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Holditch-Type Theorem for Non-Linear Points in Generalized Complex Plane $\mathbb{C}_\mathrm{p}$

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Abstract

The generalized complex number system and generalized complex plane were studied by Yaglom [1, 2] and Harkin [3]. Moreover, Holditch-type theorem for linear points in $\mathbb{C}_\mathrm{p}$ were given by Erişir et al. [4]. The aim of this paper is to find the answers of the questions ”How is the polar moments of inertia calculated for trajectories drawn by non-linear points in $\mathbb{C}_\mathrm{p}$?”, ”How is Holditch-type theorem expressed for these points in $\mathbb{C}_\mathrm{p}$?” and finally ”Is this paper a new generalization of [4]?”.

1. Introduction and preliminaries

H. Holditch expressed the Holditch theorem in the article entitled ”Geometrical Theorem” in 1858. Holditch theorem is stated that ”If the end points of a chord, with constant length $a+b$, draw any closed curve, any point on this chord draw different closed curve. So, the area between these curves is always πab”, [5]. The most important point of this classic Holditch theorem in Euclidean plane is that the area between these curves is independent of the selection of the curves. Thus, this theorem has attracted a lot of attention and been generalized with various methods and different perspectives. Then, Steiner calculated the area formula of the trajectory in a moving plane drawn by a point in the fixed plane in terms of Steiner points, [6].

Blaschke and Müller considered trajectories drawn by three points and generalized the Holditch theorem in Euclidean plane, [7]. Then, Hering expressed the Holditch theorem with respect to the length of the envelope curve with the aid of non-linear three points, [8]. Considering the above studies, there are many studies concerned with the Holditch theorem, [9, 10, 11].

The polar moment of inertia instead of area in Holditch theorem can be calculated by similar processes. Holditch theorem expressed in terms of the polar moment of inertia is called as ”Holditch-type theorem”.

Müller calculated the polar moment of inertia of the trajectory drawn by a point in the Euclidean plane. Moreover, Müller gave a conclusion that the geometric locus of all fixed points on the moving plane which has same polar moment of inertia is the circle with center which is Steiner point, [12]. Then, considering the study [12], there are lots of studies related to Holditch-type theorem, [13, 14, 15, 16, 17].

In the Euclidean plane, the Cauchy formula of the closed envelope of a family of the straight lines g and the length of the envelope of trajectories of straight lines were given by Blaschke and Müller, [7]. In the Lorentzian plane, the Cauchy formula for the envelope of a family of lines was given by Yüce and Kuruoğlu. Moreover, they proved the length of the envelope of trajectories of non-null lines and gave the Holditch theorem for the length of the envelope of trajectories for Lorentzian motion, [18].

The generalized complex number system is defined as

$$\mathbb{C}_\mathrm{p} = \left\{ x + iy \ : \ x, y \in \mathbb{R}, \quad i^2 = \mathrm{p} \in \mathbb{R} \right\}$$

and expressed by Yaglom and Harkins, [1, 2, 3]. This system involves in complex $(\mathrm{p} = -1)$, dual $(\mathrm{p} = 0)$ and hyperbolic $(\mathrm{p} = +1)$ number systems and also different planes for other values of p.

Considering the studies given by Yaglom and Harkins, some studies were done in the generalized complex plane. Gürses and Yüce considered the one parameter planar motion in Affine-Cayley Klein planes and p-complex plane $\mathbb{C}_J = \left\{ x + Jy \ : \ x, y \in \mathbb{R}, \quad J^2 = \mathrm{p}, \quad \mathrm{p} \in \{-1, 0, 1\} \right\} \subset \mathbb{C}_\mathrm{p}$, [19, 20]. Moreover, Erişir et. al. calculated the Steiner area formula and proved Holditch theorem in the generalized complex plane $\mathbb{C}_\mathrm{p}$,

