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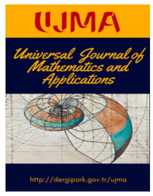

Sasakian Statistical Manifolds with Semi-Symmetric Metric Connection

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Abstract

In the present paper, firstly we express the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the torsion-free connection ∇ and obtain the relation between the curvature tensors $\tilde{R}$ of $\tilde{\nabla}$ and R of ∇. After, we obtain these relations for $\tilde{\nabla}$ and the dual connection ∇^*. Also, we give the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^* on Sasakian statistical manifolds, respectively. We obtain the relations between the Ricci tensor (and scalar curvature) of semi-symmetric metric connection $\tilde{\nabla}$ and the Ricci tensors (and scalar curvatures) of the connections ∇ and ∇^*. Finally, we construct an example of a 3-dimensional Sasakian manifold with statistical structure admitting the semi-symmetric metric connection in order to verify our results.

1. Introduction

The theory of statistical manifolds, the so called information geometry, has started with a paper of Rao in 1945 [1] and after that, the information geometry, which is typically deals with the study of various geometric structures on a statistical manifold, has begun as a study of the geometric structures possessed by a statistical model of probability distributions. Nowadays, the information geometry has an important application area, such as, information theory, stochastic processes, dynamical systems and times series, statistical physics, quantum systems and the mathematical theory of neural networks [2], [3].
In 1985, the notion of dual connection (or conjugate connection) in affine geometry, has been first introduced into statistics by Amari [4]. A statistical model equipped with a Riemannian metric together with a pair of dual affine connections is called a *statistical manifold*. For more information about statistical manifolds and information geometry, we refer to [5], [6], [7], [8], [9], [10] and etc.
Also, if Φ is a tensor field of type $(1,1)$, η is a 1-form and ξ is a vector field on a $(2n+1)$-dimensional differentiable manifold M, then almost contact structure (Φ,η,ξ) which is related to almost complex structures and satisfies the conditions $\Phi^2=-I+\eta\otimes\xi$, $\eta(\xi)=1$ has been determined by Sasaki in 1960 [11]. With the aid of this definition, different types of this manifold such as Sasakian manifold, Kenmotsu manifold, trans-Sasakian manifold and etc. have been defined and studied by many mathematicians [11], [12], [13] and etc.
According to these notions, the differential geometry of statistical manifolds are being studying by geometers by adding different geometric structures to these manifolds. For instance, in [14] quaternionic Kähler-like statistical manifold have been studied and in [15], the authors have introduced the notion of Sasakian statistical structure and obtained the condition for a real hypersurface in a holomorphic statistical manifold to admit such a structure. In [16], the notion of a Kenmotsu statistical manifold is introduced and they have showed that, a Kenmotsu statistical manifold of constant Φ-sectional curvature is constructed from a special Kahler manifold, which is an important example of holomorphic statistical manifold. Also, the projection of a dualistic structure has been defined on a twisted product manifold induces dualistic structures on the base and the fiber manifolds, and conversely in [3].
This paper is organized as follows:
In Section 2, we recall some basic notions about statistical structures and semi-symmetric metric connection. After Preliminaries, by expressing the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the torsion-free connection ∇, we obtain the relation between the curvature tensors $\tilde{R}$ of $\tilde{\nabla}$ and R of ∇ in Section 3 and then, we obtain these relations for $\tilde{\nabla}$ and the dual connection ∇^*. In Section 4,

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we give the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^* on Sasakian statistical manifolds, respectively. Also, we obtain the relations between the Ricci tensor (and scalar curvature) of semi-symmetric metric connection $\tilde{\nabla}$ and the Ricci tensors (and scalar curvatures) of the connections ∇ and ∇^*. At the end of this section, we construct an example of a 3-dimensional Sasakian manifold with statistical structure admitting the semi-symmetric metric connection in order to verify our results.

2. Preliminaries

In this section, we recall some notions about statistical structures and semi-symmetric metric connection, respectively. Throughout this paper, we assume that M is a $(2n+1)$-dimensional manifold, g is a Riemannian metric, $\hat{\nabla}$ is the Levi-Civita connection associated with g and $\Gamma(TM^{(p,q)})$ means the set of tensor fields of type (p,q) on M.
A pair (∇, g) is called a *statistical structure* on M, if ∇ is torsion-free and

$$(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z),\ \forall X,Y,Z \in \Gamma(TM) \tag{2.1}$$

holds, where the equation (2.1) is generally called *Codazzi equation*. In this case, (M,∇,g) is called a *statistical manifold.*
Let (∇, g) be a statistical structure on M. Then the connection ∇^* which is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$

is called *conjugate or dual connection* of ∇ with respect to g. If (∇, g) is a statistical structure on M, then (∇^*, g) is a statistical structure on M, too.
For a statistical structure (∇, g), one can define the *difference tensor field* $K \in \Gamma(TM^{(1,2)})$ as

