**Communications in Advanced Mathematical Sciences**
**Vol. I, No. 1, 39-44, 2018**

# A note on the "saturation" of poisson-exponential cumulative function in Hausdorff sense

Nikolay Kyurkchiev[1,2], Anton Iliev[1,2*]

**Abstract**
In this paper we study the important "saturation" characteristic for the Poisson–exponential cumulative distribution function in the Hausdorff sense. The results have independent significance in the study of issues related to lifetime analysis, insurance mathematics, biochemical kinetics, population dynamics and debugging theory. Numerical examples, illustrating our results are presented using programming environment Mathematica.

**Keywords:** Poisson–exponential cumulative distribution function (Pcdf), Hausdorff distance, Upper and lower bounds
**2010 AMS:** Primary 41A46

[1] *Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski, Plovdiv, Bulgaria*
[2] *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria*
***Corresponding author**: aii@uni-plovdiv.bg
**Received:** 17 July 2018, **Accepted:** 16 September 2018, **Available online:** 30 September 2018

## 1. Introduction

The Poisson–exponential cumulative distribution function (Pcdf) is given by (see for instance [1]):

$$M(t;\lambda;\beta)=\frac{e^{\lambda e^{-\beta t}}-e^{\lambda}}{1-e^{\lambda}} \tag{1.1}$$

where $\beta>0;\lambda>0$.

For other extensions and estimations, see [2] – [3]. Some applications of the (Pcdf) to rainfall and aircraft data with zero occurrence can be found in [3].

In this note we study the saturation of the Poisson–exponential cumulative distribution family of functions (1) to asymptote $t=1$ in the Hausdorff sense.

**Definition 1.1.** *[4] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions $f,g$ on $\Omega\subseteq\mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega\times\mathbb{R}$. More precisely,*

$$\rho(f,g)=\max\{\sup_{A\in F(f)}\inf_{B\in F(g)}||A-B||,\ \sup_{B\in F(g)}\inf_{A\in F(f)}||A-B||\},$$

*wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)||=\max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B||=max(|t_A-t_B|,|x_A-x_B|)$.*

We propose a software modules (intellectual properties) within the programming environment CAS Mathematica for the analysis.

## 2. Main Results

Without loosing of generality we will look at the following (Pcdf):

$$M^*(t) = \frac{e^{\lambda e^{-\beta t}} - e^{\lambda}}{1 - e^{\lambda}}, \tag{2.1}$$

with

$$t_0 = -\frac{1}{\beta} \ln\left(\frac{1}{\lambda} \ln\left(\frac{1+e^{\lambda}}{2}\right)\right); \quad M^*(t_0) = \frac{1}{2}. \tag{2.2}$$

To evaluate the important saturation characteristic $d$ of the (Pcdf) to asymptote $t = 1$ in the Hausdorff sense we will use the following representation:

$$M^*(t_0 + d) = 1 - d. \tag{2.3}$$

The following theorem gives upper and lower bounds for $d$

**Theorem 1**. Let

$$p = -\frac{1}{2},$$

$$q = 1 - \frac{\beta}{1-e^{\lambda}} \frac{1+e^{\lambda}}{2} \ln\left(\frac{1+e^{\lambda}}{2}\right).$$

For the Hausdorff distance $d$ the following inequalities hold for:

$$2.1q > e^{1.05} \approx 1.36079$$

$$d_l = \frac{1}{2.1q} < d < \frac{\ln(2.1q)}{2.1q} = d_r. \tag{2.4}$$

**Proof.** Let us examine the function:

$$F(d) = M^*(t_0 + d) - 1 + d. \tag{2.5}$$

From $F'(d) > 0$ we conclude that function $F$ is increasing.
Consider the function

$$G(d) = p + qd. \tag{2.6}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.
Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).
In addition $G'(d) > 0$.
Further, for $2.1q > e^{1.05}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

The model ((2)–(3)) for $\beta = 5$, $\lambda = 0.8$, $t_0 = 0.10302$ is visualized on Fig. 2. From the nonlinear equation (4) and inequalities (5) we have: $d = 0.177469$, $d_l = 0.114887$, $d_r = 0.248593$.

