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Some New Cauchy Sequence Spaces

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Abstract

In this paper, our goal is to introduce some new Cauchy sequence spaces. These spaces are defined by Cauchy transforms. We shall use notations $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ for these new sequence spaces. We prove that these new sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ are the $BK-$spaces and isomorphic to the spaces l_∞, c and c_0, respectively. Besides the bases of these spaces, $\alpha-$, $\beta-$ and $\gamma-$ duals of these spaces will be given. Finally, the matrix classes $(C(s,t):l_p)$ and $(C(s,t):c)$ have been characterized.

1. Preliminaries, background and notation

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called as a sequence space. We shall write l_∞, c, c_0 and l_p for the spaces of all bounded, convergent, null and absolutely $p-$summable sequences which are given by

$$l_\infty=\left\{x=(x_k)\in w:\sup_{k\to\infty}|x_k|<\infty\right\},$$

$$c=\left\{x=(x_k)\in w:\lim_{k\to\infty}x_k \text{ exists}\right\},$$

$$c_0=\left\{x=(x_k)\in w:\lim_{k\to\infty}x_k=0\right\}$$

and

$$l_p=\left\{x=(x_k)\in w:\sum_{k=0}^{\infty}|x_k|^p<\infty, 1\le p<\infty\right\}.$$

Also by bs, cs and l_1, we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.
A sequence space λ with a linear topology is called an $K-$ space provided of the maps $p_i:\lambda\to\mathbb{C}$ defined by $p_i(x)=x_i$ is continuous for all $i\in\mathbb{N}$; where $\mathbb{C}$ denotes the set of complex number and $\mathbb{N}=\{0,1,2,...\}$. Let λ be an $K-$ space. Then λ is called an $FK-$ space provided λ is a complete linear metric space. An $FK-$ space provided whose topology is normable is called a $BK-$ space [1].
Let X, Y be any two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real numbers, where n, $k\in\mathbb{N}$. Then, we write $Ax=((Ax)_n)$, the $A-$transform of x, if $A_n(x)=\sum_k a_{nk}x_k$ converges for each $n\in N$. If for every sequence $x=(x_k)\in X$, $A-$transform of x sequence Ax is in Y. Then we say that A defines a matrix transformation from X into Y and denote it by $A:X\to Y$. By $(X:Y)$ we mean the class of all infinite matrices such that $A:X\to Y$.
Let F denote the collection of all finite subsets on $\mathbb{N}$ and K, $\mathbb{N}\subset F$. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A=\{x=(x_k)\in w:Ax\in X\} \tag{1.1}$$

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which is a sequence space.
The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been employed by many authors recently. They introduced the sequence spaces $(c_0)_{T^r} = t_0^r$ and $(c)_{T^r} = t_c^r$ in [2], $(c_0)_{E^r} = e_0^r$ and $(c)_{E^r} = e_c^r$ in [3], $(c_0)_C = \bar{c}_0$ and $c_C = \bar{c}$ in [3], $(l_p)_{E^r} = e_p^r$ in [4], $(l_\infty)_{R^t} = r_\infty^t$, $c_{R^t} = r_c^t$ and $(c_0)_{R^t} = r_0^r$ in [5], $(l_p)_C = X_p$ in [6] and $(l_p)_{N_q}$ in [7] where T^r, E^r, C, R^t and N_q denote the Taylor, Euler, Cesaro, Riesz and Nörlund means, respectively. In recent years, constructing a new sequence space by means of the domain of an infinite matrix was used by Candan [8, 9], Altay [10], Altay and Başar [11], Aydın and Başar [12], Başar [13, 14], Başar, Altay and Mursaleen [15], Polat and Başar [16].
Following [2]-[7], [17] by the same way, to introduce the new Cauchy sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ is the purpose of this paper.

2. The Cauchy matrix of inverse formula and Cauchy sequence spaces

Given two vectors s and t such that $s_i \neq -t_j$ for all i and j , the $n\times n$ matrix $C = C(s,t)$ is a Cauchy (generalized Hilbert) matrix [18] where $C(s,t) = c_{ij} = [\frac{1}{s_i+t_j}]_{i,j=0}^{n-1}$. The inverse of Cauchy's Matrix [19] is given by

$$C^{-1}(s,t) = c_{ij}^{-1} = \frac{\prod\limits_{1\le k\le n} (s_j+t_k)(s_k+t_i)}{(s_j+t_i)\left[\prod\limits_{\substack{1\le k\le n\\ j\neq k}} (s_j-s_k)\right]\left[\prod\limits_{\substack{1\le k\le n\\ i\neq k}} (t_i-t_k)\right]}. \tag{2.1}$$