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[21]. Then, they calculated the polar moment of inertia of trajectories under the one-parameter planar motion and proved Holditch-type theorem in $\mathbb{C}_\mathrm{p}$, [4]. Moreover, Erişir and Güngör gave the Cauchy-length formula and proved Holditch theorem for non-linear points in $\mathbb{C}_\mathrm{p}$, [22].
Now, using the above studies, we give some operations on this system.
The addition, substraction and product on this generalized complex plane $\mathbb{C}_\mathrm{p}$ are

$$Z_1 \pm Z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = x_1 \pm x_2 + i(y_1 \pm y_2)$$

and

$$M^\mathrm{p}(Z_1, Z_2) = (x_1x_2 + \mathrm{p}y_1y_2) + i(x_1y_2 + x_2y_1)$$

where $Z_1 = (x_1 + iy_1),\ Z_2 = (x_2 + iy_2) \in \mathbb{C}_\mathrm{p}$, [2, 3]. In addition, the p$-$magnitude of $Z = x + iy \in \mathbb{C}_\mathrm{p}$ is

$$|Z|_\mathrm{p} = \sqrt{|M^\mathrm{p}(Z, \bar{Z})|} = \sqrt{|x^2 - \mathrm{p}y^2|}.$$

The unit circle in $\mathbb{C}_\mathrm{p}$ is the set of points in the form $|Z|_\mathrm{p} = 1$. So, now we consider the special values of p in $\mathbb{C}_\mathrm{p}$ as follows.
1) Let us consider $\mathrm{p} < 0$. Thus, the generalized complex number system matches up with the elliptical complex number system. For $\mathrm{p} = -1$, the unit circle in $\mathbb{C}_\mathrm{p}$ corresponds to the Euclidean unit circle and the plane $\mathbb{C}_{-1}$ matches up with Euclidean plane.
2) If we consider $\mathrm{p} = 0$, the plane $\mathbb{C}_0$ matches up with Gallilean plane. The unit circle in $\mathbb{C}_\mathrm{p}$ corresponds to Gallilean circle.
3) We take $\mathrm{p} > 0$. In this case, the generalized complex number system is equal to the hyperbolic complex number system. If we take $\mathrm{p} = 1$, the plane $\mathbb{C}_1$ corresponds to the Lorentzian plane, (Figure 1.1), [3].

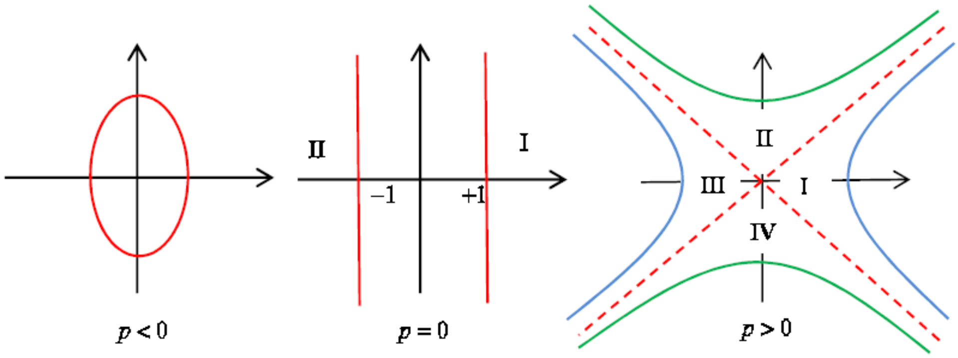

Figure 1.1: Unit Circles in $\mathbb{C}_\mathrm{p}$

So, we can give the following definition.

Definition 1.1. *Let us consider a circle in the generalized complex plane* $\mathbb{C}_\mathrm{p}$. *This circle has the center* $M(a,b)$ *and the radius* r. *So, the equation of this circle is*

$$\left|(x-a)^2 - \mathrm{p}(y-b)^2\right| = r^2$$

where $i^2 = \mathrm{p} \in \mathbb{R}$, *[3].*

Now, we mention the angle in $\mathbb{C}_\mathrm{p}$. Let us consider $\sigma \equiv {}^{y}\!/_{x}$ and $Z = x + iy$. So, we can write

$$\tan \mathrm{p}\theta_\mathrm{p} = \frac{\sin \mathrm{p}\theta_\mathrm{p}}{\cos \mathrm{p}\theta_\mathrm{p}},$$