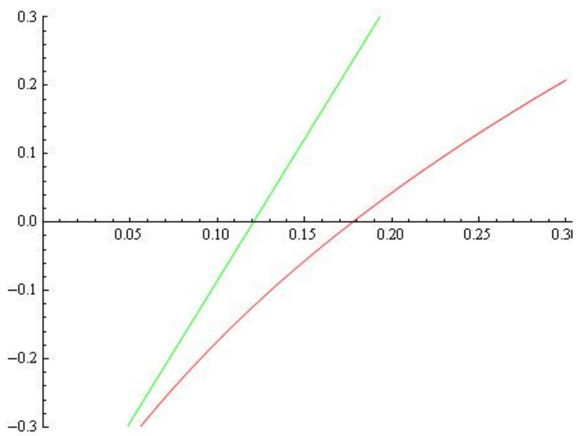


**Figure 2.1.** The functions $F(d)$ and $G(d)$.

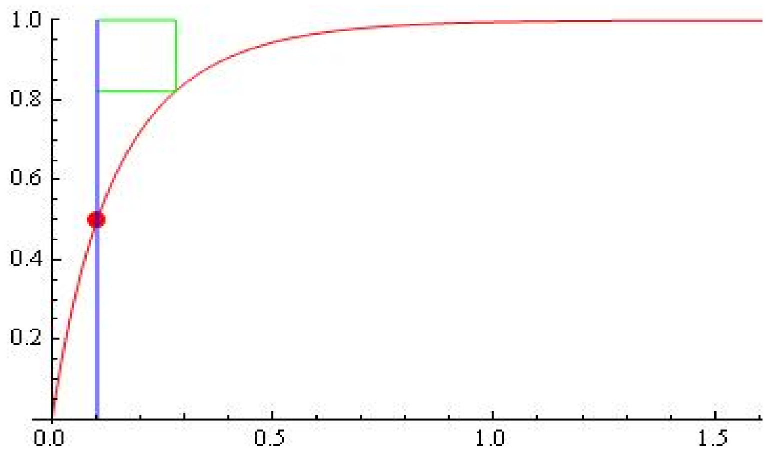


**Figure 2.2.** The model ((2)–(3)) for $\beta = 5$, $\lambda = 0.8$, $t_0 = 0.10302$; H–distance $d = 0.176469$, $d_l = 0.114887$, $d_r = 0.248593$.

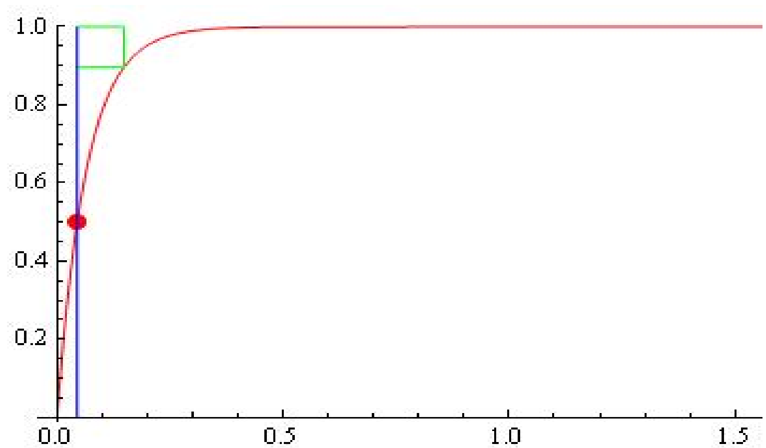


**Figure 2.3.** The model ((2)–(3)) for $\beta = 15$, $\lambda = 0.1$, $t_0 = 0.0445643$; H–distance $d = 0.10359$, $d_l = 0.0547734$, $d_r = 0.159092$.

```
Clear[λ];
Clear[β];
Manipulate[Dynamic@Show[Plot[f[t], {t, 0, 1},
    LabelStyle → Directive[Blue, Bold],
    PlotLabel → (Exp[λ * Exp[-β * t]] - Exp[λ]) / (1 - Exp[λ])],
   PlotRange → {0, 1}],
 {{λ, 0.1}, 0.01, 30, Appearance → "Open"},
 {{β, 0.1}, 0.01, 100, Appearance → "Open"},
 Initialization :→
  (f[t_] := (Exp[λ * Exp[-β * t]] - Exp[λ]) / (1 - Exp[λ]))]
```

