Universal Journal of Mathematics and Applications, 1 (4) (2018) 254-257

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: http://dx.doi.org/10.32323/ujma.422271

Compact Totally Real Minimal Submanifolds in a Bochner-Kaehler Manifold

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Article Info

Keywords: *Bochner-Kaehler manifold, Ricci curvature*
2010 AMS: *53B25, 53C55*
Received: *9 May 2018*
Accepted: *16 July 2018*
Available online: *20 December 2018*

Abstract

In this paper, we establish the following results: Let M be an $m-$dimensional compact totally real minimal submanifold immersed in a locally symmetric Bochner-Kaehler manifold $\tilde{M}$ with Ricci curvature bounded from below. Then either M is a totally geodesic or

$$\inf r \leq \frac{1}{2}\left(\frac{1}{2}m(m-1)\tilde{k}-\frac{1}{3}(m+1)\tilde{c}\right),$$

where r is the scalar curvature of M.

1. Introduction

The Bochner tensor was originally introduced in 1948 by S. Bochner as a Kaehler analogue of the Weyl conformal curvature tensor. Kaehler manifolds with vanishing Bochner tensor are known as Bochner-Kaehler manifolds, [1]. The Bochner tensor has interesting connections to several areas of mathematics and Bochner-Kaehler manifolds have been studied quite intensively in the last two decades, see for instance, [1, 2, 3].

In this work, we make us of Yau's [4] maximum principle to compact study totally real minimal submanifold with Ricci curvature bounded from below and obtain the following results:

Main Theorem. Let M be an $m-$dimensional compact totally real minimal submanifold immersed in a locally symmetric Bochner-Kaehler manifold $\tilde{M}$ with Ricci curvature bounded from below. Then either M is totally geodesic or $\inf r \leq \frac{1}{2}\left(\frac{1}{2}m(m-1)\tilde{k}-\frac{1}{3}(m+1)\tilde{c}\right)$ where r is the scalar curvature of M.

We use the same notation and terminologies as in [5] unless otherwise stated.

Let $\tilde{M}$ be an $n-$dimensional Kaehler manifold and denote by g_{AB}, F_{AB}, $\tilde{K}_{ABCD}$ and $\tilde{K}$, the metric tensor, the complex structure tensor, the curvature tensor, the Ricci tensor and the scalar curvature of $\tilde{M}$, respectively. Suppose that the Boechner curvature tensor of $\tilde{M}$ vanishes, then we have

$$\begin{aligned}\tilde{K}_{ABCD} &= -g_{AD}L_{BC}+g_{BD}L_{AC}-L_{AD}g_{BC}+L_{BD}g_{AC}-F_{AD}M_{BC} \\ &\quad +F_{BD}M_{AC}-M_{AD}F_{BC}+M_{BD}F_{AC}+2(M_{AB}F_{CD}+F_{AB}M_{CD}),\end{aligned} \tag{1.1}$$

where

$$L_{BC}=\tilde{K}_{BC}/(2n+4)-\tilde{K}g_{BC}/2(2n+2)(2n+4),\ \tilde{K}_{BC}=g^{AD}\tilde{K}_{ABDC},$$
$$\tilde{K}=g^{BC}\tilde{K}_{BC},\ M_{BC}=-L_{BD}F^{D}_{C},\ F^{D}_{C}=g^{BD}F_{CB}.$$

L_{BC} are components of a hybrid tensor of type $(0,2)$. That is

$$L_{BC}F^{B}_{A}F^{C}_{D}=L_{AD}.$$

In order to avoid repetitions it will be agreed that our indices have the following ranges throughout this paper:

$$A,B,C,D,...=1,2,...,m,1^*,2^*,...,m^*,$$
$$i,j,k,l,...=1,2,...,m;\alpha,\beta,\gamma,...=1^*,2^*,...,m^*.$$

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In the following sections, $\tilde{M}$ is always supposed to be a Bochner-Kaehler manifold, that is, $\tilde{M}$ is a Kaehler manifold with curvature tensor $\tilde{K}_{ABCD}$ given by (1.1).

