname{sgn}\left(t^2(s-q)\right) = \operatorname{sgn}(s-q).$

Thus, to show that $\operatorname{sgn}(V'(t))$ is a constant, it is enough to show that $\operatorname{sgn}(V_3(t))$ is also a constant.

Step 3. Treating the $\operatorname{sgn}(V_3(t))$.

By 3) of Lemma 2.4, there is a ξ between the smallest and largest among the $h(s), h(r)$ and $h(q)$ such that

(2.21) $V_3(t) = [h(s), h(r), h(q); g \circ h^{-1}] = \dfrac{1}{2}(g(h^{-1}(y)))''|_{y=\xi}.$

Let $F(y) := g(h^{-1}(y))$. Then

$$\begin{aligned}
F'(y) &= \frac{g'(x)}{h'(x)}, \text{ where } x = h^{-1}(y),
\end{aligned}$$

(2.22) $\qquad F''(y) = \left(\dfrac{g'(x)}{h'(x)}\right)' \cdot \dfrac{1}{h'(x)},$ where $x = h^{-1}(y).$

From (2.13) and (2.12) we have

(2.23) $\dfrac{g'(x)}{h'(x)} = \dfrac{1}{t}\dfrac{\int_x^p (p-u)\left(2\phi''(ut) + ut\phi'''(ut)\right)du}{\int_x^p (p-u)\phi''(ut)du}.$

Differentiating and simplifying yield

(2.24) $\left(\dfrac{g'(x)}{h'(x)}\right)' = \dfrac{-(p-x)\phi''(xt)}{t\left(\int_p^x (u-p)\phi''(ut)du\right)^2}\int_x^p (p-u)\phi''(ut)\left(\kappa(ut) - \kappa(xt)\right)du.$

From $\phi''(x) > (<)0$ and assumption that $\kappa(x) = x\phi'''(x)/\phi''(x)$ is strictly monotone increasing for all $x > 0$ it follows that

(2.25) $\left(\dfrac{g'(x)}{h'(x)}\right)' < 0$ for $0 < x \le p.$

This together with $h'(x) > (<)0$ yields

(2.26) $F''(y) = \left(\dfrac{g'(x)}{h'(x)}\right)' \cdot \dfrac{1}{h'(x)} < (>)0$ for $0 < x \le p,$ where $x = h^{-1}(y).$

Now we can treat the $\operatorname{sgn}(V_3(t))$.

That $h'(x) > (<)0$ implies that $h^{-1}(x)$ is reversible and strictly increasing (decreasing), from

$$\min(h(s), h(r), h(q)) < \xi < \max(h(s), h(r), h(q))$$

it follows that

$$0 < \min(s, r, q) < x = h^{-1}(\xi) < \max(s, r, q) < p.$$

(2.25) together with $h'(x) > (<)0, 0 < x < p$ yields

$$(2.27) \quad V_3(t) = \frac{1}{2}(g(h^{-1}(y)))''\Big|_{y=\xi} = \left(\frac{g'(x)}{h'(x)}\right)' \cdot \frac{1}{h'(x)}\Big|_{x=h^{-1}(\xi)} < (>)0.$$

Step 4. Final conclusion.

(2.19), (2.20) in conjunction with (2.27) yield

$$\operatorname{sgn}(V'(t)) = \operatorname{sgn}(V(t)) \cdot \operatorname{sgn}(V_1(t)) \cdot \operatorname{sgn}(V_2(t)) \cdot \operatorname{sgn}(V_3(t)) = \operatorname{sgn}(s-q).$$

This shows that, for fixed real values $p, q, r, s > 0$, $V(t)$ is monotone on $\mathbb{R}_+$.

This lemma is proved. ∎

Equipped with the above lemmas, we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Since p and q, r and s both are symmetric, we assume that $p \geq q$, $r \geq s$. Denote by $\boldsymbol{\Delta}(t) := \Phi_{p,q}(t) - \Phi_{r,s}(t)$.