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y,\ \forall X,Y \in \Gamma(TM), \tag{2.2}$$

where K satisfies

$$\begin{aligned} K(X,Y) &= K(Y,X), \\ g(K(X,Y),Z) &= g(Y,K(X,Z)). \end{aligned}$$

Furthermore, we have

$$K = \hat{\nabla} - \nabla^* = \frac{1}{2}(\nabla - \nabla^*). \tag{2.3}$$

For a more detailed treatment, we refer to [7], [15] and [17].
On the other hand in [18], Hayden introduced a metric connection with a non-zero torsion on a Riemannian manifold and this connection is called a Hayden connection. In [19], the authors have introduced the semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = w(Y)X - w(X)Y,$$

where the 1-form w is defined by

$$w(X) = g(X,U),$$

for vector fields X, Y and U on M. Also, a semi-symmetric connection $\tilde{\nabla}$ is called a semi-symmetric metric connection if it further satisfies $\tilde{\nabla}g = 0$. If $\hat{\nabla}$ is the Levi-Civita connection of a Riemannian manifold M, then the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and $\hat{\nabla}$ is

$$\tilde{\nabla}_X Y = \hat{\nabla}_X Y + w(Y)X - g(X,Y)U, \tag{2.4}$$

where $w(Y) = g(Y,U)$.

3. Curvature of semi-symetric metric connection on statistical manifolds

In this section, firstly we'll express the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the torsion-free connection ∇ and obtain the relation between the curvature tensors $\tilde{R}$ of $\tilde{\nabla}$ and R of ∇. After, we'll obtain these relations for $\tilde{\nabla}$ and the dual connection ∇^*.
Let M be an n-dimensional Riemannian manifold and (∇, g) be a statistical structure on M.
From (2.2) and (2.4), we obtain the relation between the connections $\tilde{\nabla}$ and ∇ as

$$\tilde{\nabla}_X Y = \nabla_X Y + w(Y)X - g(X,Y)U - K(X,Y). \tag{3.1}$$

The Riemannian curvature tensor $\tilde{R}$ of M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z, \tag{3.2}$$

for all $X,Y,Z \in \Gamma(TM)$. From (3.1), we have

$$\begin{aligned} \tilde{\nabla}_X \tilde{\nabla}_Y Z = {} & \nabla_X \nabla_Y Z + w(\nabla_Y Z)X + w(\nabla_X Z)Y - w(K(X,Z))Y - w(K(Y,Z))X + w(Z)\nabla_X Y - w(Z)K(X,Y) + w(Y)w(Z)X \\ & - g(X,Y)w(Z)U - g(Y,Z)w(U)X + g(Y,Z)w(X)U \\ & - g(X,\nabla_Y Z)U + g(Z,\nabla_X U)Y - g(\nabla_X Y,Z)U - g(Y,\nabla_X Z)U - g(Y,Z)\nabla_X U \\ & - g(Z,K(X,U))Y + g(K(X,Y),Z)U + g(Y,K(X,Z))U + g(X,K(Y,Z))U \\ & + g(Y,Z)K(X,U) - K(X,\nabla_Y Z) - \nabla_X K(Y,Z) + K(X,K(Y,Z)) \end{aligned} \tag{3.3}$$

and

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + w(Z)\nabla_X Y - w(Z)\nabla_Y X - g(\nabla_X Y, Z)U + g(\nabla_Y X, Z)U - K(\nabla_X Y, Z) + K(\nabla_Y X, Z). \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), we obtain the Riemannian curvature tensor $\tilde{R}$ of M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ as

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \{w(X)U - w(U)X - \nabla_X U + K(X,U)\}g(Y,Z) - \{w(Y)U - w(U)Y - \nabla_Y U + K(Y,U)\}g(X,Z)\\ &\quad - g(w(X)U - \nabla_X U + K(X,U), Z)Y + g(w(Y)U - \nabla_Y U + K(Y,U), Z)X\\ &\quad - (\nabla_X K)(Y,Z) + (\nabla_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)).\end{aligned}$$

Here, R is the Riemannian curvature tensor of M with respect to the torsion-free connection ∇ which is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.
Similarly, from (2.3) and (2.4), we obtain the relation between the connections $\tilde{\nabla}$ and ∇^* as

