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Three Dimensional Quasi-Para-Sasakian Manifolds Satisfying Certain Curvature Conditions

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Abstract
The object of the present paper is to study some classes of 3-dimensional quasi-para-Sasakian manifolds with β =const. We investigated 3-dimensional quasi-para-Sasakian manifolds with β =const. satisfying the curvature conditions $P.Q = 0$, $Q.P = 0$, $P.R = 0$, where P is the projective curvature tensor, Q is the Ricci operator and R is the Riemannian curvature tensor. Also, a 3-dimensional concircularly flat quasi-para-Sasakian manifold with β =const. is studied. Finally, an example of 3-dimensional proper quasi-para-Sasakian manifold with β =const. is given.

Keywords: *quasi-para-Sasakian manifold; Einstein manifold; concircularly flat; Projective curvature tensor.*

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1. Introduction

Almost paracontact metric structures are the natural odd-dimensional analogue to almost paraHermitian structures, just like almost contact metric structures correspond to the almost Hermitian ones. The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [10] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in one of Zamkovoy's papers [23]. Comparing with the huge literature in almost contact geometry, it seems that there are necessary new studies in almost paracontact geometry. Therefore, paracontact metric manifolds have been studied in recent years by many authors, emphasizing similarities and differences with respect to the most well known contact case. Interesting papers connecting these fields are, for example, [6], [5], [19], [23], and references therein.

Z. Olszak studied normal almost contact metric manifolds of dimension 3 [16]. He derived certain necessary and sufficient conditions for an almost contact metric structure on manifold to be normal and curvature properties of such structures and normal almost contact metric structures on a manifold of constant curvature were studied. Recently, J. Wełyczko studied curvature and torsion of Frenet-Legendre curves in 3-dimensional normal almost paracontact metric manifolds [20] and then normal almost paracontact metric manifolds were studied in [1], [11], [12]. In [14], the curvature conditions of Para-Sasakian manifolds were studied.

The notion of quasi-Sasakian manifolds, introduced by D. E. Blair in [3], unifies Sasakian and cosymplectic manifolds. By definition, a quasi-Sasakian manifold is a normal almost contact metric manifold whose fundamental 2-form $\Phi := g(\cdot, \phi\cdot)$ is closed. Quasi-Sasakian manifolds can be viewed as an odd-dimensional counterpart of Kaehler structures. These manifolds studied by several authors (e.g. [9], [15], [18]).

The projective curvature tensor is an important tensor from the differential geometric point of view. The projective curvature tensor P is invariant under a geodesic preserving transformation on a semi-Riemannian manifold. It is well known that P is not a generalized curvature tensor and hence it possesses different geometric properties than other generalized curvature tensors. If there exists a one-to-one correspondence between each

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coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the semi-Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat.

In [11], 3-dimensional ξ-projectively flat and ϕ-projectively flat normal almost paracontact metric manifolds are studied and in [12], cyclic parallel Ricci tensor, η-parallel Ricci tensor, Ricci-semisymmetry and locally ϕ-symmetry for normal almost paracontact metric manifolds are studied. But concircularly flatness of normal almost paracontact metric manifolds has not studied yet by anyone.

Due to an important topic, study of curvature properties in contact and paracontact geometry has become a very attractive field of research nowadays.

Motivated by the above considerations, we study 3-dimensional quasi-para-Sasakian manifolds with $\beta =$const. satisfying certain curvature conditions and concircularly flat 3-dimensional quasi-para-Sasakian manifolds with $\beta =$const. and we want to fill the gap of the non-existence the study of this kind of manifolds. So, we investigate the geometric effects of the manifolds when satisfies the curvature conditions $P.Q = 0$, $Q.P = 0$, $P.R = 0$.

The paper is organized in the following way.

Section 2 is preliminary section, where we recall the definition of almost paracontact metric manifolds and quasi-para-Sasakian manifolds.

In Section 3, we study 3-dimensional quasi-para-Sasakian manifolds with $\beta =$const. satisfying the curvature condition $P.Q = 0$. We proved that the square of the Ricci tensor S^2 is the linear combination of the Ricci tensor S and the metric tensor g.