[3]. In addition, the generalized Euler formula

$$e^{i\theta_\mathrm{p}} = \cos \mathrm{p}\theta_\mathrm{p} + i \sin \mathrm{p}\theta_\mathrm{p}$$

where $i^2 = \mathrm{p}$ in $\mathbb{C}_\mathrm{p}$. Thus, the polar and exponential forms of the generalized complex number Z is

$$z = r_\mathrm{p}(\cos \mathrm{p}\theta_\mathrm{p} + i \sin \mathrm{p}\theta_\mathrm{p}) = r_\mathrm{p} e^{i\theta_\mathrm{p}}$$

where θ_p and $r_\mathrm{p} = |Z|_\mathrm{p}$ are p$-$argument and p$-$magnitude of generalized complex number Z, respectively. The p$-$rotation matrix obtained by $e^{i\theta_\mathrm{p}}$ is

$$A(\theta_\mathrm{p}) = \left[\begin{array}{cc} \cos \mathrm{p}\theta_\mathrm{p} & \mathrm{p}\sin \mathrm{p}\theta_\mathrm{p} \\ \sin \mathrm{p}\theta_\mathrm{p} & \cos \mathrm{p}\theta_\mathrm{p} \end{array}\right].$$

Moreover, the derivatives of the $\mathrm{p}-$trigonometric functions $\cos\mathrm{p}$ and $\sin\mathrm{p}$ can be written by

$$\frac{d}{d\alpha}(\cos\mathrm{p}\alpha)=\mathrm{p}\sin\mathrm{p}\alpha,\qquad \frac{d}{d\alpha}(\sin\mathrm{p}\alpha)=\cos\mathrm{p}\alpha,$$

[3].
Throughout this study, we consider one-parameter planar motion $\mathbb{K}_{\mathrm{p}}/\mathbb{K}'_{\mathrm{p}}$ in generalized complex plane $\mathbb{C}_{\mathrm{p}}$. Moreover, we study in the branch I of $\mathbb{C}_{\mathrm{p}}$.
Now, we mention Cauchy formula in $\mathbb{C}_{\mathrm{p}}$ which is used in this study. This formula in $\mathbb{C}_{\mathrm{p}}$ was studied by Erişir and Güngör in [22].
Let g be a line in the branch I of $\mathbb{C}_{\mathrm{p}}$. So, the Hesse form of this line g in $\mathbb{C}_{\mathrm{p}}$ is written by

$$h=x_1\cos\mathrm{p}\psi_{\mathrm{p}}-\mathrm{p}x_2\sin\mathrm{p}\psi_{\mathrm{p}}$$

where $\left(h,\psi_{\mathrm{p}}\right)$ is the Hesse coordinates in $\mathbb{C}_{\mathrm{p}}$ and $h=h(\psi_{\mathrm{p}})$ is the distance to the origin O from the right line and the point $X\left(x_1,x_2\right)$ is the contact point of the line g with the envelope curve (g). Moreover, the Cauchy-length formula in $\mathbb{C}_{\mathrm{p}}$ is written by

$$L=\frac{1}{\sqrt{|\mathrm{p}|}}\int_{t_0}^{t_1}\left|\mathrm{p}h-\ddot{h}\right|d\psi_{\mathrm{p}}.$$

Similarly, we give the length of the enveloping curve (g) according to the fixed generalized complex plane $\mathbb{K}'_{\mathrm{p}}$. So, we can write the Hesse form of the line g according to the fixed generalized complex plane $\mathbb{K}'_{\mathrm{p}}$ as

$$h'=x'_1\cos\mathrm{p}\psi'_{\mathrm{p}}-\mathrm{p}x'_2\sin\mathrm{p}\psi'_{\mathrm{p}}$$

where h' is the distance to the origin O' from the right line g. If the necessary operations are considered, it is obtained that

$$h'=h-u_1\cos\mathrm{p}\psi_{\mathrm{p}}+\mathrm{p}u_2\sin\mathrm{p}\psi_{\mathrm{p}}.$$

