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Multiplicative (generalized)-derivations and left ideals in semiprime rings

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Abstract

Let R be a semiprime ring with center $Z(R)$. A mapping $F : R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if there exists a map $f : R \to R$ (not necessarily a derivation nor an additive map) such that $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$. The objective of the present paper is to study the following identities: (i) $F(x)F(y) \pm [x, y] \in Z(R)$, (ii) $F(x)F(y) \pm x \circ y \in Z(R)$, (iii) $F([x, y]) \pm [x, y] \in Z(R)$, (iv) $F(x \circ y) \pm (x \circ y) \in Z(R)$, (v) $F([x, y]) \pm [F(x), y] \in Z(R)$, (vi) $F(x \circ y) \pm (F(x) \circ y) \in Z(R)$, (vii) $[F(x), y] \pm [G(y), x] \in Z(R)$, (viii) $F([x, y]) \pm [F(x), F(y)] = 0$, (ix) $F(x \circ y) \pm (F(x) \circ F(y)) = 0$, (x) $F(xy) \pm [x, y] \in Z(R)$ and (xi) $F(xy) \pm x \circ y \in Z(R)$ for all x, y in some appropriate subset of R, where $G : R \to R$ is a multiplicative (generalized)-derivation associated with the map $g : R \to R$.

Keywords: Semiprime ring, left ideal, derivation, multiplicative derivation, generalized derivation, multiplicative (generalized)-derivation

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1. Introduction

Throughout the paper R will denote an associative ring with center $Z(R)$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ and is called semiprime if for any $a \in R$, $aRa = \{0\}$ implies that $a = 0$. We shall write for any pair of elements $x, y \in R$ the commutator $[x, y] = xy - yx$ and skew-commutator $x \circ y = xy + yx$. We will frequently use the basic commutator and skew-commutator identities: (i) $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = y[x, z] + [x, y]z$ and (ii) $x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$, $xy \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$ for all $x, y, z \in R$. Let S be a nonempty subset of R. A map $F : R \to R$ is called centralizing on S if $[F(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$ for all $x \in S$. The first well-known result on commuting maps is Posner's second theorem in [15]. This theorem states that the existence of a nonzero commuting derivation on a prime ring R implies R to be commutative. By a derivation, we mean an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The concept of derivation was extended to generalized derivation in [6] by Brešar. An additive mapping $g : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $g(xy) = g(x)y + xd(y)$ holds for all $x, y \in R$. In [13], Hvala gave the algebraic study of generalized derivation in prime rings. Obviously every derivation is a generalized derivation of R.

Many papers in literature have investigated the commutativity of prime and semiprime rings satisfying certain functional identities involving derivations or generalized derivations (see [1], [3], [4], [5], [9], [10], [11], [16], [17]).

In [5], Ashraf and Rehman proved that if R is a prime ring with a nonzero ideal I of R and d is a derivation of R such that either $d(xy) - xy \in Z(R)$ for all $x, y \in I$ or $d(xy) + xy \in Z(R)$ for all $x, y \in I$, then R is commutative. Recently, Ashraf et al. [3] have studied the situations replacing derivation d with a generalized derivation F. More precisely, they proved that the prime ring R must be commutative, if R satisfies any one of the following conditions : (i) $F(xy) - xy \in Z(R)$ for all $x, y \in I$, (ii) $F(xy) + xy \in Z(R)$ for all $x, y \in I$, (iii) $F(xy) - yx \in Z(R)$ for all $x, y \in I$, (iv) $F(xy) + yx \in Z(R)$ for all $x, y \in I$, (v) $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, (vi) $F(x)F(y) + xy \in Z(R)$ for all $x, y \in I$; where F is a generalized derivation of R associated with a nonzero derivation d and I is a nonzero two-sided ideal of R.

On the other hand, in [9], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal K and d is a derivation of R such that $d([x, y]) = \pm[x, y]$ for all $x, y \in K$, then K is a central ideal. In particular, if $K = R$, then R is commutative. Recently, Quadri et al. [16] generalized this result replacing derivation d with a generalized derivation in a prime ring R. More precisely, they proved the following:

Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that any one of the following holds : (i) $F([x, y]) = [x, y]$ for all $x, y \in I$; (ii) $F([x, y]) = -[x, y]$ for all $x, y \in I$; (iii) $F(x \circ y) = (x \circ y)$ for all $x, y \in I$; (iv) $F(x \circ y) = -(x \circ y)$ for all $x, y \in I$; then R is commutative.