(i) In the case of $(s-p)(q-r)(s-q)(r-p) = 0$. For instance, if $p = r, q \neq s$ then

$$\begin{aligned}
\boldsymbol{\Delta}(t) &= \Phi_{p,q}(t) - \Phi_{r,s}(t) = \Phi_{p,q}(t) - \Phi_{p,s}(t) \\
&= t^2(q-s)\frac{\dfrac{\phi(pt)-\phi(qt)}{pt-qt} - \dfrac{\phi(pt)-\phi(st)}{pt-st}}{qt-st} = t^2(q-s)[pt, qt, st; \phi].
\end{aligned}$$

If $\Phi_{p,q}(t) \leq \Phi_{r,s}(t)$ for all $t \in [c_1, c_2]$, then $\Phi_{p,q}(c_1) \leq \Phi_{r,s}(c_1)$ and $\Phi_{p,q}(c_2) \leq \Phi_{r,s}(c_2)$. Conversely, if $\Phi_{p,q}(c_1) \leq \Phi_{r,s}(c_1)$ and $\Phi_{p,q}(c_2) \leq \Phi_{r,s}(c_2)$, then $\boldsymbol{\Delta}(c_1) \leq 0$ and $\boldsymbol{\Delta}(c_2) \leq 0$. From $\boldsymbol{\Delta}(c_1) = c_1^2(q-s)[pc_1, qc_1, sc_1; \phi] \leq 0$ and $[pt, qt, st; \phi] > (<)0$ due to $\phi''(x) > (<)0$ it follows that $q - s \leq 0$. This yields $\boldsymbol{\Delta}(t) \leq 0$ for all $t \in [c_1, c_2]$.

In the same way, our required result is also true in other cases.

(ii) In the case of $(s-p)(q-r)(s-q)(r-p) \neq 0$. Then

$$\begin{aligned}
\boldsymbol{\Delta}(t) &= \Phi_{p,q}(t) - \Phi_{r,s}(t) = (\Phi_{p,q}(t) - \Phi_{p,r}(t)) - (\Phi_{r,s}(t) - \Phi_{r,p}(t)) \\
&= \frac{\Phi_{p,q}(t) - \Phi_{p,r}(t)}{q-r}\left(q - r - (s-p)\frac{\dfrac{\Phi_{r,s}(t) - \Phi_{r,p}(t)}{s-p}}{\dfrac{\Phi_{p,q}(t) - \Phi_{p,r}(t)}{q-r}}\right)
\end{aligned}$$

$$(2.28) \qquad \quad := t^2[pt, qt, rt; \phi]U(t),$$

where

$$(2.29) \quad U(t) = (q-r) - (s-p)V(t),$$

here $V(t)$ is defined by (2.8). From Lemma 2.5, we see that $U(t)$ is also monotone on $(0, \infty)$.

On the other hand, by (2.28) $U(t)$ can be written as

$$(2.30) \quad U(t) = \frac{\boldsymbol{\Delta}(t)}{t^2[pt, qt, rt; \phi]} = \frac{\Phi_{p,q}(t) - \Phi_{r,s}(t)}{t^2[pt, qt, rt; \phi]}.$$

That $\phi''(x) > (<)0$ implies that $[pt, qt, rt; \phi] > (<)0$. It follows from (2.30) that the inequality $\Phi_{p,q}(t) \leq \Phi_{r,s}(t)$ for all $t \in [c_1, c_2]$, namely, $\boldsymbol{\Delta}(t) \leq 0$ for all $t \in [c_1, c_2]$ if and only if $U(t) \leq (\geq)0$ for all $t \in [c_1, c_2]$. By Lemma 2.3 this is equivalent to $U(c_1) \leq (\geq)0$ and $U(c_2) \leq (\geq)0$, which are also equivalent to

$$\begin{aligned}
\boldsymbol{\Delta}(c_1) &= \Phi_{p,q}(c_1) - \Phi_{r,s}(c_1) = c_1^2[pc_1, qc_1, rc_1; \phi]U(c_1) \leq 0, \\
\boldsymbol{\Delta}(c_2) &= \Phi_{p,q}(c_2) - \Phi_{r,s}(c_2) = c_2[pc_2, qc_2, rc_2; \phi]U(c_2) \leq 0.
\end{aligned}$$

This shows that $\Phi_{p,q}(t) \le \Phi_{r,s}(t)$ holds for all $t \in [c_1, c_2]$ if and only if $\Phi_{p,q}(c_1) \le \Phi_{r,s}(c_1)$ and $\Phi_{p,q}(c_2) \le \Phi_{r,s}(c_2)$.