$C(s,t)$ denotes the Cauchy mean defined by the matrix $C(s,t) = (c_{ij})$, $c_{ij} = [\frac{1}{s_i+t_j}]_{i,j=1}^{n}$ for each $n\in\mathbb{N}$.
We introduce the Cauchy sequence spaces,

$$C_\infty(s,t) = \left\{x = (x_k)\in w : \sup_n \left|\sum_{k=1}^{n} \frac{1}{s_n+t_k} x_k\right| < \infty\right\},$$

$$C(s,t) = \left\{x = (x_k)\in w : \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{s_n+t_k} x_k \text{ exists}\right\}$$

and

$$C_0(s,t) = \left\{x = (x_k)\in w : \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{s_n+t_k} x_k = 0\right\}.$$

By means of the notation (1.1), we may redefine the spaces $C_0(s,t)$ and $C(s,t)$ as follows:

$$C_0(s,t) = (c_0)_{C(s,t)} \text{ and } C(s,t) = (c)_{C(s,t)}. \tag{2.2}$$

If λ is any arbitrary normed or paranormed sequence space, then we call the matrix domain $\lambda_{C(s,t)}$ as the Cauchy sequence space. We define the sequence $y = (y_k)$ which will be frequently used, as the $C(s,t)-$ transform of a sequence $x = (x_k)$ i.e.,

$$y_n = \sum_{k=1}^{n} \frac{1}{s_n+t_k} x_k. \tag{2.3}$$

It can be shown easily that $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ are linear and normed spaces by the following norm:

$$\|x\|_{C_0(s,t)} = \|C(s,t)\|_{C_\infty(s,t)} = \sup_n \left|\sum_{k=1}^{n} \frac{1}{s_n+t_k} x_k\right|. \tag{2.4}$$

Theorem 2.1. *The sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ are Banach spaces with the norm (2.4).*

Proof. Let $(x^p) = \left(x_0^{(p)}, x_1^{(p)}, x_2^{(p)}, \ldots\right)$ be a Cauchy sequence in $C_\infty(s,t)$ for all $p\in\mathbb{N}$. Then, there exists $n_0 = n_0(\varepsilon)$ for every $\varepsilon > 0$ such that $\|x^p - x^r\|_\infty < \varepsilon$ for all $p, r > n_0$. Hence, $|C(s,t)\ (x^p - x^r)| < \varepsilon$ for all $p, r > n_0$ and for each $k\in\mathbb{N}$.
Therefore, $(C(s,t)x_k^p) = \left(\left(C(s,t)x^0\right)_k, \left(C(s,t)x^1\right)_k, \left(C(s,t)x^2\right)_k, \ldots\right)$ is a Cauchy sequence in the set of complex numbers $\mathbb{C}$. Since $\mathbb{C}$ is complete, it is convergent we write $\lim\limits_{p\to\infty}(C(s,t)x^p)_k = (C(s,t)x)_k$ and $\lim\limits_{m\to\infty}(C(s,t)x^m)_k = (C(s,t)x)_k$ for each $k\in\mathbb{N}$. Hence, we have $\lim\limits_{m\to\infty}\left|C(s,t)x_k^p - x_k^m\right| = \left|C(s,t)\left(x_k^p - x_k\right) - C(s,t)\left(x_k^m - x_k\right)\right| \le \varepsilon$ for all $n\ge n_0$. This implies that $\|x^p - x^m\|\to\infty$ for $p, m\to\infty$. Now, we should show that $x\in C_\infty(s,t)$. We have

$$\|x\|_\infty = \|C(s,t)x\|_\infty = \sup_n \left|\sum_{k=1}^{n} \frac{1}{s_n+t_k} x_k\right| = \sup_n \left|\sum_{k=1}^{n} \frac{1}{s_n+t_k} (x_k - x_k^p + x_k^p\right|$$

$$\le \sup_n \left|C(s,t)\left(x_k^p - x_k\right)\right| + \sup_n \left|C(s,t)x_k^p\right| \le \|x^p - x\|_\infty + \left|C(s,t)x_k^p\right| < \infty$$

for $p, k\in\mathbb{N}$. This implies that $x = (x_k)\in C_\infty(s,t)$. Thus, $C_\infty(s,t)$ the space is a Banach space with the norm (2.4). □

It can be shown that $C_0(s,t)$ and $C(s,t)$ are closed subspaces of $C_\infty(s,t)$ which leads us to the consequence that the spaces are also the Banach spaces with the norm (2.4). Furthermore, since $C_\infty(s,t)$ is a Banach space with continuous coordinates, i.e., $\left\|C(s,t)\left(x_k^p - x\right)\right\|_\infty \to 0$ imples $\left|C(s,t)\left(x_k^p - x_k\right)\right| \to 0$ for all $k \in N$, it is also a $BK-$ space.

