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$H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava Hypergeometric Function

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Abstract
Formulas and identities involving many well known special functions (such as the Gamma and Beta functions, Gauss hypergeometric function, and so on) play important roles in themselves and their diverse applications. In this paper, we will add τ_1, τ_2, τ_3 parameters to the H_A Srivastava hypergeometric function and we introduce new $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function. Then, we present some properties of the $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function.

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1. Introduction

In this paper, $\mathbb{N}$, $\mathbb{Z}^-$, and $\mathbb{C}$ denote the sets of positive integers, negative integers, complex numbers, respectively. Also, $\mathbb{N}_0$ and $\mathbb{Z}_0^-$ represent the sets of positive integers and complex numbers by excluding origin, respectively.$\left(\mathbb{N}_0 := \mathbb{N} \cup \{0\} \text{ and } \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}.\right)$

The classical Gamma function $\Gamma(x)$ is defined by [13, 16, 17, 19, 20] ,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (\mathrm{Re}(x) > 0).$$

The familar Beta function $B(x, y)$ is a function of two complex variables x and y, is defined by the first kind of Eulerian integral [13, 16, 17, 19, 20],

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt. \tag{1.1}$$

Srivastava [18] noticed the existence of three additional complete triple hypergeometric functions of the second order; of which H_A is defined as [6, 7, 9, 19–23]

$$H_A[\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p} (\beta_1)_{m+n} (\beta_2)_{n+p}}{(\gamma_1)_m (\gamma_2)_{n+p}} \frac{x_1^m}{m!} \frac{x_2^n}{n!} \frac{x_3^p}{p!}$$
$$(|x_1| < r, |x_2| < s, |x_3| < t, r + s + t = 1 + st). \tag{1.2}$$

There, $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the set of complex numbers and the set of nonpositive integers, respectively. Here, $(\lambda)_n$ denotes the Pochhammer symbol which is defined by $(\lambda \in \mathbb{C})$ [1, 8, 10, 16, 17],

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ (\lambda)(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}, \tag{1.3}$$

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where, Γ is being the well-known Gamma function.

Virchenko et al. [24, 25]studied and investigated the following generalized τ-hypergeometric function:

$$ {}_2R_1^{\tau}(a,b;c;z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!} \tag{1.4}$$

$$(\tau>0\,,\ |z|<1\,,\ \mathrm{Re}(c)>0,\ \mathrm{Re}(b)>0)$$

They gave the Euler type integral representation as follows [24, 25]:

$$ {}_2R_1(a,b;c;\tau;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt^{\tau})^{-a}\,dt \tag{1.5}$$

$$(\tau>0;\ |\arg(1-z)|<\pi,\ \mathrm{Re}(c)>\mathrm{Re}(b)>0).$$

The special case when $\tau=1$ in (1.4) and (1.5) give the familiar representations of Gauss's hypergeometric functions [1, 3, 5, 8, 14, 15].

Furthermore , Al-Shammery and Kalla [3] introduced and studied various properties of second τ-Appell's hypergeometric functions as follows :

$$F_2^{\tau_1,\tau_2}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2) = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \tag{1.6}$$

$$\times\sum_{m_1,m_2=0}^{\infty}\frac{(\alpha)_{m_1+m_2}\Gamma(\beta_1+\tau_1 m_1)\Gamma(\beta_2+\tau_2 m_2)}{\Gamma(\gamma_1+\tau_1 m_1)\Gamma(\gamma_2+\tau_2 m_2)}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}$$

$$(\tau_1,\tau_2>0,\ |x_1|+|x_2|<1)\,.$$

The interested reader may be referred to several recent papers on the subject [5, 14, 15].The special case when τ_1,$\tau_2=1$ in (1.6) gives the familiar representations of second Appell hypergeometric function F_2.[2, 4, 11, 12, 19, 20].

The main aim of this paper is to introduce $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function. After, we will present some properties of this function such as integral representations, derivative formula and recurrence relations.