Recently in [11], Dhara proved the following result: Let R be a semiprime ring, I be a nonzero ideal of R and F be a generalized derivation of R with associated derivation d satisfying $F([x, y]) \pm [x, y] = 0$ or $F(x \circ y) \pm (x \circ y) = 0$ for all $x, y \in I$, then R must contain a nonzero central ideal, provided $d(I) \neq (0)$. In case R is prime satisfying $F([x, y]) \pm [x, y] \in Z(R)$ or $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x, y \in I$, then R must be commutative, provided $d(Z) \neq (0)$.

In this line of investigation, recently, Asma et al. [1] have studied the following situations: (i) $F(xy) \in Z(R)$, (ii) $F([x, y]) = 0$, (iii) $(F(xy) \pm yx) \in Z(R)$ and (iv)

$(F(xy) \pm [x, y]) \in Z(R)$; for all x, y in some nonzero left ideal of semiprime ring R, where F is a generalized derivation of R.

Recently, Dhara and Ali [10] studied the above mentioned results of Ashraf et al. [3] in semiprime rings replacing two-sided ideal I with left sided ideal λ and generalized derivation with multiplicative (generalized)-derivation.

Let us introduce the background of investigation about multiplicative (generalized)-derivation. A mapping $D : R \to R$ which satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$ is called a multiplicative derivation of R. Of course these mappings are not additive. To the best of my knowledge, the concept of multiplicative derivations appeared for the first time in the work of Daif [7]. Then the complete description of those maps was given by Goldmann and Šemrl in [12].

Further, Daif and Tammam-El-Sayiad [8] extended the notion of multiplicative derivation to multiplicative generalized derivation as follows: a mapping $F : R \to R$ is called a multiplicative generalized derivation if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In [10], Dhara and Ali make a slight generalization of Daif and Tammam-El-Sayiad's definition of multiplicative generalized derivation by considering d as any map. In [10], the authors defined that a mapping $F : R \to R$ (not necessarily additive) is said to be multiplicative (generalized)-derivation if $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$, where f is any mapping (not necessarily a derivation nor an additive map). For examples of such maps we refer to [10]. Moreover, multiplicative (generalized)-derivation with $f = 0$ covers the notion of multiplicative centralizers (not necessarily additive). Obviously, every generalized derivation is a multiplicative (generalized)-derivation on R.

In this line of investigation, it is more interesting to study the identities replacing generalized derivation with multiplicative (generalized)-derivation. In the present paper, our main object is to investigate the cases when a multiplicative (generalized)-derivations F and G satisfies the identities: (i) $F(x)F(y) \pm [x, y] \in Z(R)$, (ii) $F(x)F(y) \pm x \circ y \in Z(R)$, (iii) $F([x, y]) \pm [x, y] \in Z(R)$, (iv) $F(x \circ y) \pm (x \circ y) \in Z(R)$, (v) $F([x, y]) \pm [F(x), y] \in Z(R)$, (vi) $F(x \circ y) \pm (F(x) \circ y) \in Z(R)$, (vii) $[F(x), y] \pm [G(y), x] \in Z(R)$, (viii) $F([x, y]) \pm [F(x), F(y)] = 0$, (ix) $F(x \circ y) \pm (F(x) \circ F(y)) = 0$, (x) $F(xy) \pm [x, y] \in Z(R)$ and (xi) $F(xy) \pm x \circ y \in Z(R)$ for all x, y in some appropriate subset of R.

2. Main Results

2.1. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x)F(y) \pm [x, y] \in Z(R)$ for all $x, y \in \lambda$, then $\lambda[\lambda, \lambda] = (0)$ and $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.

Proof. First we consider the case

$$F(x)F(y) + [x, y] \in Z(R) \quad \text{for all} \quad x, y \in \lambda. \tag{2.1}$$

Substituting yz for y in (2.1), we have

$$\begin{aligned} F(x)F(yz) + [x, yz] &= F(x)F(y)z + F(x)yf(z) + y[x, z] + [x, y]z \\ &= (F(x)F(y)z + [x, y])z + y[x, z] + F(x)yf(z) \in Z(R) \quad \text{for all} \quad x, y, z \in \lambda. \end{aligned} \tag{2.2}$$

Commuting both sides with z in (2.2) and using (2.1), we obtain

$$[F(x)yf(z), z] + [y[x, z], z] = 0 \quad \text{for all} \quad x, y, z \in \lambda. \tag{2.3}$$

Putting $x = xz$ in the above relation, we get

$$[F(x)zyf(z), z] + [xf(z)yf(z), z] + [y[x, z], z]z = 0 \quad \text{for all} \quad x, y, z \in \lambda. \tag{2.4}$$

Replacing y by zy in (2.3), we obtain

(2.5) $\quad [F(x)zyf(z), z] + z[y[x, z], z] = 0 \text{ for all } x, y, z \in \lambda.$