Theorem 2.2. *The sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ are linearly isomorphic to the spaces l_∞, c and c_0 respectively, i.e., $C_\infty(s,t) \cong l_\infty$, $C(s,t) \cong c$ and $C_0(s,t) \cong c_0$.*

Proof. To prove the fact $C_0(s,t) \cong c_0$, we should show the existence of a linear bijection between the spaces $C_0(s,t)$ and c_0. Consider the transformation F defined, with the notation (2.3), from $C_0(s,t)$ to c_0. The linearity of F is clear. Further, it is trivial that $x = 0$ whenever $Fx = 0$ and hence F is injective.

Let $y \in c_0$ and define the sequence $x = (x_k)$ by $x_k = \sum_{j=1}^{k} c_{kj}^{-1} y_j$ for each $k \in \mathbb{N}$. Wherein c_{kj}^{-1} is as defined in (2.1). Then

$$\lim_{n\to\infty}\left(C(s,t)x\right)_n = \lim_{n\to\infty}\sum_{k=1}^{n} c_{nk}x_k = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{s_n+t_k}\sum_{j=1}^{k} c_{kj}^{-1}y_j = \lim_{n\to\infty} y_n = 0.$$

Thus, we have that $x \in C_0(s,t)$ .In addition, note that

$$\|x\|_{C_0(s,t)} = \sup_{n\in\mathbb{N}}\left|\sum_{k=1}^{n}\frac{1}{s_n+t_k}\sum_{j=1}^{k} c_{kj}^{-1}y_j\right| = \sup_{n\in\mathbb{N}}|y_n| = \|y\|_{c_0} < \infty.$$

Consequently, F is surjective and is norm preserving. Hence, F is a linear bijection therefore we say that the spaces $C_0(s,t)$ to c_0 are linearly isomorphic. In the same way, it can be shown that $C(s,t)$ and $C_\infty(s,t)$ are linearly isomorphic to c and l_∞, respectively, and so we omit the detail. □

Theorem 2.3. *The sequence space $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ includes the sequence spaces l_∞, c and c_0 respectively i.e. $l_\infty \subset C_\infty(s,t)$, $c \subset C(s,t)$ and $c_0 \subset C_0(s,t)$.*

Proof. We only prove the conclusion $l_\infty \subset C_\infty(s,t)$ and the rest follows in a similar way. Let $x \in l_\infty$. Then, using (2.3) and (2.4), we obtain that

$$\begin{aligned}\|x\| = \|C(s,t)x\|_\infty &= \sup_n\left|\sum_{k=1}^{n}\frac{1}{s_n+t_k}x_k\right| \\ &\le \sup_n|x_k|\sup_n|C(s,t)| = \|x\|_{C_\infty(s,t)}\end{aligned}$$

it means that $x \in C_\infty(s,t)$.

□

3. The basis for the spaces $C(s,t)$ and $C_0(s,t)$

Firstly, let us define the Schauder basis. A sequence $(b_n)_{n\in\mathbb{N}}$ in a normed sequence space λ is called a Schauder basis (or briefly basis) [20], if for every $x \in \lambda$ there is a unique sequence (α_n) of scalars such that

$$\lim_{n\to\infty}\|x - (\alpha_0x_0 + \alpha_1x_1 + \ldots + \alpha_nx_n)\| = 0.$$

In this section, we shall give the Schauder basis for the spaces $C(s,t)$ and $C_0(s,t)$.

