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Optimal Asymptotic Tests for Nakagami Distribution

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Abstract: Nakagami distribution is often used to model positive valued data with right skewness. The distribution includes some familiar distributions as special cases such as Rayleigh and Half-normal distributions. In real life applications, one of the simpler model may be sufficient to describe data. The aim of this paper is to adapt tests of goodness of fit of the Rayleigh distribution against Nakagami distribution. In this study likelihood ratio, $C(\alpha)$ and score tests are specifically obtained. These tests are then compared in terms of type I error and power of test by a Monte Carlo simulation study.

Nakagami Dağılımı için Optimal Asimtotik Testler

Anahtar Kelimeler
Olabilirlik Oran testi,
Skor testi,
$C(\alpha)$ testi,
Rayleigh dağılımı,
Nakagami dağılımı

Özet: Nakagami dağılımı sağa çarpık pozitif değerli verileri modellemek için sıklıkla kullanılır. Bu dağılım Rayleigh ve Yarı-Normal dağılım gibi bazı bilinen dağılımları içerir. Gerçek hayat uygulamalarında daha basit modellerden biri veriyi tanımlamak için yeterli olabilir. Bu çalışmanın amacı Rayleigh dağılımına karşı Nakagami dağılımına uyum iyiliği testlerini uyarlamaktır. Bu çalışmada özel olarak olabilirlik oran, $C(\alpha)$ ve skor testleri elde edilmiştir. Daha sonra bu testler I. tip hata ve testin gücü bakımından Monte Carlo simülasyon çalışmasıyla karşılaştırılmıştır.

1. Introduction

Characterization of wireless channels plays an important role in designing a reliable wireless communication system. Fading is occured during transmittion of signals from transmitter to receiver. In the literature there are many statistical distribution to determine behavior of fading of signals such as Rician, Rayleigh, lognormal, Weibull distributions. Nakagami distribution is one of the most common distribution among these distributions and the distribution was proposed by Nakagami [1] to model fading of radio signals.

Applications of Nakagami distribution have been carried out in many scientific fields such as engineering, hydrology and medicine. For example, Sarkar et al. [2, 3] use the distribution to derivate unit hydrographs for estimating runoff in hydrology. Shankar et al. [4] and Tsui et al. [5] apply the Nakagami distribution to model ultrasound data in medical imaging studies. Also, the statistical characteristics of “Moving Pictures Expert Group” (MPEG) is modelled by Nakagami distribution [6]. Carcole and Sato [7] and Nakahara and Carcole [8] show the usefullness of the Nakagami distribution to deal with high-frequency seismogram envelopes. Ozonur et al. (9) analyze performance of the goodness of fit tests for Nakagami distribution.

The probability density function of Nakagami distribution is

$$f(x;m,\Omega)=\frac{2m^m}{\Gamma(m)\Omega^m}x^{2m-1}e^{-\frac{m}{\Omega}x^2},\quad x>0 \tag{1}$$

where $m\geq 0.5$ is the shape parameter and $\Omega>0$ is the scale parameter.

The mean and variance of the distribution are given by following equations respectively:

$$E(X)=\frac{\Gamma(m+0.5)}{\Gamma(m)}\left(\frac{\Omega}{m}\right)^{1/2}$$

$$V(X)=\Omega\left(1-\frac{1}{m}\left(\frac{\Gamma(m+0.5)}{\Gamma(m)}\right)^2\right).$$

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Nakagami distribution includes some distributions as special cases. For example, Nakagami distribution becomes Rayleigh distribution when $m=1$, and one-sided Gaussian distribution when $m=1/2$. Nakagami probability densities are plotted for different/various m and Ω parameter combinations in Figure 1.

