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On norm-preserving isomorphisms of $L^p(\mu, H)$

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Abstract

Given an arbitrary positive measure space $(X, \mathcal{A}, \mu)$ and a Hilbert space H. In this article we give a new proof for the characterization theorem of the surjective linear isometries of the space $L^p(\mu, H)$ (for $1 \leq p < \infty$, $p \neq 2$) which is essentially different from the existing one, and depends on the p-projections of $L^p(\mu, H)$. We generalize the known characterization of the p-projections of $L^p(\mu, H)$ for σ-finite measure to the arbitrary case. They are proved to be the multiplication operations by the characteristic functions of the locally measurable sets, or that of the clopen (closed-open) subsets of the hyperstonean space the measure μ determines.

Keywords: Measure space, Bochner space, perfect measure, hyperstonean space, linear isometries.

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1. Introduction

The isometric theory of Banach spaces still fascinates some mathematicians. One of the main problems in this area is to characterize the linear isometries on or between these spaces. It might seem to be relatively easy at first glance but, generally speaking, it is indeed very difficult a problem to solve. It would be very unrealistic to expect that there might be a complete solution of this problem. However, for some subclasses there has been a great progress in that direction. For instance, the surjective isometries of $C(X)$ or L^p type classical Banach spaces this problem is solved completely, but the case of into isometries is still far from being settled.

Let $(X, \mathcal{A}, \mu)$ be a positive measure space and H a Hilbert space. For any $1 \leq p < \infty$, $p \neq 2$, the Bochner space $L^p(X, \mathcal{A}, \mu; H)$ will be denoted by $L^p(\mu, H)$, if there is no

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ambiguity about the underlying measurable space. For definitions and properties of these spaces we refer to [6].

A *regular set isomorphisms* on $\mathcal{A}$, defined modulo null sets, means a mapping on $\mathcal{A}$ to $\mathcal{A}$ with (i) $\varphi(A') = \varphi X \setminus \varphi A$ for every A in $\mathcal{A}$, where A' denotes the complement of A, (ii) $\varphi(\bigcup A_n) = \bigcup \varphi A_n$ for any sequence $\langle A_n \rangle$ in $\mathcal{A}$ mutually disjoint sets, and (iii) $\mu(\varphi A) = 0$ if, and only if, $\mu(A) = 0$. Any such mapping defines a function Φ on the set of measurable functions which we call the *induced* map. It is characterized by $\Phi(\chi_A e) = \chi_{\varphi A} e$, $A \in \mathcal{A}$, $e \in H$, whereχ_A denotes the characteristic function of A (see [7, pp. 453-454]).

The characterization of the surjective linear isometries of L^p spaces was started by Banach [1] for the Lebesgue measure λ on the closed interval $[0, 1]$. He proved that for every linear isometry T of $L^p(\lambda)$, $1 \le p < \infty$, $p \ne 2$, there exists a measurable function σ of $[0, 1]$ such that for $f \in L^p(\lambda)$

$$(Tf)(x) = h(x) f(\sigma(x)) \text{ a.e. on } [0, 1].$$

If ϕ is the regular set isomorphism defined by $\phi(A) = \sigma^{-1}(A)$ on the Borel algebra of $[0, 1]$, then the above representation becomes

$$(1.1) \qquad (Tf)(x) = h(x)\Phi(f)(x)) \text{ a.e. on } [0, 1].$$

In [11], Lamperti proves that for any σ-finite measure space $(\Omega, \mathcal{A}, \mu)$, the linear isometries of $L^p(\mu)$ onto itself, $1 \le p < \infty$, $p \ne 2$, are indeed of the above form (1) except that the isomorphism ϕ of the σ-algebra $\mathcal{A}$, need not be defined by a point mapping. Moreover, if the measure ν is defined by $\nu(A) = \mu[\phi^{-1}(A)]$, $A \in \mathcal{A}$, then

