Hacettepe Journal of Mathematics and Statistics
Volume 46 (3) (2017), 373 – 388

Convergence and data dependence results for quasi-contractive type operators in hyperbolic spaces

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Abstract

In this paper, we simplify the iterative scheme introduced by Fukhar-ud-din and Berinde [Iterative Methods for the Class of Quasi-Contractive Type Operators and Comparison of their Rate of Convergence in Convex Metric Spaces, Filomat 30 (1), 223–230, 2016] and study convergence and data dependency of the new proposed scheme of a quasi-contractive operator on a hyperbolic space. It is shown that our results provide better convergence rate.

Keywords: Data dependency, strong convergence, iterative scheme, hyperbolic space, rate of convergence.

2000 AMS Classification: 47H09, 47H10, 54H25.

Received : 29.04.2016 *Accepted* : 26.07.2016 *Doi* : 10.15672/HJMS.20174720334

1. Introduction and preliminaries

Takahashi [25] defined the concept of convexity in a metric space (M,d) as under:
A map $W : M^2 \times [0,1] \to M$ is a convex structure in M if

$$d(u, W(u_1, u_2, \eta)) \leq \eta d(u, u_1) + (1-\eta) d(u, u_2) \tag{1.1}$$

holds for all u, u_1, $u_2 \in M$ and $\eta \in [0,1]$. A nonempty subset C of a convex metric space is convex if $W(u_1, u_2, \eta) \in C$ for all u_1, $u_2 \in C$ and $\eta \in [0,1]$.

From the definition of convex structure W on M, it is obvious that

$$d(u, W(u_1, u_2, \eta)) \geq (1-\eta) d(u, u_2) - \eta d(u, u_1) \tag{1.2}$$

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for all u, u_1, $u_2 \in M$ and $\eta \in [0, 1]$.

Kohlenbach [20] enriched the concept of convex metric space as "hyperbolic metric space" by adding the following conditions.

$$d(W(u_1, u_2, \eta_1), W(u_1, u_2, \eta_2)) = |\eta_1 - \eta_2| d(u_1, u_2), \tag{1.3}$$

$$W(u_1, u_2, \eta) = W(u_2, u_1, 1 - \eta), \tag{1.4}$$

$$d(W(u_1, u_3, \eta), W(u_2, u_4, \eta)) \leq \eta d(u_1, u_2) + (1 - \eta) d(u_3, u_4), \tag{1.5}$$

for all u_1, u_2, u_3, $u_4 \in M$ and η, η_1, $\eta_2 \in [0, 1]$.

Normed spaces and their subsets are hyperbolic spaces as well as convex metric spaces. The class of hyperbolic spaces is properly contained in the class of convex metric spaces (see [18], [20]). It is worth to mention that an important and interesting example of convex metric spaces is a $CAT(0)$ space (see [4], [6], [13], [16]).

Search for new iterative scheme to solve nonlinear problems is one of the main themes of fixed point theory. For various iterative schemes, convergency, stability and data dependency have been investigated in ([1], [2], [5], [7], [9], [11], [12], [14], [15], [17], [23], [24]).

Let C be a nonempty convex subset of a convex metric space M and $T : C \to C$ be an operator. Let $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}, \{\alpha_n^{(4)}\}$ and $\{\alpha_n^{(5)}\}$ be sequences in $[0, 1]$ satisfying certain control condition (s).

Recently, Fukhar-ud-din and Berinde [8] have proposed the following scheme for a given $x_1^{(1)} \in C$:

$$\begin{cases} x_{n+1}^{(1)} = W\left(Tx_n^{(2)}, W\left(Tx_n^{(3)}, x_n^{(1)}, \frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}}\right), \alpha_n^{(1)}\right), \\ x_n^{(2)} = W\left(Tx_n^{(3)}, W\left(Tx_n^{(1)}, x_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), \alpha_n^{(4)}\right), \\ x_n^{(3)} = W\left(Tx_n^{(1)}, x_n^{(1)}, \alpha_n^{(3)}\right), n \geq 1. \end{cases} \tag{1.6}$$

It has been proved by Fukhar-ud-din and Berinde that the sequence $\{x_n^{(1)}\}$ in (1.6) converges to a unique fixed point s_* of a quasi-contractive operator T which satisfies the following condition:

$$d(Tu_1, Tu_2) \leq \delta d(u_1, u_2) + \epsilon d(Tu_1, u_1) \tag{1.7}$$

for any u_1, $u_2 \in M$, $\delta \in (0, 1)$ and $\epsilon \geq 0$. They have also shown that iterative scheme (1.6) converges faster to the mentioned fixed point s_* than Ishikawa [10], Mann [21] and Xu and Noor [28] iterative schemes.

Since lightest workload and quick achievement are always desired, therefore a faster and simpler iterative scheme is still in demand.

For this we would like to construct an iterative scheme with a better convergence rate than that of (1.6). In this context, we consider a simplified form of (1.6): Using (1.4) and "$W(u_1, u_2, 0) = u_2$ for any u_1, u_2 in M ([26], Proposition 1.2 (a))" in (1.6), we get for a given $s_1^{(1)} \in C$ that

$$\begin{cases} s_{n+1}^{(1)} = Ts_n^{(2)}, \\ s_n^{(2)} = W\left(Ts_n^{(3)}, W\left(Ts_n^{(1)}, s_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), \alpha_n^{(4)}\right), \\ s_n^{(3)} = W\left(Ts_n^{(1)}, s_n^{(1)}, \alpha_n^{(3)}\right), n \geq 1. \end{cases} \tag{1.8}$$

In this paper, we first show that iterative scheme (1.8) converges to a unique fixed point s_* of a quasi-contractive type operator (1.7) under different control conditions on parametric sequences. Next, we show that iterative schemes (1.6) and (1.8) are equivalent when converging to the same unique fixed point of the afore mentioned operator. We also show that iterative scheme (1.8) converges faster than the scheme (1.6) as well as some classical iterative schemes. Finally, we prove a data dependence result for the iterative scheme (1.8).

1.1. Definition. (see [3]) Let $T, \widetilde{T}: M \to M$ be two operators. We say that the operator $\widetilde{T}$ is an approximate operator of T if for all $v \in M$ and for some fixed $\varepsilon > 0$, we have

$$d(Tv, \widetilde{T}v) \leq \varepsilon.$$

1.2. Definition. (Rate of convergence; cf. [19]) Let $\{a_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2$ be sequences of real numbers with $\lim_{n\to\infty} a_n^{(1)} = a^{(1)}$ and $\lim_{n\to\infty} a_n^{(2)} = a^{(2)}$, where $a_n^{(2)} \neq a^{(2)}$, for all $n \in \mathbb{N}$.

We say that:

(i) $\{a_n^{(1)}\}_{n=0}^{\infty}$ converges faster than $\{a_n^{(2)}\}_{n=0}^{\infty}$ if $\lim_{n\to\infty} \frac{d\left(a_n^{(1)}, a^{(1)}\right)}{d\left(a_n^{(2)}, a^{(2)}\right)} = 0$,

(ii) $\{a_n^{(1)}\}_{n=0}^{\infty}$ converges slower than $\{a_n^{(2)}\}_{n=0}^{\infty}$ if $\lim_{n\to\infty} \frac{d\left(a_n^{(1)}, a^{(1)}\right)}{d\left(a_n^{(2)}, a^{(2)}\right)} = +\infty$,

(iii) $\{a_n^{(1)}\}_{n=0}^{\infty}$ converges at the same rate as $\{a_n^{(2)}\}_{n=0}^{\infty}$ if

$$0 < \underline{\lim}_{n\to\infty} \frac{d\left(a_n^{(1)}, a^{(1)}\right)}{d\left(a_n^{(2)}, a^{(2)}\right)} \leq \overline{\lim}_{n\to\infty} \frac{d\left(a_n^{(1)}, a^{(1)}\right)}{d\left(a_n^{(2)}, a^{(2)}\right)} < +\infty,$$

