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AW(k)-type Salkowski Curves in the Euclidean 3-Space IE^3

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ABSTRACT

We deal with AW(k)-type $(k = 1,2, \text{and } 3)$ Salkowski (anti-Salkowski) curves with constant $\kappa \neq 0$ $(\tau \neq 0)$ in the Euclidean 3-space. We show that there is no AW(1)-type Salkowski curve and AW(1)-type anti-Salkowski curve in IE^3. Also, we handle weak AW(2)-type and weak AW(3)-type Salkowski (anti-Salkowski) curves. Also, we show that there is no weak AW(2)-type Salkowski curve in IE^3.

Keywords: AW(k)-type, Salkowski curve

1. INTRODUCTION

In study [1], AW(k)-type curves and submanifolds were defined by Arslan and West. Thereafter, many works related to this topic were handled by so many authors. In [2,3], the authors gave some features of the AW(k)-type curves in IE^m. In [4], the authors considered AW(k)-type (k=1,2, and 3) surfaces and curves. They gave some examples of surfaces and curves that satisfy AW(k)-type situations as well. In [5,6], the authors studied these types of curves with the parallel transport frame in Euclidean spaces IE^3 and IE^4. In [7], Yoon studied these types of curves in the Lie group G. Also, the same author characterized AW(k)-type general helices in G.

The term "helix" is used in other sciences apart from mathematics because of its properties and applicability. Helices arise in DNA molecules, structures of proteins, and carbon nano tubes, and the like. [8,9]. Also, helices are used in fractal geometry, e.g. hyper helices [10]. Moreover, helices are used for the design of highways or kinematic motion simulations [11]. In the sense of geometry, a helix is a kind of curve having a non-zero constant curvature and a non-zero constant torsion [12,13]. This curve is called as W-curve by F. Klein and S. Lie in [14].

A regular curve having constant Frenet curvature ratios is a ccr-curve [15]. In [16], authors give a characterization of these curves in IE^m. In that study, authors show that every W-curve is a ccr-curve. As is well-known, generalized helices in IE^3 are characterized by the fact that the quotient $\frac{\tau}{\kappa}$ is constant. It is in this sense that ccr-curves are generalization to IE^m of generalized helices in IE^3.

Salkowski curves (anti-Salkowski curves), having constant curvature (having constant torsion), were introduced by Salkowski in 1909 in [17]. Thereafter, in [18,19], author studied non-lightlike Salkowski curves in IE_1^3. In [20], the authors investigated Salkowski type Manheim curves in IE^3. In [21], the author characterized the

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Salkowski curve whose principal normal vector and a fixed line constitute a constant angle.

This article is arranged as follows: In section 2, some basic concepts of curves in IE^3 are given and the AW(k)-type curve concept is presented. In sections 3 and 4, AW(k)-type Salkowski curves and AW(k)-type anti-Salkowski curves are handled respectively.

2. BASIC CONCEPTS

Let $\gamma = \gamma(s) : I \to IE^3$ be a curve parametrized with the arc length function s in the Euclidean space IE^3. If the derivatives $\gamma'(s), \gamma''(s)$, and $\gamma'''(s)$ of γ are linearly independant and $\gamma'(s), \gamma''(s), \gamma'''(s), \gamma^{(iv)}(s)$ are linearly dependent for all $s \in I$, then γ is a Frenet curve of osculating order 3. For each curve of osculating order 3, we can correlate an orthonormal 3-frame $\{T, N, B\}$ called as Frenet frame of the curve γ along γ ($\gamma'(s) = T(s)$) and functions $\kappa, \tau : I \to IR$ called as the Frenet curvatures of the curve γ. Then the famous Frenet frame formula of γ is given as follows:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

[22].

Now, we introduce AW(k)-type Frenet curves.

Proposition 2.1. Let γ be a Frenet curve of osculating order 3 in IE^3. Thus, one can conclude:

$$\begin{aligned} \gamma'(s) &= T(s), \\ \gamma''(s) &= T'(s) = \kappa(s)N(s), \\ \gamma'''(s) &= -\kappa^2(s)T(s) + \kappa'(s)N(s) + \kappa(s)\tau(s)B(s), \\ \gamma^{(iv)}(s) &= -3\kappa(s)\kappa'(s)T(s) \\ &\quad + \{\kappa''(s) - \kappa^3(s) - \kappa(s)\tau^2(s)\}N(s) \\ &\quad + \{2\kappa'(s)\tau(s) + \kappa(s)\tau'(s)\}B(s). \end{aligned}$$

[2].

Notation 2.2. Let us write

$$\begin{aligned} N_1(s) &= \kappa(s)N(s), \\ N_2(s) &= \kappa'(s)N(s) + \kappa(s)\tau(s)B(s), \\ N_3(s) &= \{\kappa''(s) - \kappa^3(s) - \kappa(s)\tau^2(s)\}N(s) \\ &\quad + \{2\kappa'(s)\tau(s) + \kappa(s)\tau'(s)\}B(s). \end{aligned} \tag{1}$$

[2].