So, we obtain that

$$L'=\frac{1}{\sqrt{|\mathrm{p}|}}\left|\mathrm{p}h\delta_{\mathrm{p}}-A\cos\mathrm{p}\psi_{\mathrm{p}}+\mathrm{p}B\sin\mathrm{p}\psi_{\mathrm{p}}\right|$$

where $A=\int_{t_0}^{t_1}(\mathrm{p}u_1-\ddot{u}_1)d\theta_{\mathrm{p}}$ and $B=\int_{t_0}^{t_1}(\mathrm{p}u_2-\ddot{u}_2)d\theta_{\mathrm{p}}$.
Moreover, we know that

$$L'=\sqrt{|\mathrm{p}|}\left(\int_{t_0}^{t_1}\bar{q}d\theta_{\mathrm{p}}+L_Q^g\right)$$

where $L_Q^g=q_2\cos\mathrm{p}\psi_{\mathrm{p}}-q_1\sin\mathrm{p}\psi_{\mathrm{p}}\Big|_{t_0}^{t_1}$ is the length of orthogonal projection of the line segment Q_1Q_2 of the moving pole curve (Q) on the line g. Moreover, $\bar{q}=h-q_1\cos\mathrm{p}\psi_{\mathrm{p}}+\mathrm{p}q_2\sin\mathrm{p}\psi_{\mathrm{p}}$ is distance of the pole point Q to the line g in the generalized complex plane in $\mathbb{C}_{\mathrm{p}}$, [22].
In addition, the following theorem can be given.

Theorem 1.2. *All the fixed lines with Hesse coordinates (h,ψ_{p}) of the generalized moving complex plane $\mathbb{K}_{\mathrm{p}}$ whose envelope of trajectories have the same length $L'=c$ are tangent to the cycles with center $S_G=\left(\frac{A}{\mathrm{p}\delta_{\mathrm{p}}},\frac{B}{\mathrm{p}\delta_{\mathrm{p}}}\right)$ and radius $\frac{c}{\sqrt{|\mathrm{p}|}\delta_{\mathrm{p}}}$ in the generalized moving plane $\mathbb{K}_{\mathrm{p}}$, [22].*

2. Main theorems and proofs

In this section, we prove the Holditch-type theorem for non-linear points in the generalized complex plane $\mathbb{C}_{\mathrm{p}}$ for one-parameter planar motion with $S=S_G$. We firstly express and prove following theorem.

Theorem 2.1. *Let the non-linear points $X=(0,0)$, $Y=(a+b,0)$ and $Z=(a,c)$ be fixed on the generalized moving plane $\mathbb{K}_{\mathrm{p}}$ in $\mathbb{C}_{\mathrm{p}}$. In addition, the points X, Y and Z move along the trajectories k_X, k_Y and k_Z on $\mathbb{K}'_{\mathrm{p}}$ with moments T_X, T_Y and T_Z, respectively. So, the relationship between the polar moments of inertia T_X, T_Y and T_Z is*

$$T_Z=\frac{aT_Y+bT_X}{a+b}-\delta_{\mathrm{p}}\left(\mathrm{p}c^2+ab\right)-2\sqrt{|\mathrm{p}|}cL_{XY}$$

where L_{XY} is the length of the enveloping curve of (XY).

Proof. Let the points X, Y and Z be non-linear points. Moreover, we consider that these points $X=(0,0)$, $Y=(a+b,0)$ and $Z=(a,c)$. We know that the polar moments of inertia of any point X in $\mathbb{C}_{\mathrm{p}}$ is given

$$T_Z=T_0+\delta_{\mathrm{p}}\left(x_1{}^2-\mathrm{p}x_2{}^2-2x_1s_1+2\mathrm{p}x_2s_2\right)$$

in [4]. So, if we use this formula for the points X, Y and Z, we find that

$$T_X = T_0 \tag{2.1}$$

$$T_Y = T_X + \delta_{\mathrm{p}} \left((a+b)^2 - 2(a+b)s_1 \right) \tag{2.2}$$

$$T_Z = T_X + \delta_{\mathrm{p}} \left(a^2 - \mathrm{p}c^2 - 2as_1 + 2\mathrm{p}cs_2 \right). \tag{2.3}$$