(iv) $\{a_n^{(1)}\}_{n=0}^{\infty}$ and $\{a_n^{(2)}\}_{n=0}^{\infty}$ converge completely at an incomparable rate if

$$\underline{\lim}_{n\to\infty} \frac{d\left(a_n^{(1)}, a^{(1)}\right)}{d\left(a_n^{(2)}, a^{(2)}\right)} = 0 \text{ and } \overline{\lim}_{n\to\infty} \frac{d\left(a_n^{(1)}, a^{(1)}\right)}{d\left(a_n^{(2)}, a^{(2)}\right)} = +\infty.$$

1.3. Definition. (A modified definition of rate of convergence; see [22]) Let $\{a_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2$ be iterative sequences converging to the same fixed point s_*. The sequence $\{a_n^{(1)}\}_{n=0}^{\infty}$ converges faster than $\{a_n^{(2)}\}_{n=0}^{\infty}$ to s_* if

$$\lim_{n\to\infty} \frac{d\left(a_n^{(1)}, s_*\right)}{d\left(a_n^{(2)}, s_*\right)} = 0.$$

1.4. Lemma. *(see [24]) Let $\{\sigma_n^{(1)}\}_{n=0}^{\infty}$ be a non-negative sequence of real numbers. If there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has the inequality*

$$\sigma_{n+1}^{(1)} \leq (1 - \sigma_n^{(2)})\sigma_n^{(1)} + \sigma_n^{(2)}\sigma_n^{(3)},$$

where $\sigma_n^{(2)} \in (0, 1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \sigma_n^{(2)} = \infty$, and $\sigma_n^{(3)} \geq 0$, $\forall n \in \mathbb{N}$, then the following inequality holds:

$$0 \leq \limsup_{n\to\infty} \sigma_n^{(1)} \leq \limsup_{n\to\infty} \sigma_n^{(3)}.$$

1.5. Lemma. *(see [27]) Let $\{\sigma_n^{(1)}\}_{n=0}^{\infty}$ and $\{\sigma_n^{(2)}\}_{n=0}^{\infty}$ be non-negative real sequences satisfying the following inequality:*

$$\sigma_{n+1}^{(1)} \leq (1 - \sigma_n^{(3)})\sigma_n^{(1)} + \sigma_n^{(2)},$$

where $\sigma_n^{(3)} \in (0,1)$, *for all* $n \geq n_0$, $\sum\limits_{n=0}^{\infty} \sigma_n^{(3)} = \infty$, *and* $\frac{\sigma_n^{(2)}}{\sigma_n^{(3)}} \to 0$ *as* $n \to \infty$. *Then* $\lim_{n\to\infty} \sigma_n^{(1)} = 0$.

2. Main Results

In the rest of the paper, we assume that C is a nonempty, closed and convex subset of a hyperbolic metric space M, $T : C \to C$ is a quasi-contractive operator satisfying (1.7), $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\alpha_n^{(1)} + \alpha_n^{(2)}\}$, $\{\alpha_n^{(3)}\}$, $\{\alpha_n^{(4)}\}$, $\{\alpha_n^{(5)}\}$ and $\{\alpha_n^{(4)} + \alpha_n^{(5)}\}$ are appropriately chosen sequences in $[0,1]$ and $F(T)$ denotes the set of fixed points of T.

2.1. Theorem. *Let* $\{s_n^{(1)}\}_{n=1}^{\infty}$ *be the iterative scheme (1.8) with* $s_1^{(1)} \in C$, $\sum\limits_{n=1}^{\infty} \alpha_n^{(4)} = \infty$ *(or* $\sum\limits_{n=1}^{\infty} \alpha_n^{(5)} = \infty$*). If* T *is a quasi-contractive operator and* $F(T) \neq \emptyset$, *then* $\{s_n^{(1)}\}_{n=1}^{\infty}$ *converges strongly to a unique fixed point* s_* *of* T.

Proof. An application of inequalities (1.1) and (1.8) to the sequences $\{s_n^{(3)}\}$ and $\{s_n^{(2)}\}$ in (1.8) yield

$$\begin{aligned} d\left(s_n^{(3)}, s_*\right) &= d(W(Ts_n^{(1)}, s_n^{(1)}, \alpha_n^{(3)}), s_*) \\ &\leq \alpha_n^{(3)} d\left(Ts_n^{(1)}, s_*\right) + \left(1 - \alpha_n^{(3)}\right) d\left(s_n^{(1)}, s_*\right) \\ &= \left[1 - \alpha_n^{(3)}(1-\delta)\right] d\left(s_n^{(1)}, s_*\right), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} d\left(s_n^{(2)}, s_*\right) &= d\left(W\left(Ts_n^{(3)}, W\left(Ts_n^{(1)}, s_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), \alpha_n^{(4)}\right), s_*\right) \\ &\leq \alpha_n^{(4)} d\left(Ts_n^{(3)}, s_*\right) + \left(1 - \alpha_n^{(4)}\right) d\left(W\left(Ts_n^{(1)}, s_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), s_*\right) \\ &\leq \alpha_n^{(4)} d\left(Ts_n^{(3)}, s_*\right) \\ &\quad + \left(1 - \alpha_n^{(4)}\right)\left\{\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}} d\left(Ts_n^{(1)}, s_*\right) + \left(\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right) d\left(s_n^{(1)}, s_*\right)\right\} \\ &= \alpha_n^{(4)} d\left(Ts_n^{(3)}, s_*\right) + \alpha_n^{(5)} d\left(Ts_n^{(1)}, s_*\right) + \left(1 - \alpha_n^{(4)} - \alpha_n^{(5)}\right) d\left(s_n^{(1)}, s_*\right) \\ &\leq \alpha_n^{(4)} \delta d\left(s_n^{(3)}, s_*\right) + \alpha_n^{(5)} \delta d\left(s_n^{(1)}, s_*\right) + \left(1 - \alpha_n^{(4)} - \alpha_n^{(5)}\right) d\left(s_n^{(1)}, s_*\right). \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$d(s_n^{(2)}, s_*) \leq \left\{\alpha_n^{(4)} \delta \left[1 - \alpha_n^{(3)}(1-\delta)\right] + \alpha_n^{(5)} \delta + 1 - \alpha_n^{(4)} - \alpha_n^{(5)}\right\} d\left(s_n^{(1)}, s_*\right). \tag{2.3}$$

Since $\delta \in (0,1)$ and $\alpha_n^{(3)} \in [0,1]$ for all $n \geq 1$,

$$1 - \alpha_n^{(3)}(1-\delta) \leq 1. \tag{2.4}$$

Using (2.4) in (2.3), we get

$$d\left(s_n^{(2)}, s_*\right) \leq \left[1 - \left(\alpha_n^{(4)} + \alpha_n^{(5)}\right)(1-\delta)\right] d\left(s_n^{(1)}, s_*\right). \tag{2.5}$$

Again applying inequalities (1.1) and (1.7) to the sequence $\{s_n^{(1)}\}$ in (1.8) (or putting $\alpha_n^{(1)} = 1$ and $\alpha_n^{(2)} = 0$ in inequality (2.4) in the proof of ([8], Theorem 3)), we obtain

$$(2.6) \qquad d\left(s_{n+1}^{(1)}, s_*\right) \leq \delta d\left(s_n^{(2)}, s_*\right).$$

Now inserting (2.5) into (2.6), we get that

$$(2.7) \qquad d(s_{n+1}^{(1)}, s_*) \leq \delta\left[1 - \left(\alpha_n^{(4)} + \alpha_n^{(5)}\right)(1-\delta)\right] d\left(s_n^{(1)}, s_*\right).$$