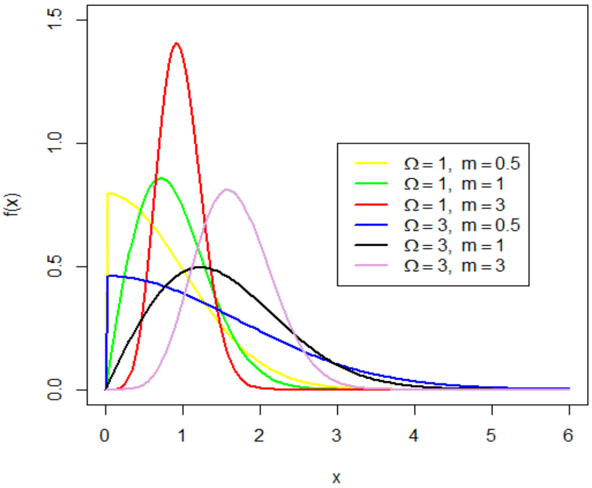

Figure 1. Nakagami densities for various parameter combinations.

In real applications, a one-parameter distribution may be sufficient to analyze data avoiding unnecessary complications. Aim of the study is to adapt tests of goodness of fit of the Rayleigh distribution against Nakagami distribution. Specifically we adapt likelihood ratio test, Neyman's $C(\alpha)$ test and Rao's score test in the study. Monte Carlo simulation study is conducted to compare these tests in terms of empirical size and power and concluding remarks are given.

$C(\alpha)$ test is developed by regressing the residuals of the score function for the parameters of interest on the score function for the nuisance parameters. The nuisance parameters are then replaced by some $\sqrt{n}$ consistent estimates (n is number of observations). $C(\alpha)$ statistic reduces to the score statistic, when maximum likelihood estimators of nuisance parameters, which are $\sqrt{n}$ consistent, are used [10]. Many authors have shown that the $C(\alpha)$ or score test is asymptotically equivalent to the likelihood ratio test [11, 12]. $C(\alpha)$ or score tests have some advantages such as maintaining a preassigned level of significance approximately [13], requiring estimates of the parameters only under the null hypothesis. Also, the tests often are calculated easily [14].

These tests are all asymptotically optimal. They provide tests with good properties in large samples [15]. Although there are various studies including these goodness of fit tests [16, 17], these tests have not been taken into consideration for Nakagami distribution. Goodness of fit problem of the distribution is considered in the study due to pervasive usage in many scientific areas as mentioned above. In this context, the main focus of this study is to test goodness of fit of Rayleigh distribution against Nakagami distribution.

2. Test Statistics

In this section asymptotically optimal goodness of fit tests such as likelihood ratio, Neyman's $C(\alpha)$ test and Rao's score test are obtained to compare Rayleigh distribution against Nakagami distribution.

2.1. Likelihood Ratio Test

$X_1,\ldots,X_n$ is a random sample from a Nakagami distribution with probability density function given in Equation (1) with the parameter vector $\gamma=(m,\Omega)^{\mathrm{T}}$. Our interest is to test $H_0: m=1$ against the alternative $H_1: m\neq 1$. The log-likelihood function under the model (1) is given by

$$l(\gamma;x)=n\left(\log\left(2m^m\right)-\log\Gamma(m)-m\log(\Omega)\right)+(2m-1)\sum_{i=1}^{n}\log x_i-\frac{m}{\Omega}\sum_{i=1}^{n}x_i^2. \qquad (2)$$

The likelihood ratio test (LR) for testing H_0 against H_1 is given as follows:

$$LR=2\left(l(\hat{\gamma};X)-l(\hat{\gamma}_0;X)\right) \qquad (3)$$

where $\hat{\gamma}=(\hat{m},\hat{\Omega})^{\mathrm{T}}$ and $\hat{\gamma}_0=(m=1,\hat{\Omega}_0)^{\mathrm{T}}$ are the maximum likelihood estimators of γ under the full model and under the null hypothesis respectively. By solving the following equations

$$n\log(m)+n-n\Psi(m)-n\log\Omega+2\sum_{i=1}^{n}\log x_i-\frac{1}{\Omega}\sum_{i=1}^{n}x_i^2=0$$

$$\frac{-nm}{\Omega}+\frac{m\sum_{i=1}^{n}x_i^2}{\Omega^2}=0$$

the maximum likelihood estimates of parameters under alternative hypothesis are obtained as $\hat{\Omega}=\dfrac{\sum_{i=1}^{n}x_i^2}{n}$ and $\hat{m}=\dfrac{1+\sqrt{1+4A/3}}{4A}$ where $A=\log\left(\dfrac{\sum_{i=1}^{n}x_i^2}{n}\right)-\dfrac{2}{n}\sum_{i=1}^{n}\log(x_i)$.