2. Totally real submanifolds in $\tilde{M}$

We call M as a totally real submanifold of $\tilde{M}$ if M admits an isometric immersion into $\tilde{M}$ such that for all $x \in M$, $F(T_x(M)) \subset v_x$, where $T_x(M)$ denotes the tangent space of M at x and F the complex structure of $\tilde{M}$. If the real dimension of M is m, then $m \leq n$, n is the complex dimension of $\tilde{M}$. We choose a local field of orthonormal frames

$$e_1,\dots,e_m,e_{m+1},\dots,e_n\;;\qquad e_{1^*}=Fe_{1,\dots,}e_{m^*}=Fe_{m,\dots,}e_{n^*}=Fe_n,$$

in $\tilde{M}$ in a such a way that, restricted to M, $e_1,\dots,e_m$ are tangents to M. With respect to this frame field, F and g have the components

$$(F_{AB})=\begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix},\quad (g_{AB})=(I_{2n}),$$

where I_k denotes the identity matrix of degree k.
We consider the case $n=m$ only in this paper.
The equation of Gauss of M in $\tilde{M}$ is written as

$$K_{ijkl}=\tilde{K}_{ijkl}+\sum_{\alpha}\left(h^{\alpha}_{ik}h^{\alpha}_{jl}-h^{\alpha}_{il}h^{\alpha}_{jk}\right). \tag{2.1}$$

K_{ijkl} is the curvature tensor and h^{α}_{ij} is the second fundamental tensor of M. Since M is a totally real submanifold in $\tilde{M}$, with respect to the above frame we have the relation $h^{i^*}_{jk}=h^{j^*}_{ik}$. Let $\tilde{K}$ be the curvature tensor field of $\tilde{M}$ so that $\tilde{K}_{ABCD}=g\left(\tilde{K}(e_C,e_D)e_B,e_A\right)$. Then (1.1) is equivalent to

$$\begin{aligned}\tilde{K}(X,Y)Z=&\,L(Y,Z)X-L(X,Z)Y+\langle Y,Z\rangle NX-\langle X,Z\rangle NY\\&+M(Y,Z)FX-M(X,Z)FY+\langle FY,Z\rangle PX\\&-\langle FX,Z\rangle PY-2\left(M(X,Y)FZ+\langle FX,Y\rangle PZ\right),\end{aligned} \tag{2.2}$$

where NX,PX are defined by $g(NX,Y)=L(X,Y)$, $g(PX,Y)=M(X,Y)$ and $\langle,\rangle$ denotes the inner product with respect to g. Let $\tilde{K}(X)$ be the holomorphic sectional curvature spanned by a unit vector X and FX. By (1.1) or (2.2) we have

$$\tilde{K}(X)=\tilde{K}(X,FX,FX,X)=\left\langle\tilde{K}(X,FX)FX,X\right\rangle=8L(X,X),$$

Let $\tilde{\rho}(X,Y)$ denote the sectional curvature of $\tilde{M}$ determined by section $\{X,Y\}$ spanned by two orthonormal vector $\{X,Y\}$. If X,Y are both tangent to the totally real submanifold M then we have

$$\tilde{\rho}(X,Y)=L(X,X)+L(Y,Y)=\frac{1}{8}\left(\tilde{K}(X)+\tilde{K}(Y)\right). \tag{2.3}$$