2. $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-Hypergeometric Function

By adding parameters τ_1,τ_2,τ_3 to a known $H_A(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)$ Srivastava hypergeometric function, new $H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)$ Srivastava's triple τ-hypergeometric function defined as:

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3) = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \tag{2.1}$$

$$\times\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p}\,\Gamma(\beta_1+\tau_1 m+n)\,\Gamma(\beta_2+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m)\,\Gamma(\gamma_2+\tau_2 n+\tau_3 p)}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}$$

$$(\tau_1,\tau_2,\tau_3>0;\ |x_1|<r,\ |x_2|<s, |x_3|<t,\ r+s+t=1+st\,)\,.$$

The special case when τ_1,τ_2,$\tau_3=1$ in (2.1) gives the familiar representations of Srivastava's H_A triple hypergeometric function (1.2) [6, 7, 9, 18–23]

2.1 Integral Representations of $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's τ-Hypergeometric Function

In this section we give some integral representations of $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function. Let us start following theorem.

Theorem 2.1. *The following integral representation holds true:*

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3) = \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \tag{2.2}$$

$$\times\int_0^{\infty}\int_0^{\infty} t^{\alpha-1}s^{\beta_1-1}e^{-t-s}\,{}_0F_1^{\tau_1}(-;\gamma_1;ts^{\tau_1}x_1)\,\Phi_1^{\tau_2,\tau_3}(\beta_2;\gamma_2;sx_2,tx_3)\,dtds$$

$$(\tau_1,\tau_2,\tau_3>0,\ \mathrm{Re}(x_2)<1,\ \mathrm{Re}(x_3)<1,\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta_1)>0\,)$$

where

$$\Phi_1^{\tau_2,\tau_3}(\beta_2;\gamma_2;sx_2,tx_3)=\sum_{n,p=0}^{\infty}\frac{(\beta_2)_{\tau_2n+\tau_3p}}{(\gamma_2)_{\tau_2n+\tau_3p}}\frac{(sx_2)^n}{n!}\frac{(tx_3)^p}{p!}. \tag{2.3}$$

Proof. Using the equation (1.3) in (2.1), we have

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)}$$
$$\times\sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha+m+p)\Gamma(\beta_1+\tau_1m+n)(\beta_2)_{\tau_2n+\tau_3p}}{(\gamma_1)_{\tau_1m}(\gamma_2)_{\tau_2n+\tau_3p}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.$$

If the values of $\Gamma(\alpha+m+p)$ and $\Gamma(\beta_1+\tau_1m+n)$ from equation (1.1) is replaced, then the desired result can be obtained. □

Theorem 2.2. *The following integral representations hold true:*

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{B(\alpha,s-\alpha)} \tag{2.4}$$
$$\times\int_0^1 t^{\alpha-1}(1-t)^{s-\alpha-1}H_A^{\tau_1,\tau_2,\tau_3}[s,\beta_1,\beta_2;\gamma_1,\gamma_2;tx_1,x_2,tx_3]\,dt,$$

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{B(\beta_1,s-\beta_1)} \tag{2.5}$$
$$\times\int_0^1 t^{\beta_1-1}(1-t)^{s-\beta_1-1}H_A^{\tau_1,\tau_2,\tau_3}(\alpha,s,\beta_2;\gamma_1,\gamma_2;t^{\tau_1}x_1,tx_2,x_3)\,dt,$$

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{B(\beta_2,s-\beta_2)} \tag{2.6}$$
$$\times\int_0^1 t^{\beta_2-1}(1-t)^{s-\beta_2-1}H_A^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,s;\gamma_1,\gamma_2;x_1,t^{\tau_2}x_2,t^{\tau_3}x_3]\,dt,$$

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{B(s,\gamma_1-s)} \tag{2.7}$$
$$\times\int_0^1 t^{s-1}(1-t)^{\gamma_1-s-1}H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;s,\gamma_2;t^{\tau_1}x_1,x_2,x_3)\,dt,$$

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{B(s,\gamma_2-s)} \tag{2.8}$$
$$\times\int_0^1 t^{s-1}(1-t)^{\gamma_2-s-1}H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,s;x_1,t^{\tau_2}x_2,t^{\tau_3}x_3)\,dt,$$