From the equations (2.1) and (2.2), we have

$$s_1 = \frac{a+b}{2} + \frac{T_X - T_Y}{2\delta_{\mathrm{p}}(a+b)}. \tag{2.4}$$

Moreover, from the equations (2.3) and (2.4), we find that

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_{\mathrm{p}} \left(\mathrm{p}c^2 + ab \right) + 2\mathrm{p}\delta_{\mathrm{p}}cs_2.$$

The other hand, from $S = S_G$ we know that

$$s_2 = \frac{B}{\mathrm{p}\delta_{\mathrm{p}}}.$$

Finally, if L' is written for $X = (0,0)$, $Y = (a+b,0)$ and $Z = (a,c)$ we obtain that

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_{\mathrm{p}} \left(\mathrm{p}c^2 + ab \right) - 2\sqrt{|\mathrm{p}|}cL_{XY} \tag{2.5}$$

□

So, the following conclusion can be given.

Conclusion 2.2. *Let us take that X, Y and Z are linear points during the motion with $S = S_G$ in $\mathbb{C}_{\mathrm{p}}$. Namely, we have $c = 0$. From the equation (2.5) the relation between the polar moments of inertia of trajectory drawn by the points X, Y and Z is*

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_{\mathrm{p}}ab.$$

This formula is the formula given relationship between polar moments of inertia for the linear three points in [4]. So, the formula (2.5) is generalization of the formula in [4].

Note: For the value $\mathrm{p} = 0$, the formula (2.5) is obtain that

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_{\mathrm{p}}ab. \tag{2.6}$$

This formula is also the formula between polar moments of inertia for the linear three points in [4]. Namely, for $\mathrm{p} = 0$, the formula of polar moment of inertia for linear three points is same the formula of moment for non-linear three points. The reason of this is the metric in the plane C_0. From the definition of metric in C_0 $(\mathrm{p} = 0)$ the distance between the points X and R (the orthogonal projection of the point Z on the line segment XY), (a), is same the distance between the points X and Z. Similarly, the distance between the points Y and R, (b), is same the distance between the points Y and Z. So, for $\mathrm{p} = 0$ the equation (2.6) is valid the polar moments of inertia for both linear three points and non-linear three points.
In addition, we give the following conclusions.

Conclusion 2.3. *If the points X and Y move along the same trajectories k_X with moment T_X, the formula (2.5) is obtained that*

$$T_Z = T_X - \delta_{\mathrm{p}} \left(\mathrm{p}c^2 + ab \right) - 2\sqrt{|\mathrm{p}|}cL_{XY}.$$

Conclusion 2.4. *The relationship between the length of envelope curve (g) and the length of the enveloping curve of (XY) is*

$$L' = \sqrt{|\mathrm{p}|} \left(h\delta_{\mathrm{p}} + \left(\frac{T_Y - T_X}{2\delta_{\mathrm{p}}(a+b)} - \frac{a+b}{2} \right) \delta_{\mathrm{p}} \cos \mathrm{p}\psi_{\mathrm{p}} - \sqrt{|\mathrm{p}|}L_{XY} \sin \mathrm{p}\psi_{\mathrm{p}} \right)$$

Finally, we can give the main theorem from the equation (2.5).

Theorem 2.5. Main Theorem (Holditch-Type Theorem): *Let us consider motion with $S = S_G$ and the points $X = (0,0)$, $Y = (a+b,0)$ and the point $Z = (a,c)$ non-linear with X and Y fixed on $\mathbb{K}_p$. In a specific time interval, while the points X and Y move along the same trajectories k_X with moment T_X, the point Z non-linear with the points X and Y draws different trajectory k_Z with the moment T_Z. The moment of section between the curves k_X (k_Y) and k_Z depends on the distances of the point R to the endpoints X and Y, the distance of the point Z to the line $\overline{\mathrm{XY}}$, the length of the enveloping curve and the rotation angle of the motion. This moment is independent of the choice of curves.*

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