Theorem 3.1. *Let $k \in \mathbb{N}$ be a fixed natural number and $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathbb{N}}$ where $b_n^{(k)} = \left[c_{nk}^{-1}\right]_{k=1}^{n}$, $(n \in \mathbb{N})$. Wherein c_{nk}^{-1} is as defined in (2.1). Then the following assertions are true:*

i. *The sequence $\left\{b_n^{(k)}\right\}$ is a basis for the space $C_0(s,t)$ and every $x \in C_0(s,t)$ has a unique representation of the form $x = \sum_k \lambda_k b^{(k)}$ where $\lambda_k = (C(s,t)x)_k$ for all $k \in \mathbb{N}$. For simplicity, here and thereafter an unlimited sum symbol runs from zero to infinity.*

ii. *The set $\left\{e, b^{(0)}, b^{(1)}, \ldots, b^{(k)}, \ldots\right\}$ is a basis for the space $C(s,t)$ and every $x \in C(s,t)$ has a unique representation of the form $x = le + \sum_k(\lambda_k - l)b^{(k)}$ where $l = \lim_{k\to\infty}(C(s,t)x)_k$ and $\lambda_k = (C(s,t)x)_k$ for all $k \in \mathbb{N}$.*

4. The $\alpha-$, $\beta-$ and $\gamma-$ Duals of the Spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$

In this section, we state and prove the theorems determining the $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$. For the sequence spaces λ and μ, we define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu) = \{z = (z_k) \in w : xz = (x_kz_k) \in \mu \text{ for all } x \in \lambda\}.$$

The $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence spaces λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by Garling [21], by $\lambda^\alpha = S(\lambda, l_1)$, $\lambda^\beta = S(\lambda, cs)$ and $\lambda^\gamma = S(\lambda, bs)$. We shall begin with the lemmas due to Stieglitz and Tietz [22], which are needed in the proof of the Theorems 4.4-4.6.

Lemma 4.1. *$A \in (c_0 : l_1) = (c : l_1)$ if and only if, for $(\alpha_k) \subset \mathbb{R}$*

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty \tag{4.1}$$

Lemma 4.2. *$A \in (c_0 : c)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty \tag{4.2}$$

and

$$\lim_{n \to \infty} a_{nk} = \alpha_k, \ (k \in \mathbb{N}).$$

Lemma 4.3. *$A \in (c_0 : l_\infty)$ if and only if (4.2) holds.*

In the following theorems, we denote by K and F finite subsets of $\mathbb{N}$.

Theorem 4.4. *Let $a = (a_k) \in w$ and define the matrix $B = \left(c_{nk}^{-1} a_n\right)$ for all k, $n \in \mathbb{N}$. The $\alpha-$ dual of the sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ is the set $D = \left\{ a = (a_k) \in w : \sup_{K \in F} \sum_n \left| \sum_{k \in K} c_{nk}^{-1} a_n \right| < \infty \right\}$. Wherein c_{nk}^{-1} is as defined in (2.1) for each k, $n \in \mathbb{N}$.*

Proof. Let $a = (a_n) \in w$ and consider the matrix B whose rows are the products of the rows of the matrix $C^{-1}(s,t)$ and sequence $a = (a_n)$. Bearing in mind the relation (2.3), we immediately derive that

$$a_n x_n = \sum_{k=1}^{n} c_{nk}^{-1} a_n y_k = (By)_n, \ (n \in \mathbb{N}). \tag{4.3}$$

We therefore observe by (4.3) that $ax = (a_n x_n) \in l_1$ whenever $x \in C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ if and only if $By \in l_1$ whenever $y \in l_\infty$, c, and c_0. Then, by means of Lemma 4.1, we get $\sup_{K \in F} \sum_n \left| \sum_{k \in K} c_{nk}^{-1} a_n \right| < \infty$ which yields the consequences that $\{C_\infty(s,t)\}^\alpha = \{C(s,t)\}^\alpha = \{C_0(s,t)\}^\alpha = D$. □

Theorem 4.5. *Let us consider the sets B_1, B_2, B_3 and B_4 defined as follows:*

$$B_1 = \left\{ a = (a_k) \in w : \sup_n \sum_{k=1}^{n} \left| \sum_{j=k}^{n} c_{jk}^{-1} a_j \right| < \infty \right\},$$

$$B_2 = \left\{ a = (a_k) \in w : \sum_{j=k}^{\infty} c_{jk}^{-1} a_j \ exists \ for \ each \ k \in \mathbb{N} \right\},$$

$$B_3 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=1}^{n} \left| \sum_{j=k}^{n} c_{jk}^{-1} a_j \right| = \sum_{k=1}^{n} \left| \lim_{n \to \infty} \sum_{j=k}^{n} c_{jk}^{-1} a_j \right| \right\}$$

and

$$B_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=1}^{n} \sum_{j=k}^{n} c_{jk}^{-1} a_j \ exists \right\}.$$

Wherein c_{jk}^{-1} is as defined in (2.1) for each j, $k \in \mathbb{N}$. Then $\{C_0(s,t)\}^\beta = B_1 \cap B_2$, $\{C(s,t)\}^\beta = B_1 \cap B_2 \cap B_4$ and $\{C_\infty(s,t)\}^\beta = B_2 \cap B_3$.