This completes the proof. ∎

2.6. Remark. From the part one of proof above, it is easy to see that $\Phi_{p,q}(t)$ defined by (2.1) is strictly increasing (decreasing) with respect to either p or q if $\phi''(x) > (<)0$.

3. Proofs of Main Results

By Theorem 2.2 and the following simple lemma, we can prove Theorem 1.5.

3.1. Lemma([11]). *The function* $(x, y) \to L(x, y)$ $(x, y \in \mathbb{R}_+)$ *is strictly increasing in either x or y.*

Proof of Theorem 1.5. Since $e^{1/L(p,q)}S_{p,q}(a,b)$ is symmetric with respect to a and b, we assume that $a > b$ and denote by $t := \ln\sqrt{a/b}$. Then $t > 0$. By some simple transformations and using $\sinh x = \left(e^x - e^{-x}\right)/2$ we have

$$(3.1)\quad \ln\left(e^{1/L(p,q)}S_{p,q}(a,b)\right) = \frac{\ln(a^p - b^p) - \ln(a^q - b^q)}{p - q} = \ln\sqrt{ab} + \frac{\ln\sinh(pt) - \ln\sinh(qt)}{p - q}.$$

Let $\phi(t) = \ln\sinh t$ and

$$(3.2)\quad \Phi_{p,q}(t) := \begin{cases} \frac{\ln\sinh(pt) - \ln\sinh(qt)}{p - q} & \text{if } p \ne q, \\ \frac{t\cosh pt}{\sinh pt} & \text{if } p = q. \end{cases}$$

Then the comparison inequality (1.19) is equivalent to

$$(3.3)\quad \Phi_{p,q}(t) \le \Phi_{r,s}(t) \quad \text{for all } 0 < t < \infty.$$

Some direct computations yield

$$\phi'(t) = \frac{\cosh t}{\sinh t}, \quad \phi''(t) = -\frac{1}{\sinh^2 t} < 0, \quad \phi'''(t) = \frac{2\cosh t}{\sinh^3 t}$$

$$\left(\frac{t\phi'''(t)}{\phi''(t)}\right)' = \left(-\frac{2t\cosh t}{\sinh t}\right)' = -\frac{\sinh 2t - 2t}{\sinh^2 t} < 0 \text{ for all } t > 0.$$

According to Theorem 2.2, for fixed $p, q, r, s > 0$ and $c_2 > c_1 > 0$, the comparison inequality (1.19) holds for all $a, b \in \mathbb{R}_+$ with $a \ne b$ if and only if both the following

$$(3.4)\qquad\qquad \Phi_{p,q}(c_1) \le \Phi_{r,s}(c_1), \ c_1 \to 0^+,$$
$$(3.5)\qquad\qquad \Phi_{p,q}(c_2) \le \Phi_{r,s}(c_2), \ c_2 \to \infty$$

hold.

Letting $c_1 \to 0^+$ yields

$$\lim_{c_1 \to 0^+} \Phi_{p,q}(c_1) = \lim_{c_1 \to 0^+} \frac{\ln\sinh(pc_1) - \ln\sinh(qc_1)}{p - q} = \frac{\ln p - \ln q}{p - q} = \frac{1}{L(p, q)}.$$

Then (3.4) is equivalent to

$$(3.6)\quad L(p, q) \ge L(r, s),$$

which is the first inequality of (1.20).