Repeating the above process, we obtain

$$(2.8) \qquad d\left(s_{n+1}^{(1)}, s_*\right) \leq \delta^n \prod_{k=1}^{n}\left[1 - \left(\alpha_k^{(4)} + \alpha_k^{(5)}\right)(1-\delta)\right] d\left(s_1^{(1)}, s_*\right).$$

In view of the fact $1 + x \leq e^x$ for $x \geq 0$, the inequality (2.8) becomes

$$(2.9) \qquad d\left(s_{n+1}^{(1)}, s_*\right) \leq \delta^n e^{-(1-\delta)\sum_{k=1}^{n}\left(\alpha_k^{(4)} + \alpha_k^{(5)}\right)} d\left(s_1^{(1)}, s_*\right).$$

Since $\alpha_n^{(4)}$, $\alpha_n^{(5)} \in [0,1]$ for all $n \geq 1$, therefore we get

$$\alpha_n^{(4)} \leq \alpha_n^{(4)} + \alpha_n^{(5)} \text{ (or } \alpha_n^{(5)} \leq \alpha_n^{(4)} + \alpha_n^{(5)}\text{)}$$

which leads to

$$(2.10) \qquad \sum_{k=1}^{n} \alpha_k^{(4)} \leq \sum_{k=1}^{n}\left(\alpha_k^{(4)} + \alpha_k^{(5)}\right) \text{ (or } \sum_{k=1}^{n} \alpha_k^{(5)} \leq \sum_{k=1}^{n}\left(\alpha_k^{(4)} + \alpha_k^{(5)}\right)\text{)}.$$

Now taking limit on both sides of (2.10) and then using the assumption $\sum_{n=1}^{\infty} \alpha_n^{(4)} = \infty$ (or $\sum_{n=1}^{\infty} \alpha_n^{(5)} = \infty$), we get that $\sum_{n=1}^{\infty}\left(\alpha_n^{(4)} + \alpha_n^{(5)}\right) = \infty$. It thus follows from the inequality (2.9) that $s_n^{(1)} \to s_* \in F(T)$ as $n \to \infty$.

Next we prove that $F(T) = \{s_*\}$. Assume that there exists another fixed point s'_* of T. Now

$$d\left(s_*, s'_*\right) \leq d\left(Ts_*, Ts'_*\right) \leq \delta d\left(s_*, s'_*\right),$$

implies $s_* = s'_*$ as $\delta \in (0,1)$. □

2.2. Theorem. *Let $\{x_n^{(1)}\}_{n=0}^{\infty}$ be defined by (1.6) for $x_1^{(1)} \in C$ with $\sum_{n=1}^{\infty} \alpha_n^{(1)} = \infty$ (or $\sum_{n=1}^{\infty} \alpha_n^{(2)} = \infty$) and let $\{s_n^{(1)}\}_{n=0}^{\infty}$ be defined by (1.8) for $s_1^{(1)} \in C$. If for a quasi-contractive operator T, there exists a unique fixed s_* of T, then the following statements are equivalent:*

(i) $\{x_n^{(1)}\}_{n=0}^{\infty}$ strongly converges to s_*;

(ii) $\{s_n^{(1)}\}_{n=0}^{\infty}$ strongly converges to s_*.

Proof. (i)⇒(ii): It is established in the proof of Theorem 2.1.

(ii)⇒(i): Assume that $\lim_{n\to\infty} s_n^{(1)} = s_*$. Using (1.1), (1.5), (1.6), (1.7), and (1.8), we have the estimates:

$$\begin{aligned} d\left(s_{n+1}^{(1)}, x_{n+1}^{(1)}\right) &= d\left(Ts_n^{(2)}, W\left(Tx_n^{(2)}, W\left(Tx_n^{(3)}, x_n^{(1)}, \frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}}\right), \alpha_n^{(1)}\right)\right) \\ &\leq \alpha_n^{(1)} d\left(Ts_n^{(2)}, Tx_n^{(2)}\right) \\ &\quad + \left(1-\alpha_n^{(1)}\right) d\left(Ts_n^{(2)}, W\left(Tx_n^{(3)}, x_n^{(1)}, \frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}}\right)\right) \\ &\leq \alpha_n^{(1)} d\left(Ts_n^{(2)}, Tx_n^{(2)}\right) \\ &\quad + \left(1-\alpha_n^{(1)}\right) \frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}} d\left(Ts_n^{(2)}, Tx_n^{(3)}\right) \\ &\quad + \left(1-\alpha_n^{(1)}\right)\left(1-\frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}}\right) d\left(Ts_n^{(2)}, Tx_n^{(1)}\right) \\ &\leq \alpha_n^{(1)} \delta d\left(s_n^{(2)}, x_n^{(2)}\right) + \left(\alpha_n^{(1)} + \alpha_n^{(2)}\right) \epsilon d\left(Ts_n^{(2)}, s_n^{(2)}\right) \\ &\quad + \alpha_n^{(2)} \delta d\left(s_n^{(2)}, x_n^{(3)}\right) + \left(1-\alpha_n^{(1)}-\alpha_n^{(2)}\right) d\left(Ts_n^{(2)}, x_n^{(1)}\right), \end{aligned} \tag{2.11}$$

$$\begin{aligned} d\left(s_n^{(2)}, x_n^{(2)}\right) &= d\left(W\left(Ts_n^{(3)}, W\left(Ts_n^{(1)}, s_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), \alpha_n^{(4)}\right),\right. \\ &\quad \left. W\left(Tx_n^{(3)}, W\left(Tx_n^{(1)}, x_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), \alpha_n^{(4)}\right)\right) \\ &\leq \alpha_n^{(4)} d\left(Ts_n^{(3)}, Tx_n^{(3)}\right) \\ &\quad + \left(1-\alpha_n^{(4)}\right) d\left(W\left(Ts_n^{(1)}, s_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), W\left(Tx_n^{(1)}, x_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right)\right) \\ &\leq \alpha_n^{(4)} d\left(Ts_n^{(3)}, Tx_n^{(3)}\right) + \left(1-\alpha_n^{(4)}\right) \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}} d\left(Ts_n^{(1)}, Tx_n^{(1)}\right) \\ &\quad + \left(1-\alpha_n^{(4)}\right)\left(1-\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right) d\left(s_n^{(1)}, x_n^{(1)}\right) \\ &\leq \alpha_n^{(4)} \delta d\left(s_n^{(3)}, x_n^{(3)}\right) + \alpha_n^{(4)} \epsilon d\left(Ts_n^{(3)}, s_n^{(3)}\right) + \alpha_n^{(5)} \epsilon d\left(Ts_n^{(1)}, s_n^{(1)}\right) \\ &\quad + \left(\alpha_n^{(5)}\delta + 1 - \alpha_n^{(4)} - \alpha_n^{(5)}\right) d\left(s_n^{(1)}, x_n^{(1)}\right), \end{aligned} \tag{2.12}$$

$$\begin{aligned} d\left(s_n^{(3)}, x_n^{(3)}\right) &= d\left(W\left(Ts_n^{(1)}, s_n^{(1)}, \alpha_n^{(3)}\right), W\left(Tx_n^{(1)}, x_n^{(1)}, \alpha_n^{(3)}\right)\right) \\ &\leq \alpha_n^{(3)} d\left(Ts_n^{(1)}, Tx_n^{(1)}\right) + \left(1-\alpha_n^{(3)}\right) d\left(s_n^{(1)}, x_n^{(1)}\right) \\ &\leq \alpha_n^{(3)} \delta d\left(s_n^{(1)}, x_n^{(1)}\right) + \alpha_n^{(3)} \epsilon d\left(Ts_n^{(1)}, s_n^{(1)}\right) + \left(1-\alpha_n^{(3)}\right) d\left(s_n^{(1)}, x_n^{(1)}\right) \\ &= \left[1-\alpha_n^{(3)}(1-\delta)\right] d\left(s_n^{(1)}, x_n^{(1)}\right) + \alpha_n^{(3)} \epsilon d\left(Ts_n^{(1)}, s_n^{(1)}\right). \end{aligned} \tag{2.13}$$