Subtracting (2.5) from (2.4), we get

(2.6) $\quad [xf(z)yf(z), z] + [[y[x, z], z], z] = 0 \text{ for all } x, y, z \in \lambda.$

Putting $x = xz$, the above relation yields that

(2.7) $\quad [xzf(z)yf(z), z] + [[y[x, z], z], z]z = 0 \text{ for all } x, y, z \in \lambda.$

Right multiplying (2.6) by z and then subtracting it from (2.7), we get

(2.8) $\quad [x[f(z)yf(z), z], z] = 0 \text{ for all } x, y, z \in \lambda.$

Now we substitute $f(z)yf(z)x$ for x in (2.8), to get

$$\begin{aligned} 0 &= [f(z)yf(z)x[f(z)yf(z), z], z] \\ &= f(z)yf(z)[x[f(z)yf(z), z], z] + [f(z)yf(z), z]x[f(z)yf(z), z] \\ &\text{for all } x, y, z \in \lambda. \end{aligned} \tag{2.9}$$

Using (2.8), it reduces to

(2.10) $\quad [f(z)yf(z), z]x[f(z)yf(z), z] = 0 \text{ for all } x, y, z \in \lambda.$

Since λ is a left ideal of R, it follows that $x[f(z)yf(z), z]Rx[f(z)yf(z), z] = (0)$ for all $x, y, z \in \lambda$. Since R is semiprime, we have

(2.11) $\quad x[f(z)yf(z), z] = 0 \text{ for all } x, y, z \in \lambda,$

that is,

(2.12) $\quad x(f(z)yf(z)z - zf(z)yf(z)) = 0 \text{ for all } x, y, z \in \lambda.$

Replacing y by $yf(z)u$ in (2.12), we obtain

(2.13) $\quad x(f(z)yf(z)uf(z)z - zf(z)yf(z)uf(z)) = 0 \text{ for all } u, x, y, z \in \lambda.$

Using (2.12), this can be written as

(2.14) $\quad x(f(z)yzf(z)uf(z) - f(z)yf(z)zuf(z)) = 0 \text{ for all } u, x, y, z \in \lambda,$

which gives

(2.15) $\quad xf(z)y[f(z), z]uf(z) = 0 \text{ for all } u, x, y, z \in \lambda.$

This implies that $x[f(z), z]y[f(z), z]u[f(z), z] = 0$ for all $u, x, y, z \in \lambda$ and so $(\lambda[f(z), z])^3 = (0)$ for all $z \in \lambda$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [2]), it follows that $\lambda[f(z), z] = (0)$ for all $z \in \lambda$.

Now replacing y by yz in (2.3), we get

(2.16) $\quad [F(x)yzf(z), z] + [yz[x, z], z] = 0 \text{ for all } x, y, z \in \lambda.$

Right multiplying (2.3) by z and then subtracting from (2.16), we get

(2.17) $\quad [F(x)y[f(z), z], z] + [y[x, z]_2, z] = 0 \text{ for all } x, y, z \in \lambda.$

By using $\lambda[f(z), z] = (0)$ for all $z \in \lambda$, (2.17) yields $[y[x, z]_2, z] = 0$ for all $x, y, z \in \lambda$. Substituting y by xy, we obtain $0 = [xy[x, z]_2, z] = x[y[x, z]_2, z] + [x, z]y[x, z]_2 = [x, z]y[x, z]_2$ and hence $y[x, z]_2Ry[x, z]_2 = (0)$ for all $x, y, z \in \lambda$. Since R is semiprime ring, $\lambda[x, z]_2 = (0)$ for all $x, z \in \lambda$. Linearizing the last relation with respect to z, we have $(0) = \lambda[[x, u], v] + \lambda[[x, v], u]$ for all $x, u, v \in \lambda$. Now we put $u = uv$ and get $(0) = \lambda([[x, u], v]v + [u[x, v], v]) + \lambda([[x, v], u]v + u[[x, v], v]) = \lambda[u[x, v], v]$ for all $x, u, v \in \lambda$. Now we put $u = xu$ in this last relation and then get $(0) = \lambda[xu[x, v], v] = \lambda x[u[x, v], v] + \lambda[x, v]u[x, v] = \lambda[x, v]u[x, v]$ for all $x, u, v \in \lambda$. Thus $\lambda[x, v]R\lambda[x, v] = (0)$ for all $x, v \in \lambda$. Since R is semiprime, it yields $\lambda[\lambda, \lambda] = (0)$, as desired.

