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An Arbitrary Order Differential Equations on Times Scale

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Abstract

Here existence and stability results of ψ-Hilfer fractional differential equations on time scales is obtained. Here sufficient condition for existence and uniqueness of solution by using Schauder's fixed point theorem (FPT) and Banach FPT is produced. In addition, generalized Ulam stability of the proposed problem is also discussed. problem.

1. Introduction

In the past decade, fractional differential equations(FDEs) appeared as rich and beautiful field of research due to their applications to the physical and life sciences and it is witnessed by blossoming literature, for instance see [1]-[6].
Consider the dynamic equation on time scales with ψ-Hilfer fractional derivative (HFD) of the form

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(t) = \mathfrak{g}(\tau,\mathfrak{u}(\tau)), \quad \tau = [0,b] := J \subseteq \mathbb{T}, \\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(0) = \mathfrak{u}_0, \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{1.1}$$

where ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ is ψ-HFD defined on $\mathbb{T}$, $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\mathfrak{I}^{1-\gamma;\psi}$ is ψ-fractional interal of order $1-\gamma(\gamma = \alpha+\beta-\alpha\beta)$. Let $\mathbb{T}$ be a time scale, that is nonempty subset of Banach space and $\mathfrak{g} : J \times \mathbb{T} \to R$ is a right-dense function.
Time scales calculus allows us to study the dynamic equations, which include both difference and differential equations, both of which are very important in implementing applications; for further information about the theoretical and potential applications of time scales, refer [7]-[9]
The dynamical behaviour of FDEs on time scales is currently undergoing active investigations. Several authors deliberate the existence and uniqueness solutions for problems involving classical fractional derivative [10, 11]. Motivated by the above works here we discuss the existence theory and stability criteria of FDEs on times scale. In order to solve the proposed problem ψ-HFD is utilized. The emergent and properties of ψ-HFD and the qualitative analysis is briefly studied in [12]-[14]. Further considerable attention paid to Ulam stability results for FDEs. For Ulam-Hyers stability theory of FDEs and its recent development, one can refer to [15]-[17]. Further the solution of generalized Ulam-Hyers-Rassias(UHR) is obtained.

2. Preliminaries

Throughout this study, let $C(J)$ be continuous function with norm

$$\|\mathfrak{u}\|_C = \max\{|\mathfrak{u}(\tau)| : \tau \in J\}.$$

We denote the space $C_\gamma(J)$ as follows

$$C_\gamma(J) := \left\{\mathfrak{g}(\tau) : J \to R |\,(\psi(\tau)-\psi(0))^\gamma \mathfrak{g}(\tau) \in C(J)\right\}, 0 \le \gamma < 1$$

the weighted space $C_\gamma(J)$ of the functions $\mathfrak{g}$ on the interval J.Thus, $C_\gamma(J)$ is the Banach space provided the norm

$$\|\mathfrak{g}\|_{C_\gamma} = \left\|(\psi(\tau)-\psi(0))^\gamma \mathfrak{g}(\tau)\right\|_C.$$

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Definition 2.1. *Let time scale be $\mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(\tau) := \inf\{s \in \mathbb{T} : s > \tau\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}$.*

Proposition 2.2. *Suppose $\mathbb{T}$ is a time scale and $[a,b] \subset \mathbb{T}$, $\mathfrak{g}$ is increasing continuous function on $[a,b]$. If the extension of $\mathfrak{g}$ is given in the following form:*

$$\mathscr{G}(s) = \begin{cases} \mathfrak{g}(s); & s \in \mathbb{T} \\ \mathfrak{g}(\tau); & s \in (\tau, \sigma(\tau)) \notin \mathbb{T}. \end{cases}$$

Then we have

$$\int_a^b \mathfrak{g}(t)\Delta t \leq \int_a^b \mathscr{G}(t)dt.$$

Definition 2.3. *Let $\mathbb{T}$ be a time scale, $J \in \mathbb{T}$. The left-sided R-L fractional integral of order $\alpha \in R^+$ of function $\mathfrak{g}(\tau)$ is defined by*

$$\left({}^{\mathbb{T}}\mathfrak{J}^{\alpha}\mathfrak{g}\right)(\tau) = \int_0^\tau \psi'(s)\frac{(\psi(\tau)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\mathfrak{g}(s)\Delta s.$$