$$(1.2) \qquad |h(x)|^p = d\nu/d\mu \text{ a.e. on } \Omega.$$

In [3], Cambern generalizes this result to the Bochner spaces. He proves that if $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space and H is a separable Hilbert space, then for any linear isometry of $L^p(\mu, H)$ onto itself, $1 \le p < \infty$, $p \ne 2$, in addition to the maps h and Φ in Lamperti's characterization now there also exists a weakly measurable operator-valued function U defined on X, where $U_x = U(x)$ is an isometry of H onto itself for almost all $x \in X$, such that for $F \in L^p(\mu, H)$,

$$(1.3) \qquad (Tf)(x) = U_x(h(x)\Phi(F)(x)) \text{ a.e. on } X.$$

In [9], Greim and Jamison obtain the same representation for an arbitrary Hilbert space, but the measure is still σ-finite.

In [5] this characterization is extended to perfect measures. In view of the fact that any arbitrary measure space can be replaced by a perfect one without disturbing the L^p spaces for $1 \le p < \infty$, perhaps possibly enlarging the L^∞ space [4], for Hilbert spaces the above result is then the most general result one can get for the surjective isometries.

The purpose of this article is two-fold. Our first goal is to obtain a complete description of the *p-projections* of $L^p(\mu, H)$ (which is known if μ is σ-finite). We will prove that a p-projection of $L^p(\mu, H)$ is the characteristic function of a *locally* measurable set (i.e. its intersection with every set of finite measure is measurable), and of a measurable set if μ is *decomposable,* in particular *perfect.* (For the definition *of a* decomposable measure see [10, p.317].) Our second goal is to use this result to give a second proof for the characterization of the isometries of $L^p(\mu, H)$ which is fundamentally different from the one given in [5], and we shall also demonstrate that in order to prove this characterization theorem one does not have to replace the given measure space by a perfect one, but this will be possible after we characterize the so-called pseudocharacteristic functions on the σ-rign of all σ-finite measurable sets.

2. Norm-Preserving Isomorphisms of $L^p(\mu, H)$

Let us recall that a compact Hausdorff space X is called *extremally disconnected* if the closure of every open subset is open, and a nonnegative extended real-valued Borel measure μ[¶] on X is called *perfect* if

(i) every nonempty open set contains a clopen set with finite positive measure,
(ii) every nowhere dense Borel set has measure zero (equivalently, every closed set with empty interior has measure zero).

A *perfect measure space* $(X, \mathcal{B}, \mu)$ will mean that X is an extremally disconnected Hausdorff space, $\mathcal{B}$ is the Borel algebra on X and μ is a perfect measure. A hyperstonean space is an extremally disconnected compact Hausdorff space on which there is a perfect Borel measure.

Let F be a Banach space, and $1 \leq p < \infty$, $p \neq 2$. A linear mapping P on F is said to be a *p-projection* if $P^2 = P$ and

$$\|x\|^p = \|Px\|^p + \|x - Px\|^p \text{ for all } x \in F.$$

The set $\mathbb{P}_p(F)$ of p-projections on F is a complete Boolean algebra and its Stonean space is hyperstonean [2, pp.11, 25-26].

Given a measure space $(S, \mathcal{A}, \mu)$ and $1 \leq p < \infty$, $p \neq 2$. For any measurable set A, the mapping $f \longrightarrow f\chi_A, f \in L^p(\mu, H)$, is a p-projection on $L^p(\mu, H)$, and if μ is σ-finite the converse is also true [8, pp.124-126].

In this section we shall fix a perfect measure space $(\Omega, \mathcal{B}, \mu)$ which may be assumed to have the property that every locally null set is actually null and show that the p-projections on the Bochner space $L^p(\mu, H), 1 \leq p < \infty$, $p \neq 2$, are of the above form.

Any Borel subset B of Ω is equivalent to a clopen subset of Ω in the sense that there exists a clopen subset U of Ω such that $B \triangle U = (B \setminus U) \cup (U \setminus B)$ is locally null [2, p.31]. Thus any characteristic function χ_A, with A measurable, equals a.e. to the characteristic function of a clopen set.