Similarly we can prove the result for the case $F(x)F(y) - [x, y] \in Z(R)$ for all $x, y \in \lambda$. □

2.2. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x)F(y) \pm (x \circ y) \in Z(R)$ for all $x, y \in \lambda$, then $\lambda[\lambda, \lambda] = (0)$ and $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.

Proof. First we consider that

$$F(x)F(y) - (x \circ y) \in Z(R) \quad \text{for all} \quad x, y \in \lambda. \tag{2.18}$$

Substituting yz for y in (2.18), we have

$$\begin{aligned} &F(x)F(yz) - (x \circ yz) = F(x)F(y)z + F(x)yf(z) - (x \circ y)z + y[x, z] \\ &= (F(x)F(y) - x \circ y)z + y[x, z] + F(x)yf(z) \in Z(R) \quad \text{for all} \quad x, y, z \in \lambda. \end{aligned} \tag{2.19}$$

Commuting both sides with z in (2.19) and using (2.18), we obtain

$$[F(x)yf(z), z] + [y[x, z], z] = 0 \quad \text{for all} \quad x, y, z \in \lambda. \tag{2.20}$$

This is same as (2.3) in Theorem 2.1. Then by same argument of Theorem 2.1, we conclude the result.

Similarly, we can prove the result for the case $F(x)F(y) + (x \circ y) \in Z(R)$ for all $x, y \in \lambda$. □

2.3. Corollary. Let R be a semiprime ring and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If R satisfies any one of the following conditions:

(1) $F(x)F(y) \pm [x, y] \in Z(R)$ for all $x, y \in R$;
(2) $F(x)F(y) \pm (x \circ y) \in Z(R)$ for all $x, y \in R$;

then R must be commutative.

Note that the map $G(r) = F(r) \pm r$ for all $r \in R$ is a multiplicative (generalized)-derivation of R.

2.4. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x, y]) \pm [x, y] = 0$ for all $x, y \in \lambda$, then $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.

Proof. By hypothesis, we have

$$G([x, y]) = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.21}$$

Replacing y by yx in (2.21) and using (2.21), we obtain

$$0 = G([x, yx]) = G([x, y]x) = G([x, y])x + [x, y]f(x) = [x, y]f(x) \quad \text{for all} \quad x, y \in \lambda. \tag{2.22}$$

This gives that

$$[x, y]f(x) = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.23}$$

Substituting $f(x)y$ for y in (2.23), we get

$$[x, f(x)]yf(x) = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.24}$$

Replace y by yx in (2.24), to get

(2.25) $[x, f(x)]yxf(x) = 0$ for all $x, y \in \lambda$.

Right multiplying (2.24) by x and then subtracting from (2.25), we obtain

(2.26) $[x, f(x)]y[f(x), x] = 0$ for all $x, y \in \lambda$.

This implies that $\lambda[f(x), x]R\lambda[f(x), x] = (0)$ for all $x \in \lambda$. Hence the semiprimeness of R forces that $\lambda[f(x), x] = (0)$ for all $x \in \lambda$. □

2.5. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x \circ y) \pm (x \circ y) = 0$ for all $x, y \in \lambda$, then $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.

Proof. By hypothesis, we have

(2.27) $G(x \circ y) = 0$ for all $x, y \in \lambda$.

Replacing y by yx in (2.27) and using (2.27), we obtain

(2.28)
$$0 = G(x \circ yx) = G((x \circ y)x) = G(x \circ y)x + (x \circ y)f(x) = (x \circ y)f(x) \quad \text{for all} \quad x, y \in \lambda.$$

This implies that

(2.29) $(x \circ y)f(x) = 0$ for all $x, y \in \lambda$.

Substituting $f(x)y$ for y in (2.29) and using (2.29), we obtain

(2.30) $0 = (x \circ f(x)y)f(x) = f(x)(x \circ y)f(x) + [x, f(x)]yf(x)$ for all $x, y \in \lambda$.

This implies that

(2.31) $[x, f(x)]yf(x) = 0$ for all $x, y \in \lambda$.

Replace y by yx in (2.31), to get

(2.32) $[x, f(x)]yxf(x) = 0$ for all $x, y \in \lambda$.

Right multiplying (2.31) by x and then subtracting from (2.32), we obtain

(2.33) $[x, f(x)]y[f(x), x] = 0$ for all $x, y \in \lambda$.