Here the notation $\Psi(m)$ is digamma function, i.e., $\Psi(m)=\frac{\partial}{\partial m}\log\Gamma(m)$. The statistic LR asymptotically follows a chi-square distribution with one degree of freedom under the null hypothesis.

2.2. $C(\alpha)$ and Score Tests

Rao [10] introduced score test as an alternative to likelihood ratio test. Neyman [18] proposed $C(\alpha)$ test as a generalization of Rao's score test. The $C(\alpha)$ and score test statistics are based on score functions. Score function and information matrix are given respectively by $S(\gamma)=\frac{\partial l(\gamma;X)}{\partial\gamma_i}$ and $I(\gamma)=-E\left[\frac{\partial^2 l(\gamma;X)}{\partial\gamma_i\partial\gamma_j}\right]$. Score function can be partitioned as

$$S(\gamma)=\begin{pmatrix}S_m(\gamma)\\ S_\Omega(\gamma)\end{pmatrix}$$

where $S_m(\gamma)$ and $S_\Omega(\gamma)$ represent the first derivative of log-likelihood function with respect to the parameter m and Ω respectively. Similarly, information matrix is obtained as follows:

$$I(\gamma)=\begin{pmatrix}I_{mm}(\gamma) & I_{m\Omega}(\gamma)\\ I_{\Omega m}(\gamma) & I_{\Omega\Omega}(\gamma)\end{pmatrix},$$

where elemets of $I(\gamma)$ denote the minus expected value of the second derivatives of the log-likelihood with respect to parameters. We obtained elements of information matrix as

$$I_{mm}=-E\left[\frac{\partial^2 l(\gamma;X)}{\partial m^2}\right]=n\Psi'(m)-n/m,$$

$$I_{m\Omega}=I_{\Omega m}=-E\left[\frac{\partial^2 l(\gamma;X)}{\partial m\partial\Omega}\right]=0,$$

$$I_{\Omega\Omega}=-E\left[\frac{\partial^2 l(\gamma;X)}{\partial\Omega^2}\right]=nm/\Omega^2,$$

where trigamma function $\Psi'(m)$ is derivative of digamma function $\Psi(m)$. Define $\Sigma(\gamma)$ by

$$\Sigma(\gamma)=I_{mm}(\gamma)-I_{m\Omega}(\gamma)I_{\Omega\Omega}^{-1}(\gamma)I_{\Omega m}(\gamma).$$

$\Sigma(\gamma)$ is the asymptotic covariance matrix of $S_m(\gamma)$ and that $\Sigma(\gamma)^{-1}$ is the asymptotic covariance matrix of $\hat{m}$ [11]. The Rao's score statistic is as follows:

$$T_S=\frac{S_m(\hat{\gamma}_0)^2}{\Sigma(\hat{\gamma}_0)} \quad (4)$$

which has an asymptotic Chi-square distribution with 1 degree of freedom.

Neyman's $C(\alpha)$ statistic is given by

$$T_C=\frac{\left(S_m(\tilde{\gamma}_0)-I_{m\Omega}(\tilde{\gamma}_0)I_{\Omega\Omega}^{-1}(\tilde{\gamma}_0)S_\Omega(\tilde{\gamma}_0)\right)^2}{\Sigma(\hat{\gamma}_0)}. \quad (5)$$

If the parameter Ω, under null hypothesis, is replaced by its moment estimate $\hat{\Omega}_{mm}=4\bar{x}^2/\pi$, which are $\sqrt{n}$ consistent, the distribution of T_C is also asymptotically Chi-square distribution with 1 degree of freedom. On the other hand, if Ω is replaced by its maximum likelihood estimate $\hat{\Omega}_{ml}=\sum_{i=1}^{n}x_i^2\Big/n$, the T_C statistic becomes the score statistic.