$$\tilde{\nabla}_X Y = \nabla^*_X Y + w(Y)X - g(X,Y)U + K(X,Y). \tag{3.5}$$

for all $X, Y, Z \in \Gamma(TM)$. From (3.5), we have

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z &= \nabla^*_X\nabla^*_Y Z + w(\nabla^*_Y Z)X + w(\nabla^*_X Z)Y + w(K(X,Z))Y + w(K(Y,Z))X + w(Z)\nabla^*_X Y + w(Z)K(X,Y) + w(Y)w(Z)X\\ &\quad - g(X,Y)w(Z)U - g(Y,Z)w(U)X + g(Y,Z)w(X)U\\ &\quad - g(X, \nabla^*_Y Z)U + g(Z, \nabla^*_X U)Y - g(\nabla^*_X Y, Z)U - g(Y, \nabla^*_X Z)U - g(Y,Z)\nabla^*_X U\\ &\quad + g(Z, K(X,U))Y - g(K(X,Y),Z)U - g(Y, K(X,Z))U - g(X, K(Y,Z))U\\ &\quad - g(Y,Z)K(X,U) + K(X, \nabla^*_Y Z) + \nabla^*_X K(Y,Z) + K(X, K(Y,Z))\end{aligned} \tag{3.6}$$

and

$$\tilde{\nabla}_{[X,Y]}Z = \nabla^*_{[X,Y]}Z + w(Z)\nabla^*_X Y - w(Z)\nabla^*_Y X - g(\nabla^*_X Y, Z)U + g(\nabla^*_Y X, Z)U + K(\nabla^*_X Y, Z) - K(\nabla^*_Y X, Z). \tag{3.7}$$

Using (3.6) and (3.7) in (3.2), we obtain the Riemannian curvature tensor $\tilde{R}$ of M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ as

$$\begin{aligned}\tilde{R}(X,Y)Z &= R^*(X,Y)Z + \{w(X)U - w(U)X - \nabla^*_X U - K(X,U)\}g(Y,Z) - \{w(Y)U - w(U)Y - \nabla^*_Y U - K(Y,U)\}g(X,Z)\\ &\quad - g(w(X)U - \nabla^*_X U - K(X,U), Z)Y + g(w(Y)U - \nabla^*_Y U - K(Y,U), Z)X\\ &\quad + (\nabla^*_X K)(Y,Z) - (\nabla^*_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)).\end{aligned}$$

Here, R^* is the Riemannian curvature tensor of M with respect to the dual connection ∇^* which is defined by $R^*(X,Y)Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z$.
Hence, we can give the following Proposition:

Proposition 3.1. *Let (∇, g) be a statistical structure on a Riemannian manifold M. Then, the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^*, respectively, are*

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \{w(X)U - w(U)X - \nabla_X U + K(X,U)\}g(Y,Z) - \{w(Y)U - w(U)Y - \nabla_Y U + K(Y,U)\}g(X,Z)\\ &\quad - g(w(X)U - \nabla_X U + K(X,U), Z)Y + g(w(Y)U - \nabla_Y U + K(Y,U), Z)X\\ &\quad - (\nabla_X K)(Y,Z) + (\nabla_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)) \qquad (3.8)\\ &= R^*(X,Y)Z + \{w(X)U - w(U)X - \nabla^*_X U - K(X,U)\}g(Y,Z) - \{w(Y)U - w(U)Y - \nabla^*_Y U - K(Y,U)\}g(X,Z)\\ &\quad - g(w(X)U - \nabla^*_X U - K(X,U), Z)Y + g(w(Y)U - \nabla^*_Y U - K(Y,U), Z)X\\ &\quad + (\nabla^*_X K)(Y,Z) - (\nabla^*_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)), \qquad (3.9)\end{aligned}$$

for all vector fields X, Y and Z on M.

4. Semi-symmetric metric connection on Sasakian statistical manifolds

A $(2n+1)$-dimensional differentiable manifold M is said to admit an *almost contact Riemannian structure* (Φ, η, ξ, g), where Φ is a (1,1)-tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\begin{aligned}&\Phi\xi = 0,\ \eta(\xi) = 1,\ g(\xi, X) = \eta(X),\\ &\Phi^2 X = -X + \eta(X)\xi,\\ &g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y),\end{aligned} \tag{4.1}$$

for any vector fields X, Y on M . In addition, if (Φ, η, ξ, g) satisfy the equations

$$\begin{aligned}&d\eta = 0,\ \hat{\nabla}_X\xi = \Phi X,\\ &(\hat{\nabla}_X\Phi)Y = \eta(Y)X - g(X,Y)\xi,\end{aligned} \tag{4.2}$$

then M is called a *Sasakian manifold* (for detail, see [15] and [20]).
Also in [15], the authors have defined the notion of Sasakian statistical structure and have obtained the necessary and sufficient conditions for a statistical structure on an almost contact metric manifold to be a Sasakian statistical structure as follows:

Definition 4.1. *A quadruple (∇, g, Φ, ξ) is called a Sasakian statistical structure on M, if (∇, g) is a statistical structure, (g, Φ, ξ) is a Sasakian structure on M and the formula $K(X, \Phi Y) + \Phi K(X,Y) = 0$ holds for any vector fields X and Y on M.*