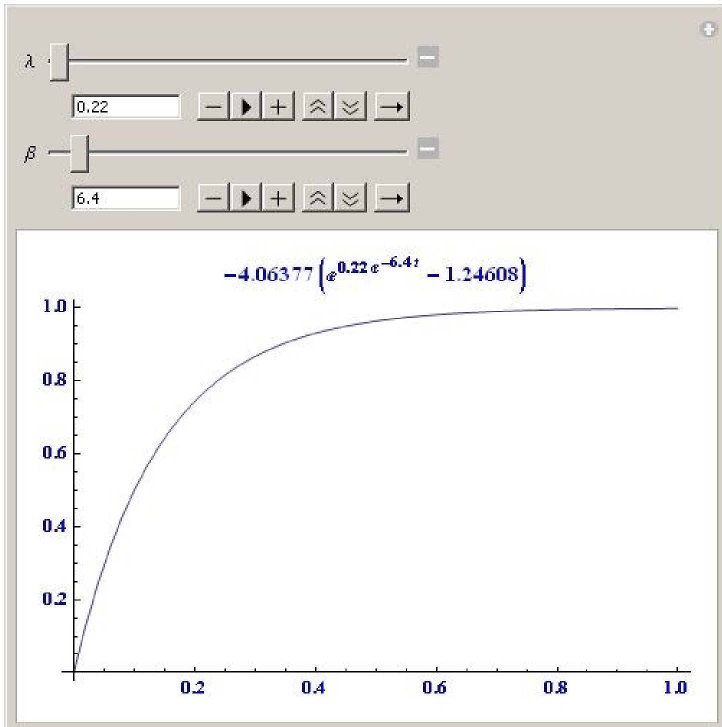


**Figure 2.4.** An example of the usage of dynamical and graphical representation for the family $M(t)$. For example $\lambda = 0.22$, $\beta = 6.4$. The plots are prepared using CAS Mathematica.

The model ((2)–(3)) for $\beta = 15$, $\lambda = 0.1$, $t_0 = 0.0445643$ is visualized on Fig. 3. From the nonlinear equation (4) and inequalities (5) we have: $d = 0.10359$, $d_l = 0.0547734$, $d_r = 0.159092$.

From the above examples, it can be seen that the proven bottom estimate (see Theorem 1) for the value of the Hausdorff distance is reliable when assessing the important characteristic - "saturation".

This characteristic (as we have already shown in our previous publications) has its equal participation together with the other two characteristics - "confidence intervals" and "confidence bounds" in the area of the Software Reliability Theory. Constructions of "confidence curves" and "confidence bounds" as basics elements from Software Reliability Theory are not easy to be calculated in comparison to estimations pointed in the theorem proven here.

Some software reliability models, can be found in [5]–[6].

**Remark.** Ramos, Percontini, Cordeiro and Silva [7] studied the following Burr XII–Negative–Binomial Distribution with applications to lifetime data:

$$M_1(t) = \frac{(1-\beta)^{-s} - \left(1 - \beta\left(1 + \left(\frac{t}{a}\right)^c\right)^{-k}\right)^{-s}}{(1-\beta)^{-s} - 1}$$

where $a, k, s, c > 0$ and $\beta \in (0, 1)$.

The reader can get accurate bounds for the saturation feature using the technique described in this article.

For some approximation and modeling aspects see [8]–[21].

We hope that the results will be useful for specialists in this scientific area.

**Acknowledgments.**

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

This work has been supported by the project FP17-FMI008 of Department for Scientific Research, Paisii Hilendarski University of Plovdiv.

## References

[1] V. Cancho, F. Louzada and G. Barriga, The Poisson–exponential lifetime distribution, Comp. Stat. Data Anal., vol. 55, pp. 677–686, 2011.

[2] G. Rodrigues, F. Louzada and P. Ramos, Poisson–exponential distribution: different methods of estimation, J. of Appl. Stat., vol. 45 (1), pp. 128–144, 2018.