In Section 4, 3-dimensional quasi-para-Sasakian manifolds with $\beta =$const. satisfying the curvature condition $Q.P = 0$ are studied. We found that the trace of square of the Ricci operator of a 3-dimensional quasi-para-Sasakian manifold is equal to $-2\beta^2$ times trace of the Ricci operator. In Section 5, we proved that if a 3-dimensional quasi-para-Sasakian manifold with $\beta =$const. satisfies the curvature condition $P.R = 0$, then the manifold is an Einstein manifold. Einstein manifolds play an important role in semi-Riemannian geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [2]. In Section 6, we showed that a 3-dimensional concircularly flat quasi-para-Sasakian manifold with $\beta =$const. has a scalar curvature $r = -6\beta^2$. Finally, an example of 3-dimensional proper quasi-para-Sasakian manifold with $\beta =$const. is given.

2. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold and ϕ is a $(1,1)$ tensor field, ξ is a vector field and η is a one-form on M. Then (ϕ, ξ, η) is called an *almost paracontact structure* on M if

(i) $\phi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1,$

(ii) the tensor field ϕ induces an almost paracomplex structure on the distribution $D = \ker \eta$, that is the eigendistributions $D^{\pm}$, corresponding to the eigenvalues ± 1, have equal dimensions, $\dim D^+ = \dim D^- = n$.

The manifold M is said to be an *almost paracontact manifold* if it is endowed with an almost paracontact structure [23].

Let M be an almost paracontact manifold. M will be called an *almost paracontact metric manifold* if it is additionally endowed with a pseudo-Riemannian metric g of a signature $(n+1, n)$, i.e.

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.1}$$

For such manifold, we have

$$\eta(X) = g(X, \xi),\ \phi(\xi) = 0,\ \eta \circ \phi = 0. \tag{2.2}$$

Moreover, we can define a skew-symmetric tensor field (a 2-form) Φ by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.3}$$

usually called *fundamental form*.

For an almost paracontact manifold, there exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a *ϕ-basis*.

On an almost paracontact manifold, one defines the $(1,2)$-tensor field $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi, \tag{2.4}$$

where $[\phi,\phi]$ is the *Nijenhuis torsion* of ϕ

$$[\phi,\phi](X,Y) = \phi^2 [X,Y] + [\phi X,\phi Y] - \phi [\phi X,Y] - \phi [X,\phi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be *normal* [23]. The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$

$$J(X,\lambda \frac{d}{dt}) = (\phi X + \lambda\xi, \eta(X)\frac{d}{dt}),$$

is integrable.

If $d\eta(X,Y) = g(X,\phi Y) = \Phi(X,Y)$, then (M,ϕ,ξ,η,g) is said to be *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}_\xi$, denotes the Lie derivative. It is known [23] that h anti-commutes with ϕ and satisfies $h\xi = 0$, tr$h$ =tr$h\phi = 0$ and $\nabla\xi = -\phi + \phi h$, where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M,g).

Moreover $h = 0$ if and only if ξ is Killing vector field. In this case (M,ϕ,ξ,η,g) is said to be a *K-paracontact manifold*. Similarly as in the class of almost contact metric manifolds [4], a normal almost paracontact metric manifold will be called *para-Sasakian* if $\Phi = d\eta$ [8]. The para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3. A paracontact metric manifold will be called *paracosymplectic* if $d\Phi = 0$, $d\eta = 0$ [6], more generally α-para-Kenmotsu if $d\Phi = 2\alpha\eta \wedge \Phi$, $d\eta = 0$, $\alpha = const. \neq 0$.

Definition 2.1. An almost paracontact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is called *quasi-para-Sasakian if the structure is normal and its fundamental* 2-form Φ is closed.

Proposition 2.1. *[20] For a 3-dimensional almost paracontact metric manifold M the following three conditions are mutually equivalent*

(a) M is normal,
(b) there exist functions α, β on M such that

$$(\nabla_X\phi)Y = \beta(g(X,Y)\xi - \eta(Y)X) + \alpha(g(\phi X,Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

(c) there exist functions α, β on M such that

$$\nabla_X\xi = \alpha(X - \eta(X)\xi) + \beta\phi X. \tag{2.6}$$

Corollary 2.1. *[20]For a normal almost paracontact metric structure (ϕ,ξ,η,g) on M, we have $\nabla_\xi\xi = 0$ and $d\eta = -\beta\Phi$. The functions α, β realizing (2.5) as well as (2.6) are given by*

$$2\alpha = \textit{ Trace } \{X \longrightarrow \nabla_X\xi\}, \;\; 2\beta = \textit{Trace } \{X \longrightarrow \phi\nabla_X\xi\}. \tag{2.7}$$