Proof. We only give the proof for the space $C_0(s,t)$. Since the proof may give by a similar way for the spaces $C(s,t)$ and $C_\infty(s,t)$, we omit others. Consider the equation

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} \left[\sum_{j=1}^{k} c_{kj}^{-1} y_j \right] a_k = \sum_{k=1}^{n} \left[\sum_{j=k}^{n} c_{kj}^{-1} a_j \right] y_k = (By)_n,$$

where $B = (b_{nk})$ is defined by $b_{nk} = \sum_{j=k}^{n} c_{nj}^{-1} a_j$, $(n, k \in \mathbb{N})$. Thus, we deduce from Lemma 4.2 with (4.2) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k \in) C_0(s,t)$ if and only if $By \in c$ whenever $y = (y_k) \in c_0$. Therefore, we observe using relations (4.1) and (4.2), we conclude that $\lim_{n \to \infty} c_{nk}^{-1}$ exists for each n, $k \in \mathbb{N}$ and $\sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} c_{nk}^{-1} \right| < \infty$. Thus, we obtain $\{C_0(s,t)\}^\beta = B_1 \cap B_2$. □

Theorem 4.6. *The $\gamma-$ dual of the sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ is the set B_1.*

Proof. This theorem can be proved using the same technique as in the proof of Theorem 4.4 with Lemma 4.3 instead of Lemma 4.2. So, we omit the details. □

5. Some matrix mappings related to Cauchy sequence spaces

Lemma 5.1. *[22, p. 57] The matrix mappings between $BK-$ spaces are continuous.*

Lemma 5.2. *[22, p. 128] $A \in (c : l_p)$ if and only if*

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} a_{nk} \right|^p < \infty, \ (1 \leq p < \infty) \tag{5.1}$$

Theorem 5.3. *$A \in (C(s,t) : l_p)$ if and only if the following conditions are satisfied*

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |g_{nk}| < \infty \,, \tag{5.2}$$

$$\lim_{n \to \infty} g_{nk} \text{ exists for all } k \in \mathbb{N}, \tag{5.3}$$

$$\lim_{n \to \infty} \sum_{k=0}^{n} g_{nk} \text{ converges for all } n \in \mathbb{N}, \tag{5.4}$$

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} g_{nk} \right|^p < \infty \,, (1 \leq p < \infty) \tag{5.5}$$

and

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |g_{nk}|^p < \infty, \ (p = \infty) \tag{5.6}$$

where $g_{nk} = \sum_{j=k}^{n} c_{kj}^{-1} a_{nj}$ and c_{kj}^{-1} is defined by (2.1).

Proof. Let $1 \leq p < +\infty$. Assume that conditions (5.2)-(5.6) are satisfied and take any $x \in C(s,t)$. Then $(a_{nk}) \in (C(s,t))^{\beta}$ for all $k, n \in \mathbb{N}$, which implies that Ax exists. We define the matrix $G = (g_{nk})$ for all $n, k \in \mathbb{N}$. Then, since condition (5.1) is satisfied for the matrix G, we have $G \in (c : l_p)$. Now consider the following equality obtained from the s th partial sum of the series $\sum_k a_{nk} x_k$:

$$\sum_{k=1}^{s} a_{nk} x_k = \sum_{k=1}^{s} \sum_{j=k}^{s} c_{jk}^{-1} a_{nj} y_k$$

$(s, n \in \mathbb{N})$. Therefore, we derive from that as $s \to \infty$ that

$$\sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} g_{nk} y_k \tag{5.7}$$

$(n \in \mathbb{N})$. Whence taking l_p- norm we get

$$\|Ax\|_{l_p} = \|Gy\|_{l_p} < \infty.$$

This means that $A \in (C(s,t) : l_p)$. Now let $p = \infty$. Assume that conditions (5.2)-(5.6) are satisfied and take any $x \in C(s,t)$. Then $(a_{nk}) \in (C(s,t))^{\beta}$ for all $k, n \in \mathbb{N}$, which implies that Ax exists. Whence taking $l_\infty-$ norm (5.7)

$$\|Ax\|_{l_\infty} = \sup_{n \in N} \left| \sum_k g_{nk} \right| \leq \|y\|_{l_\infty} \sup_{n \in N} \sum_k |g_{nk}| < \infty.$$