Theorem 4.2. *Let (∇, g) be a statistical structure and (g, Φ, ξ) an almost contact metric structure on M. (∇, g, Φ, ξ) is a Sasakian statistical structure if and only if the following formulas hold:*

$$\nabla_X \Phi Y - \Phi \nabla^*_X Y = g(Y,\xi)X - g(Y,X)\xi \tag{4.3}$$

$$\nabla_X \xi = \Phi X + g(\nabla_X \xi, \xi)\xi. \tag{4.4}$$

So, we can give the following Example:

Example 4.3. *Let (Φ, η, ξ, g) be an almost contact Riemannian structure on M. Set the connection $\check{\nabla}$ as*

$$\check{\nabla}_X Y = \hat{\nabla}_X Y + 3\eta(X)\eta(Y)\xi, \tag{4.5}$$

for any $X, Y \in \Gamma(TM)$. Then, $\check{\nabla}$ is torsion-free and satisfies the Codazzi equation (2.1). So, $(\check{\nabla}, g)$ is a statistical structure on the almost contact Riemannian manifold (M, Φ, η, ξ, g).
Also, from (2.2), (2.3) and (4.5) we have $K(X,Y) = 3\eta(X)\eta(Y)\xi$ and $\check{\nabla}^*_X Y = \hat{\nabla}_X Y - 3\eta(X)\eta(Y)\xi$. So, the equations (4.3) and (4.4) hold for the connection $\check{\nabla}$. Hence $(\check{\nabla}, g, \Phi, \eta, \xi)$ is a Sasakian statistical structure on M.

Now, firstly we'll give the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^* on Sasakian statistical manifolds with the aid of Proposition 3.1. For this, we use the equation

$$\tilde{\nabla}_X Y = \hat{\nabla}_X Y + \eta(Y)X - g(X,Y)\xi, \tag{4.6}$$

which has been obtained by Yano [21] on almost contact manifolds. Here, $\tilde{\nabla}$ is the semi-symmetric metric connection and $\hat{\nabla}$ is the Levi-Civita connection on (M,g), η is a 1-form and ξ is a vector field defined by $w(X) = g(X,\xi)$. If we write η instead of w and ξ instead of U in the equations (3.8) and (3.9) and use (2.2), (2.3), (4.1) and (4.2), then we have the following Theorem:

Theorem 4.4. *Let $(M, \nabla, g, \Phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors R and R^* of the connections ∇ and ∇^*, respectively, are*

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + \{\Phi^2 X - \Phi X\}g(Y,Z) - \{\Phi^2 Y - \Phi Y\}g(X,Z) \\ &\quad + g(\Phi X, Z)Y - g(\Phi Y, Z)X - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X \\ &\quad - (\nabla_X K)(Y,Z) + (\nabla_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)) \qquad (4.7) \\ &= R^*(X,Y)Z + \{\Phi^2 X - \Phi X\}g(Y,Z) - \{\Phi^2 Y - \Phi Y\}g(X,Z) \\ &\quad + g(\Phi X, Z)Y - g(\Phi Y, Z)X - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X \\ &\quad + (\nabla^*_X K)(Y,Z) - (\nabla^*_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)), \qquad (4.8) \end{aligned}$$

for all vector fields X, Y and Z on M.

Corollary 4.5. *Let $(M, \nabla, g, \Phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, we have*

$$\begin{aligned} \tilde{R}(X,Y)\xi &= R(X,Y)\xi + \eta(X)\Phi Y - \eta(Y)\Phi X - (\nabla_X K)(Y,\xi) + (\nabla_Y K)(X,\xi) \qquad (4.9) \\ &= R^*(X,Y)\xi + \eta(X)\Phi Y - \eta(Y)\Phi X + (\nabla^*_X K)(Y,\xi) - (\nabla^*_Y K)(X,\xi) \qquad (4.10) \end{aligned}$$

and

$$\begin{aligned} \tilde{R}(\xi,X)Y &= R(\xi,X)Y + \eta(Y)\Phi X - g(\Phi X, Y)\xi - (\nabla_\xi K)(X,Y) + (\nabla_X K)(\xi,Y) + K(\xi, K(X,Y)) - K(X, K(\xi,Y)) \qquad (4.11) \\ &= R^*(\xi,X)Y + \eta(Y)\Phi X - g(\Phi X, Y)\xi + (\nabla^*_\xi K)(X,Y) - (\nabla^*_X K)(\xi,Y) + K(\xi, K(X,Y)) - K(X, K(\xi,Y)), \qquad (4.12) \end{aligned}$$

for all vector fields X and Y on M.

Proof. We know that [15], on a Sasakian statistical manifold, the equation $\nabla_X \xi = \Phi X + \eta(\nabla_X \xi)\xi$ holds. So, from (2.2) we get $K(X,\xi) = \eta(\nabla_X \xi)\xi$. Using this, we have $K(X, K(Y,\xi)) = \eta(\nabla_X \xi)\eta(\nabla_Y \xi)\xi$ and so, we obtain that

$$K(X, K(Y,\xi)) = K(Y, K(X,\xi)). \tag{4.13}$$

Using (4.1) and (4.13) in (4.7) and (4.8), we reach the equations (4.9)-(4.12) and the proof completes. □

Now, let us give the relations between the Ricci tensor $\tilde{S}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the Ricci tensors S and S^* of the connections ∇ and ∇^*, respectively.