Proposition 2.2. *[20] For a 3-dimensional almost paracontact metric manifold M, the following three conditions are mutually equivalent*

(a) M is quasi-para-Sasakian,
(b)there exists a function β on M such that

$$(\nabla_X\phi)Y = \beta(g(X,Y)\xi - \eta(Y)X), \tag{2.8}$$

(c)there exists a function β on M such that

$$\nabla_X\xi = \beta\phi X. \tag{2.9}$$

A 3-dimensional normal almost paracontact metric manifold is

- quasi-para-Sasakian if and only if $\alpha = 0$ and β is certain function [8], [20], in particular para-Sasakian if $\beta = -1$ [20], [23],
- paracosymplectic if $\alpha = \beta = 0$ [6],
- α-para-Kenmotsu if $\alpha \neq 0$ and α is const. and $\beta = 0$ [13].

Namely, the class of para-Sasakian and paracosymplectic manifolds are contained in the class of quasi-para-Sasakian manifolds.

Theorem 2.1. *[11]Let (M, ϕ, ξ, η, g) be a 3-dimensional normal almost paracontact metric manifold. Then the following curvature identities hold*

$$\begin{aligned} & R(X,Y)Z \\ = \; & (2(\xi(\alpha)+\alpha^2+\beta^2)+\frac{1}{2}r)(g(Y,Z)X-g(X,Z)Y) \\ & -(\xi(\alpha)+3(\alpha^2+\beta^2)+\frac{1}{2}r)((g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ & +\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y)+(\phi Z(\beta)-Z(\alpha))(\eta(Y)X-\eta(X)Y) \\ & +(\phi Y(\beta)-Y(\alpha))(\eta(Z)X-g(X,Z)\xi) \\ & -(\phi X(\beta)-X(\alpha))(\eta(Z)Y-g(Y,Z)\xi) \\ & +(\phi grad\beta+grad\alpha)(\eta(Y)g(X,Z)-\eta(X)g(Y,Z)). \end{aligned} \tag{2.10}$$

$$\begin{aligned} S(Y,Z) = \; & -(\xi(\alpha)+\alpha^2+\beta^2+\frac{1}{2}r)g(\phi Y,\phi Z) \\ & +\eta(Z)(\phi Y(\beta)-Y(\alpha)) \\ & +\eta(Y)(\phi Z(\beta)-Z(\alpha))-2(\alpha^2+\beta^2)\eta(Y)\eta(Z), \end{aligned} \tag{2.11}$$

where R, S and r are resp. Riemannian curvature, Ricci tensor and scalar curvature of M.

If we take $\alpha=0$ and β is const. different from zero in the above theorem, we can give following result.

Theorem 2.2. *Let (M, ϕ, ξ, η, g) be a 3-dimensional quasi-para-Sasakian manifold. Then the following identities hold*

$$R(X,Y)\xi = \beta^2(\eta(X)Y-\eta(Y)X), \tag{2.12}$$
$$R(\xi,X)Y = \beta^2(\eta(Y)X-g(X,Y)\xi), \tag{2.13}$$
$$S(X,\xi) = -2\beta^2\eta(X), \tag{2.14}$$
$$Q\xi = -2\beta^2\xi, \tag{2.15}$$
$$\eta(R(X,Y)Z) = \beta^2(\eta(Y)g(X,Z)-\eta(X)g(Y,Z)) \tag{2.16}$$

where $g(QX,Y)=S(X,Y)$ and Q is the Ricci operator and S denotes the Ricci tensor of type $(0,2)$ on M.

3. Quasi-para-Sasakian manifolds satisfying $P.Q=0$

In this section, we will consider 3-dimensional quasi-para-Sasakian manifolds with β =const. which satisfies the condition $P.Q=0$. First we will recall some well-known results which we will use in our new results.

For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [17]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X-S(X,Z)Y] \tag{3.1}$$

for all $X,Y,Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor.

Proposition 3.1. *[7] Let A be symmetric $(0,2)$-tensor at point x of a semi Riemannian manifold (M,g), dim $M \geq 3$ and let T =$g \barwedge A$ be the Kulkarni-Nomizu product of g and A. Then the relation*

$$T.T = \alpha Q(g,T), \; \alpha \in \mathbb{R}$$

is satisfied at x if and only if the condition

$$A^2 = \alpha A + \lambda g, \; \alpha \in \mathbb{R}$$

holds at x.