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{B(\beta_2,\gamma_2-\beta_2)} \tag{2.9}$$
$$\times\int_0^1 t^{\beta_2-1}(1-t)^{\gamma_2-\beta_2-1}(1-t^{\tau_2}x_2)^{-\alpha}(1-t^{\tau_3}x_3)$$
$$\times\,{}_2F_1^{\tau_1}\left(\alpha,\beta_1;\gamma_1;\frac{x_1}{(1-t^{\tau_2}x_2)^{\tau_1}(1-t^{\tau_3}x_3)}\right)dt,$$

and,

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{B(\beta_2,\gamma_2-\beta_2)B(\beta_1,\gamma_1-\beta_1)}$$
$$\times\int_0^1\int_0^1 t^{\beta_2-1}u^{\beta_1-1}(1-t)^{\gamma_2-\beta_2-1}(1-u)^{\gamma_1-\beta_1-1}(1-t^{\tau_2}x_2)^{-\alpha}(1-t^{\tau_3}x_3)^{-\alpha}$$
$$\times(1-\frac{x_1u^{\tau_1}}{(1-t^{\tau_2}x_2)^{\tau_1}(1-t^{\tau_3}x_3)})dtdu. \tag{2.10}$$

Proof. From the equations (1.3) and (2.1), we get

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)$$
$$\sum_{m,n,p=0}^{\infty}\frac{(s)_{m+p}(\alpha)_{m+p}(\beta_1)_{\tau_1m+n}(\beta_2)_{\tau_2n+\tau_3p}}{(s)_{m+p}(\gamma_1)_{\tau_1m}(\gamma_2)_{\tau_2n+\tau_3p}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.$$

If we consider following equalitiy and using definition of Beta function [13, 16, 17, 19, 20] in equation (1.1),

$$\frac{(\alpha)_{m+p}}{(s)_{m+p}}=\frac{B(\alpha+m+p,s-\alpha)}{B(\alpha,s-\alpha)}$$

the (2.4) is obtained.

The equations (2.5) − (2.9) are obtained in similar ways.

If the following equality [24, 25],

$${}_2F_1^{\tau}(\alpha,\beta;\gamma;x)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-u^{\tau}x)\,du$$

is substituted into the statement (2.9), then equation (2.10) is derived. □

Theorem 2.3. *The following Euler integral representation holds true:*

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3) \tag{2.11}$$
$$=\frac{1}{B(\beta_1,\gamma_1-\beta_1)B(\beta_2,\gamma_2-\beta_2)B(\beta_2,\gamma_2-\beta_2)}$$
$$\times\int_0^1\int_0^1\int_0^1 u^{\beta_1-1}v^{\beta_1-1}w^{\beta_2-1}(1-u)^{\gamma_1-\beta_1-1}[(1-v)(1-w)]^{\gamma_2-\beta_2-1}$$
$$\times(1-x_1u^{\tau_1}-x_3v^{\tau_3})^{-\alpha}(1-x_2w^{\tau_2})^{-\beta_1}dudvdw,$$
$$(\tau_1,\tau_2,\tau_3>0,\ \mathrm{Re}(\gamma_1-\beta_1)<1,\ \mathrm{Re}(\beta_1)>1,\ \mathrm{Re}(\gamma_1+\gamma_2-\beta_1)>1,\ \mathrm{Re}(\beta_2)>0\ ).$$

Proof. From the definitions of $H_A^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]$ Srivastava's triple τ-hypergeometric function (2.1) and the second τ-Appell's hypergeometric functions (1.6), we get

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\sum_{n=0}^{\infty}\frac{(\beta_1)_n(\beta_2)_{\tau_2n}}{(\gamma_2)_{\tau_2n}}$$
$$\times F_2^{\tau_1,,\tau_3}(\alpha,\beta_1+n,\beta_2+\tau_2n;\gamma_1,\gamma_2+\tau_2n;x_1,x_3)\frac{x_2^n}{n!}. \tag{2.12}$$

In [2], we have following equation:

$$F_2^{\tau_1,,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_3)=\frac{1}{B(\beta_1,\gamma_1-\beta_1)B(\beta_2,\gamma_2-\beta_2)}$$
$$\times\int_0^1\int_0^1 t^{\beta_1-1}s^{\beta_2-1}(1-t)^{\gamma_1-\beta_1-1}(1-s)^{\gamma_2-\beta_2-1}(1-x_1t^{\tau_1}-x_3s^{\tau_3})\,dtds. \tag{2.13}$$