Since $\Phi_{p,q}(t)$ is symmetric with respect to p and q, we assume that $p \ge q$; likewise, we can assume that $r \ge s$; that is,

$$(3.7)\quad q = \min(p, q), \quad s = \min(r, s).$$

In order to obtain the second one of (1.20) from (3.5), we distinguish four cases.

Case 1: $(p-q)(r-s) \neq 0$. A simple transformation yields

$$\Phi_{p,q}(t) = \frac{\ln \sinh(pt) - \ln \sinh(qt)}{p-q} = \frac{\ln\left(e^{pt/2}(1-e^{-pt})/2\right) - \ln\left(e^{qt/2}(1-e^{-qt})/2\right)}{p-q}$$

$$= \frac{t}{2} + \frac{\ln(1-e^{-pt}) - \ln(1-e^{-qt})}{p-q},$$

so we have

$$\lim_{t \to \infty} \frac{\Phi_{p,q}(t) - \Phi_{r,s}(t)}{e^{-t\min(q,s)}} = \lim_{t \to \infty} \frac{\frac{\ln(1-e^{-pt})-\ln(1-e^{-qt})}{p-q} - \frac{\ln(1-e^{-rt})-\ln(1-e^{-st})}{r-s}}{e^{-t\min(q,s)}}$$

$$= \lim_{t \to \infty} \frac{\frac{1}{p-q}\left(\frac{pe^{-pt}}{1-e^{-pt}} - \frac{qe^{-qt}}{1-e^{-qt}}\right) - \frac{1}{r-s}\left(\frac{re^{-rt}}{1-e^{-rt}} - \frac{se^{-st}}{1-e^{-st}}\right)}{-\min(q,s)e^{-t\min(q,s)}}$$

$$(3.8) \qquad = \begin{cases} \frac{-1}{r-s} < 0 & \text{if } q > s; \\ \frac{1}{p-q} - \frac{1}{r-s} & \text{if } q = s; \\ \frac{1}{p-q} > 0 & \text{if } q < s. \end{cases}$$

By Lemma 3.1, the first inequality of (1.20) or (3.6) leads to $p \geq r$ if $q = s$, which yields $\frac{1}{p-q} - \frac{1}{r-s} = \frac{r-p}{(p-q)(r-s)} \leq 0$. It follows from (3.8) that

$$(3.9) \qquad \lim_{t \to \infty} \frac{\Phi_{p,q}(t) - \Phi_{r,s}(t)}{e^{-t\min(q,s)}} \begin{cases} < 0, & q > s; \\ \leq 0, & q = s; \\ > 0, & q < s, \end{cases}$$

which implies that, in this case, (3.5) is equivalent to

$$(3.10) \quad \min(p,q) \geq \min(r,s).$$

Case 2: $p = q, r > s$. We claim that $q \neq s$. If not, that is, $q = s$, then from the first inequality of (1.20) or (3.6) it follows that

$$s = L(s,s) = L(p,q) \geq L(r,s),$$

which in conjunction with $L(r,s) \geq \min(r,s) = s$ yields $s = r$. This is a contradiction. Thus,

$$\lim_{t \to \infty} \frac{\Phi_{q,q}(t) - \Phi_{r,s}(t)}{e^{-t\min(q,s)}} = \lim_{t \to \infty} \frac{\frac{qe^{-qt}}{1-e^{-qt}} - \frac{\ln(1-e^{-rt})-\ln(1-e^{-st})}{r-s}}{e^{-t\min(q,s)}}$$

$$= \lim_{t \to \infty} \frac{\frac{-q^2 e^{-qt}}{\left(1-e^{-qt}\right)^2} - \frac{1}{r-s}\left(\frac{re^{-rt}}{1-e^{-rt}} - \frac{se^{-st}}{1-e^{-st}}\right)}{-\min(q,s)e^{-t\min(q,s)}}$$

$$(3.11) \qquad = \begin{cases} \frac{-1}{r-s} < 0 & \text{if } q > s, \\ q > 0 & \text{if } q < s. \end{cases}$$

This shows that (3.5) is equivalent to (3.10) in this case.