Definition 2.4. *Suppose $\mathbb{T}$ is a time scale, $[0,b]$ is an interval of $\mathbb{T}$. The R-L fractional derivative of order $\alpha \in [n-1,n)$, $n \in \mathbb{Z}^+$ of function $\mathfrak{g}(\tau)$ is defined by*

$$\left({}^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}\right)(\tau) = \left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^n \int_0^\tau \psi'(s)\frac{(\psi(\tau)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\mathfrak{g}(s)\Delta s.$$

Definition 2.5. *[2] The ψ-HFD of order α and type β of function $\mathfrak{g}(\tau)$ is defined by*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{g}(t) = \left({}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta({}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi}\mathfrak{g})\right)(\tau),$$

where ${}^{\mathbb{T}}\Delta := \frac{d}{d\tau}$.

Remark 2.6.

1. *Here ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ is also written as*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta {}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta^{\gamma;\psi},\ \gamma = \alpha+\beta-\alpha\beta.$$

2. *Let $\beta = 0$, it transfers into R-L derivative given by ${}^{\mathbb{T}}\Delta^{\alpha} := {}^{\mathbb{T}}\Delta^{\alpha,0}$.*
3. *Let $\beta = 0$, it turns to be Caputo fractional derivative given by ${}^{\mathbb{T}}_{c}\Delta^{\alpha} := {}^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\ {}^{\mathbb{T}}\Delta$.*

Next, we review some lemmas which will be used to extabilish our existence results.

Lemma 2.7. *If $\alpha > 0$ and $\beta > 0$, there exist*

$$\left[{}^{\mathbb{T}}\mathfrak{J}^{\alpha}(\psi(s)-\psi(0))^{\beta-1}\right](\tau) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(\tau)-\psi(0))^{\beta+\alpha-1}$$

Lemma 2.8. *Let $\alpha \geq 0$, $\beta \geq 0$ and $\mathfrak{g} \in L^1(J)$. Then*

$${}^{\mathbb{T}}\mathfrak{J}^{\alpha}\ {}^{\mathbb{T}}\mathfrak{J}^{\beta}\mathfrak{g}(\tau) \overset{a.e}{=} {}^{\mathbb{T}}\mathfrak{J}^{\alpha+\beta}\mathfrak{g}(\tau).$$

Lemma 2.9. *If $\mathfrak{g} \in C_\gamma(J)$ and ${}^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g} \in C^1_\gamma(J)$, then*

$${}^{\mathbb{T}}\mathfrak{J}^{\alpha}\ {}^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}(\tau) = \mathfrak{g}(\tau) - \frac{\left({}^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g}\right)(0)}{\Gamma(\alpha)}(\psi(\tau)-\psi(0))^{\alpha-1}.$$

Lemma 2.10. *Suppose $\alpha > 0$, $a(\tau)$ is a nonnegative function locally integrable on $0 \leq \tau < b$ (some $b \leq \infty$), and let $g(\tau)$ be a nonnegative, nondecreasing continuous function defined on $0 \leq \tau < b$, such that $g(\tau) \leq K$ for some constant K. Further let $\mathfrak{u}(\tau)$ be a nonnegative locally integrable on $0 \leq \tau < b$ function with*

$$|\mathfrak{u}(\tau)| \leq a(\tau) + g(\tau)\int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\mathfrak{u}(s)\Delta s,$$

with some $\alpha > 0$. Then

$$|\mathfrak{u}(\tau)| \leq a(\tau) + \int_0^\tau \left[\sum_{n=1}^{\infty}\frac{(g(\tau)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi'(s)(\psi(\tau)-\psi(s))^{n\alpha-1}\right]\mathfrak{u}(s)\Delta s.$$

Theorem 2.11. *(Schauder FPT) Let $\mathscr{E}$ be a Banach space and $\mathscr{Q}$ be a nonempty bounded convex and closed subset of $\mathscr{E}$ and $\mathscr{N} : \mathscr{Q} \to \mathscr{Q}$ is compact, and continuous map. Then $\mathscr{N}$ has at least one fixed point in $\mathscr{Q}$.*