Definition 2.3. Frenet curves of osculating order 3 are

i) of type weak AW(2) if they enable

$$N_3(s) = \langle N_3(s), N_2^*(s) \rangle N_2^*(s), \tag{2}$$

ii) of type weak AW(3) if they enable

$$N_3(s) = \langle N_3(s), N_1^*(s) \rangle N_1^*(s), \tag{3}$$

where

$$\begin{aligned} N_1^*(s) &= \frac{N_1(s)}{\|N_1(s)\|} = N(s), \\ N_2^*(s) &= \frac{N_2(s) - \langle N_2(s), N_1^*(s) \rangle N_1^*(s)}{\left\| N_2(s) - \langle N_2(s), N_1^*(s) \rangle N_1^*(s) \right\|} = B(s) \end{aligned} \tag{4}$$

[2].

Definition 2.4. Frenet curves of osculating order 3 are

i) of type AW(1) if they enable

$$N_3(s) = 0, \tag{5}$$

ii) of type AW(2) if they enable

$$\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s) \rangle N_2(s) \tag{6}$$

iii) of type AW(3) if they enable

$$\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s) \rangle N_1(s) \tag{7}$$

[2].

3. AW(k)-TYPE SALKOWSKI CURVES IN IE^3

Here, we handle AW(k)-type Salkowski curves in IE^3.

Let γ be a Salkowski curve parametrized with the arc length function, and $\{T, N, B\}$ be the Frenet frame of the curve in IE^3. Since γ is a Salkowski curve, the curvature κ of the curve is a non-zero constant. Thus, the equations (1) become

$$N_1(s) = \kappa N(s),$$
$$N_2(s) = \kappa\tau(s)B(s), \quad (8)$$
$$N_3(s) = \{-\kappa^3 - \kappa\tau^2(s)\}N(s) + \{\kappa\tau'(s)\}B(s).$$

From Definition 2.3, we have the following theorems:

Theorem 3.1. There is no weak AW(2)-type Salkowski curve with constant $\kappa \neq 0$ in IE^3.

Proof. Let γ be a Salkowski curve which is parametrized by the arc length function and given with constant $\kappa \neq 0$ in IE^3. From the equations (2), (4) and (8), we get $-\kappa(\kappa^2 + \tau^2) = 0$, which is a contradiction. Thus, there is no weak AW(2)-type Salkowski curve with constant $\kappa \neq 0$ in IE^3.

Theorem 3.2. Let γ be a Salkowski curve which is parametrized by the arc length function and given with constant $\kappa \neq 0$ in IE^3. If γ is of type weak AW(3), γ is a W-curve.

Proof. Let γ be a unit speed Salkowski curve with constant $\kappa \neq 0$ and γ is of type weak AW(3) in IE^3. From the equations (3), (4) and (8), we get $\kappa\tau' = 0$. Since $\kappa \neq 0$, $\tau' = 0$, i.e. τ is a constant. Hence, γ is a W-curve.

From Definition 2.4, we have the following theorems:

Theorem 3.3. There is no AW(1)-type Salkowski curve with constant $\kappa \neq 0$ in IE^3.

Proof. Let γ be a Salkowski curve which is parametrized by the arc length function and given with constant $\kappa \neq 0$ in IE^3. From the equations (5) and (8), we obtain $-\kappa(\kappa^2 + \tau^2) = 0$ and $\kappa\tau' = 0$. Since $\kappa \neq 0$, we get $\tau' = 0$, i.e. τ is a constant. Considering τ is a constant in $-\kappa(\kappa^2 + \tau^2) = 0$, we get a contradiction. Thus, there is no AW(1)-type Salkowski curve in IE^3.

Theorem 3.4. Let γ be a Salkowski curve which is parametrized by the arc length function and given with constant $\kappa \neq 0$ in IE^3. If γ is of type AW(2), then γ is a circle.

Proof. Let γ be a unit speed Salkowski curve with constant $\kappa \neq 0$ and γ is of type AW(2). Using the equations (6) and (8), we obtain $-\kappa^3\tau^2(\kappa^2 + \tau^2) = 0$. Since $\kappa \neq 0$, we get $\tau = 0$, which means that γ is a circle.

Theorem 3.5. Let γ be a Salkowski curve which is parametrized by the arc length function and given with constant $\kappa \neq 0$ in IE^3. If γ is of type AW(3), then γ is a W-curve.

Proof. Let γ be a unit speed Salkowski curve with constant $\kappa \neq 0$ and γ is of type AW(3). Using the equations (7) and (8), we obtain $\kappa^3\tau' = 0$. Since $\kappa \neq 0$, we get $\tau' = 0$, i.e. τ is a constant. Thus, γ is a W-curve.