Case 3: $p > q, r = s$. If $q \neq s$, then

$$\lim_{t \to \infty} \frac{\Phi_{p,q}(t) - \Phi_{s,s}(t)}{e^{-t\min(q,s)}} = \lim_{t \to \infty} \frac{\frac{\ln(1-e^{-pt})-\ln(1-e^{-qt})}{p-q} - \frac{se^{-st}}{1-e^{-st}}}{e^{-t\min(q,s)}}$$

$$= \lim_{t \to \infty} \frac{\frac{1}{p-q}\left(\frac{pe^{-pt}}{1-e^{-pt}} - \frac{qe^{-qt}}{1-e^{-qt}}\right) - \frac{-s^2 e^{-st}}{\left(1-e^{-st}\right)^2}}{-\min(q,s)e^{-t\min(q,s)}}$$

$$(3.12) \qquad = \begin{cases} -s < 0 & \text{if } q > s, \\ \frac{1}{p-q} > 0 & \text{if } q < s. \end{cases}$$

If $q = s$, from $\phi''(t) < 0$ and Remark 2.6 it follows that $\Phi_{p,q}(t) < \Phi_{q,q}(t)$, which yields

$$\Phi_{p,q}(t) - \Phi_{s,s}(t) < \Phi_{q,q}(t) - \Phi_{s,s}(t) = 0 \text{ for all } t > 0.$$

These also show that (3.5) is equivalent to (3.10) in this case.

Case 4: $p = q, r = s$. We have

$$(3.13) \quad \lim_{t \to \infty} \frac{\Phi_{q,q}(t) - \Phi_{s,s}(t)}{e^{-t\min(q,s)}} = \lim_{t \to \infty} \frac{\frac{qe^{-qt}}{1-e^{-qt}} - \frac{se^{-st}}{1-e^{-st}}}{e^{-t\min(q,s)}} = \begin{cases} -s < 0 & \text{if } q > s, \\ 0 & \text{if } q = s, \\ q > 0 & \text{if } q < s. \end{cases}$$

This shows that (3.5) is similarly equivalent to (3.10) in this case.

Consequently, (3.5) is equivalent to (3.10) in all cases, that is, the second inequality of (1.20).

In conclusion, then necessary and sufficient condition for (1.19) to hold is: (3.6) and (3.10), that is, (1.20).

This completes the proof. ∎

It is regrettable that Theorem 2.2 is also not applicable to prove Theorem 1.6, because when $\phi(x) = \ln(x \cosh x)$ the condition "$\kappa(x) = x\phi'''(x)/\phi''(x)$ is strictly monotone for all $x > 0$" is not satisfied. Fortunately, we can use Theorem 1.5 to prove Theorem 1.6. To this end, we also need other two lemmas.

3.2. Lemma. *For fixed $m > 0$ the function $x \to L(x, m-x)$ $(0 < x < m)$ is strictly increasing in x on $(0, m/2]$ and strictly decreasing on $[m/2, m)$.*

Proof. A direct derivation yields

$$\frac{dL(x, m-x)}{dx} = \frac{2(\ln x - \ln(m-x)) - \frac{m(2x-m)}{x(m-x)}}{(\ln x - \ln(m-x))^2} = \frac{g(x)}{(\ln x - \ln(m-x))^2},$$

$$\frac{dg(x)}{dx} = -\frac{m(2x-m)^2}{x^2(m-x)^2} < 0.$$

It follows that $g(x) > g(m/2) = 0$ if $x \in (0, m/2)$ and $g(x) < g(m/2) = 0$ if $x \in (m/2, m)$, which completes the proof. ∎

3.3. Lemma. *Denote by*

$$(3.14) \qquad \Omega_1 = \{(p,q;r,s) : p+q \geq r+s, p,q,r,s \in \mathbb{R}_+\},$$
$$(3.15) \qquad \Omega_2 = \{(p,q;r,s) : \min(p,q) \geq \min(r,s), p,q,r,s \in \mathbb{R}_+\},$$
$$(3.16) \qquad \Omega_3 = \{(p,q;r,s) : L(p,q) \geq L(r,s), p,q,r,s \in \mathbb{R}_+\}.$$

Then $\Omega_1 \cap \Omega_2 \subseteq \Omega_3$.