Combining (2.11)-(2.13), we have

$$\begin{aligned} d\left(s_{n+1}^{(1)}, x_{n+1}^{(1)}\right) \leq & \left\{\alpha_n^{(1)}\alpha_n^{(4)}\delta^2\left[1-\alpha_n^{(3)}(1-\delta)\right]\right. \\ & \left.+\alpha_n^{(1)}\delta\left(\alpha_n^{(5)}\delta+1-\alpha_n^{(4)}-\alpha_n^{(5)}\right)\right\} d\left(s_n^{(1)}, x_n^{(1)}\right) \\ & +\left\{\alpha_n^{(1)}\alpha_n^{(4)}\delta^2\alpha_n^{(3)}\epsilon+\alpha_n^{(1)}\delta\alpha_n^{(5)}\epsilon\right\} d\left(Ts_n^{(1)}, s_n^{(1)}\right) \\ & +\left(\alpha_n^{(1)}+\alpha_n^{(2)}\right)\epsilon d\left(Ts_n^{(2)}, s_n^{(2)}\right)+\alpha_n^{(1)}\delta\alpha_n^{(4)}\epsilon d\left(Ts_n^{(3)}, s_n^{(3)}\right) \\ & +\alpha_n^{(2)}\delta d\left(s_n^{(2)}, x_n^{(3)}\right)+\left(1-\alpha_n^{(1)}-\alpha_n^{(2)}\right) d\left(Ts_n^{(2)}, x_n^{(1)}\right). \end{aligned} \tag{2.14}$$

Using now (1.1), (1.7), and triangle inequality for metric, we get that

$$d\left(Ts_n^{(1)}, s_n^{(1)}\right) \leq d\left(Ts_n^{(1)}, s_*\right)+d\left(s_*, s_n^{(1)}\right) \leq (1+\delta) d\left(s_n^{(1)}, s_*\right), \tag{2.15}$$

$$d\left(Ts_n^{(2)}, s_n^{(2)}\right) \leq d\left(Ts_n^{(2)}, s_*\right)+d\left(s_*, s_n^{(2)}\right) \leq (1+\delta) d\left(s_n^{(2)}, s_*\right), \tag{2.16}$$

$$d\left(Ts_n^{(3)}, s_n^{(3)}\right) \leq d\left(Ts_n^{(3)}, s_*\right)+d\left(s_*, s_n^{(3)}\right) \leq (1+\delta) d\left(s_n^{(3)}, s_*\right), \tag{2.17}$$

$$\begin{aligned} d\left(Ts_n^{(2)}, x_n^{(1)}\right) & \leq d\left(Ts_n^{(2)}, s_n^{(1)}\right)+d\left(s_n^{(1)}, x_n^{(1)}\right) \\ & \leq d\left(Ts_n^{(2)}, s_*\right)+d\left(s_n^{(1)}, s_*\right)+d\left(s_n^{(1)}, x_n^{(1)}\right) \\ & \leq \delta d\left(s_n^{(2)}, s_*\right)+d\left(s_n^{(1)}, s_*\right)+d\left(s_n^{(1)}, x_n^{(1)}\right), \end{aligned} \tag{2.18}$$

$$\begin{aligned} d\left(s_n^{(2)}, x_n^{(3)}\right) & \leq d\left(s_n^{(2)}, s_n^{(3)}\right)+d\left(s_n^{(3)}, x_n^{(3)}\right) \\ & \leq d\left(s_n^{(2)}, s_*\right)+d\left(s_n^{(3)}, s_*\right) \\ & \quad+\left[1-\alpha_n^{(3)}(1-\delta)\right] d\left(s_n^{(1)}, x_n^{(1)}\right)+\alpha_n^{(3)}\epsilon d\left(Ts_n^{(1)}, s_n^{(1)}\right), \end{aligned} \tag{2.19}$$

$$\begin{aligned} d\left(s_n^{(2)}, s_*\right) & \leq d\left(W\left(Ts_n^{(3)}, W\left(Ts_n^{(1)}, s_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), \alpha_n^{(4)}\right), s_*\right) \\ & \leq \alpha_n^{(4)} d\left(Ts_n^{(3)}, s_*\right)+\left(1-\alpha_n^{(4)}\right) d\left(W\left(Ts_n^{(1)}, s_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), s_*\right) \\ & \leq \alpha_n^{(4)}\delta d\left(s_n^{(3)}, s_*\right)+\left\{\alpha_n^{(5)}\delta+1-\alpha_n^{(4)}-\alpha_n^{(5)}\right\} d\left(s_n^{(1)}, s_*\right), \end{aligned} \tag{2.20}$$

$$\begin{aligned} d\left(s_n^{(3)}, s_*\right) & \leq d\left(W\left(Ts_n^{(1)}, s_n^{(1)}, \alpha_n^{(3)}\right), s_*\right) \\ & \leq \left[1-\alpha_n^{(3)}(1-\delta)\right] d\left(s_n^{(1)}, s_*\right). \end{aligned} \tag{2.21}$$

Inserting (2.15)-(2.21) into (2.14) and simplifying, we get that

$$\begin{aligned} d\left(s_{n+1}^{(1)}, x_{n+1}^{(1)}\right) \leq & \left[1-\left(\alpha_n^{(1)}+\alpha_n^{(2)}\right)(1-\delta)\right] d\left(s_n^{(1)}, x_n^{(1)}\right) \\ & +\left\{\left[\left(\alpha_n^{(4)}\alpha_n^{(3)}\delta+\alpha_n^{(5)}+\alpha_n^{(4)}\right)\alpha_n^{(1)}+\alpha_n^{(2)}\alpha_n^{(3)}\right]\delta\epsilon(1+\delta)\right. \\ & +1-\alpha_n^{(1)}-\alpha_n^{(2)}+\alpha_n^{(2)}\delta+\left(1-\alpha_n^{(1)}\right)\delta \\ & \left.+\left(\alpha_n^{(1)}+\alpha_n^{(2)}\right)\epsilon(1+\delta)\right\} d\left(s_n^{(1)}, s_*\right). \end{aligned} \tag{2.22}$$

Define

$$
\begin{aligned}
\sigma_n^{(1)} &= d\left(s_n^{(1)}, x_n^{(1)}\right), \\
\sigma_n^{(3)} &= \left(\alpha_n^{(1)} + \alpha_n^{(2)}\right)(1-\delta) \in (0,1), \\
\sigma_n^{(2)} &= \left\{\left[\alpha_n^{(1)}\alpha_n^{(4)}\alpha_n^{(3)}\delta + \alpha_n^{(1)}\alpha_n^{(5)} + \alpha_n^{(1)}\alpha_n^{(4)} + \alpha_n^{(2)}\alpha_n^{(3)}\right]\delta\epsilon(1+\delta) + 1 - \alpha_n^{(1)}\right. \\
&\left. -\alpha_n^{(2)} + \alpha_n^{(2)}\delta + \left(1-\alpha_n^{(1)}\right)\delta + \left(\alpha_n^{(1)} + \alpha_n^{(2)}\right)\epsilon(1+\delta)\right\} d\left(s_n^{(1)}, s_*\right).
\end{aligned}
$$

Hence, (2.22) satisfies all the requirements of Lemma 1.5 and so we get $\lim_{n\to\infty} d\left(s_n^{(1)}, x_n^{(1)}\right) = 0$. Also, we have

$$
d(x_n^{(1)}, s_*) \leq d(s_n^{(1)}, x_n^{(1)}) + d(s_n^{(1)}, s_*),
$$

which implies $\lim_{n\to\infty} d\left(x_n^{(1)}, s_*\right) = 0$. □

2.3. Theorem. *Suppose that we have the following conditions on the sequences* $\{\alpha_n^{(1)}\}_{n=0}^{\infty}$, $\{\alpha_n^{(2)}\}_{n=0}^{\infty}$, $\{\alpha_n^{(3)}\}_{n=0}^{\infty}$, $\{\alpha_n^{(4)}\}_{n=0}^{\infty}$, $\{\alpha_n^{(5)}\}_{n=0}^{\infty} \subset [0,1]$*:*

(C1): $0 < \alpha < \alpha_n^{(4)} \leq 1$ *or* $0 < \alpha < \alpha_n^{(5)} \leq 1$ *for all* $n \geq 1$,

(C2): $0 \leq \alpha_n^{(1)} + \alpha_n^{(2)} < \frac{1}{1+\delta}$ *for all* $n \geq 1$, $\delta \in [0,1)$ *and* $\lim_{n\to\infty}\left(\alpha_n^{(1)} + \alpha_n^{(2)}\right) = 0$.