3. Simulation Study

In this section, we compare performance of the test statistics T_C, T_S, LR in terms of Type I errors and powers of tests by using statistical software R 3.4.1.

Firstly, critical values of the goodness of fit tests are obtained by simulating 10000 samples of size n from Nakagami distribution with $m=1$. Using critical values, type I errors are calculated by generating 5000 random samples from Nakagami distribution with $m=1$ for various combinations of levels, sample sizes and scale parameters. In the simulation study, we consider sample sizes $n=20,30,50$ and scale parameters $\Omega=0.3, 0.5, 1, 1.5, 3, 5$. Type I errors of test statistis are summarized in Table 1. On the other hand, we obtain powers of test statistics based on 10000 replicated samples of size n from Nakagami distribution with different Ω and m parameters. In power study we only consider nominal level 0.05. Powers of test statistics are given in Table 2-4.

As seen in Table 1, type I errors of all the three tests close to nominal levels irrespective of values of scale parameter Ω and sample size.

As seen in Table 2-4, as the sample size increases powers of all test statistics increase. Also, powers of test statistics increase as the value of m moves away from null and power results are not affected by scale parameter Ω. The power results in Table 2-4 show that the $C(\alpha)$ test statistic T_C is the least powerful. Although, as the sample size increases its power increases, this statistic still remains least powerful. Moreover, likelihood ratio test statistic is the most powerful test. For large sample size, T_c and T_S are close to each other in respect of power, however LR is still the most powerful.

Table 1. Type I errors of goodness of fit tests for different scale parameters, nominal levels and sample sizes

		$n=20$			$n=30$			$n=50$		
Ω	α	T_C	T_S	LR	T_C	T_S	LR	T_C	T_S	LR
0.3	0.10	0.0972	0.0982	0.0956	0.1016	0.1004	0.0974	0.0986	0.1004	0.1040
	0.05	0.0542	0.0538	0.0564	0.0512	0.0520	0.0428	0.0516	0.0516	0.0484
	0.01	0.0072	0.0074	0.0076	0.0114	0.0110	0.0100	0.0084	0.0078	0.0104
0.5	0.10	0.1016	0.1020	0.1004	0.0972	0.0956	0.0912	0.0872	0.0894	0.0932
	0.05	0.0552	0.0560	0.0492	0.0476	0.0478	0.0474	0.0482	0.0486	0.0514
	0.01	0.0086	0.0086	0.0106	0.0106	0.0108	0.0102	0.0098	0.0100	0.0086
1	0.10	0.0976	0.0974	0.0956	0.1012	0.1016	0.1020	0.1066	0.1072	0.1048
	0.05	0.0502	0.0508	0.0506	0.0472	0.0460	0.0480	0.0512	0.0520	0.0500
	0.01	0.0094	0.0096	0.0136	0.0106	0.0100	0.0118	0.0082	0.0072	0.0052
1.5	0.10	0.0948	0.0936	0.0896	0.0992	0.0976	0.1042	0.1062	0.1072	0.1054
	0.05	0.0510	0.0512	0.0480	0.0514	0.0512	0.0504	0.0520	0.0510	0.0530
	0.01	0.0134	0.0134	0.0140	0.0102	0.0096	0.0110	0.0144	0.0142	0.0104
3	0.10	0.0940	0.0922	0.1016	0.1104	0.1102	0.113	0.0536	0.0528	0.0542
	0.05	0.0458	0.0462	0.0498	0.0464	0.0492	0.0546	0.0102	0.0100	0.0108
	0.01	0.0102	0.0106	0.0118	0.0088	0.0090	0.0094	0.1004	0.1002	0.0972
5	0.10	0.0968	0.0992	0.0986	0.1034	0.1028	0.1014	0.0092	0.0096	0.0110
	0.05	0.0446	0.0442	0.0462	0.0486	0.0498	0.0490	0.0500	0.0474	0.0484
	0.01	0.0126	0.0124	0.0142	0.0090	0.0092	0.0104	0.1002	0.0992	0.0996