Theorem 4.6. *Let $(M, \nabla, g, \Phi, \eta, \xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, the relations between the Ricci tensors of semi-symmetric metric connection $\tilde{\nabla}$ and the connections ∇ and ∇^*, respectively, are*

$$\begin{aligned} \tilde{S}(X,Y) &= S(X,Y) - (2n-1)g(\Phi X, \Phi Y + Y) \qquad (4.14) \\ &\quad - \sum_{i=1}^{2n+1} g((\nabla_X K)(e_i,e_i) - (\nabla_{e_i} K)(X,e_i) - K(X, K(e_i,e_i)) + K(e_i, K(X,e_i)), Y) \\ &= S^*(X,Y) - (2n-1)g(\Phi X, \Phi Y + Y) \qquad (4.15) \\ &\quad + \sum_{i=1}^{2n+1} g((\nabla^*_X K)(e_i,e_i) - (\nabla^*_{e_i} K)(X,e_i) + K(X, K(e_i,e_i)) - K(e_i, K(X,e_i)), Y), \end{aligned}$$

for all vector fields X and Y on M.

Proof. Using (4.1) and (4.7) in the equation $\tilde{S}(X,Y) = \sum_{i=1}^{2n+1} g(\tilde{R}(X,e_i)e_i,Y)$, we get (4.14). Similarly, using (4.1) and (4.8) in the equation $\tilde{S}(X,Y) = \sum_{i=1}^{2n+1} g(\tilde{R}(X,e_i)e_i,Y)$, we get (4.15). □

Here, let us give the relations between the scalar curvature $\tilde{\tau}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the scalar curvatures τ and τ^* of the connections ∇ and ∇^*.

Theorem 4.7. *Let $(M,\nabla,g,\Phi,\eta,\xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, the relations between the scalar curvature of semi-symmetric metric connection $\tilde{\nabla}$ and the connections ∇ and ∇^*, respectively, are*

$$\tilde{\tau} = \tau + 2n - 4n^2 - \sum_{i,j=1}^{2n+1} g((\nabla_{e_j}K)(e_i,e_i) - (\nabla_{e_i}K)(e_j,e_i) - K(e_j,K(e_i,e_i)) + K(e_i,K(e_j,e_i)),e_j) \tag{4.16}$$

$$= \tau^* + 2n - 4n^2 + \sum_{i,j=1}^{2n+1} g((\nabla^*_{e_j}K)(e_i,e_i) - (\nabla^*_{e_i}K)(e_j,e_i) + K(e_j,K(e_i,e_i)) - K(e_i,K(e_j,e_i)),e_j). \tag{4.17}$$

Proof. Using (4.14) and (4.15) in the equation $\tilde{\tau} = \sum_{j=1}^{2n+1} \tilde{S}(e_j,e_j)$, we get (4.16) and (4.17), respectively. □

Finally, let us construct an example of a 3-dimensional Sasakian manifold with statistical structure admitting the semi-symmetric metric connection in order to verify our results.

Example 4.8. *Let us consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ z \neq 0\}$, where (x,y,z) are the standart coordinates in $\mathbb{R}^3$. We choose the vector fields $\{e_1,e_2,e_3\}$ as*

$$e_1 = \frac{\partial}{\partial x},\ e_2 = -x\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right) + \frac{\partial}{\partial y},\ e_3 = \frac{1}{2}\frac{\partial}{\partial z},$$

which are linearly independent at each point of M.
Let g be the Riemannian metric defined by $g(e_i,e_j) = 0$, $i \neq j$, $i,j = 1,2,3$ and $g(e_k,e_k) = 1$, $k = 1,2,3$.
Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M.
Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2,\ \phi e_2 = e_1,\ \phi e_3 = 0. \tag{4.18}$$

Using the linearity of ϕ and g, we have $\eta(e_3) = 1$, $\phi^2 Z = -Z + \eta(Z)e_3$ and $g(\phi Z,\phi U) = g(Z,U) - \eta(Z)\eta(U)$, for any $U,Z \in \chi(M)$. Thus, for $e_3 = \xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on M.
Now, we have

$$[e_1,e_2] = -e_1 + 2e_3,\ [e_1,e_3] = 0,\ [e_2,e_3] = 0.$$

The Levi-Civita connection $\hat{\nabla}$ of the metric tensor g is given by Koszul's formula which is defined as

$$2g(\hat{\nabla}_X Y,Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).$$

Taking $e_3 = \xi$ and using Koszul's formula, we get the following

$$\begin{aligned} &\hat{\nabla}_{e_1}e_1 = e_2,\ \hat{\nabla}_{e_1}e_2 = -e_1 + e_3,\ \hat{\nabla}_{e_1}e_3 = -e_2,\\ &\hat{\nabla}_{e_2}e_1 = -e_3,\ \hat{\nabla}_{e_2}e_2 = 0,\ \hat{\nabla}_{e_2}e_3 = e_1,\\ &\hat{\nabla}_{e_3}e_1 = -e_2,\ \hat{\nabla}_{e_3}e_2 = e_1,\ \hat{\nabla}_{e_3}e_3 = 0. \end{aligned} \tag{4.19}$$