If T *is a quasi-contractive operator and* $F(T) \neq \emptyset$, *then* $\{s_n^{(1)}\}_{n=0}^{\infty}$ *in (1.8) converges to* s_* *faster than* $\{x_n^{(1)}\}_{n=0}^{\infty}$ *in (1.6), provided that* $s_1^{(1)} = x_1^{(1)} \in C$.

Proof. Using (1.1), (1.2), (1.6) and (1.7), we have the estimate:

$$
\begin{aligned}
& d\left(x_{n+1}^{(1)}, s_*\right) \\
= \; & d\left(W\left(Tx_n^{(2)}, W\left(Tx_n^{(3)}, x_n^{(1)}, \frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}}\right), \alpha_n^{(1)}\right), s_*\right) \\
\geq \; & \left(1-\alpha_n^{(1)}\right) d\left(W\left(Tx_n^{(3)}, x_n^{(1)}, \frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}}\right), s_*\right) - \alpha_n^{(1)} d\left(Tx_n^{(2)}, s_*\right) \\
\geq \; & \left(1-\alpha_n^{(1)}\right)\left\{\left(1-\frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}}\right) d\left(x_n^{(1)}, s_*\right) - \frac{\alpha_n^{(2)}}{1-\alpha_n^{(1)}} d\left(Tx_n^{(3)}, s_*\right)\right\} \\
& -\alpha_n^{(1)}\left\{\delta d\left(x_n^{(2)}, s_*\right) + \epsilon d\left(Ts_*, s_*\right)\right\} \\
\geq \; & \left(1-\alpha_n^{(1)}-\alpha_n^{(2)}\right) d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(2)} \delta d\left(x_n^{(3)}, s_*\right) - \alpha_n^{(1)} \delta d\left(x_n^{(2)}, s_*\right) \\
\geq \; & \left(1-\alpha_n^{(1)}-\alpha_n^{(2)}\right) d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(2)} \delta \alpha_n^{(3)} d\left(Tx_n^{(1)}, s_*\right) \\
& -\alpha_n^{(2)} \delta\left(1-\alpha_n^{(3)}\right) d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(1)} \delta \alpha_n^{(4)} d\left(Tx_n^{(3)}, s_*\right) \\
& -\alpha_n^{(1)} \delta\left(1-\alpha_n^{(4)}\right) d\left(W\left(Tx_n^{(1)}, x_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), s_*\right) \\
\geq \; & \left(1-\alpha_n^{(1)}-\alpha_n^{(2)}\right) d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(2)} \delta \alpha_n^{(3)} \delta d\left(x_n^{(1)}, s_*\right) \\
& -\alpha_n^{(2)} \delta\left(1-\alpha_n^{(3)}\right) d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(1)} \delta \alpha_n^{(4)} \delta d\left(x_n^{(3)}, s_*\right) \\
& -\alpha_n^{(1)} \delta\left(1-\alpha_n^{(4)}\right)\left\{\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}} d\left(Tx_n^{(1)}, s_*\right) + \left(1-\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right) d\left(x_n^{(1)}, s_*\right)\right\} \\
\geq \; & \left(1-\alpha_n^{(1)}-\alpha_n^{(2)}\right) d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(2)} \alpha_n^{(3)} \delta^2 d\left(x_n^{(1)}, s_*\right) \\
& -\left(1-\alpha_n^{(3)}\right) \alpha_n^{(2)} \delta d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(1)} \alpha_n^{(4)} \delta^2 \alpha_n^{(3)} d\left(Tx_n^{(1)}, s_*\right) \\
& -\alpha_n^{(1)} \alpha_n^{(4)} \delta^2\left(1-\alpha_n^{(3)}\right) d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(1)} \alpha_n^{(5)} \delta^2 d\left(x_n^{(1)}, s_*\right) \\
& -\alpha_n^{(1)}\left(1-\alpha_n^{(4)}-\alpha_n^{(5)}\right) \delta d\left(x_n^{(1)}, s_*\right) \\
\geq \; & \left(1-\alpha_n^{(1)}-\alpha_n^{(2)}\right) d\left(x_n^{(1)}, p\right) - \alpha_n^{(2)} \alpha_n^{(3)} \delta^2 d\left(x_n^{(1)}, s_*\right) \\
& -\left(1-\alpha_n^{(3)}\right) \alpha_n^{(2)} \delta d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(1)} \alpha_n^{(4)} \alpha_n^{(3)} \delta^3 d\left(x_n^{(1)}, s_*\right) \\
& -\alpha_n^{(1)} \alpha_n^{(4)}\left(1-\alpha_n^{(3)}\right) \delta^2 d\left(x_n^{(1)}, s_*\right) - \alpha_n^{(1)} \alpha_n^{(5)} \delta^2 d\left(x_n^{(1)}, s_*\right) \\
& -\alpha_n^{(1)}\left(1-\alpha_n^{(4)}-\alpha_n^{(5)}\right) d\left(x_n^{(1)}, s_*\right) \\
= \; & \left\{1-\left(\alpha_n^{(1)}+\alpha_n^{(2)}\right) - \alpha_n^{(2)} \delta\left[1-\alpha_n^{(3)}(1-\delta)\right] - \alpha_n^{(1)} \alpha_n^{(4)} \delta^2\left[1-\alpha_n^{(3)}(1-\delta)\right]\right. \\
& \left.-\alpha_n^{(1)} \delta\left[\alpha_n^{(5)} \delta + 1 - \alpha_n^{(4)} - \alpha_n^{(5)}\right]\right\} d\left(x_n^{(1)}, s_*\right) \\
\geq \; & \left\{1-\left(\alpha_n^{(1)}+\alpha_n^{(2)}\right) - \alpha_n^{(2)} \delta\right. \\
& \left.-\alpha_n^{(1)} \delta\left[1-\left(\alpha_n^{(4)}+\alpha_n^{(5)}\right)(1-\delta)\right]\right\} d\left(x_n^{(1)}, s_*\right) \; \left(\text{as } 1-\alpha_n^{(3)}(1-\delta) \text{ in } [0,1]\right) \\
\geq \; & \left[1-\left(\alpha_n^{(1)}+\alpha_n^{(2)}\right)(1+\delta)\right] d\left(x_n^{(1)}, s_*\right) \; \left(\text{as } 1-\left(\alpha_n^{(4)}+\alpha_n^{(5)}\right)(1-\delta) \leq 1 \text{ for all } n \geq 1\right) \\
\geq \; & \cdots \geq \prod_{k=1}^{n}\left[1-\left(\alpha_k^{(1)}+\alpha_k^{(2)}\right)(1+\delta)\right] d\left(x_1^{(1)}, s_*\right).
\end{aligned}
$$

That is,

$$d\left(x_{n+1}^{(1)}, s_*\right) \geq \prod_{k=1}^{n}\left[1-\left(\alpha_k^{(1)}+\alpha_k^{(2)}\right)(1+\delta)\right] d\left(x_1^{(1)}, s_*\right). \tag{2.23}$$