Proof. Without loss of generality, we assume that $p \geq q, r \geq s$, that is, $\min(p,q) = q$, $\min(r,s) = s$. We now prove that $(p,q;r,s) \in \Omega_3$ for every $(p,q;r,s) \in \Omega_1 \cap \Omega_2$.

Indeed, $(p,q;r,s) \in \Omega_1$ implies that $r \leq p+q-s$, and from Lemma 3.1 it follows that

$$L(r,s) \leq L(p+q-s, s).$$

While $(p,q;r,s) \in \Omega_2$ together with assumption $p \geq q$ implies that $0 < s \leq q \leq (p+q)/2$, and from Lemma 3.2 it follows that

$$L(r,s) \leq L(p+q-s, s) \leq L(p+q-q, q) = L(p,q),$$

which shows that $(p,q;r,s) \in \Omega_3$.

This completes the proof. ∎

Proof of Theorem 1.6. First of all, we show that (1.21) holds if $(p, q; r, s) \in \Omega_1 \cap \Omega_2 \cap \Omega_3$.

Indeed, $(p, q; r, s) \in \Omega_2 \cap \Omega_3$ implies that

$$L(p, q) \geq L(r, s) \quad \text{and} \quad \min(p, q) \geq \min(r, s),$$

which is equivalent to

$$L(2p, 2q) \geq L(2r, 2s) \quad \text{and} \quad \min(2p, 2q) \geq \min(2r, 2s).$$

Thus, by Theorem 1.5 the comparison inequality

$$(3.17) \quad e^{1/L(2p, 2q)} S_{2p, 2q}(a, b) \leq e^{1/L(2r, 2s)} S_{2r, 2s}(a, b)$$

holds for all $a, b \in \mathbb{R}_+$ with $a \neq b$ if and only if $(p, q; r, s) \in \Omega_2 \cap \Omega_3$.

Note $S_{2p, 2q}(a, b) = \sqrt{S_{p,q}(a, b) G_{p,q}(a, b)}$, by a simple equivalent transformation inequality (3.17) can be written as

$$e^{1/L(p, q)} S_{p,q}(a, b) G_{p,q}(a, b) \leq e^{1/L(r, s)} S_{r,s}(a, b) G_{r,s}(a, b),$$

which is equivalent to

$$(3.18) \quad e^{1/L(p, q)} G_{p,q}(a, b) \leq e^{1/L(r, s)} G_{r,s}(a, b) \frac{S_{r,s}(a, b)}{S_{p,q}(a, b)}.$$

Meanwhile, by Theorem 1.1 the comparison inequality $S_{r,s}(a, b) \leq S_{p,q}(a, b)$ or

$$(3.19) \quad \frac{S_{r,s}(a, b)}{S_{p,q}(a, b)} \leq 1$$

holds for all $a, b \in \mathbb{R}_+$ with $a \neq b$ if and only if $(p, q; r, s) \in \Omega_1 \cap \Omega_2$.

It follows from (3.18) and (3.19) that

$$(3.20) \quad e^{1/L(p, q)} G_{p,q}(a, b) \leq e^{1/L(r, s)} G_{r,s}(a, b) \frac{S_{r,s}(a, b)}{S_{p,q}(a, b)} \leq e^{1/L(r, s)} G_{r,s}(a, b)$$

hold if and only if $(p, q; r, s) \in \Omega_1 \cap \Omega_2 \cap \Omega_3$. This shows that (1.21) holds if $(p, q; r, s) \in \Omega_1 \cap \Omega_2 \cap \Omega_3$.

By Lemma 3.3, the proof is completed. ∎

4. COMPANION INEQUALITIES FOR BIVARIATE MEANS

In this section, we will give the companion inequalities for Stolarsky means and Gini means.