In (2.12) , if we use (2.13) and necessary arrangements are made, we get

$$H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)=\frac{1}{B(\beta_1,\gamma_1-\beta_1)\,B(\beta_2,\gamma_2-\beta_2)}$$
$$\times\int_0^1\int_0^1 u^{\beta_1-1}w^{\beta_2-1}(1-u)^{\gamma_1-\beta_1-1}(1-w)^{\gamma_2-\beta_2-1}(1-x_1u^{\tau_1}-x_3v^{\tau_3})^{-\alpha}$$
$$\times\,{}_2R_1^{\tau_2}(\beta_1,\beta_2;\gamma_2;x_2uw(1-u))\,dudv. \tag{2.14}$$

Thereafter, from (1.5) the integral representation of ${}_2R_1$ is written in (2.14), (2.11) can be obtained. □

Theorem 2.4. *The following integral representation holds true:*

$$H_A^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\gamma_2;\gamma_1,\gamma_2;x_2x_3,x_2,x_3]=\frac{(1-x_3)^{-\alpha}(1-x_2)^{-\beta_1}}{\Gamma(\alpha)\,\Gamma(\beta_1)}$$
$$\times\int_0^\infty\int_0^\infty e^{-u-v}u^{\alpha-1}v^{\beta_1-1}\,{}_0F_1^{\tau_1}\left(-;\gamma_1;\frac{x_2x_3uv^{\tau_1}}{(1-x_3)(1-x_2)^{\tau_1}}\right)dudv. \tag{2.15}$$

Proof. If we put $\beta_2=\gamma_2$, $x_1=x_2x_3$ and using $\Phi_1^{\tau_2,\tau_3}(\gamma_2;\gamma_2;sx_2,tx_3)=e^{x_2s+x_3t}$ in (2.2), we have

$$H_A^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\gamma_2;\gamma_1,\gamma_2;x_2x_3,x_2,x_3]=\frac{1}{\Gamma(\alpha)\,\Gamma(\beta_1)} \tag{2.16}$$
$$\times\int_0^\infty\int_0^\infty e^{-s(1-x_2)-t(1-x_3)}t^{\alpha-1}s^{\beta_1-1}\,{}_0F_1^{\tau_1}(-;\gamma_1;ts^{\tau_1}x_2x_3)\,dtds.$$

Setting $t(1-x_3)=u$, $s(1-x_2)=v$ in (2.16),we are led to the desired integral representation of (2.15). □

2.2 Partial Differential Equations

Here, we will give partial differential equations for $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function.

Theorem 2.5. *The following system of partial differential equations hold true:*

$$\left\{\begin{array}{c}\left[\theta(\tau_1\theta+\gamma_1-1)_{\tau_1}-x_1(\theta+\phi+\alpha)(\tau_1\theta+\varphi+\beta_1)_{\tau_1}\right]H_A^{\tau_1,\tau_2,\tau_3}=0\\ \left[\varphi(\tau_2\varphi+\tau_3\phi+\gamma_2-\tau_2)_{\tau_2}-x_2(\tau_1\theta+\varphi+\beta_1)(\tau_2\varphi+\tau_3\phi+\beta_2)_{\tau_2}\right]H_A^{\tau_1,\tau_2,\tau_3}=0\\ \left[\phi(\tau_2\varphi+\tau_3\phi+\gamma_2-\tau_3)_{\tau_3}-x_3(\theta+\phi+\alpha)(\tau_2\varphi+\tau_3\phi+\beta_2)\right]H_A^{\tau_1,\tau_2,\tau_3}=0\end{array}\right. \tag{2.17}$$

where, $\theta=x_1\dfrac{d}{dx_1}$, $\varphi=x_2\dfrac{d}{dx_2}$ *and* $\phi=x_3\dfrac{d}{dx_3}$.

Proof. Using the $H_A^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]$ Srivastava's triple τ-hypergeometric function in equation (2.1) and multiplying this equation $\theta(\theta+\gamma_1-1)_{\tau_1}$, we have

$$\begin{aligned}&\theta(\tau_1\theta+\gamma_1-1)_{\tau_1}H_A^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]\\ &=\sum_{m,n,p=0}^{\infty}\frac{m(\tau_1m+\gamma_1-1)_{\tau_1}(\alpha)_{m+p}(\beta_1)_{\tau_1m+n}(\beta_2)_{\tau_2n+\tau_3p}}{(\gamma_1)_{\tau_1m}(\gamma_2)_{\tau_2n+\tau_3p}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\qquad(m\to m+1)\\ &=\sum_{m,n,p=0}^{\infty}\frac{(\tau_1m+\tau_1+\gamma_1-1)_{\tau_1}(\alpha)_{m+1+p}(\beta_1)_{\tau_1m+\tau_1+n}(\beta_2)_{\tau_2n+\tau_3p}}{(\gamma_1)_{\tau_1m+\tau_1}(\gamma_2)_{\tau_2n+\tau_3p}}\frac{x_1^{m+1}}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.\end{aligned} \tag{2.18}$$