2.1. Theorem. *For any clopen subset B of Ω, the function $\chi_B : f \longrightarrow f\chi_B$ is a p-projection on $L^p(\mu, H)$, and conversely, every p-projection on $L^p(\mu, H)$ is of this form.*

Proof. Property (i) of μ, together with an application of Zorn's lemma, can be used to prove that there exists a disjoint family $\{\Omega_i : i \in I\}$ of clopen subsets of Ω with positive *finite* measure such that their union is dense in Ω. Therefore, the closed set $\Omega \setminus \bigcup_i \Omega_i$ has measure zero. From this it follows that

$$L^p(\mu, H) = \sum_i \oplus L^p(\Omega_i, H), \ (p\text{-direct sum}).$$

Now let P be a p-projection on $L^p(\mu, H)$. Then by a theorem in [2, p.20], for each $i \in I$ there exists a p-projection P_i on the Banach *subspace* $L^p(\Omega_i, H)$ of $L^p(\mu, H)$ such that $P = \sum_i \oplus P_i$ (direct sum), that is, $P(f) = \sum_i P_i(f_i)$ for all $f \in L^p(\mu, H)$ where each $f_i = f\chi_{\Omega_i}$.

Since for each i, μ is finite on Ω_i, $P_i = \chi_{B_i}$ for some clopen subset B_i contained in Ω_i. Hence $P = \sum_i \oplus P_i = \sum_i \oplus \chi_{B_i} = \chi_B$, where $B = cl(\cup B_i)$. This completes the proof. $\square$

Throughout the section, T will denote a fixed surjective linear isometry on $L^1(\mu, H)$.

2.2. Theorem. *(i) For each measurable set A, the mapping $P = T\chi_A T^{-1}$ is a p-projection on $L^p(\mu, H)$.*

[¶]All measures throughout this paper will be nonnegative.

(ii) The mapping φ defined on the Boolean algebra $\mathcal{K}(\Omega)$ by the equation $T\chi_A T^{-1} = \chi_{\varphi A}$ is an isomorphism of $\mathcal{K}(\Omega)$ onto itself. Moreover, for any sequence $\{A_n\}$ in $\mathcal{K}(\Omega)$, $\varphi(\bigvee_n A_n) = \bigvee_n \varphi(A_n)$ (i.e., $\varphi(cl(\bigcup_n A_n)) = cl(\bigcup_n \varphi(A_n))$.

Proof. (i) Obviously $P^2 = P$ and since T^{-1} is also an isometry and χ_A is a p-projection, for each f in $L^p(\mu, H)$ we have

$$\begin{aligned}\|Pf\|^p + \|f - Pf\|^p &= \left\|T\chi_A T^{-1} f\right\|^p + \left\|TT^{-1}f - T\chi_A T^{-1} f\right\|^p \\ &= \left\|\chi_A T^{-1} f\right\|^p \left\|T^{-1}f - \chi_A T^{-1} f\right\|^p \\ &= \left\|T^{-1} f\right\|^p \\ &= \|f\|^p .\end{aligned}$$

This completes the proof of (i).

(ii) Let B be a clopen subset of Ω, then, $T^{-1}\chi_B T$, being a p-projection, equals χ_A for some A in $\mathcal{K}(\Omega)$. Therefore $\chi_B T = T\chi_A$ which means that $B = \varphi A$, hence φ maps $\mathcal{K}(\Omega)$ onto itself.

Now let A, B be any two sets in $\mathcal{K}(\Omega)$. Then

$$T\chi_{A\cap B} = T(\chi_A \chi_B) = \chi_{\varphi B} T \chi_A = \chi_{\varphi A}\chi_{\varphi B} T = \chi_{\varphi A \cap \varphi B} T$$

which implies that $\varphi(A \cap B) = \varphi A \cap \varphi B$.