Remark 3.1. The Kulkarni-Nomizu product $A \barwedge B$ is defined by [2] for symmetric $(0,2)$-tensor fields A and B on M by following

$$\begin{aligned} A \barwedge B(X_1,...,X_4) = \; & A(X_1,X_4)B(X_2,X_3)-A(X_1,X_3)B(X_2,X_4) \\ & +A(X_2,X_3)B(X_1,X_4)-A(X_2,X_4)B(X_1,X_3). \end{aligned}$$

Theorem 3.1. *If a 3-dimensional quasi-para-Sasakian manifold $(M^3, \phi, \xi, \eta, g)$ with β =const. satisfies the condition $P.Q = 0$, then the square of the Ricci tensor S^2 is the linear combination of the Ricci tensor S and the metric tensor g.*

Proof. Let us consider a 3-dimensional quasi-para-Sasakian manifold with β =const. which satisfies the condition $P.Q = 0$, namely

$$P(X, Y)QZ - Q(P(X, Y)Z) = 0, \tag{3.2}$$

for any vector fields X, Y and Z. Setting $Y = \xi$ in (3.2), we obtain

$$P(X, \xi)QZ - Q(P(X, \xi)Z) = 0. \tag{3.3}$$

Putting $Y = \xi$ and $Z = QZ$ in (3.1) and using (2.13), (2.14), we get

$$P(X, \xi)QZ = \beta^2 g(X, QZ)\xi + \frac{1}{2}S(X, QZ)\xi. \tag{3.4}$$

On the other hand, setting again $Y = \xi$ in (3.1), we have

$$Q(P(X, \xi)Z) = -2\beta^4 g(X, Z)\xi - \beta^2 S(X, Z)\xi. \tag{3.5}$$

By virtue of (3.4) and (3.5), we get from (3.3) that

$$\beta^2 g(X, QZ)\xi + \frac{1}{2}S(X, QZ)\xi + 2\beta^4 g(X, Z)\xi + \beta^2 S(X, Z)\xi = 0. \tag{3.6}$$

The equation

$$S^2(X, Z) = S(QX, Z) = -4\beta^2 S(X, Z) - 4\beta^4 g(X, Z)$$

follows from taking the inner product of (3.6) with ξ. So, we get the result. □

From Proposition *3.1* and Theorem *3.1*, we can give following corollary.

Corollary 3.1. *If a 3-dimensional quasi-para-Sasakian manifold $(M^3, \phi, \xi, \eta, g)$ with β =const. satisfies the condition $P.Q = 0$, then $T.T = \alpha Q(g, T)$, where $T = g \barwedge S$ and $\alpha = -4\beta^2, \lambda = -4\beta^4$.*

4. Quasi-para-Sasakian manifolds satisfying $Q.P = 0$

In this section, we will consider 3-dimensional quasi-para-Sasakian manifolds with β =const. which satisfies the condition $Q.P = 0$.

Theorem 4.1. *If a 3-dimensional quasi-para-Sasakian manifold $(M^3, \phi, \xi, \eta, g)$ with β =const. satisfies the condition $Q.P = 0$, then the trace of square of the Ricci operator of a quasi-para-Sasakian manifold is equal to $-2\beta^2$ times trace of the Ricci operator.*

Proof. Let us consider a 3-dimensional quasi-para-Sasakian manifold with β =const. which satisfies the condition $(Q.P)(X, Y)Z = 0$, namely

$$Q(P(X, Y)Z) - P(QX, Y)Z - P(X, QY)Z - P(X, Y)QZ = 0, \tag{4.1}$$

for any vector fields X, Y and Z.

By setting $Y = \xi$ in (4.1), we have

$$Q(P(X, \xi)Z) = P(QX, \xi)Z + P(X, Q\xi)Z + P(X, \xi)QZ = 0. \tag{4.2}$$

We calculated the left hand side of (4.2) in (3.5). Hence for calculating the right hand side of (4.2), we use (3.1). So, we can write

$$\begin{aligned}
-2\beta^4 g(X, Z)\xi - \beta^2 S(X, Z)\xi \quad = \quad & R(QX, \xi)Z - \frac{1}{2}(S(\xi, Z)QX - S(QX, Z)\xi) \\
& + R(X, Q\xi)Z - \frac{1}{2}(S(Q\xi, Z)X - S(X, Z)Q\xi) \\
& + R(X, \xi)QZ - \frac{1}{2}(S(\xi, QZ)X - S(X, QZ)\xi).
\end{aligned}$$