[3] F. Louzada, P. Ramos and P. Ferreira, Exponential–Poisson distribution: estimation and applications to rainfall and aircraft data with zero occurrence, Communication in Statistics–Simulation and Computation, 2018.

[4] F. Hausdorff, Set Theory (2 ed.) (Chelsea Publ., New York, (1962 [1957]) (Republished by AMS-Chelsea 2005), ISBN: 978–0–821–83835–8.

[5] N. Pavlov, A. Iliev, A. Rahnev and N. Kyurkchiev, Some software reliability models: Approximation and modeling aspects, LAP LAMBERT Academic Publishing, 2018, ISBN: 978-613-9-82805-0.

[6] N. Pavlov, A. Iliev, A. Rahnev and N. Kyurkchiev, Nontrivial Models in Debugging Theory (Part 2), LAP LAMBERT Academic Publishing, 2018, ISBN: 978-613-9-87794-2.

[7] M. Ramos, A. Percontini, G. Cordeiro and R. Silva, The Burr XII Negative Binomial Distribution with applications to Lifetime Data, Int. J. of Stat. and Prob., vol. 4 (1), pp. 109–124, 2015.

[8] N. Kyurkchiev and S. Markov, Sigmoid functions: Some Approximation and Modelling Aspects. (LAP LAMBERT Academic Publishing, Saarbrucken, 2015); ISBN 978-3-659-76045-7.

[9] N. Kyurkchiev and S. Markov, On the Hausdorff distance between the Heaviside step function and Verhulst logistic function. J. Math. Chem., vol. 54 (1), pp. 109–119, DOI:10.1007/S10910-015-0552-0.

[10] A. Iliev, N. Kyurkchiev and S. Markov, On the Approximation of the step function by some sigmoid functions, Mathematics and Computers in Simulation, 2015, DOI:10.1016/j.matcom.2015.11.005.

[11] N. Kyurkchiev, A new transmuted cumulative distribution function based on the Verhulst logistic function with application in population dynamics. Biomath Communications, vol. 4 (1), 2017.

[12] N. Kyurkchiev, A new class of activation function based on the correcting amendments Int. J. for Sci., Res. and Developments, vol. 6 (2), pp. 565–568, 2018.

[13] N. Kyurkchiev, The new transmuted C.D.F. based on Gompertz function, Biomath Communications, vol. 5 (1), 2018.

[14] V. Kyurkchiev and N. Kyurkchiev, A Family of Recurrence Generated Functions Based on "Half-Hyperbolic Tangent Activation Function". Biomedical Statistics and Informatics, vol. 2 (3), pp. 87–94, 2017.

[15] A. Iliev, N. Kyurkchiev and S. Markov, A Note on the New Activation Function of Gompertz Type, Biomath Communications, vol. 4 (2), 2017.

[16] N. Kyurkchiev, A. Iliev, Extension of Gompertz-type Equation in Modern Science: 240 Anniversary of the birth of B. Gompertz, LAP LAMBERT Academic Publishing, 2018, ISBN: 978-613-9-90569-0.

[17] V. Kyurkchiev, A. Malinova, O. Rahneva, P. Kyurkchiev, On the Burr XII-Weibull Software Reliability Model, Int. J. of Pure and Appl. Math., vol. 119 (4), pp. 639—650, 2018.

[18] V. Kyurkchiev, A. Malinova, O. Rahneva, P. Kyurkchiev, Some Notes on the Extended Burr XII Software Reliability Model, Int. J. of Pure and Appl. Math., vol. 120 (1), pp. 127–136, 2018.

[19] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, Application of a New Class Cumulative Lifetime Distribution to Software Reliability Analysis, Communications in Applied Analysis, vol. 22 (4), pp. 555-–565, 2018.

[20] V. Kyurkchiev, H. Kiskinov, O. Rahneva, G. Spasov, A Note on the Exponentiated Exponential-Poisson Software Reliability Model, Neural, Parallel, and Scientific Computations, vol. 26, 2018. (to appear)

[21] N. Kyurkchiev, A. Iliev and S. Markov, Some Techniques for Recurrence Generating of Activation Functions: Some Modeling and Approximation Aspects, LAP LAMBERT Academic Publishing, 2017, ISBN: 978-3-330-33143-3.