It follows from (2.8) and (2.23) that

$$\frac{d\left(s_{n+1}^{(1)}, s_*\right)}{d\left(x_{n+1}^{(1)}, s_*\right)} \leq \frac{\delta^n \prod_{k=1}^{n}\left[1-\left(\alpha_k^{(4)}+\alpha_k^{(5)}\right)(1-\delta)\right] d\left(s_1^{(1)}, s_*\right)}{\prod_{k=1}^{n}\left[1-\left(\alpha_k^{(1)}+\alpha_k^{(2)}\right)(1+\delta)\right] d\left(x_1^{(1)}, s_*\right)}. \tag{2.24}$$

Now using (C_1) and assumption $s_1^{(1)} = x_1^{(1)}$ in (2.24), we get that

$$\frac{d\left(s_{n+1}^{(1)}, s_*\right)}{d\left(x_{n+1}^{(1)}, s_*\right)} \leq \frac{\delta^n\left[1-\alpha(1-\delta)\right]^n}{\prod_{k=1}^{n}\left[1-\left(\alpha_k^{(1)}+\alpha_k^{(2)}\right)(1+\delta)\right]}.$$

Let $\theta_n = \frac{\delta^n[1-\alpha(1-\delta)]^n}{\prod_{k=1}^{n}\left[1-\left(\alpha_k^{(1)}+\alpha_k^{(2)}\right)(1+\delta)\right]}$. By the assumption in ($C_2$), we obtain

$$\begin{aligned}
\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} &= \lim_{n\to\infty}\frac{\frac{\delta^{n+1}[1-\alpha(1-\delta)]^{n+1}}{\prod_{k=1}^{n+1}\left[1-\left(\alpha_k^{(1)}+\alpha_k^{(2)}\right)(1+\delta)\right]}}{\frac{\delta^{n}[1-\alpha(1-\delta)]^{n}}{\prod_{k=1}^{n}\left[1-\left(\alpha_k^{(1)}+\alpha_k^{(2)}\right)(1+\delta)\right]}} = \lim_{n\to\infty}\frac{\delta\left[1-\alpha(1-\delta)\right]}{\left[1-\left(\alpha_k^{(1)}+\alpha_k^{(2)}\right)(1+\delta)\right]} \\
&= \delta\left[1-\alpha(1-\delta)\right] < 1.
\end{aligned}$$

Thus $\lim_{n\to\infty}\theta_n = 0$ which implies in the sense of Definition 1.3 that $\{s_n^{(1)}\}_{n=1}^{\infty}$ converges faster than $\{x_n^{(1)}\}_{n=1}^{\infty}$. □

By ([8], Theorem 7) and Theorem 2.3, we get the following.

2.4. Corollary. *Suppose that (C_1) and (C_2) of Theorem 2.3 hold. If T is a quasi-contractive operator and $F(T) \neq \emptyset$, then $\{s_n^{(1)}\}_{n=1}^{\infty}$ in (1.8) converges to s_* faster than all the three classical iterative schemes, namely, Xu-Noor, Ishikawa and Mann iterations provided that all these schemes have the same initial guess in C.*

2.5. Example. Let $C = [-\frac{1}{2}, \frac{1}{2}]$ be endowed with the usual metric. Define operator $T : [-\frac{1}{2}, \frac{1}{2}] \to [-\frac{1}{2}, \frac{1}{2}]$ by $Tx = \frac{1}{2}e^{x-1/2}$ with a fixed point $s_* = 0.5$. Clearly, T satisfies condition (1.7) with $\delta \in (\frac{1}{2\sqrt[4]{e}}, \frac{1}{2}) \subset (0, 1)$. For $\delta = 2/5$, $x_1^{(1)} = 0$, $\alpha_n^{(4)} = \frac{2n+1}{3n+2}$ and $\alpha_n^{(1)} = \alpha_n^{(2)} = \alpha_n^{(3)} = \alpha_n^{(5)} = \frac{1}{3n+2}$ for all $n \geq 1$, the convergence results for different iterative schemes to $s_* = 0.5$ are shown in Figure 1 and Table 1 below.

2.6. Theorem. *Let $\widetilde{T} : C \to C$ be an approximate operator of T satisfying condition (1.7). Let $\{s_n^{(1)}\}_{n=1}^{\infty}$ be the iterative sequence generated by (1.8) with $\frac{1}{2} \leq \alpha_n^{(4)}$ (or $\frac{1}{2} \leq \alpha_n^{(5)}$) for all $n \geq 1$, $\lim_{n\to\infty}\alpha_n^{(3)} = 0$ and $\{\widetilde{s}_n^{(1)}\}_{n=1}^{\infty}$ be a sequence generated by:*

$$\begin{cases}
\widetilde{s}_{n+1}^{(1)} = \widetilde{T}\widetilde{s}_n^{(2)}, \\
\widetilde{s}_n^{(2)} = W\left(\widetilde{T}\widetilde{s}_n^{(3)}, W\left(\widetilde{T}\widetilde{s}_n^{(1)}, \widetilde{s}_n^{(1)}, \frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right), \alpha_n^{(4)}\right), \\
\widetilde{s}_n^{(3)} = W\left(\widetilde{T}\widetilde{s}_n^{(1)}, \widetilde{s}_n^{(1)}, \alpha_n^{(3)}\right), n \geq 1.
\end{cases} \tag{2.25}$$

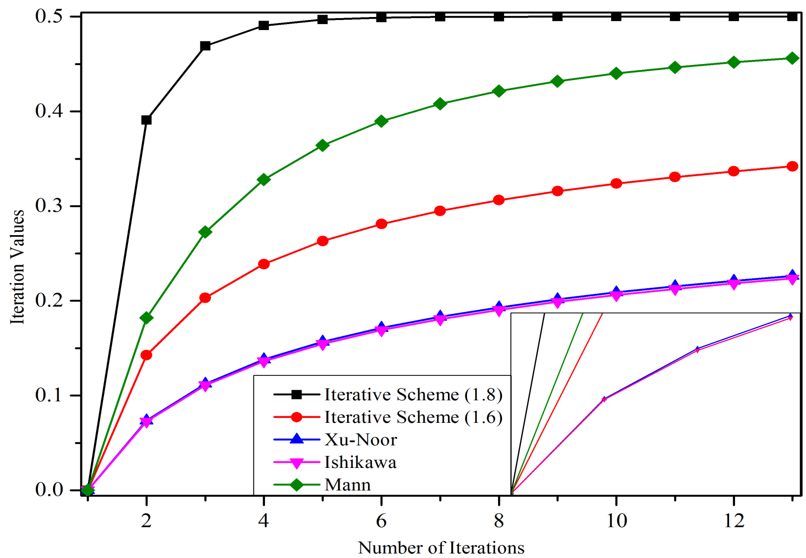

Figure 1. Rate of convergence comparison for various iterative schemes.

Table 1. Comparison (rate of convergence among various iterative methods for Example 2.5).