Then, we have $A \in (C(s,t) : l_\infty)$.
Conversely, assume that $A \in (C(s,t) : l_p)$. Then, since $C(s,t)$ and l_p are $BK-$ spaces, it follows from Lemma 5.1 that there exists a real constant $K > 0$ such that

$$\|Ax\|_{l_p} = K \|x\|_{C(s,t)}$$

for all $x \in C(s,t)$. Since inequality **??** also holds for the sequence $x = (x_k) = \sum_{k \in F} b^{(k)} \in C(s,t)$ where $b_n^{(k)} = \left[c_{nk}^{-1}\right]_{k=1}^{n}$, $(n \in \mathbb{N})$. Wherein c_{nk}^{-1} is as defined in 2.1. We have $\|Ax\|_{l_p} = [\sum_n |\sum_{k \in F} g_{nk}|^p]^{\frac{1}{p}} \leq K \|x\|_{C(s,t)} = K$ which shows the necessity of 5.5. □

Theorem 5.4. *$A \in (C(s,t) : c)$ if and only if conditions are satisfied*

$$g_{nk} \text{ exists for all } n,\, k \in \mathbb{N}, \tag{5.8}$$

$$\sup_n \sum_k |g_{nk}| < \infty \text{ for all } n,\, k \in \mathbb{N}, \tag{5.9}$$

$$\lim_n g_{nk} = \alpha_k \text{ for all } k \in \mathbb{N} \tag{5.10}$$

and

$$\lim_n \sum_k g_{nk} = \alpha \tag{5.11}$$

where $g_{nk} = \sum_{j=k}^{n} c_{kj}^{-1} a_{nj}$ and c_{kj}^{-1} is defined by (2.1).

Proof. Assume that A satisfies conditions (5.8)-(5.11). Let us take an arbitrary an $x = (x_k)$ in $C(s,t)$ such that $x_k \to l$ as $k \to \infty$. Then Ax exists and it is trivial that the sequence $y = (y_k)$ associated with the sequence $x = (x_k)$ by relation (2.3) belongs to c and is such that $y_k \to l$ as $k \to \infty$. At this stage, it follows from (5.4) and (5.6) that

$$\sum_{i=0}^{k} |\alpha_i| \leq \sup_n \sum_i |g_{ni}| < \infty$$

for every $k \in \mathbb{N}$. This yield $\alpha_k \in l_1$. Considering $\sum_k a_{nk} x_k = \sum_k g_{nk} y_k$ we write

$$\sum_k a_{nk} x_k = \sum_k g_{nk}(y_k - l) + l \sum_k g_{nk} y_k \tag{5.12}$$

In this situation, letting $n \to \infty$ in (5.6), we establish that the first term on the right-hand side tends to $\sum_k \alpha_k (y_k - l)$ by (5.3) and (5.4) and the second term tends to $l\alpha$ by (5.12). Taking these facts into account, we deduce from (5.12) as $n \to \infty$ that $(Ax)_n = \sum_k \alpha_k (y_k - l) + l\alpha$ which shows that $A \in (C(s,t) : c)$.
Conversely, assume that $A \in (C(s,t) : c)$. Then, since the inclusion $c \subset l_\infty$ holds the necessity of (5.10), (5.12) is immediately obtained from $\sup_n \sum_k |b_{nk}| < \infty$. To prove the necessity of (5.11) consider the sequence $x = b^{(k)} = \left\{ b_n^{(k)} \right\}_{n \in \mathbb{N}}$ in $C(s,t)$ which defined above for every fixed $k \in \mathbb{N}$. Since Ax exists and belongs to c for every $x \in C(s,t)$, one can easily see that $Ab^{(k)} = \left\{ b_n^{(k)} \right\}_{n \in \mathbb{N}}$ for each $k \in \mathbb{N}$, which yields the necessity of (5.11). Similarly, by setting $x = e$ in (5.7), we obtain $Ax = \{\sum_k g_{nk}\}_{n \in \mathbb{N}}$, which belongs to the space c, and this shows the necessity of (5.12). Where $g_{nk} = \sum_{j=k}^{n} c_{kj}^{-1} a_{nj}$ and c_{kj}^{-1} is defined by (2.1). This step conludes the proof. □

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