If we employ (2.13), (2.14) and (2.15) in the last equation, we obtain $-2\beta^2 S(X,Z)\xi = g(Q^2X,Z)\xi$. By taking the inner product with ξ, we get

$$-2\beta^2 g(QX,Z) = g(Q^2X,Z). \tag{4.3}$$

Replacing $X,\ Z$ by e_i in the last equation and taking summation over i (Let $\{e_1,e_2,e_3\}$ be an ϕ-basis of the tangent space at any point of the manifold), we find

$$\sum_{i=1}^{3} -2\beta^2 \varepsilon_i g(Qe_i,,e_i) = \sum_{i=1}^{3} \varepsilon_i g(Q^2e_i,,e_i). \tag{4.4}$$

From (4.4), one can easily get the requested result. □

5. Quasi-para-Sasakian manifolds satisfying $P.R = 0$

In this section, we will consider 3-dimensional quasi-para-Sasakian manifolds with β =const. which satisfies the condition $P.R = 0$.

Theorem 5.1. *If a 3-dimensional quasi-para-Sasakian manifold (M^3,ϕ,ξ,η,g) with β =const. satisfies the condition $P.R = 0$, then the manifold is Einstein manifold.*

Proof. Let us consider a 3-dimensional quasi-para-Sasakian manifold with β =const. satisfies $(P(X,Y).R)(U,V)W = 0$, namely

$$P(X,Y)R(U,V)W - R(P(X,Y)U,V)W - R(U,P(X,Y)V)W - R(U,V)P(X,Y)W = 0, \tag{5.1}$$

for any vector fields X,Y,U,V and W.

By setting ξ instead of X and U in (5.1), we have

$$-R(P(\xi,Y)\xi,V)W - R(\xi,P(\xi,Y)V)W - R(\xi,V)P(\xi,Y)W + P(\xi,Y)R(\xi,V)W = 0. \tag{5.2}$$

For calculating the first and second term of (5.2), we write $X = Z = \xi$ in (3.1) and after using (2.13), (2.14),we derive

$$R(P(\xi,Y)\xi,V)W = 0, \tag{5.3}$$
$$R(\xi,P(\xi,Y)V)W = 0. \tag{5.4}$$

For the third and fourth term of (5.2), using (3.1), (2.13) and (2.14), after some calculations, we get

$$R(\xi,V)P(\xi,Y)W = (\beta^4 g(Y,W) + \frac{1}{2}\beta^2 S(Y,W))(-V+\eta(V)\xi), \tag{5.5}$$
$$P(\xi,Y)R(\xi,V)W = -\beta^4\eta(W)g(Y,V)\xi - \frac{1}{2}\beta^2\eta(W)S(Y,V)\xi. \tag{5.6}$$

If we employ (5.3), (5.4), (5.5) and (5.6) in (5.2), we deduce that

$$\begin{aligned} &-\beta^4\eta(W)g(Y,V)\xi - \frac{1}{2}\beta^2\eta(W)S(Y,V)\xi + \beta^4 g(Y,W)V \\ &-\beta^4 g(Y,W)\eta(V)\xi + \frac{1}{2}\beta^2 S(Y,W)V - \frac{1}{2}\beta^2 S(Y,W)\eta(V)\xi \\ =\ & 0. \end{aligned} \tag{5.7}$$

After writing $W = \xi$ in the last equation, we have

$$\begin{aligned} &-\beta^4 g(Y,V)\xi - \frac{1}{2}\beta^2 S(Y,V)\xi + \beta^4\eta(Y)V \\ &-\beta^4\eta(Y)\eta(V)\xi + \frac{1}{2}\beta^2 S(Y,\xi)V - \frac{1}{2}\beta^2 S(Y,\xi)\eta(V)\xi \\ =\ & 0. \end{aligned} \tag{5.8}$$

If we take the inner product of (5.8) with ξ and use (2.14), we get

$$S(Y,V) = -2\beta^2 g(Y,V).$$

So, the proof of the theorem ends. □

6. Concircularly flat quasi-para-Sasakian manifolds

In this section, we will consider 3-dimensional concircularly flat quasi-para-Sasakian manifolds.