Using Theorem 1.1 and 1.5, we first give the improvement of Theorem 1.3 as a corollary of our main results.

4.1. Corollary. *Let* $p, q, r, s \in \mathbb{R}_+$. *Then the following companion inequalities*

$$(4.1) \quad S_{r,s}(a, b) \leq S_{p,q}(a, b) \leq \exp\left(\frac{1}{L(r, s)} - \frac{1}{L(p, q)}\right) S_{r,s}(a, b)$$

hold for all $a, b \in \mathbb{R}_+$ *with* $a \neq b$ *if and only if (1.22) holds.*

Proof. By Theorem 1.1 the first inequality of (4.1) holds if and only if $(p, q; r, s) \in \Omega_1 \cap \Omega_3$. The second one of (4.1) is equivalent to (1.19), which holds, by Theorem 1.5, if and only if $(p, q; r, s) \in \Omega_2 \cap \Omega_3$. Hence (4.1) hold if and only if $(p, q; r, s) \in \Omega_1 \cap \Omega_2 \cap \Omega_3$.

By Lemma 3.3 the assertion follows. ∎

For the Gini means, we have the same result.

4.2. Corollary. *Let $p, q, r, s \in \mathbb{R}_+$. Then the following companion inequalities*

$$(4.2) \qquad G_{r,s}(a,b) \leq G_{p,q}(a,b) \leq \exp\left(\frac{1}{L(r,s)} - \frac{1}{L(p,q)}\right) G_{r,s}(a,b).$$

hold for all $a, b \in \mathbb{R}_+$ with $a \neq b$ if and only if (1.22) holds.

Proof. **Necessity.** If (4.2) holds, then by comparison theorem for Gini means 1.2 the first inequality of (4.2) implies that $(p, q; r, s) \in \Omega_1 \cap \Omega_2$. In the second one of (4.2), letting $b \to a$ yields $a \leq a \exp\left(\frac{1}{L(r,s)} - \frac{1}{L(p,q)}\right)$, which implies that $L(p,q) \geq L(r,s)$, that is, $(p, q; r, s) \in \Omega_3$. Hence (4.2) holds implies that $(p, q; r, s) \in \Omega_1 \cap \Omega_2 \cap \Omega_3$. By Lemma 3.3 $(p, q; r, s) \in \Omega_1 \cap \Omega_2$.

Sufficiency. If (1.22) holds, by Theorem 1.2 and Theorem 1.6 we obtain that

$$G_{p,q}(a,b) \geq G_{r,s}(a,b) \quad \text{and} \quad e^{1/L(p,q)} G_{p,q}(a,b) \leq e^{1/L(r,s)} G_{r,s}(a,b),$$

respectively, which are equivalent to the first and second inequality of (4.2), respectively.

This completes the proof. ∎

The following is a pair of companion inequalities for Gini means and power mean.

4.3. Corollary. *Let $t \neq 0$. Then the companion inequalities*

$$(4.3) \qquad A_t(a,b) \leq G_{p,q}(a,b) < 2^{1/t} A_t(a,b)$$

hold for all $a, b \in \mathbb{R}_+$ with $a \neq b$ if and only if $p, q \geq 0$ and $p + q \geq t > 0$.

Proof. **Necessity.** Firstly, we have $t > 0$. If not, that is, $t < 0$, then

$$A_t(a,b) \leq G_{p,q}(a,b) < 2^{1/t} A_t(a,b) < A_t(a,b),$$

which yields a contradiction. Therefore $t > 0$.