Next we show that $\varphi(A \cup B) = \varphi A \cup \varphi B$. First let us assume that $A \cap B = \emptyset$. Then,

$$T\chi_{A\cup B} = T(\chi_A + \chi_B) = (\chi_{\varphi A} + \chi_{\varphi B})T = \chi_{\varphi A \cup \varphi B} T$$

from which it follows that $\varphi(A \cup B) = \varphi A \cup \varphi B$ as claimed.

Now let A, B be any two sets in $\mathcal{K}(\Omega)$, then by the preceding result,

$$\begin{aligned}\varphi(A \cup B) &= [\varphi(A \setminus B) \cup \varphi(A \cap B)] \cup [\varphi(A \cap B) \cup \varphi(B \setminus A)] \\ &= \varphi A \cup \varphi B.\end{aligned}$$

For the last claim in the theorem, let $\{A_n\}$ be a sequence in $\mathcal{K}(\Omega)$. first let us assume that they are mutually disjoint. Let $A = \bigcup_n A_n$ and for each n, $B_n = \bigcup_{i \geq n+1} A_i$. Then $\bigcap_n B_n = \emptyset$ and for $f \in L^p(\mu, H)$, for each n we have

$$\left\| \chi_A^{(t)} f(t) - \sum_{i=1}^{n} \chi_{A_i}^{(t)} f(t) \right\|^p = \chi_{B_n}(t) \left\|f(t)\right\|^p \leq \left\|f(t)\right\|^p$$

for all t in Ω, and since $\|f(t)\|^p$ is integrable, by the dominated convergence theorem [10, p.172] we obtain

$$\lim_n \left\| \chi_A - \sum_{i=1}^{n} \chi_{A_i} f \right\|^p = \lim_n \int_\Omega \chi_{B_n}(.) \left\|f(.)\right\|^p d\mu = 0$$

which means that $\chi_A f = \sum_{i=1}^{\infty} \chi_{A_i} f$ in $L^p(\mu, H)$.

From this, it follows that the series $\sum_{i=1} \chi_{A_i}$ converges pointwise to χ_A (as operators on $L^p(\mu, H)$). Similarly, the series $\sum_i \varphi A_i$ converges pointwise to the operator $\chi_{\bigcup_i A_i}$. Therefore, since each open set and its closure differ by a null set,

$$\begin{aligned}T\chi_{clA} = T\chi_A = T(\sum_n \chi_{A_n}) &= \sum_n T\chi_{A_n} \\ = \sum_n (\chi_{\varphi A_n} T) &= (\chi_{\bigcup_n \varphi A_n})T = (\chi_{cl \bigcup_n \varphi A_n})T\end{aligned}$$

which implies that

$$\varphi(clA) = cl(\bigcup_n \varphi A_n). \tag{2.1}$$

Now let $\{A_n\}$ be any sequence in $\mathcal{K}(\Omega)$ and define $C_1 = A_1$, $C_n = A_n \setminus \bigcup_{i=1}^{n-1}$ for all $n \geq 2$. Then, C_n 's are mutually disjoint and $\bigcup_n C_n = \bigcup_n A_n$, and since φ maps disjoint sets to disjoint set and $C_n \subset A_n$, $\forall n$, it follows that

$$\bigcup_n \varphi(C_n) \subset \bigcup_n \varphi(A_n) \tag{2.2}$$

On the other hand, since $A_n \subset \bigcup_{i=1}^{n} C_i$ we have $\varphi A_n \subset \bigcup_{i=1}^{n} \varphi C_i \subset \bigcup_{i=1}^{\infty} \varphi C_i$ $\forall n$, therefore $\bigcup_n \varphi A_n \subset \bigcup_n \varphi C_n$. Combining this inclusion with (5) we obtain

$$\bigcup_{n=1}^{\infty} \varphi A_n = \bigcup_{n=1}^{\infty} \varphi C_n. \tag{2.3}$$

Hence from (4) and (6) one obtains

$$\begin{aligned}\varphi(clA) &= \varphi(cl\bigcup_n A_n) = \varphi(cl\bigcup_n C_n) = cl(\bigcup_n \varphi C_n)\\ &= cl(\bigcup_n \varphi C_n)\end{aligned}$$

as claimed. This completes the proof of the theorem. □

An element f of $L^p(\mu, H)$ is an equivalence class rather than a function. The support of each function in this class is equivalent to the same clopen set which we shall call the support of f and denote it by *supp*(f).