From the above, it can be easily seen that (ϕ,ξ,η,g) is a Sasakian structure on M. Consequently, (M,ϕ,ξ,η,g) is a 3-dimensional Sasakian manifold.
Now, the components of the curvature tensors, Ricci tensors and scalar curvature with respect to the Levi-Civita connection $\hat{\nabla}$ are obtained by

$$\begin{aligned} &\hat{R}(e_1,e_2)e_1 = 4e_2,\ \hat{R}(e_1,e_2)e_2 = -4e_1,\ \hat{R}(e_1,e_2)e_3 = 0,\\ &\hat{R}(e_1,e_3)e_1 = -e_3,\ \hat{R}(e_1,e_3)e_2 = 0,\ \hat{R}(e_1,e_3)e_3 = e_1,\\ &\hat{R}(e_2,e_3)e_1 = 0,\ \hat{R}(e_2,e_3)e_2 = -e_3,\ \hat{R}(e_2,e_3)e_3 = e_2, \end{aligned}$$

$$\begin{aligned} &\hat{S}(e_1,e_1) = -3,\ \hat{S}(e_1,e_2) = 0,\ \hat{S}(e_1,e_3) = 0,\\ &\hat{S}(e_2,e_1) = 0,\ \hat{S}(e_2,e_2) = -3,\ \hat{S}(e_2,e_3) = 0,\\ &\hat{S}(e_3,e_1) = 0,\ \hat{S}(e_3,e_2) = 0,\ \hat{S}(e_3,e_3) = 2 \end{aligned}$$

and

$$\hat{\tau} = -4,$$

respectively.
Here, let us add a statistical structure to this Sasakian manifold. From (2.2) and (4.19), we have

$$\begin{aligned}
&\nabla_{e_1}e_1=e_2+K(e_1,e_1),\ \nabla_{e_1}e_2=-e_1+e_3+K(e_1,e_2),\ \nabla_{e_1}e_3=-e_2+K(e_1,e_3),\\
&\nabla_{e_2}e_1=-e_3+K(e_2,e_1),\ \nabla_{e_2}e_2=K(e_2,e_2),\ \nabla_{e_2}e_3=e_1+K(e_2,e_3),\\
&\nabla_{e_3}e_1=-e_2+K(e_3,e_1),\ \nabla_{e_3}e_2=e_1+K(e_3,e_2),\ \nabla_{e_3}e_3=K(e_3,e_3).
\end{aligned}\tag{4.20}$$

So, the components of the curvature tensors, Ricci tensors and scalar curvature with respect to the torsion-free connection ∇ are obtained by

$$\begin{aligned}
&R(e_1,e_2)e_1=4e_2+K(e_1,e_1)-3K(e_1,e_3)-K(e_2,e_2)+\nabla_{e_1}K(e_2,e_1)-\nabla_{e_2}K(e_1,e_1),\\
&R(e_1,e_2)e_2=-4e_1+2K(e_1,e_2)-3K(e_2,e_3)+\nabla_{e_1}K(e_2,e_2)-\nabla_{e_2}K(e_1,e_2),\\
&R(e_1,e_2)e_3=K(e_1,e_1)+K(e_1,e_3)+K(e_2,e_2)-2K(e_3,e_3)+\nabla_{e_1}K(e_2,e_3)-\nabla_{e_2}K(e_1,e_3),\\
&R(e_1,e_3)e_1=-e_3-K(e_1,e_2)-K(e_2,e_3)+\nabla_{e_1}K(e_3,e_1)-\nabla_{e_3}K(e_1,e_1),\\
&R(e_1,e_3)e_2=K(e_1,e_1)+K(e_1,e_3)-K(e_3,e_3)+\nabla_{e_1}K(e_3,e_2)-\nabla_{e_3}K(e_1,e_2),\\
&R(e_1,e_3)e_3=e_1+K(e_2,e_3)+\nabla_{e_1}K(e_3,e_3)-\nabla_{e_3}K(e_1,e_3),\\
&R(e_2,e_3)e_1=-K(e_2,e_2)+K(e_3,e_3)+\nabla_{e_2}K(e_3,e_1)-\nabla_{e_3}K(e_2,e_1),\\
&R(e_2,e_3)e_2=-e_3+K(e_2,e_1)+\nabla_{e_2}K(e_3,e_2)-\nabla_{e_3}K(e_2,e_2),\\
&R(e_2,e_3)e_3=e_2-K(e_3,e_1)+\nabla_{e_2}K(e_3,e_3)-\nabla_{e_3}K(e_2,e_3),
\end{aligned}\tag{4.21}$$