Table 2. Powers of goodness of fit tests for $n=20$ and $\alpha=0.05$

		m								
Ω	Statistics	1	1.3	1.5	1.8	2	2.3	2.5	2.8	3
0.3	T_C	0.0454	0.0684	0.1362	0.3088	0.4586	0.6532	0.6946	0.8194	0.8920
	T_S	0.0458	0.0792	0.1502	0.3526	0.4936	0.6844	0.7206	0.8506	0.9138
	LR	0.0486	0.1508	0.2976	0.5426	0.6560	0.8038	0.8744	0.9512	0.9746
0.5	T_C	0.0490	0.0652	0.1530	0.2962	0.4226	0.5870	0.7164	0.8320	0.8636
	T_S	0.0480	0.0742	0.1718	0.3242	0.4576	0.6106	0.7524	0.8590	0.8864
	LR	0.0464	0.1564	0.2996	0.5104	0.6518	0.7940	0.8818	0.9472	0.9640
1	T_C	0.0560	0.0756	0.1254	0.3298	0.4122	0.5660	0.7108	0.8320	0.8642
	T_S	0.0572	0.0846	0.1466	0.3642	0.4560	0.6176	0.7428	0.8588	0.8890
	LR	0.0536	0.1664	0.2792	0.5158	0.6494	0.8080	0.8766	0.9466	0.9692
1.5	T_C	0.0500	0.0698	0.1436	0.3080	0.4190	0.6262	0.7198	0.8402	0.8644
	T_S	0.0496	0.0804	0.1596	0.3398	0.4570	0.6576	0.7536	0.8598	0.8876
	LR	0.0468	0.1578	0.2850	0.5126	0.6460	0.8204	0.8804	0.9418	0.9672
3	T_C	0.0474	0.0696	0.1394	0.3342	0.3922	0.5772	0.7278	0.8304	0.8874
	T_S	0.0454	0.0796	0.1550	0.3658	0.4450	0.6182	0.7554	0.8514	0.9042
	LR	0.0506	0.1630	0.2758	0.5224	0.6554	0.7958	0.8814	0.9376	0.9682
5	T_C	0.0564	0.0702	0.1532	0.2902	0.4248	0.6378	0.7084	0.8440	0.8864
	T_S	0.0586	0.0808	0.1692	0.3302	0.4558	0.6700	0.7312	0.8606	0.9050
	LR	0.0580	0.1624	0.2858	0.4880	0.6324	0.8122	0.8786	0.9416	0.9642

Table 3. Powers of goodness of fit tests for $n=30$ and $\alpha=0.05$

		m								
Ω	Statistics	1	1.3	1.5	1.8	2	2.3	2.5	2.8	3
0.3	T_C	0.0534	0.1204	0.2604	0.5394	0.7076	0.8706	0.9368	0.9780	0.9904
	T_S	0.0528	0.1378	0.2768	0.5586	0.7268	0.8800	0.9448	0.9802	0.9926
	LR	0.0468	0.2102	0.4170	0.7018	0.8410	0.9496	0.9764	0.9938	0.9970
0.5	T_C	0.0498	0.1236	0.2822	0.5478	0.7000	0.8882	0.9336	0.9772	0.9926
	T_S	0.0502	0.1352	0.3020	0.5662	0.7192	0.9040	0.9390	0.9808	0.9944
	LR	0.0480	0.2082	0.4172	0.7212	0.8234	0.9586	0.9758	0.9950	0.9982
1	T_C	0.0468	0.1204	0.2894	0.5290	0.7120	0.8762	0.9412	0.9794	0.9922
	T_S	0.0462	0.1316	0.3074	0.5522	0.7342	0.8866	0.9508	0.9842	0.9932
	LR	0.0498	0.2052	0.4320	0.6876	0.8440	0.9456	0.9772	0.9962	0.9986
1.5	T_C	0.0552	0.1212	0.2654	0.5354	0.6954	0.8790	0.9290	0.9824	0.9914
	T_S	0.0530	0.1392	0.2814	0.5612	0.7198	0.8922	0.9390	0.9844	0.9930
	LR	0.0544	0.2152	0.4112	0.7050	0.8442	0.9500	0.9776	0.9958	0.9990
3	T_C	0.0450	0.1210	0.2760	0.5600	0.7036	0.8910	0.9258	0.9784	0.9908
	T_S	0.0474	0.1314	0.2928	0.5814	0.7300	0.9002	0.9360	0.982	0.9936
	LR	0.0512	0.2064	0.4170	0.7010	0.8518	0.9506	0.9790	0.9944	0.9982
5	T_C	0.0546	0.1240	0.2490	0.5460	0.6962	0.8502	0.9486	0.981	0.9900
	T_S	0.0554	0.1358	0.2718	0.5738	0.7136	0.8696	0.9542	0.984	0.9924
	LR	0.0516	0.2024	0.4090	0.7062	0.8334	0.9474	0.9812	0.9946	0.9988