The following lemmas will be needed for our next theorem.

2.3. Lemma. *For each $f \in L^p(\mu, H)$, $\varphi(supp\ f) = supp(Tf)$.*

Proof. Fix $f \in L^p(\mu, H)$, and let $A = supp(f)$, $S = supp(Tf)$. Since $Tf = T(\chi_A f) = \chi_{\varphi A} Tf$ we conclude that $S \subset \varphi A$. Now let $B \subset \varphi A \setminus S$ be any clopen set with finite measure, and let $u \in H$, $u \neq 0$. Then there exists a function g in $L^p(\mu, H)$ such that $Tg = \chi_B u$. Since $\chi_B u$ and Tf have disjoint supports and T^{-1} maps functions with disjoint supports to functions with disjoint supports [3, p.12] g and f have disjoint supports. Thus, since A and *supp*(g) are disjoint

$$0 = T(\chi_A g) = \chi_{\varphi A} Tg = \chi_{\varphi A} \chi_B u = \chi_B u$$

which means that $B = \emptyset$. Hence $S = \varphi A$, proving our lemma. □

2.4. Corollary. *Each $Y_i = \varphi \Omega_i$ is σ-finite.*

2.5. Lemma. *For each $i \in I$, let $Y_i = \varphi \Omega_i$, and $Y = \bigcup_i Y_i$. Then*

(i) Y' is a closed null set and T maps $L^p(\Omega_i, H)$ onto $L^p(Y_i, H)$;
(ii) φ maps clopen sets in Ω_i onto clopen sets in Y_i.

Proof. For $f \in L^p(\Omega_i, H)$,

$$supp(Tf) = \varphi(supp(f)) \subset \varphi(\Omega_i) = Y_i$$

which shows that for each i, T maps $L^p(\Omega_i, H)$ into $L^p(Y_i, H)$.

Since the sets Y_i are mutually disjoint,

$$L^p(Y,H) = \sum_i \oplus L^p(Y_i,H).$$

Thus, T maps $L^p(\mu,H) = \sum_i \oplus L^p(\Omega_i,H)$ onto $L^p(Y,H)$, which implies that Y' is a closed null set, and that T maps $L^p(\Omega_i,H)$ onto $L^p(Y_i,H)$ for all $i \in I$. This completes the proof of (i).

For (ii) we fix $i \in I$ and show that each clopen set B_i in Y_i is the image under φ of a clopen set in Ω_i. Fix a clopen set $B \subset Y_i$ and let $u \in H$, $u \neq 0$. Then, there exists an f in $L^p(\Omega_i,H)$ such that $Tf = \chi_B u$. By Lemma 2.3,

$$\varphi(supp(f)) = supp(Tf) = supp\chi_B u = B,$$

proving (ii). □

We can show very easily that, for each $i \in I$, the mapping $\mu \circ \varphi^{-1}$ is countably additive on the algebra of all clopen subsets of Y_i; that is, for any sequence $\{B_n\}$ of mutually disjoint clopen subsets of Y_i whose union is also a clopen subset of Y_i,

$$\mu \circ \varphi^{-1}(\bigcup B_n) = \sum_n \mu \circ \varphi^{-1}(B_n)$$

Thus, since Y_i is σ-finite, $\mu \circ \varphi^{-1}$ extends uniquely to a perfect regular Borel measure on the Borel algebra of Y_i [12, p. 120]. We will denote this extension also by $\mu \circ \varphi^{-1}$. Then we define a measure ν and $\mathcal{A}$ by

$$\nu(A) = \sum_i \mu \circ \varphi^{-1}(A \cap \Omega_i), \ A \in \mathcal{A}.$$