Since λ is a left ideal of R, it follows that $\lambda[f(x), x]R\lambda[f(x), x] = (0)$ for all $x \in \lambda$. Semiprimeness of R yields that $\lambda[f(x), x] = (0)$ for all $x \in \lambda$. □

2.6. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x, y]) \pm [x, y] \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x), x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

Proof. By hypothesis, we have $G([x, y]) \in Z(R)$ for all $x, y \in \lambda$. If $G([x, y]) = 0$ for all $x, y \in \lambda$, then by Theorem 2.4, $\lambda[f(x), x] = (0)$ for all $x \in \lambda$, as desired. Assume that there exist some $x, y \in \lambda$ such that $0 \neq G([x, y]) \in Z(R)$. This gives $Z(R) \neq (0)$. Let $z \in Z(R)$. Replacing y by yz in our hypothesis, we have

(2.34) $G([x, y]z) = G([x, y])z + [x, y]f(z) = G([x, y])z + [x, y]f(z) \in Z(R)$,

which implies $[x, y]f(z) \in Z(R)$ for all $x, y \in \lambda$. Thus $0 = [[x, y]f(z), r]$ for all $x, y \in \lambda$ and $r \in R$. Replacing x with yx, we get $0 = [[yx, y]f(z), r] = [y[x, y]f(z), r] = [y, r][x, y]f(z)$, Since $[x, y]f(z) \in Z(R)$ for all $x, y \in \lambda$. Replacing r with sr, we get $0 = [y, sr][x, y]f(z) = s[y, r][x, y]f(z) + [y, s]r[x, y]f(z) = [y, s]r[x, y]f(z)$ for all $x, y \in \lambda$ and $r, s \in R$ and hence

$(0) = [y,x]f(z)R[x,y]f(z)$ for all $x,y \in \lambda$. Since R is semiprime, above relation yields $0 = [x,y]f(z)$ for all $x,y \in \lambda$. Replacing y with $f(z)y$, we obtain $0 = [x,f(z)y]f(z) = f(z)[x,y]f(z) + [x,f(z)]yf(z) = [x,f(z)]yf(z)$and hence $(0) = y[x,f(z)]Ry[x,f(z)]$ for all $x,y \in \lambda$. Semiprimeness of R yields $\lambda[\lambda, f(Z)] = (0)$. □

2.7. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x,y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x),x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

Proof. By hypothesis, we have $G(x \circ y) \in Z(R)$ for all $x,y \in \lambda$. If $G(x \circ y) = 0$ for all $x,y \in \lambda$, then by Theorem 2.5, $\lambda[f(x),x] = (0)$ for all $x \in \lambda$, as desired. Assume that there exist some $x,y \in \lambda$ such that $0 \neq G(x \circ y) \in Z(R)$. This gives $Z(R) \neq (0)$. Let $z \in Z(R)$. Substituting yz for y in our hypothesis, we have

$$G(x \circ yz) = G(x \circ y)z + (x \circ y)f(z) = (x \circ y)f(z) \in Z(R). \tag{2.35}$$

This implies that $(x \circ y)f(z) \in Z(R)$ for all $x,y \in \lambda$ and hence

$$[(x \circ y)f(z), r] = 0 \text{ for all } x,y \in \lambda, \text{ for all } r \in R. \tag{2.36}$$

Replacing x by yx in (2.36) and then using the fact that $(x \circ y)f(z) \in Z(R)$ for all $x,y \in \lambda$, we get

$$0 = [y(x \circ y)f(z), r] = [y,r](x \circ y)f(z) \text{ for all } x,y \in \lambda, \tag{2.37}$$

that is

$$[y,r](x \circ y)f(z) = 0 \text{ for all } x,y \in \lambda, \text{ for all } r \in R. \tag{2.38}$$

Substituting sx for x in (2.38) and using $(x \circ y)f(z) \in Z(R)$ for all $x,y \in \lambda$, we obtain

$$\begin{aligned} 0 &= [y,r](sx \circ y)f(z) = [y,r]s(x \circ y)f(z) - [y,r][s,y]xf(z) \\ &= [y,r](x \circ y)f(z)s + [r,y][s,y]xf(z) \text{ for all } x,y \in \lambda, \text{ for all } r,s \in R. \end{aligned} \tag{2.39}$$

Using (2.38), the above relation yields that

$$[r,y][s,y]xf(z) = 0 \text{ for all } x,y \in \lambda, \text{ for all } r,s \in R. \tag{2.40}$$

Replacing r with rt and using (2.40) we have

$$[r,y]t[s,y]xf(z) = 0 \text{ for all } x,y \in \lambda, \text{ for all } r,s,t \in R. \tag{2.41}$$

In the same manner, replacing s with sp, we obtain

$$[r,y]t[s,y]pxf(z) = 0 \text{ for all } x,y \in \lambda, \text{ for all } r,s,t,p \in R. \tag{2.42}$$

Now replacing x with xy and right multiplying (2.42) by y respectively, and then subtract one from another to get

$$[r,y]t[s,y]px[f(z),y] = 0 \text{ for all } x,y \in \lambda, \text{ for all } r,s,t,p \in R. \tag{2.43}$$