3. Existence results

Lemma 3.1. *Here* $\mathfrak{u}$ *is solution of* (1.1) *if and only if* $\mathfrak{u}$ *satisfies the following integral equation*

$$\mathfrak{u}(\tau)=\frac{\mathfrak{u}_0}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))\Delta s,\quad t>0. \tag{3.1}$$

For further investigation, we give the following assumptions:

(H1) The function $\mathfrak{g}:J\times R\to R$ is a rd-continuous.
(H2) There exists a positive constants $L>0$ such that

$$|\mathfrak{g}(\tau,\mathfrak{u})-\mathfrak{g}(\tau,\mathfrak{v})|\leq L|\mathfrak{u}-\mathfrak{v}|.$$

(H3) There exists an increasing function $\varphi\in C_{1-\gamma}(J)$ and there exists $\lambda_\varphi>0$such that for any $\tau\in J$,

$$^{\mathbb{T}}\mathfrak{I}^{\alpha}\varphi(\tau)\leq\lambda_\varphi\varphi(\tau).$$

Theorem 3.2. *Assume that (H1)-(H2) are fulfilled. Then, equation* (1.1) *has at least one solution.*

Proof. Consider the operator $\mathscr{P}:C_{1-\gamma,\psi}(J)\to C_{1-\gamma,\psi}(J)$. The equivalent Volterra integral equation (3.1) which can be written in the operator form

$$(\mathscr{P}\mathfrak{u})(\tau)=\mathfrak{u}_0(\tau)+\frac{1}{\Gamma(\alpha)}\int_0^\tau\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))\Delta s$$

with

$$\mathfrak{u}_0(\tau)=\frac{\mathfrak{u}_0}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}.$$

Define $B_r=\left\{\mathfrak{u}\in C_{1-\gamma,\psi}(J):\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}\leq r\right\}$.
Set $\tilde{\mathfrak{g}}(s)=\mathfrak{g}(s,0)$,

$$\sigma=\frac{|\mathfrak{u}_0|}{\Gamma(\gamma)}+\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b)-\psi(0)\right)^{\alpha}\|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}$$

and

$$\omega=\frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b)-\psi(0)\right)^{\alpha}.$$

To verify Theorem 2.11, we divide the proof into three steps.
Step 1: We check that $\mathscr{P}(B_r)\subset B_r$.

$$\begin{aligned}
&\left|\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}(\mathscr{P}\mathfrak{u})(\tau)\right|\\
&\leq\frac{|\mathfrak{u}_0|}{\Gamma(\gamma)}+\frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s))|\Delta s\\
&\leq\frac{|\mathfrak{u}_0|}{\Gamma(\gamma)}+\frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s))-\mathfrak{g}(s,0)|\Delta s\\
&\quad+\frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,0)|\Delta s\\
&\leq\frac{|\mathfrak{u}_0|}{\Gamma(\gamma)}+\frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}L|\mathfrak{u}|\Delta s\\
&\quad+\frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\tilde{\mathfrak{g}}|\Delta s\\
&\leq\frac{|\mathfrak{u}_0|}{\Gamma(\gamma)}+\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b)-\psi(0)\right)^{\alpha}\|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}+\frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b)-\psi(0)\right)^{\alpha}\|u\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

Hence

$$\|(\mathscr{P}\mathfrak{u})\|\leq\sigma+\omega r\leq r.$$

Which yields that $\mathscr{P}(B_r)\subset B_r$.
Next, the completely continuous of operator $\mathscr{P}$ is proved.
Step 2: The operator $\mathscr{P}$ is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $C_{1-\gamma,\psi}(J)$.