We have completed almost all but few details of the proof of the following theorem:

2.6. Theorem. *There exists a locally strongly measurable operator-valued function U and a measurable scalar-valued function h on Ω such that for each $x \in \Omega$, $U_x = U(x)$ is an isometry of H onto itself and that for every f in $L^p(\mu,H)$,*

$$(Tf)(x) = U_x(h(x)\Phi(f)(x)) \ \textit{a.e. on } \Omega$$

where Φ is the isomorphism of $L^p(\mu,H)$ onto itself induced by φ. Moreover,

$$|h|^p = \frac{d\nu}{d\mu},$$

(the Radon-Nikodým derivative). Conversely, every mapping of the above form is an isometry of $L^p(\mu,H)$ onto itself.

Proof. For each $i \in I$, by a theorem of Greim and Jamison [9, p.513], there exists a strongly measurable function $U^{(i)}$ from Y_i into the set of linear surjective isometries of H and a scalar function h_i on Y_i such that for every f in $L^p(\Omega_i,H)$,

$$(Tf)(y) = U_y^{(i)}(h_i(y)\Phi(f)(y)) \text{ a.e. on } Y_i$$

and furthermore,

$$|h_i|^p = \frac{d(\mu \circ \varphi^{-1})}{d\mu_i}$$

where μ_i denotes the restriction of μ to $\mathcal{A}(Y_i)$.

Each measurable set A is equivalent to a unique clopen set A_c, and so, we may extend φ to a regular set isomorphism from $\mathcal{A}$ onto itself, defined modulo null sets, by the equation

$$\varphi(A) = \varphi(A_c), \ A \in \mathcal{A}$$

Since for each $i \in I$, φ maps $\mathcal{A}(\Omega_i)$ (the trace of $\mathcal{A}$ on Ω_i), isomorphically (modulo null sets) onto $\mathcal{A}(Y_i)$, the induced mapping Φ is an isomorphism of $L^p(\Omega_i, H)$ onto $L^p(Y_i, H)$.

Now we let $U = \sum_i U^{(i)}$, $h = \sum_i h_i$ on Y, and on Y' we let $U_y = I$ (the identity operator on H) and $h(y) = 1$. Obviously U is locally strongly measurable, h is measurable, and for each f in $L^p(\mu, H)$,

$$(Tf)(x) = U_x(h(x)\Phi(f)(x))$$

a.e. on Ω, and furthermore,

$$|h|^p = \frac{d\nu}{d\mu}.$$

This completes the proof of our theorem. □

3. The Case of Nonperfect Measures

In the proof of Theorem 1, the isomorphism φ on $\mathcal{K}(\Omega)$ played a crucial role, and it was constructed by the help of the p-projections on $L^p(\mu, H)$ which were characterized as the characteristic functions of the sets in $\mathcal{K}(\Omega)$.

In this section, instead of a perfect measure space we will work with an *arbitrary* measure space $(X, \mathcal{A}, \mu)$, and prove that a similar construction is possible.

Let $\mathbb{A}_\sigma$ denote the Boolean ring of σ-finite measurable sets, (two sets A, B are regarded to be the same if their symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is locally null, i.e., its intersection with every set in $\mathcal{A}_\sigma$ has measure zero), with the ring operations $U + V = U \triangle V$ and $U.V = U \cap V$.

Following [2] we shall call a function $\gamma : \mathcal{A}_\sigma \longrightarrow \mathcal{A}_\sigma$ a *pseudocharacteristic function* (PCF) if $\gamma(AB) = A\gamma(B) = B\gamma(A)$ for all A, B in $\mathcal{A}_\sigma$. Clearly, for each $A \in \mathcal{A}$, the mapping $\gamma_A : B \longrightarrow B \cap A$, $B \in \mathcal{A}_\sigma$ is a PCF on $\mathcal{A}_\sigma$, but examples show that the converse is not always true. However, as we shall see soon that something very close to this is true.