4. AW(k)-TYPE ANTI-SALKOWSKI CURVES IN IE^3

Here, we handle AW(k)-type anti-Salkowski curves in IE^3.

Let γ be a unit speed anti-Salkowski curve, and $\{T, N, B\}$ be Frenet frame of the curve γ in IE^3. Since γ is an anti-Salkowski curve with a non-zero constant torsion, the equations (1) become

$$
\begin{aligned}
N_1(s) &= \kappa(s)N(s), \\
N_2(s) &= \kappa'(s)N(s) + \kappa(s)\tau B(s), \\
N_3(s) &= \{\kappa''(s) - \kappa^3(s) - \kappa(s)\tau^2\}N(s) \\
&\quad + \{2\kappa'(s)\tau\}B(s).
\end{aligned} \tag{9}
$$

From Definition 2.3, we have Theorem 4.1 and Theorem 4.2:

Theorem 4.1. Let γ be an anti-Salkowski curve which is parametrized by the arc length function and given with constant $\tau \neq 0$ in IE^3. If γ is of type weak AW(2), the differential equation

$\kappa''(s) - \kappa^3(s) - \kappa(s)\tau^2 = 0$

with the curvatures of γ holds.

Proof. Let γ be a unit speed anti-Salkowski curve with constant $\tau \neq 0$ and γ is of type weak AW(2) in IE^3. From the equations (2), (4) and (9), we get the solution.

Theorem 4.2. Let γ be an anti-Salkowski curve which is parametrized by the arc length function and given with constant $\tau \neq 0$ in IE^3. If γ is of type weak AW(3), γ is a W-curve.

Proof. Let γ be a unit speed anti-Salkowski curve with constant $\tau \neq 0$ and γ is of type weak AW(3) in IE^3. From the equations (3), (4) and (9), we get $2\kappa'\tau = 0$. Since $\tau \neq 0$, $\kappa' = 0$, i.e. κ is a constant. Hence, γ is a W-curve.

From Definition 2.4, we have the following theorems:

Theorem 4.3. There is no AW(1)-type anti-Salkowski curve with constant $\tau \neq 0$ in IE^3.

Proof. Let γ be an anti-Salkowski curve which is parametrized by the arc length function and given with constant $\tau \neq 0$ in IE^3. From the equations (5) and (9), we obtain $\kappa'' - \kappa^3 - \kappa\tau^2 = 0$ and $2\kappa'\tau = 0$. Since $\tau \neq 0$, $\kappa' = 0$, i.e. κ is a constant. Considering κ is a constant in $\kappa'' - \kappa^3 - \kappa\tau^2 = 0$, we get a contradiction. Thus, there is no AW(1)-type anti-Salkowski curve with constant $\tau \neq 0$ in IE^3.

Theorem 4.4. Let γ be an anti-Salkowski curve which is parametrized by the arc length function and given with constant $\tau \neq 0$ in IE^3. If γ is of type AW(2), then the differential equation

$2(\kappa'(s))^2 = \kappa''(s)\kappa(s) - \kappa^4(s) - \kappa^2(s)\tau^2 = 0$

with the curvatures of γ holds.

Proof. Let γ be a unit speed anti-Salkowski curve with constant $\tau \neq 0$ and γ is of type AW(2). Using the equations (6) and (9), we get

$$\|N_2\|^2 N_3 = ((\kappa')^2 + \kappa^2\tau^2)\begin{pmatrix} (\kappa'' - \kappa^3 - \kappa\tau^2)N \\ + 2\kappa'\tau B \end{pmatrix}$$

and

$$\langle N_3, N_2 \rangle N_2 = \kappa'(\kappa'' - \kappa^3 + \kappa\tau^2)(\kappa' N + \kappa\tau B).$$

From the definition of AW(2)-type, we get

$$((\kappa')^2 + \kappa^2\tau^2)(\kappa'' - \kappa^3 - \kappa\tau^2)$$
$$= (\kappa')^2(\kappa'' - \kappa^3 + \kappa\tau^2)$$

and

$$2\kappa'\tau((\kappa')^2 + \kappa^2\tau^2) = \kappa\kappa'\tau(\kappa'' - \kappa^3 + \kappa\tau^2),$$

which completes the proof.

Theorem 4.5. Let γ be an anti-Salkowski curve which is parametrized by the arc length function and given with constant $\tau \neq 0$ in IE^3. If γ is of type AW(3), then γ is a W-curve.

Proof. Let γ be a unit speed anti-Salkowski curve with constant $\tau \neq 0$ and γ is of type AW(3). Using the equations (7) and (9), we obtain $2\kappa^2\kappa'\tau = 0$. Since $\kappa \neq 0$ and $\tau \neq 0$, we get $\kappa' \neq 0$, i.e. κ is a constant. Thus, γ is a W-curve.

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