In particular, we have

$$x[f(z),y]Rx[f(z),y]Rx[f(z),y] = (0) \text{ for all } x,y \in \lambda, \tag{2.44}$$

that is $(x[f(z),y]R)^3 = (0)$ for all $x,y \in \lambda$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [2]), it follows that $x[f(z),y]R = (0)$, that is $x[f(z),y] = 0$ for all $x,y \in \lambda$ and $z \in Z(R)$. Thus we have $\lambda[\lambda, f(Z)] = (0)$. □

2.8. Corollary. Let R be a semiprime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x,y]) \pm [x,y] \in Z(R)$ for all $x,y \in R$ or $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x,y \in R$, then either f is commuting on R or $f : Z(R) \to Z(R)$.

2.9. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x, y]) \pm [F(x), y] \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x), x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

Proof. By our hypothesis, we have

$$F([x, y]) \pm [F(x), y] = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.45}$$

Then replacing y by yx in (2.45), we get

$$\begin{aligned} 0 &= F([x, yx]) \pm [F(x), yx] = F([x, y]x) \pm ([F(x), y]x + y[F(x), x]) \\ &= F([x, y])x + [x, y]f(x) \pm ([F(x), y]x + y[F(x), x]) \\ &\quad \text{for all} \quad x, y \in \lambda. \end{aligned} \tag{2.46}$$

Using (2.45) in the above relation, we obtain

$$[x, y]f(x) \pm y[F(x), x] = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.47}$$

Substituting $f(x)y$ for y in (2.47), we get

$$f(x)[x, y]f(x) + [x, f(x)]yf(x) \pm f(x)y[F(x), x] = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.48}$$

Left multiplying (2.47) by $f(x)$ and then comparing with (2.48), we get

$$[x, f(x)]yf(x) = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.49}$$

Then by similar argument as in the proof of Theorem 2.4, we have $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.
Next, we assume that there exist some $x, y \in \lambda$ such that $0 \neq F([x, y]) \pm [F(x), y] \in Z(R)$. This implies that $Z(R) \neq (0)$. Let $z \in Z(R)$. Substituting y by yz in our hypothesis, we have

$$\begin{aligned} &F([x, y]z) \pm [F(x), y]z = F([x, y])z + [x, y]f(z) \pm [F(x), y]z \\ &= (F([x, y]) \pm [F(x), y])z + [x, y]f(z) \in Z(R), \end{aligned} \tag{2.50}$$

which implies that $[x, y]f(z) \in Z(R)$ for all $x, y \in \lambda$. Then by the same argument as in the proof of Theorem 2.6, we conclude that $\lambda[\lambda, f(Z)] = (0)$. □

2.10. Theorem. Let R be a semiprime ring, λ be a nonzero left ideal of R and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x \circ y) \pm (F(x) \circ y) \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x), x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

Proof. By hypothesis, we have

$$F(x \circ y) \pm (F(x) \circ y) = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.51}$$

Then replacing y by yx in (2.51), we have

$$\begin{aligned} 0 &= F(x \circ yx) \pm (F(x) \circ yx) = F((x \circ y)x) \pm ((F(x) \circ y)x - y[F(x), x]) \\ &= F(x \circ y)x + (x \circ y)f(x) \pm ((F(x) \circ y)x - y[F(x), x]) \quad \text{for all} \quad x, y \in \lambda. \end{aligned} \tag{2.52}$$

Using (2.51) in the above relation, we get

$$(x \circ y)f(x) \mp y[F(x), x] = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.53}$$

Substituting $f(x)y$ for y in (2.53), we have

$$f(x)(x \circ y)f(x) + [x, f(x)]yf(x) \mp f(x)y[F(x), x] = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.54}$$

Left multiplying (2.53) by $f(x)$ and then subtracting from (2.54), we obtain

$$[x, f(x)]yf(x) = 0 \quad \text{for all} \quad x, y \in \lambda. \tag{2.55}$$

Then by similar argument of Theorem 2.4, $\lambda[f(x),x]=(0)$ for all $x\in\lambda$.
Next, assume that there exist some $x,y\in\lambda$ such that $0\neq F(x\circ y)\pm(F(x)\circ y)\in Z(R)$. This gives $Z(R)\neq(0)$. Let $z\in Z(R)$. Substituting yz for y in our hypothesis, we have

$$\begin{aligned}(2.56)\qquad & F((x\circ y)z)\pm(F(x)\circ y)z=F(x\circ y)z+(x\circ y)f(z)\pm(F(x)\circ y)z\\ & =(F(x\circ y)\pm F(x)\circ y)z+(x\circ y)f(z)\in Z(R).\end{aligned}$$