$$\begin{aligned}
&\left|(\psi(\tau)-\psi(0))^{1-\gamma}((\mathscr{P}\mathfrak{u}_n)(t)-(\mathscr{P}\mathfrak{u})(\tau))\right| \\
&\le \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}_n(s))-\mathfrak{g}(s,\mathfrak{u}(s))|\Delta s \\
&\le \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\sup_{s\in J}|\mathfrak{g}(s,\mathfrak{u}_n(s))-\mathfrak{g}(s,\mathfrak{u}(s))|\Delta s \\
&\le \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}_n(s))-\mathfrak{g}(s,\mathfrak{u}(s))|ds, \quad \text{(by Proposition 2.2)} \\
&\le \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(b)-\psi(0))^{\alpha}\|\mathfrak{g}(\cdot,\mathfrak{u}_n(\cdot))-\mathfrak{g}(\cdot,\mathfrak{u}(\cdot))\|_{C_{1-\gamma,\psi}},
\end{aligned}$$

Since $\mathfrak{g}$ is continuous, Lebesgue dominated convergence theorem implies

$$\|\mathscr{P}\mathfrak{u}_n-\mathscr{P}\mathfrak{u}\|_{C_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n\to\infty.$$

Step 3: $\mathscr{P}(B_r)$ is relatively compact.
Thus $\mathscr{P}(B_r)$ is uniformly bounded. Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$, then

$$\begin{aligned}
&\left|(\mathscr{P}\mathfrak{u})(\tau_2)(\psi(\tau_2)-\psi(0))^{1-\gamma}-(\mathscr{P}\mathfrak{u})(\tau_1)(\psi(\tau_1)-\psi(0))^{1-\gamma}\right| \\
&\le \left|\frac{(\psi(\tau_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))\Delta s\right. \\
&\quad\left.-\frac{(\psi(\tau_1)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau_1}\psi'(s)(\psi(\tau_1)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))\Delta s\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\psi'(s)\left|(\psi(\tau_2)-\psi(0))^{1-\gamma}(\psi(\tau_2)-\psi(s))^{\alpha-1}-(\psi(\tau_1)-\psi(0))^{1-\gamma}(\psi(\tau_1)-\psi(s))^{\alpha-1}\right||\mathfrak{g}(s,\mathfrak{u}(s))|\Delta s \\
&\quad+\frac{(\psi(t_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s))|\Delta s \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\psi'(s)\left|(\psi(\tau_2)-\psi(0))^{1-\gamma}(\psi(\tau_2)-\psi(s))^{\alpha-1}-(\psi(\tau_2)-\psi(0))^{1-\gamma}(\psi(\tau_1)-\psi(s))^{\alpha-1}\right||\mathfrak{g}(s,\mathfrak{u}(s))|ds \\
&\quad+\frac{(\psi(\tau_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}(\psi(\tau_2)-\psi(\tau_1))^{\alpha+\gamma-1}B(\gamma,\alpha)\|\mathfrak{g}\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

Thus, right-hand part tends to zero. Hence along with the Arzëla-Ascoli theorem and from Step 1-3, it is concluded that $\mathscr{P}$ is completely continuous. Thus the proposed problem has at least one solution. □

Lemma 3.3. *Assume that (H1) and (H3) are fulfilled. If*

$$\left(\frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(b)-\psi(0))^{\alpha}\right)<1 \tag{3.2}$$

then there exists unique solution for Eq. (1.1).

Proof. Define the operator $\mathscr{P}: C_{1-\gamma,\psi}(J)\to C_{1-\gamma,\psi}(J)$.

$$(\mathscr{P}\mathfrak{u})(\tau)=\mathfrak{u}_0(\tau)+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))\Delta s$$

with $\mathfrak{u}_0(\tau)=\frac{\mathfrak{u}_0}{\Gamma(\gamma)}(\psi(\tau)-\psi(0))^{\gamma-1}$.
Let $\mathfrak{u}_1,\mathfrak{u}_2\in C_{1-\gamma,\psi}(J)$ and $\tau\in J$, then

$$\begin{aligned}
&\left|(\psi(\tau)-\psi(0))^{1-\gamma}((\mathscr{P}\mathfrak{u}_1)(\tau)-(\mathscr{P}\mathfrak{u}_2)(\tau))\right| \\
&\le \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}_1(s))-\mathfrak{g}(s,\mathfrak{u}_2(s))|\Delta s \\
&\le \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}_1(s))-\mathfrak{g}(s,\mathfrak{u}_2(s))|ds \\
&\le \frac{L(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau\psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}|\mathfrak{u}_1(s)-\mathfrak{u}_2(s)|ds \\
&\le \frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(b)-\psi(0))^{\alpha}\|\mathfrak{u}_1-\mathfrak{u}_2\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