Taking advantage of the following property of Pochhammer symbol [24, 25] in (2.18)

$$(\alpha)_{\tau_1m+\tau_2n}=(\alpha+\tau_1m)_{\tau_2n}(\alpha)_{\tau_1m}$$

and making some useful arragement in the equation (2.18), we have the desired result (2.17). □

The special case when τ_1,τ_2,$\tau_3=1$ in (2.17) gives the partial differential equations of Srivastava's H_A triple hypergeometric function [18, 21]

2.3 Derivative Formula

In this section, we derive derivative formula for $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's triple τ-hypergeometric function.

Theorem 2.6. *The following derivative formula for $H_A^{\tau_1,\tau_2,\tau_3}$ holds true:*

$$\begin{aligned}
&\frac{d^{r+s+t}}{dx_1^r dx_2^s dx_3^t} H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3) \\
&= \frac{(\alpha)_{r+t}(\beta_1)_{\tau_1 r+s}(\beta_2)_{\tau_2 s+\tau_3 t}}{(\gamma_1)_{\tau_1 r}(\gamma_2)_{\tau_2 s+\tau_3 t}} \\
&\times H_A^{\tau_1,\tau_2,\tau_3}[(\alpha+r+t),(\beta_1+\tau_1 r+s),(\beta_2+\tau_2 s+\tau_3 t); \\
&(\gamma_1+\tau_1 r),(\gamma_2+\tau_2 s+\tau_3 t);x_1,x_2,x_3]
\end{aligned} \tag{2.19}$$

Proof. By utilizing the definition of $H_A^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]$ Srivastava's triple τ-hypergeometric function in equation (2.1) and derivatives this equation r times, we get

$$\begin{aligned}
&\frac{d^r}{dx_1^r} H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3) \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+r+p}(\beta_1)_{\tau_1 m+\tau_1 r+n}(\beta_2)_{\tau_2 n+\tau_3 p}}{(\gamma_1)_{\tau_1 m+\tau_1 r}(\gamma_2)_{\tau_2 n+\tau_3 p}} \frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.
\end{aligned} \tag{2.20}$$

Then, taking derivative s times of (2.20), we have

$$\begin{aligned}
&\frac{d^{r+s}}{dx_1^r dx_2^s} H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3) \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+r+p}(\beta_1)_{\tau_1 m+\tau_1 r+n+s}(\beta_2)_{\tau_2 s+\tau_2 n+\tau_3 p}}{(\gamma_1)_{\tau_1 m+\tau_1 r}(\gamma_2)_{\tau_2 s+\tau_2 n+\tau_3 p}} \frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.
\end{aligned} \tag{2.21}$$

Finally, we take derivative t times of (2.21), we get

$$\begin{aligned}
&\frac{d^{r+s+t}}{dx_1^r dx_2^s dx_3^t} H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3) \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+r+p+t}(\beta_1)_{\tau_1 m+\tau_1 r+n+s}(\beta_2)_{\tau_2 s+\tau_2 n+\tau_3 p+\tau_3 t}}{(\gamma_1)_{\tau_1 m+\tau_1 r}(\gamma_2)_{\tau_2 s+\tau_2 n+\tau_3 p+\tau_3 t}} \frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}.
\end{aligned} \tag{2.22}$$

Using the following property of Pochhammer symbol [17**?**] in (2.22),

$$(\alpha)_{m+n} = (\alpha+m)_n(\alpha)_m$$

we have the desired result (2.19) . □

2.4 Recurrence Relations of $H_A^{\tau_1,\tau_2,\tau_3}$ Srivastava's τ-Hypergeometric Function

In this section we give some recurrence relations of $H_A^{\tau_1,\tau_2,\tau_3}$Srivastava's τ-Hypergeometric Function.