This implies that $(x\circ y)f(z)\in Z(R)$ for all $x,y\in\lambda$ and hence

$$(2.57)\qquad [(x\circ y)f(z),r]=0\ \text{ for all }\ x,y\in\lambda,\ \text{ for all }\ r\in R.$$

Then by the same argument as in the proof of Theorem 2.7, we get $\lambda[\lambda,f(Z)]=(0)$, as desired. □

2.11. Corollary. Let R be a semiprime ring and $F:R\to R$ be a multiplicative (generalized)-derivation associated with the map $f:R\to R$. If $F([x,y])\pm[F(x),y]\in Z(R)$ for all $x,y\in R$ or $F(x\circ y)\pm(F(x)\circ y)\in Z(R)$ for all $x,y\in R$, then either f is commuting on R or $f:Z(R)\to Z(R)$.

2.12. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F,G:R\to R$ are multiplicative (generalized)-derivations associated with the maps $f,g:R\to R$. If $[F(x),y]\pm[G(y),x]\in Z(R)$ for all $x,y\in\lambda$, then one of the following holds:

(1) $\lambda[g(x),x]=(0)$ for all $x\in\lambda$;
(2) $\lambda[\lambda,g(Z)]=(0)$.

Proof. By hypothesis, we have $[F(x),y]\pm[G(y),x]\in Z(R)$ for all $x,y\in\lambda$. If

$$(2.58)\qquad [F(x),y]\pm[G(y),x]=0\ \text{ for all }\ x,y\in\lambda,$$

then replacing y by yx in (2.58), we get

$$\begin{aligned}(2.59)\qquad & 0=[F(x),yx]\pm[G(yx),x]=[F(x),y]x+y[F(x),x]\pm([G(y),x]x+[yg(x),x])\\ & =([F(x),y]\pm[G(y),x])x+y[F(x),x]\pm[yg(x),x]\\ & \quad \text{for all }\ x,y\in\lambda.\end{aligned}$$

Using (2.58) in the above relation, we obtain

$$(2.60)\qquad y[F(x),x]\pm[yg(x),x]=0\ \text{ for all }\ x,y\in\lambda.$$

Substituting $g(x)y$ for y in (2.60), we get

$$(2.61)\qquad g(x)y[F(x),x]\pm g(x)[yg(x),x]\pm[g(x),x]yg(x)=0\ \text{ for all }\ x,y\in\lambda.$$

Left multiplying (2.60) by $g(x)$ and then comparing with (2.61), we get

$$(2.62)\qquad [g(x),x]yg(x)=0\ \text{ for all }\ x,y\in\lambda.$$

This is the same as (2.24) in Theorem 2.4, we obtain $\lambda[g(x),x]=(0)$.
Next, we assume that there exist some $x,y\in\lambda$ such that $0\neq[F(x),y]\pm[G(y),x]\in Z(R)$. This implies that $Z(R)\neq(0)$. Let $z\in Z(R)$. Substituting y by yz in our hypothesis, we have

$$\begin{aligned}(2.63)\qquad & [F(x),yz]\pm[G(yz),x]=[F(x),y]z\pm[G(y),x]z\\ & +[yg(z),x]=([F(x),y]\pm[G(y),x])z\pm[yg(z),x]\in Z(R),\end{aligned}$$

For any $r\in R$, this implies that

$$(2.64)\qquad [[yg(z),x],r]=0\ \text{ for all }\ x,y\in\lambda.$$

Replacing y by wy in the above expression and using it, we get

(2.65)

$$[w,r][yg(z),x]=[w,x][yg(z),r]+[[w,x],r]yg(z)=0\ \text{ for all }\ x,y,w\in\lambda,\ \text{ for all }\ r\in R.$$

Taking $x = w$ in (2.65), we obtain

(2.66) $\quad [w, r][yg(z), w] = 0$ for all $y, w \in \lambda$, for all $r \in R$.

Replacing r by $yg(z)r$ in the above relation, we get

(2.67) $\quad [yg(z), w]r[yg(z), w] = 0$ for all $y, w \in \lambda$, for all $r \in R$.

Semiprimeness of R yields that

(2.68) $\quad [yg(z), w] = 0$ for all $y, w \in \lambda$.

Substituting $g(z)y$ for y in (2.68), we obtain

(2.69) $\quad [g(z)yg(z), w] = 0$ for all $y, w \in \lambda$.

This implies that

(2.70) $\quad g(z)yg(z)w - wg(z)yg(z) = 0$ for all $y, w \in \lambda$.