Secondly, note $A_t(a,b) = G_{t,0}(a,b)$, by Theorem 1.2 the first inequality of (4.3) holds if and only if both the inequalities

$$p + q \geq t + 0,$$

$$\begin{cases} (i) \ \min(p,q) \geq \min(t,0) & \text{if } \min(p,q,t,0) \geq 0, \\ (ii) \ \max(p,q) \leq \max(t,0) & \text{if } \max(p,q,t,0) \leq 0, \\ (iii) \ \mu(p,q) \geq \mu(t,0) & \text{if } \min(p,q,t,0) < 0 < \max(p,q,t,0), \end{cases}$$

hold. Solving the inequalities in conjunction with $t > 0$ yields

$$(p, q, t) \in \{(p, q, t) : p, q \geq 0, p + q \geq t > 0\} := E.$$

Sufficiency. If $(p, q, t) \in E$, then the first inequality of (4.3) follows from Theorem 1.2.

To prove the second one of (4.3), we first show that

$$(4.4) \qquad \max(a,b) < 2^{1/t} A_t(a,b) < \infty$$

hold for $t > 0$.

For $t_1 > t_2 > 0$ we easily check that $L(2t_1, t_1) > L(2t_2, t_2)$ and $\min(2t_1, t_1) > \min(2t_2, t_2)$. By Theorem 1.5 we have $e^{1/L(2t_1,t_1)} S_{2t_1,t_1}(a,b) < e^{1/L(2t_2,t_2)} S_{2t_2,t_2}(a,b)$, that is, $2^{1/t_1} A_{t_1}(a,b) < 2^{1/t_2} A_{t_2}(a,b)$, which implies that the function $t \to 2^{1/t} A_t(a,b)$ strictly decreases on $\mathbb{R}_+$. And, simple calculations lead to

$$\lim_{t \to 0, t > 0}\left(2^{1/t} A_t(a,b)\right) = \infty, \quad \lim_{t \to \infty}\left(2^{1/t} A_t(a,b)\right) = \max(a,b).$$

Hence (4.4) holds.

On the other hand, since $G_{p,q}(a,b)$ is a mean of positive number a and b for every (p,q), we have

$$(4.5) \qquad \min(a,b) \leq G_{p,q}(a,b) \leq \max(a,b).$$

Inequality (4.4) in conjunction with (4.5) yields the second one of (4.3).

The proof is ended. ∎

Lastly, we give a new proof of a part of Theorem 1.4. For clarity, we restate this part as a corollary of our main results.

4.4. Corollary. *Let $p, q \geq 0$ and $r, s > 0$. Then the following companion inequalities*

$$(4.6) \qquad S_{r,s}(a,b) \leq G_{p,q}(a,b) < e^{1/L(r,s)} S_{r,s}(a,b).$$

hold for all $a, b \in \mathbb{R}_+$ with $a \neq b$ if $p + q \geq \max((r+s)/3, \min(r,s))$.

Proof. If $p + q \geq \max((r+s)/3, \min(r,s))$, that is, $2(p+q) + (p+q) \geq r + s$ and $\min(2(p+q), (p+q)) \geq \min(r,s)$, then by Corollary 4.1, we have

$$(4.7) \qquad S_{r,s}(a,b) \leq S_{2(p+q),(p+q)}(a,b) \leq \exp\left(\frac{1}{L(r,s)} - \frac{1}{L(2(p+q),(p+q))}\right) S_{r,s}(a,b).$$

Note $S_{2(p+q),p+q}(a,b) = A_{p+q}(a,b)$, then (4.7) can be rewritten as

$$(4.8) \qquad S_{r,s}(a,b) \leq A_{p+q}(a,b) \quad \text{and} \quad 2^{1/(p+q)} A_{p+q}(a,b) \leq e^{1/L(r,s)} S_{r,s}(a,b).$$

Putting $t = p + q$ in Corollary 4.3, since $p, q \geq 0$ and $p + q \geq t > 0$, we get

$$A_{p+q}(a,b) \leq G_{p,q}(a,b) < 2^{1/(p+q)} A_{p+q}(a,b),$$

which together with (4.8) leads to

$$S_{r,s}(a,b) \leq A_{p+q}(a,b) \leq G_{p,q}(a,b) < 2^{1/(p+q)} A_{p+q}(a,b) \leq e^{1/L(r,s)} S_{r,s}(a,b).$$

Our required result follows. ∎

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