The concircular curvature tensor $\bar{C}$ of a $(2n+1)$-dimensional manifold is defined by

$$\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{6.1}$$

for all $X, Y, Z \in T(M)$, where R is the curvature tensor and $r = tr(S)$ is scalar curvature [21], [22]. For $n \geq 1$, M is coincular flat if and only if the well known coincular curvature tensor $\bar{C}$ vanishes.

Theorem 6.1. *A 3-dimensional concircularly flat quasi-para-Sasakian manifold $(M^3, \phi, \xi, \eta, g)$ has a scalar curvature*

$$r = -6\beta^2. \tag{6.2}$$

Proof. Let us suppose that 3-dimensional quasi-para-Sasakian manifold is concircularly flat. If we take the inner product of (6.1) with W, we get

$$g(R(X,Y)Z,W) = \frac{r}{6}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

By setting $W = X = \xi$ in the last equation, we obtain

$$\begin{aligned} g(R(\xi,Y)Z,\xi) &= g(R(Y,\xi)\xi,Z) \\ &= \frac{r}{6}(g(Y,Z) - \eta(Y)\eta(Z)). \end{aligned} \tag{6.3}$$

Comparing (6.3) with (2.12), we have

$$-\beta^2(g(Y,Z) - \eta(Y)\eta(Z)) = \frac{r}{6}(g(Y,Z) - \eta(Y)\eta(Z)). \tag{6.4}$$

Considering the ϕ-basis and and putting $Y = Z = e_i$ in (6.4), we get

$$\begin{aligned} \sum_{i=1}^{3}\Big(-\beta^2\varepsilon_i(g(e_i,e_i) - \eta(e_i)\eta(e_i)) &= \frac{r}{6}\varepsilon_i(g(e_i,e_i) - \eta(e_i)\eta(e_i))\Big) \\ -2\beta^2 &= \frac{2r}{6} \end{aligned}$$

which completes the proof of the theorem. □

7. Example

Now, we will give an example of 3-dimensional proper quasi-para-Sasakian manifold.

Example 7.1. We consider the 3-dimensional manifold

$$M^3 = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$$

and the vector fields

$$\phi e_2 = e_1 = 4y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \quad \phi e_1 = e_2 = \frac{\partial}{\partial y}, \quad \xi = e_3 = \frac{\partial}{\partial x}.$$

The 1-form $\eta = dx - \frac{4y}{z}dz$ defines an almost paracontact structure on M with characteristic vector field $\xi = \frac{\partial}{\partial x}$. Let g, ϕ be the semi-Riemannian metric ($g(e_1,e_1) = -g(e_2,e_2) = g(\xi,\xi) = 1$) and the $(1,1)$-tensor field respectively given by

$$g = \begin{pmatrix} 1 & 0 & -\frac{2y}{z} \\ 0 & -1 & 0 \\ -\frac{2y}{z} & 0 & \frac{1+28y^2}{z^2} \end{pmatrix},$$

$$\phi = \begin{pmatrix} 0 & 4y & 0 \\ 0 & 0 & \frac{1}{z} \\ 0 & z & 0 \end{pmatrix},$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

Using (2.9) we have

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0, & \nabla_{e_2}e_1 = -2\xi, & \nabla_{\xi}e_1 = 2e_2, \\ \nabla_{e_1}e_2 = 2\xi, & \nabla_{e_2}e_2 = 0, & \nabla_{\xi}e_2 = 2e_1, \\ \nabla_{e_1}\xi = 2e_2, & \nabla_{e_2}\xi = 2e_1, & \nabla_{\xi}\xi = 0. \end{array}$$

Hence the manifold is a 3-dimensional quasi-para-Sasakian manifold with β is constant function. Using the above equations, we obtain

$$\begin{array}{lll} R(e_1,e_2)\xi = 0, & R(e_2,\xi)\xi = -4e_2, & R(e_1,\xi)\xi = -4e_1, \\ R(e_1,e_2)e_2 = -12e_1, & R(e_2,\xi)e_2 = -4\xi, & R(e_1,\xi)e_2 = 0, \\ R(e_1,e_2)e_1 = -12e_2, & R(e_2,\xi)e_1 = 0, & R(e_1,\xi)e_1 = 4\xi. \end{array} \tag{7.1}$$

Using (7.1), we have constant scalar curvature as follows, $r = S(e_1,e_1) - S(e_2,e_2) + S(\xi,\xi) = 8$. We want to remark that this example is neither the paracosymplectic manifold nor the para-Sasakian manifold example.

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