It is known that if $1 \leq p < \infty$, $p \neq 2$, and γ is a PCF on $\mathcal{A}_\sigma$ then the mapping $P : f \longrightarrow f\chi_{\gamma(S(f))}$ is a p-projection on the L^p-space $L^p(\mu)$ of scalar-valued measurable functions, where $s(f) =supp(f)$; and conversely, every p-projection on $L^p(\mu)$ is of this form [2, p.58]. Greim [8] generalizes this result to $L^p(\mu, H)$ for σ-*finite* μ, that is, he proves that a mapping P on $L^p(\mu, H)$ is a p-projection if and only if $P = \chi_A$ for some measurable set A.

In this section the above mentioned representations for the p-projections on $L^p(\mu)$ and $L^p(\mu, H)$ (with μ σ-finite), will be generalized to the p-projections on $L^p(\mu, H)$ for *arbitrary* μ. For this we will need the following:

3.1. Proposition. *Let γ be a PCF on $\mathcal{A}_\sigma$ (defined modulo null sets). Then*
(i) $\gamma(C) \subset \gamma(B) \subset B$ for all B, C in $\mathcal{A}_\sigma$ and $C \subset B$. In particular $\gamma(\emptyset) = \emptyset$,
(ii) for any sequence $\{B_n\}$ in $\mathcal{A}_\sigma$

$$\gamma(\bigcup_n B_n) = \bigcup_n \gamma(B_n) \text{ and } \gamma(\bigcap_n B_n) = \bigcap_n \gamma(B_n),$$

consequently γ maps disjoint sets to disjoint sets, and
(iii) $\gamma(B \setminus C) = \gamma(B) \setminus \gamma(C)$ for any B, C in $\mathcal{A}_\sigma$.

Proof. The proof of (i) is trivial.

(ii) Let $\{B_n\}$ be a sequence in $\mathcal{A}_\sigma$, $B = \bigcup_n B_n$. Then, $\gamma(B_n) = \gamma(B) \cap B_n$ for all n, therefore,

$$\gamma(\bigcap_n B_n) = \gamma(B) \cap (\bigcap_n B_n) = \bigcap_n [\gamma(B) \cap B_n] = \bigcap_n \gamma(B_n).$$

(iii) Let $A = \gamma(B \cup C)$, then since $E \subset F$ implies that $\gamma(E) = E \cap \gamma(F)$,

$$\begin{aligned}\gamma(B \setminus C) &= \gamma((B \setminus C) \cap (B \cap C)) = (B \setminus C) \cap \gamma(B \cup C) \\ &= (B \setminus C) \cap A = (B \cap A) \setminus (C \cap A) = \gamma(B) \setminus \gamma(C).\end{aligned}$$

This completes the proof. □

We will call two measurable sets *μ-disjoint* if their intersection has measure zero. One can apply Zorn's lemma to show that in any measure space there exists a maximal family of mutually μ-disjoint measurable sets with strictly positive finite measure. Any such family will be called a *μ-decomposition* for the measure space.

Let $\{F_i : i \in I\}$ be a *μ-decomposition* for the measure space $(X, \mathcal{A}, \mu)$. Then, it can be shown very easily that every σ-finite measurable set is contained a.e. in the union of a countable subfamily of $\{F_i : i \in I\}$.

Given an arbitrary measure space $(X, \mathcal{A}, \mu)$. We may and will assume that the measure space is *complete* in the sense that every subset of a null set is measurable (i.e., if $A \in \mathcal{A}$, $\mu(A) = 0$ then every subset of A is also in $\mathcal{A}$).