No. of Iter.	Iteration (1.8)	Iteration (1.6)	Xu-Noor	Ishikawa	Mann
1	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.390957	0.142637	0.073590	0.072757	0.181959
3	0.469010	0.203206	0.112493	0.110996	0.272856
⋮	⋮	⋮	⋮	⋮	⋮
13	0.500000	0.342111	0.226410	0.223637	0.456162
⋮	⋮	⋮	⋮	⋮	⋮
1000		0.472261	0.400160	0.398645	0.499498
⋮		⋮	⋮	⋮	⋮
5000		0.485258	0.430745	0.429695	0.499898
⋮		⋮	⋮	⋮	⋮

Assume that $\{s_n^{(1)}\}_{n=1}^{\infty}$ *and* $\{\widetilde{s}_n^{(1)}\}_{n=1}^{\infty}$ *converge strongly to* s_* *and* $\widetilde{s}_*$*, respectively. Then we have*

$$d\left(s_*, \widetilde{s}_*\right) \leq \frac{2\left(1+\delta\right)\varepsilon}{1-\delta},$$

where $\delta \in (0,1)$ *and* $\varepsilon > 0$ *is a fixed number.*

Proof. By (1.1), (1.5), (1.7), (1.8), (2.25), and Definition 1.1, we have that

$$\begin{aligned} d\left(s_{n+1}^{(1)},\widetilde{s}_{n+1}^{(1)}\right) &= d\left(Ts_n^{(2)},\widetilde{T}\widetilde{s}_n^{(2)}\right) \\ &\leq d\left(Ts_n^{(2)},T\widetilde{s}_n^{(2)}\right)+d\left(T\widetilde{s}_n^{(2)},\widetilde{T}\widetilde{s}_n^{(2)}\right) \\ &\leq \delta d\left(s_n^{(2)},\widetilde{s}_n^{(2)}\right)+\epsilon d\left(Ts_n^{(2)},s_n^{(2)}\right)+\varepsilon, \end{aligned} \tag{2.26}$$

$$\begin{aligned} d\left(s_n^{(2)},\widetilde{s}_n^{(2)}\right) &= d\left(W\left(Ts_n^{(3)},W\left(Ts_n^{(1)},s_n^{(1)},\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right),\alpha_n^{(4)}\right),\right. \\ &\quad \left.W\left(\widetilde{T}\widetilde{s}_n^{(3)},W\left(\widetilde{T}\widetilde{s}_n^{(1)},\widetilde{s}_n^{(1)},\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right),\alpha_n^{(4)}\right)\right) \\ &\leq \alpha_n^{(4)}d\left(Ts_n^{(3)},\widetilde{T}\widetilde{s}_n^{(3)}\right) \\ &\quad +\left(1-\alpha_n^{(4)}\right)d\left(W\left(Ts_n^{(1)},s_n^{(1)},\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right),W\left(\widetilde{T}\widetilde{s}_n^{(1)},\widetilde{s}_n^{(1)},\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right)\right) \\ &\leq \alpha_n^{(4)}d\left(Ts_n^{(3)},T\widetilde{s}_n^{(3)}\right)+\alpha_n^{(4)}d\left(T\widetilde{s}_n^{(3)},\widetilde{T}\widetilde{s}_n^{(3)}\right) \\ &\quad +\left(1-\alpha_n^{(4)}\right)\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}d\left(Ts_n^{(1)},\widetilde{T}\widetilde{s}_n^{(1)}\right) \\ &\quad +\left(1-\alpha_n^{(4)}\right)\left(1-\frac{\alpha_n^{(5)}}{1-\alpha_n^{(4)}}\right)d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right) \\ &\leq \alpha_n^{(4)}\delta d\left(s_n^{(3)},\widetilde{s}_n^{(3)}\right)+\alpha_n^{(4)}\epsilon d\left(Ts_n^{(3)},s_n^{(3)}\right)+\alpha_n^{(4)}\varepsilon+\alpha_n^{(5)}d\left(Ts_n^{(1)},T\widetilde{s}_n^{(1)}\right) \\ &\quad +\alpha_n^{(5)}d\left(T\widetilde{s}_n^{(1)},\widetilde{T}\widetilde{s}_n^{(1)}\right)+\left(1-\alpha_n^{(4)}-\alpha_n^{(5)}\right)d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right) \\ &\leq \alpha_n^{(4)}\delta d\left(s_n^{(3)},\widetilde{s}_n^{(3)}\right)+\alpha_n^{(4)}\epsilon d\left(Ts_n^{(3)},s_n^{(3)}\right)+\alpha_n^{(4)}\varepsilon+\alpha_n^{(5)}\delta d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right) \\ &\quad +\alpha_n^{(5)}\epsilon d\left(Ts_n^{(1)},s_n^{(1)}\right)+\alpha_n^{(5)}\varepsilon+\left(1-\alpha_n^{(4)}-\alpha_n^{(5)}\right)d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right), \end{aligned} \tag{2.27}$$

$$\begin{aligned} d\left(s_n^{(3)},\widetilde{s}_n^{(3)}\right) &= d\left(W\left(Ts_n^{(1)},s_n^{(1)},\alpha_n^{(3)}\right),W\left(\widetilde{T}\widetilde{s}_n^{(1)},\widetilde{s}_n^{(1)},\alpha_n^{(3)}\right)\right) \\ &\leq \alpha_n^{(3)}d\left(Ts_n^{(1)},\widetilde{T}\widetilde{s}_n^{(1)}\right)+\left(1-\alpha_n^{(3)}\right)d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right) \\ &\leq \alpha_n^{(3)}d\left(Ts_n^{(1)},T\widetilde{s}_n^{(1)}\right)+\alpha_n^{(3)}d\left(T\widetilde{s}_n^{(1)},\widetilde{T}\widetilde{s}_n^{(1)}\right) \\ &\quad +\left(1-\alpha_n^{(3)}\right)d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right) \\ &\leq \alpha_n^{(3)}\delta d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right)+\alpha_n^{(3)}\epsilon d\left(Ts_n^{(1)},s_n^{(1)}\right)+\alpha_n^{(3)}\varepsilon \\ &\quad +\left(1-\alpha_n^{(3)}\right)d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right). \end{aligned} \tag{2.28}$$

Combining (2.26)-(2.28), we obtain

$$\begin{aligned} d\left(s_{n+1}^{(1)},\widetilde{s}_{n+1}^{(1)}\right) &\le \left[1-\left(\alpha_n^{(4)}+\alpha_n^{(5)}\right)(1-\delta)\right] d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right) \\ &\quad +\left[\alpha_n^{(4)}\alpha_n^{(3)}\delta+\alpha_n^{(5)}\right]\delta\epsilon d\left(Ts_n^{(1)},s_n^{(1)}\right) \\ &\quad +\epsilon d\left(Ts_n^{(2)},s_n^{(2)}\right)+\alpha_n^{(4)}\delta\epsilon d\left(Ts_n^{(3)},s_n^{(3)}\right) \\ &\quad +\left(\alpha_n^{(4)}\alpha_n^{(3)}\delta+\alpha_n^{(4)}+\alpha_n^{(5)}\right)\delta\varepsilon+\varepsilon. \end{aligned} \tag{2.29}$$

By the assumption $\frac{1}{2}\le\alpha_n^{(4)}$ (or $\frac{1}{2}\le\alpha_n^{(5)}$), we have that

$$1\le 2\left(\alpha_n^{(4)}+\alpha_n^{(5)}\right),\ \text{for all } n\ge 1. \tag{2.30}$$

Also we have that

$$\alpha_n^{(4)}\le\alpha_n^{(4)}+\alpha_n^{(5)}\ (\text{or } \alpha_n^{(5)}\le\alpha_n^{(4)}+\alpha_n^{(5)}),\ \text{for all } n\ge 1. \tag{2.31}$$

Using (2.30) and (2.31) in (2.29), we get that

$$\begin{aligned} d\left(s_{n+1}^{(1)},\widetilde{s}_{n+1}^{(1)}\right) &\le \left[1-\left(\alpha_n^{(4)}+\alpha_n^{(5)}\right)(1-\delta)\right] d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right) \\ &\quad +\left(\alpha_n^{(4)}+\alpha_n^{(5)}\right)\left\{\left[\delta\alpha_n^{(3)}+1\right]\delta\epsilon d\left(Ts_n^{(1)},s_n^{(1)}\right)+2\epsilon d\left(Ts_n^{(2)},s_n^{(2)}\right)\right. \\ &\quad \left.+\delta\epsilon d\left(Ts_n^{(3)},s_n^{(3)}\right)+\left(\delta^2 a_n+2\delta+2\right)\varepsilon\right\}. \end{aligned} \tag{2.32}$$