Then,

$$\|\mathscr{P}\mathfrak{u}_1-\mathscr{P}\mathfrak{u}_2\|_{C_{1-\gamma,\psi}}\le\frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(b)-\psi(0))^{\alpha}\|\mathfrak{u}_1-\mathfrak{u}_2\|_{C_{1-\gamma,\psi}}.$$

From (3.2), it follows that $\mathscr{P}$ has a unique fixed point which is solution of problem (1.1). □

4. Stability analysis

Next, we shall give the definitions and the criteria generalized UHR stability.

Definition 4.1. *Equation* (1.1) *is generalized UHR stable with respect to* $\varphi \in C_{1-\gamma}(J)$ *if there exists a real number* $c_{\mathfrak{g},\varphi} > 0$ *such that for each solution* $\mathfrak{v} \in C_{1-\gamma}(J)$ *of the inequality*

$$\left| {}^{\mathbb{T}}\Delta^{\alpha,\beta}\mathfrak{v}(\tau) - \mathfrak{g}(\tau, \mathfrak{v}(\tau)) \right| \leq \varphi(t), \tag{4.1}$$

there exists a solution $\mathfrak{u} \in C^{\gamma}_{1-\gamma}(J)$ *of equation* (1.1) *with*

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq c_{\mathfrak{g},\varphi}\varphi(\tau).$$

Theorem 4.2. *Assume that (H1), (H3), (H4) and* (3.2) *are satisfied. Then, the problem* (1.1) *is generalized UHR stable.*

Proof. Let $\mathfrak{v} \in C_{1-\gamma}(J)$ be solution of the following inequality (4.1) and let $\mathfrak{u} \in C_{1-\gamma}(J)$ be the unique solution of the ψ-Hilfer type dynamics equation (1.1). By Lemma 3.1,

$$\mathfrak{u}(\tau) = \mathfrak{u}_0(\tau) + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)\,(\psi(\tau) - \psi(s))^{\alpha-1}\,\mathfrak{g}(s, \mathfrak{u}(s))\Delta s.$$

By integration of (4.1) we obtain

$$\left| \mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)\,(\psi(\tau) - \psi(s))^{\alpha-1}\,\mathfrak{g}(s, \mathfrak{v}(s))\Delta s \right| \leq \lambda_\varphi \varphi(\tau).$$

On the other hand, we have

$$\begin{aligned}
|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| &\leq \left| \mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)\,(\psi(\tau) - \psi(s))^{\alpha-1}\,\mathfrak{g}(s, \mathfrak{v}(s))\Delta s \right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)\,(\psi(\tau) - \psi(s))^{\alpha-1}\,|\mathfrak{g}(s, \mathfrak{v}(s)) - \mathfrak{g}(s, \mathfrak{u}(s))|\,\Delta s \\
&\leq \left| \mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)\,(\psi(\tau) - \psi(s))^{\alpha-1}\,\mathfrak{g}(s, \mathfrak{v}(s))\Delta s \right| \\
&\quad + \frac{L}{\Gamma(\alpha)}\int_0^t \psi'(s)\,(\psi(\tau) - \psi(s))^{\alpha-1}\,|\mathfrak{v}(s) - \mathfrak{u}(s)|\,ds \\
&\leq \lambda_\varphi \varphi(\tau) + \frac{L}{\Gamma(\alpha)}\int_0^t \psi'(s)\,(\psi(\tau) - \psi(s))^{\alpha-1}\,|\mathfrak{v}(s) - \mathfrak{u}(s)|\,ds.
\end{aligned}$$

By applying Lemma 2.10, we obtain

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq \left[(1 + \nu_1 L\lambda_\varphi)\lambda_\varphi\right]\varphi(\tau),$$

where $\nu_1 = \nu_1(\alpha)$ is a constant,then for any $\tau \in J$:

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq c_{\mathfrak{g}}\varepsilon\varphi(\tau),$$

Thus, the proof is complete. □

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