Theorem 2.7. *The following recurrence relation for $H_A^{\tau_1,\tau_2,\tau_3}$ holds true:*

$$\begin{aligned}
&H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3) \\
&= H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1-1,\gamma_2;x_1,x_2,x_3) \\
&+\frac{\alpha\beta_1 x_1}{(\gamma_1)(1-\gamma_1)} H_A^{\tau_1,\tau_2,\tau_3}((\alpha+1),\beta_1+1,\beta_2;\gamma_1+1,\gamma_2;x_1,x_2,x_3).
\end{aligned} \tag{2.23}$$

Proof. Using integral representation of $H_A^{\tau_1,\tau_2,\tau_3}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3)$ Srivastava τ-hypergeometric function in (2.2) and following contiguous relation for the function ${}_0F_1^{\tau_1}$ [24, 25],

$${}_0F_1^{\tau_1}(-;\gamma-1;x) - {}_0F_1^{\tau_1}(-;\gamma;x) - \frac{x}{\gamma(1-\gamma)}\,{}_0F_1^{\tau_1}(-;\gamma+1;x) = 0, \tag{2.24}$$

we obtain the recurrence relation (2.23). □

Theorem 2.8. *The following recurrence relation for* $H_A^{\tau_1,\tau_2,\tau_3}$ *holds true:*

$$\begin{aligned}(\beta_2-\gamma_2+1)&H_A^{\tau_1,\tau_2,\tau_3}\left(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right)\\ &=\beta_2 H_A^{\tau_1,\tau_2,\tau_3}\left(\alpha,\beta_1,\beta_2+1;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right)\\ &-(\gamma_2-1)H_A^{\tau_1,\tau_2,\tau_3}\left(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2-1,\gamma_3;x_1,x_2,x_3\right).\end{aligned} \tag{2.25}$$

Proof. The series on the right side of (2.25) are:

$$\begin{aligned}&\beta_2 H_A^{\tau_1,\tau_2,\tau_3}\left(\alpha,\beta_1,\beta_2+1;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3\right)=\frac{(\beta_2)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)}{\Gamma(\beta_1)\,\Gamma(\beta_2)}\\ &\times\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p}\,\Gamma(\beta_1+\tau_1 m+n)\,\Gamma(\beta_2+1+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m+n)\,\Gamma(\gamma_2+\tau_2 n+\tau_3 p)}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\\ &=\frac{\Gamma(\gamma_1)\,\Gamma(\gamma_2)}{\Gamma(\beta_1)\,\Gamma(\beta_2)}\\ &\times\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p}\,\Gamma(\beta_1+\tau_1 m+n)\,\Gamma(\beta_2+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m+n)\,\Gamma(\gamma_2+\tau_2 n+\tau_3 p)}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\\ &\times(\beta_2+\tau_2 n+\tau_3 p)\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}&(\gamma_2-1)H_A^{\tau_1,\tau_2,\tau_3}\left(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2-1,\gamma_3;x_1,x_2,x_3\right)=\frac{(\gamma_2-1)\Gamma(\gamma_1)\,\Gamma(\gamma_2)}{\Gamma(\beta_1)\,\Gamma(\beta_2)}\\ &\times\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p}\,\Gamma(\beta_1+\tau_1 m+n)\,\Gamma(\beta_2+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m+n)\,\Gamma(\gamma_2-1+\tau_2 n+\tau_3 p)}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\\ &=\frac{\Gamma(\gamma_1)\,\Gamma(\gamma_2)}{\Gamma(\beta_1)\,\Gamma(\beta_2)}\\ &\times\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p}\,\Gamma(\beta_1+\tau_1 m+n)\,\Gamma(\beta_2+\tau_2 n+\tau_3 p)}{\Gamma(\gamma_1+\tau_1 m+n)\,\Gamma(\gamma_2-1+\tau_2 n+\tau_3 p)}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\\ &\times(\gamma_2-1+\tau_2 n+\tau_3 p).\end{aligned} \tag{2.27}$$

Substituting (2.26) and (2.27) in the series expressions into the right hand side of (2.25), we get the desired result of $(\beta_2-\gamma_2+1)H_A^{\tau_1,\tau_2,\tau_3}[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]$. □

The special case when τ_1,τ_2,$\tau_3=1$ in (2.23) and (2.25) give the recurrence relations for Srivastava's H_A triple hypergeometric function [23]

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