Define for all $n\ge 1$,

$$\begin{aligned} \sigma_n^{(1)} &= d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right)\ge 0, \\ \sigma_n^{(2)} &= \left(\alpha_n^{(4)}+\alpha_n^{(5)}\right)(1-\delta)\in(0,1), \\ \sigma_n^{(3)} &= \frac{\left[\delta\alpha_n^{(3)}+1\right]\delta\epsilon d\left(Ts_n^{(1)},s_n^{(1)}\right)+2\epsilon d\left(Ts_n^{(2)},s_n^{(2)}\right)+\delta\epsilon d\left(Ts_n^{(3)},s_n^{(3)}\right)+\left(\delta^2\alpha_n^{(3)}+2\delta+2\right)\varepsilon}{1-\delta}. \end{aligned}$$

The assumption $\frac{1}{2}\le\alpha_n^{(4)}$ (or $\frac{1}{2}\le\alpha_n^{(5)}$) for all $n\ge 1$ implies $\sum_{n=1}^{\infty}\left(\alpha_n^{(4)}+\alpha_n^{(5)}\right)=\infty$. It is now easy to check that (2.32) fulfills all the requirements of Lemma 1.4 and so by its conclusion, we get

$$0\le\limsup_{n\to\infty} d\left(s_n^{(1)},\widetilde{s}_n^{(1)}\right)\le\limsup_{n\to\infty}\frac{\left(\delta^2\alpha_n^{(3)}+2\delta+2\right)\varepsilon}{1-\delta}.$$

By Theorem 2.1, we have that $\lim_{n\to\infty}s_n^{(1)}=s_*$. Using this fact and the assumptions $\lim_{n\to\infty}\widetilde{s}_n^{(1)}=\widetilde{s}_*$ and $\lim_{n\to\infty}\alpha_n^{(3)}=0$, we obtain

$$d\left(s_*,\widetilde{s}_*\right)\le\frac{2(1+\delta)\varepsilon}{1-\delta}.$$

□

The following example shows validity of Theorem 2.6.

2.7. Example. Let C, T, and δ be as in Example 2.5. Now, define operator $\widetilde{T}:C\to C$ by

$$\widetilde{T}x=\frac{x^5}{240}+\frac{x^4}{96}+\frac{5x^3}{96}+\frac{29x^2}{192}+\frac{233x}{768}+0.253255.$$

The Wolfram Mathematica 9 software package implies that

$$\max_{x\in C}\left|T-\widetilde{T}\right| = 0.050561.$$

Thus, for all $x \in C$ and for a fixed $\varepsilon = 0.050561 > 0$, we have

$$\left|Tx-\widetilde{T}x\right| \leq 0.050561,$$

that is, $\widetilde{T}$ is an approximate operator of T in the sense of Definition 1.1. On the other

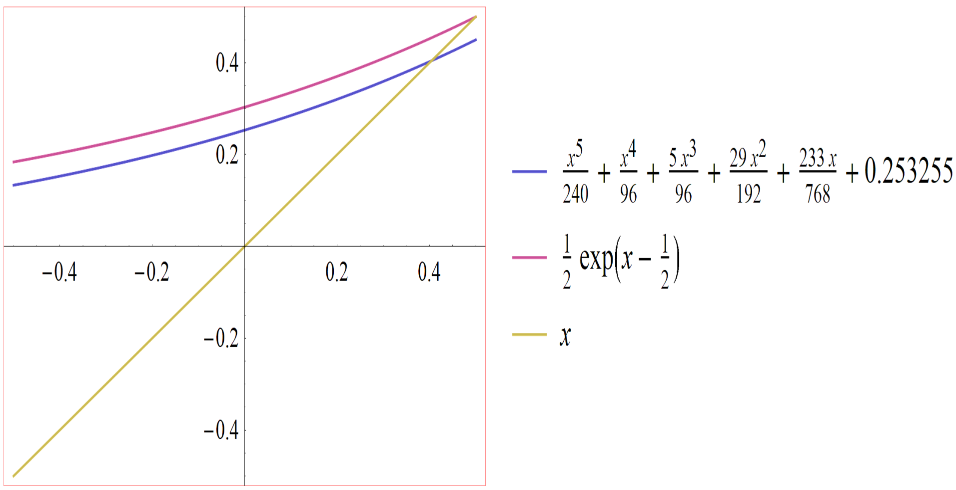

Figure 2. Graphs of T, $\widetilde{T}$, and $y = x$.

hand $\widetilde{s}_* = 0.404425 \in C$ is a fixed point for the operator $\widetilde{T}$. Indeed, if we put $\alpha_n^{(3)} = \alpha_n^{(5)} = \frac{1}{3n+2}$, $\alpha_n^{(4)} = \frac{2n+1}{3n+2}$ for all $n \geq 1$ and $\widetilde{T}x = \frac{x^5}{240} + \frac{x^4}{96} + \frac{5x^3}{96} + \frac{29x^2}{192} + \frac{233x}{768} + 0.253255$ in (2.25), then we have

(2.33)

$$\begin{cases} \widetilde{s}_1^{(1)} \in C, \\ \widetilde{s}_{n+1}^{(1)} = 0.253255 + \frac{\left(\widetilde{s}_n^{(2)}\right)^5}{240} + \frac{\left(\widetilde{s}_n^{(2)}\right)^4}{96} + \frac{5\left(\widetilde{s}_n^{(2)}\right)^3}{96} + \frac{29\left(\widetilde{s}_n^{(2)}\right)^2}{192} + \frac{233\widetilde{s}_n^{(2)}}{768}, \\ \widetilde{s}_n^{(2)} = \left(1-\frac{2n+2}{3n+2}\right)\widetilde{s}_n^{(1)} \\ \quad +\frac{2n+1}{3n+2}\left(0.253255 + \frac{\left(\widetilde{s}_n^{(3)}\right)^5}{240} + \frac{\left(\widetilde{s}_n^{(3)}\right)^4}{96} + \frac{5\left(\widetilde{s}_n^{(3)}\right)^3}{96} + \frac{29\left(\widetilde{s}_n^{(3)}\right)^2}{192} + \frac{233\widetilde{s}_n^{(3)}}{768}\right) \\ \quad +\frac{1}{3n+2}\left(0.253255 + \frac{\left(\widetilde{s}_n^{(1)}\right)^5}{240} + \frac{\left(\widetilde{s}_n^{(1)}\right)^4}{96} + \frac{5\left(\widetilde{s}_n^{(1)}\right)^3}{96} + \frac{29\left(\widetilde{s}_n^{(1)}\right)^2}{192} + \frac{233\widetilde{s}_n^{(1)}}{768}\right), \\ \widetilde{s}_n^{(3)} = \left(1-\frac{1}{3n+2}\right)\widetilde{s}_n^{(1)} \\ +\frac{1}{3n+2}\left(0.253255 + \frac{\left(\widetilde{s}_n^{(1)}\right)^5}{240} + \frac{\left(\widetilde{s}_n^{(1)}\right)^4}{96} + \frac{5\left(\widetilde{s}_n^{(1)}\right)^3}{96} + \frac{29\left(\widetilde{s}_n^{(1)}\right)^2}{192} + \frac{233\widetilde{s}_n^{(1)}}{768}\right), \; n \geq 1. \end{cases}$$

The sequence $\{\widetilde{s}_n^{(1)}\}_{n=1}^{\infty}$ generated by (2.33) converges to the fixed point $\widetilde{s}_* = 0.404425 \in C$ as follows:

No. of Iter.	Iterative Scheme (2.33)
1	0.000000
2	0.324903
3	0.384663
⋮	⋮
11	0.404425
⋮	⋮

Consequently, $|s_* - \widetilde{s}_*| = 0.095575$.
Alternatively, we can find by Theorem 2.6, the following estimate:

$$|s_* - \widetilde{s}_*| \leq \frac{2 \times (1 + \frac{2}{5}) \times 0.050561}{1 - \frac{2}{5}} = 0.235951.$$

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