$$\begin{aligned}
&S(e_1,e_1)=-3+g(2K(e_1,e_2)-2K(e_2,e_3)+\nabla_{e_1}K(e_2,e_2)-\nabla_{e_2}K(e_1,e_2)+\nabla_{e_1}K(e_3,e_3)-\nabla_{e_3}K(e_1,e_3),e_1),\\
&S(e_1,e_2)=g(2K(e_1,e_2)-2K(e_2,e_3)+\nabla_{e_1}K(e_2,e_2)-\nabla_{e_2}K(e_1,e_2)+\nabla_{e_1}K(e_3,e_3)-\nabla_{e_3}K(e_1,e_3),e_2),\\
&S(e_1,e_3)=g(2K(e_1,e_2)-2K(e_2,e_3)+\nabla_{e_1}K(e_2,e_2)-\nabla_{e_2}K(e_1,e_2)+\nabla_{e_1}K(e_3,e_3)-\nabla_{e_3}K(e_1,e_3),e_3),\\
&S(e_2,e_1)=g(-K(e_1,e_1)+2K(e_1,e_3)+K(e_2,e_2)-\nabla_{e_1}K(e_2,e_1)+\nabla_{e_2}K(e_1,e_1)+\nabla_{e_2}K(e_3,e_3)-\nabla_{e_3}K(e_2,e_3),e_1),\\
&S(e_2,e_2)=-3+g(-K(e_1,e_1)+2K(e_1,e_3)+K(e_2,e_2)-\nabla_{e_1}K(e_2,e_1)+\nabla_{e_2}K(e_1,e_1)+\nabla_{e_2}K(e_3,e_3)-\nabla_{e_3}K(e_2,e_3),e_2),\\
&S(e_2,e_3)=g(-K(e_1,e_1)+2K(e_1,e_3)+K(e_2,e_2)-\nabla_{e_1}K(e_2,e_1)+\nabla_{e_2}K(e_1,e_1)+\nabla_{e_2}K(e_3,e_3)-\nabla_{e_3}K(e_2,e_3),e_3),\\
&S(e_3,e_1)=g(K(e_3,e_2)-\nabla_{e_1}K(e_3,e_1)-\nabla_{e_2}K(e_3,e_2)+\nabla_{e_3}K(e_1,e_1)+\nabla_{e_3}K(e_2,e_2),e_1),\\
&S(e_3,e_2)=g(K(e_3,e_2)-\nabla_{e_1}K(e_3,e_1)-\nabla_{e_2}K(e_3,e_2)+\nabla_{e_3}K(e_1,e_1)+\nabla_{e_3}K(e_2,e_2),e_2),\\
&S(e_3,e_3)=2+g(K(e_3,e_2)-\nabla_{e_1}K(e_3,e_1)-\nabla_{e_2}K(e_3,e_2)+\nabla_{e_3}K(e_1,e_1)+\nabla_{e_3}K(e_2,e_2),e_3)
\end{aligned}\tag{4.22}$$

and

$$\begin{aligned}
\tau=-4&+g(K(K(e_1,e_2),e_2)-K(e_1,K(e_2,e_2))+K(e_3,K(e_1,e_3))-K(e_1,K(e_3,e_3))\\
&\quad+(\nabla_{e_1}K)(e_2,e_2)-(\nabla_{e_2}K)(e_1,e_2)+(\nabla_{e_1}K)(e_3,e_3)-(\nabla_{e_3}K)(e_1,e_3),e_1)\\
&+g(K(e_1,K(e_2,e_1))-K(e_2,K(e_1,e_1))-K(e_2,K(e_3,e_3))+K(e_3,K(e_2,e_3))\\
&\quad-(\nabla_{e_1}K)(e_2,e_1)+(\nabla_{e_2}K)(e_1,e_1)+(\nabla_{e_2}K)(e_3,e_3)-(\nabla_{e_3}K)(e_2,e_3),e_2)\\
&+g(K(e_1,K(e_3,e_1))+K(e_2,K(e_3,e_2))-K(e_3,K(e_1,e_1))-K(e_3,K(e_2,e_2))\\
&\quad-(\nabla_{e_1}K)(e_3,e_1)-(\nabla_{e_2}K)(e_3,e_2)+(\nabla_{e_3}K)(e_1,e_1)+(\nabla_{e_3}K)(e_2,e_2),e_3),
\end{aligned}\tag{4.23}$$

respectively. (Similarly, the above equations can be obtained for the dual connection ∇^*.)
Finally, from (4.6) (or from (3.1) for $w=\eta$ and $U=\xi$) and (4.19), we have