Table 4. Powers of goodness of fit tests for $n=50$ and $\alpha=0.05$

		m								
Ω	Statistics	1	1.3	1.5	1.8	2	2.3	2.5	2.8	3
0.3	T_C	0.0476	0.2328	0.4988	0.8370	0.9484	0.9922	0.9976	0.9998	1.0000
	T_S	0.0476	0.2442	0.5146	0.8422	0.9516	0.9930	0.9980	0.9998	1.0000
	LR	0.0458	0.3220	0.6178	0.9040	0.9784	0.9974	0.9998	1.0000	1.0000
0.5	T_C	0.0522	0.2248	0.4812	1.0000	0.9460	0.9934	0.9966	0.9996	1.0000
	T_S	0.0510	0.2400	0.5002	1.0000	0.9512	0.9942	0.9972	0.9996	1.0000
	LR	0.0502	0.3146	0.6028	1.0000	0.9734	0.9982	0.9992	0.9998	1.0000
1	T_C	0.0528	0.2366	0.4862	0.8290	0.9570	0.9934	0.9992	0.9998	1.0000
	T_S	0.0546	0.2472	0.4960	0.8390	0.9600	0.9950	0.9992	0.9998	1.0000
	LR	0.0524	0.3262	0.6004	0.8990	0.9778	0.9980	0.9996	1.0000	1.0000
1.5	T_C	0.0468	0.1966	0.4594	0.8606	0.9504	0.9924	0.9984	1.0000	1.0000
	T_S	0.0478	0.2104	0.4798	0.8692	0.9566	0.9944	0.9984	1.0000	1.0000
	LR	0.0510	0.3034	0.5880	0.9140	0.9776	0.9964	0.9990	1.0000	1.0000
3	T_C	0.0492	0.2336	0.4924	0.8366	0.9476	0.9924	0.9984	1.0000	1.0000
	T_S	0.0486	0.2454	0.5090	0.8476	0.9526	0.9936	0.9986	1.0000	1.0000
	LR	0.0468	0.3282	0.6010	0.9162	0.9740	0.9972	0.9996	1.0000	1.0000
5	T_C	0.0458	0.2176	0.4748	0.8344	0.9426	0.9932	0.9986	1.0000	1.0000
	T_S	0.0460	0.2318	0.4904	0.8454	0.9484	0.9938	0.9986	1.0000	1.0000
	LR	0.0482	0.3086	0.5980	0.9114	0.9688	0.9968	0.9994	1.0000	1.0000

4. Discussion and Conclusion

Three test statistics, namely, $C(\alpha)$ statistic T_C, score test statistic T_S and a likelihood ratio statistic LR are adapted to test goodness of fit of the Rayleigh distribution against two parameter Nakagami distribution. Performance of the test statistics are compared in terms of type I error and power of test by Monte Carlo simulation study. Simulation study shows that LR test statistic is the most powerful test. Powers of test statistics are not affected by scale parameter Ω and power results increase as shape parameter increases. Although T_c and T_S are close to each other in respect of power for large sample size, T_c test statistic is the least powerful test among the test statistics. Finally, our recommendation is to use LR statistic for all sample sizes and/or m parameters.

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