The equation of (2.3) has been obtained by Iwasaki and Ogitsu, [6].
Let $\rho(X,Y)$ denote the sectional curvature of M determined by orthonormal tangent vectors $\{X,Y\}$ of M. Then the equation of Gauss (2.1) and (2.3) imply

$$\rho(X,Y)=\frac{1}{8}\left(\tilde{K}(X)+\tilde{K}(Y)\right)+\langle\sigma(X,X),\sigma(Y,Y)\rangle-\|\sigma(X,Y)\|^2,$$

where σ is the second fundamental form which is related to h^{α}_{ij} by $g(\sigma(X,Y),\xi)=h^{i^*}_{jk}X^jY^k\xi^{i^*}$ for any normal $\xi=\xi^{i^*}e_{i^*}$.
Let S be the Ricci tensor of M and r the scalar curvature of M. Then

$$\begin{aligned}S(X,Y)&=(m-2)L(X,Y)+\frac{1}{8}m\tilde{k}\langle X,Y\rangle-\sum_{\alpha}g(h_{\alpha}X,h_{\alpha}Y),\\ r&=\frac{1}{4}m(m-1)\tilde{k}-\|\sigma\|^2.\end{aligned}$$

Let $\tilde{M}$ is locally symmetric. Let Δ denote the Laplacian, $\bigtriangledown'$denote the covariant differentiation with respect to connection in (tangent bundle) $\oplus$ (normal bundle) of M in $\tilde{M}$. If M is a minimal submanifold of $\tilde{M}$ the following holds (see [5] for example). Since $\tilde{M}$ is assumed to be locally symmetric:

$$\frac{1}{2}\Delta\|\sigma\|^2=\left\|\bigtriangledown'\sigma\right\|^2+\frac{1}{4}(m+1)\tilde{c}\|\sigma\|^2+\sum tr\left(h_{i^*}h_{j^*}-h_{j^*}h_{i^*}\right)^2-\sum tr\left(h_{i^*}h_{j^*}\right)^2, \tag{2.4}$$

where $\tilde{c}$ is a function on M defined by $h^{j^*}_{lk}h^{j^*}_{ik}\tilde{K}_{li}=\frac{1}{2}(m+1)\tilde{c}\|\sigma\|^2$.
In order to prove the main theorem, we need the following lemmas.

Lemma 2.1. *Let H_i, $i \geq 2$ be symmetric $n \times n$ matrices, $S_i = trH_i^2$, $S = \sum_i S_i$. Then*

$$\sum_{i,j} tr\left(H_iH_j - H_jH_i\right)^2 - \sum_{i,j} tr\left(H_iH_j\right)^2 \geq -\frac{3}{2}\|\sigma\|^4, \tag{2.5}$$

and the equality holds if and only if either all $H_i = 0$ or there exists two of H_i different from zero. Moreover, if $H_1 \neq 0, H_2 \neq 0$, $H_i = 0, i \neq 1,2$, then $S_1 = S_2$ and there exists an orthogonal $n \times n$ matrices T such that

$$TH_1^tT = \begin{pmatrix} \frac{\sqrt{S_1}}{2} & 0 & \dots & 0 \\ 0 & -\frac{\sqrt{S_1}}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad TH_2^tT = \begin{pmatrix} 0 & \frac{\sqrt{S_1}}{2} & \dots & 0 \\ \frac{\sqrt{S_1}}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

[7, 8].

Lemma 2.2. *Let N be a complete Riemannian manifold with Ricci curvature bounded from below and let f be a C^2-function bounded from above on N, then for all $\varepsilon > 0$, there exists a point $x \in N$ at which ,*
i) $\sup f - \varepsilon < f(x)$,
ii) $\|\bigtriangledown f(x)\| < \varepsilon$,
iii) $\Delta f(x) < \varepsilon$, *in [9].*

3. Proof of the main theorem

In this section, the method proof used by Ximin in [9] is applied totally real minimal submanifold immersed in a Bochner-Kaehler manifold. From (2.4) and (2.5), we obtain

$$\frac{1}{2}\Delta\|\sigma\|^2 \geq \|\sigma\|^2 \left(\frac{1}{4}(m+1)\tilde{c} - \frac{3}{2}\|\sigma\|^2\right). \tag{3.1}$$