$$\begin{aligned}
&\tilde{\nabla}_{e_1}e_1=e_2-e_3,\ \tilde{\nabla}_{e_1}e_2=-e_1+e_3,\ \tilde{\nabla}_{e_1}e_3=-e_2+e_1,\\
&\tilde{\nabla}_{e_2}e_1=-e_3,\ \tilde{\nabla}_{e_2}e_2=-e_3,\ \tilde{\nabla}_{e_2}e_3=e_1+e_2,\\
&\tilde{\nabla}_{e_3}e_1=-e_2,\ \tilde{\nabla}_{e_3}e_2=e_1,\ \tilde{\nabla}_{e_3}e_3=0.
\end{aligned}$$

and the curvature tensors, Ricci tensors and scalar curvature with respect to the semi-symmetric metric connection $\tilde{\nabla}$ are obtained as follows, respectively:

$$\begin{aligned}
&\tilde{R}(e_1,e_2)e_1=5e_2,\ \tilde{R}(e_1,e_2)e_2=-5e_1,\ \tilde{R}(e_1,e_2)e_3=0,\\
&\tilde{R}(e_1,e_3)e_1=-e_3,\ \tilde{R}(e_1,e_3)e_2=-e_3,\ \tilde{R}(e_1,e_3)e_3=e_1+e_2,\\
&\tilde{R}(e_2,e_3)e_1=e_3,\ \tilde{R}(e_2,e_3)e_2=-e_3,\ \tilde{R}(e_2,e_3)e_3=e_2-e_1,
\end{aligned}\tag{4.24}$$

$$\begin{aligned}
&\tilde{S}(e_1,e_1)=-4,\ \tilde{S}(e_1,e_2)=1,\ \tilde{S}(e_1,e_3)=0,\\
&\tilde{S}(e_2,e_1)=-1,\ \tilde{S}(e_2,e_2)=-4,\ \tilde{S}(e_2,e_3)=0,\\
&\tilde{S}(e_3,e_1)=0,\ \tilde{S}(e_3,e_2)=0,\ \tilde{S}(e_3,e_3)=2
\end{aligned}\tag{4.25}$$

and

$$\tilde{\tau}=-6.\tag{4.26}$$

Hence, one can easily see that, from (4.1), (4.18), (4.21) and (4.24), the equation (4.7) in Theorem 4.4 is verified; from (4.1), (4.18), (4.22) and (4.25), the equation (4.14) in Theorem 4.6 is verified and from (4.23) and (4.26), the equation (4.16) in Theorem 4.7 is verified for $n=1$. Similarly, obtaining the above equations for dual connection ∇^*, one can easily see that, the equations (4.8), (4.15) and (4.17) are verified, too.

Now, by choosing the difference tensor field K as in Example 4.3, we'll obtain the equations (4.20)-(4.23) in the following Example.

Example 4.9. *If we choose the difference tensor field K as $K(X,Y) = 3\eta(X)\eta(Y)\xi$, then the equations (4.20)-(4.23) are obtained as*

$$\nabla_{e_1}e_1 = e_2,\ \nabla_{e_1}e_2 = -e_1 + e_3,\ \nabla_{e_1}e_3 = -e_2,$$
$$\nabla_{e_2}e_1 = -e_3,\ \nabla_{e_2}e_2 = 0,\ \nabla_{e_2}e_3 = e_1,$$
$$\nabla_{e_3}e_1 = -e_2,\ \nabla_{e_3}e_2 = e_1,\ \nabla_{e_3}e_3 = 3e_3,$$

$$\begin{aligned} &R(e_1,e_2)e_1 = 4e_2,\ R(e_1,e_2)e_2 = -4e_1,\ R(e_1,e_2)e_3 = -6e_3,\\ &R(e_1,e_3)e_1 = -e_3,\ R(e_1,e_3)e_2 = -3e_3,\ R(e_1,e_3)e_3 = e_1 - 3e_2,\\ &R(e_2,e_3)e_1 = 3e_3,\ R(e_2,e_3)e_2 = -e_3,\ R(e_2,e_3)e_3 = e_2 + 3e_1, \end{aligned} \tag{4.27}$$

$$\begin{aligned} &S(e_1,e_1) = -3,\ S(e_1,e_2) = -3,\ S(e_1,e_3) = 0,\\ &S(e_2,e_1) = 3,\ S(e_2,e_2) = -3,\ S(e_2,e_3) = 0,\\ &S(e_3,e_1) = 0,\ S(e_3,e_2) = 0,\ S(e_3,e_3) = 2 \end{aligned} \tag{4.28}$$

and

$$\tau = -4. \tag{4.29}$$

Here, one can easily see that, from (4.1), (4.18), (4.24)-(4.26) and (4.27)-(4.29), the Theorems 4.4, 4.6 and 4.7 are verified.

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