3.2. Theorem. *(i) Let γ be a PCF on $\mathcal{A}_\sigma$. Then there exists a locally measurable subset Y of X such that*

$$\gamma(B) = B \cap Y \text{ for all } B \in \mathcal{A}_\sigma,$$

(ii) For each locally measurable subset Y of X the mapping E_γ defined by

$$E_\gamma(f) = f\chi_{\gamma(S(f))}, \; f \in L^p(\mu, H)$$

is an p-projection on $L^p(\mu, H)$, where $S(f)$ =supp(f) and $\gamma(B) = B \cap Y$; and conversely, every p-projection on $L^p(\mu, H)$ is of this form.

Proof. Let $\{F_i : i \in I\}$ be a μ-decomposition of the measure space $(X, \mathcal{A}, \mu)$, and let B be a σ-finite set. Then, there exist indices $i_1, i_2, \ldots$ in I such that $B \subset F_{i_1} \cup F_{i_2} \cup \ldots$a.e. Let $X_1 = \bigcup_k F_{i_k}$, $B_1 = B \cap X_1$, $B_0 = B \cap (X \setminus X_1)$, and for each $i \in I$ let $Y_i = \gamma(F_i)$, $Y = \bigcup_i Y_i$ and $Z = \gamma(X_1)$. Since B_0 is null, $\gamma(B_0)$ is null. Therefore,

$$\begin{aligned}\gamma(B) &= \gamma(B_1) \cup \gamma(B_0) = \gamma(B_1 \cap X_1) \cup \gamma(B_0) \\ &= B_1 \cap Z \text{ a.e.} \\ &= B \cap Z \text{ a.e.}\end{aligned}$$

Now, for each i, $B \cap (Y_i \setminus Z) = B \cap \gamma(F_i \setminus X_1) \subset B \cap (F_i \setminus X_1) \subset B \cap \gamma(X \setminus X_1) = B_0$, and taking union over i, we obtain $B \cap (Y \setminus Z) \subset B_0$, so $B \cap (Y \setminus Z)$ is null, and since $Y \supset Z$ we have

$$\begin{aligned}\gamma(B) &= (B \cap Z) \cup (B \cap (Y \setminus Z)) \text{ a.e.} \\ &= B \cap Y \text{ a.e.}\end{aligned}$$

which proves (i).

(ii) Now let E be a p-projection on $L^p(\mu)$. Then $\exists$ a PCF γ on $\mathcal{A}_\sigma$ such that

$$E(f) = f\chi_{\gamma(S(f))}, \; f \in L^1(\mu) \tag{3.1}$$

and then by the first part of the theorem, there is a locally measurable set Y such that $\gamma(B) = B \cap Y$ for all $B \in \mathcal{A}_\sigma$.

Now, (7) becomes

$$E(f) = f\chi_{\gamma(S(f))} = f\chi_{S(f)\cap Y} = f\chi_Y \text{ for all } f \in L^1(\mu).$$

Generalization of this result to the Bochner space $L^p(\mu, H)$ is routine. This completes the proof of the theorem. □

3.3. Corollary. *For each L^p-projection P there exists a locally measurable set Y such that*

(3.2) $\quad P(f) = f\chi_Y,\ f \in L^p(\mu, H),$

and conversely, for each locally measurable set Y, the mapping P defined by (8) *is an L^p-projection.*

Clearly the correspondence between the L^p-projections and the locally measurable sets is one-to-one modulo locally null sets.

Now, as in Section 2, using the above characterization of the L^p-projections we can define a set isomorphisms φ from the σ-algebra $\mathcal{M}_\iota$ of all locally measurable sets onto itself defined modulo locally null sets.

3.4. Remark. A natural question is whether or not the characterization obtained for the linear isometries of the Bochner space $L^p(\mu, H)$ onto itself, $1 \leq p < \infty$, $p \neq 2$, holds, should a Banach space replace H as the range space. In general, the answer is in the negative; however, Sourour [13, p.31] was able to replace H by a suitable Banach space.

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