We know that $\|\sigma\|^2 = \frac{1}{4}m(m-1)\tilde{k} - r$. By the condition of the theorem, we conclude that $\|\sigma\|^2$ is bounded. We define $f = \|\sigma\|^2$ and $F = (f+a)^{\frac{1}{2}}$ (where $a > 0$ is any positive constant number). F is bounded. We have

$$dF = \frac{1}{2}(f+a)^{-\frac{1}{2}} df,$$

$$\Delta F = \frac{1}{2}\left(-\frac{1}{2}(f+a)^{-\frac{3}{2}}\|df\|^2 + (f+a)^{-\frac{1}{2}}\Delta f\right),$$
$$= \frac{1}{2}\left(-2\|dF\|^2 + \Delta f\right)(f+a)^{-\frac{1}{2}},$$

i.e.,

$$\Delta F = \frac{1}{2}\left(-2\|dF\|^2 + \Delta f\right).$$

Hence, $F\Delta F = -\|dF\|^2 + \frac{1}{2}\Delta f$ or $\frac{1}{2}\Delta f = F\Delta F + \|dF\|^2$. Applying Lemma 2.2 to F, we have for all $\varepsilon > 0$, there exists a point $x \in M$ such that at x

$$\|dF(x)\| \leq \varepsilon, \tag{3.2}$$

$$\Delta F(x) < \varepsilon, \tag{3.3}$$

$$F(x) > \sup F - \varepsilon. \tag{3.4}$$

From (3.2),(3.3) and (3.4), we have

$$\frac{1}{2}\Delta f < \varepsilon^2 + F\varepsilon = \varepsilon(\varepsilon + F). \tag{3.5}$$

We take a sequence $\{e_n\}$ such that $\varepsilon_n \to 0\,(n \to \infty)$ and for all n, there exists a point $x_n \in M$ such that (3.2), (3.3) and (3.4) hold. Therefore, $\varepsilon_n(\varepsilon_n + F(x_n)) \to 0\,(n \to \infty)$ (Because F is bounded). From (3.4), we have $F(x_n) > \sup F - \varepsilon_n$. Because $\{F(x_n)\}$ is a bounded sequence. So we get $F(x_n) \to F_0$ (If necessary, we can choose a subsequence). Hence, $F_0 \geq \sup F$. So we have

$$F_0 = \sup F.$$

From the definition of F, we get

$$f(x_n) \to f = \sup f.$$

(3.1) and (3.5) imply that

$$f\left(\frac{1}{4}(m+1)\tilde{c}-\frac{3}{2}f\right)\leq\frac{1}{2}\Delta f\leq\varepsilon(\varepsilon+F),$$

and

$$f(x_n)\left(\frac{1}{4}(m+1)\tilde{c}-\frac{3}{2}f(x_n)\right)<\varepsilon_n^2+\varepsilon_nF(x_n)\leq\varepsilon_n^2+\varepsilon_nF_0,$$

let $n\to\infty$, then $\varepsilon_n\to 0$ and $f(x_n)\to f_0$. Hence,

$$f_0\left(\frac{1}{4}(m+1)\tilde{c}-\frac{3}{2}f_0\right)\leq 0.$$

i) If $f_0=0$, we have $f=\|\sigma\|^2=0$. Hence M is a totally geodesic.
ii) If $f_0>0$, we have $\frac{1}{4}(m+1)\tilde{c}-\frac{3}{2}f_0\leq 0$ and

$$f_0\geq\frac{1}{6}(m+1)\tilde{c},$$

that is, $\sup\|\sigma\|^2\geq\frac{1}{6}(m+1)\tilde{c}$. Therefore,

$$\inf r\leq\frac{1}{2}\left(\frac{1}{2}m(m-1)\tilde{k}-\frac{1}{3}(m+1)\tilde{c}\right).$$

This completes the proof.

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