Replacing y by $yg(z)x$ in the above expression, we have

(2.71) $\quad g(z)yg(z)xg(z)w - wg(z)yg(z)xg(z) = 0$ for all $x, y, w \in \lambda$.

Using (2.70), we get

(2.72) $\quad g(z)y[g(z), x]wg(z) = 0$ for all $x, y, w \in \lambda$.

This implies that $(\lambda[\lambda, g(z)])^3 = (0)$ for any $z \in Z(R)$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [2]), it follows that $\lambda[\lambda, g(z)] = (0)$. □

Using the similar arguments and taking $G = F$ or $G = -F$ in Theorem 2.12, one can prove the following theorem:

2.13. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ are multiplicative (generalized)-derivations associated with the maps $f : R \to R$. If $[F(x), y] \pm [F(y), x] \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x), x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

2.14. Corollary. Let R be a semiprime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $[F(x), y] \pm [F(y), x] \in Z(R)$ for all $x, y \in R$, then either f is commuting on R or $f : Z(R) \to Z(R)$.

2.15. Theorem. Let R be a semiprime ring with $Z(R) \neq (0)$, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x, y]) \pm [F(x), F(y)] = 0$ for all $x, y \in \lambda$, then $\lambda[\lambda, f(Z)] = (0)$.

Proof. Suppose that

(2.73) $\quad F([x, y]) \pm [F(x), F(y)] = 0$ for all $x, y \in \lambda$.

Since $Z(R) \neq (0)$, replacing y by yz in (2.73), where $z \in Z(R)$, we get

$$\begin{aligned} 0 &= F([x, yz]) \pm [F(x), F(yz)] = F([x, y]z) \pm ([F(x), y]z + y[F(x), f(z)]) \\ &+[F(x), y]f(z) = F([x, y])z + [x, y]f(z) \pm ([F(x), f(y)]z + y[F(x), f(z)]) \\ &+[F(x), y]f(z) = [x, y]f(z) + y[F(x), f(z)] + [F(x), y]f(z) \\ &\text{for all } x, y \in \lambda. \end{aligned} \tag{2.74}$$

Using (2.73) in the above relation, we obtain

(2.75) $\quad [x, y]f(z) \pm y[F(x), f(z)] + [F(x), y]f(z) = 0$ for all $x, y \in \lambda$.

Replacing ry for y in (2.75), we get

$$\begin{aligned} &r[x, y]f(z) + [x, r]yf(z) \pm ry[F(x), f(z)] + r[F(x), y]f(z) + [F(x), r]yf(z) = 0 \\ &\text{for all } x, y \in \lambda, \text{ for all } r \in R. \end{aligned} \tag{2.76}$$

Left multiplying (2.75) by r and then subtracting from (2.76), we get

(2.77) $\quad [x,r]yf(z) \pm [F(x),r]yf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

Replacing x by xz in (2.77), where $z \in Z(R)$, we have

(2.78)
$$z[x,r]yf(z) \pm z[F(x),r]yf(z) + [xf(z),r]yf(z) = 0 \text{ for all } x, y \in \lambda, \text{ for all } r \in R.$$

Using (2.77), we get

(2.79) $\quad [xf(z),r]yf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

Replacing r by sr in the above relation and using it, we get

(2.80) $\quad [xf(z),s]ryf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

Substituting y by ty in (2.80), we obtain

(2.81) $\quad [xf(z),s]rtyf(z) = 0$ for all $x, y \in \lambda$, for all $r, t \in R$

Right multiplying (2.80) by t and then subtracting from (2.81), we get

(2.82) $\quad [xf(z),s]r[yf(z),t] = 0$ for all $x, y \in \lambda$, for all $r, s, t \in R$.

Semiprimeness of R yields that $[xf(z), r] = 0$ for all $x \in \lambda$ and $r \in R$. Replacing x by $f(z)x$ in the above relation, we get

(2.83) $\quad [f(z)xf(z),r] = 0$ for all $x \in \lambda$, for all $r \in R$,

that is

(2.84) $\quad f(z)xf(z)r - rf(z)xf(z) = 0$ for all $x \in \lambda$, for all $r \in R$.

Replacing x by $xf(z)y$ in (2.84), we obtain

(2.85) $\quad f(z)xf(z)yf(z)r - rf(z)xf(z)yf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

Using (2.84) in the above relation, we get

(2.86) $\quad f(z)xrf(z)yf(z) - f(z)xf(z)ryf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

We find that $f(z)x[f(z),r]yf(z) = 0$ for all $x, y \in \lambda$, $r \in R$. Which implies that $(\lambda[\lambda, f(z)])^3 = (0)$ for any $z \in Z(R)$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [2]), we obtain $\lambda[\lambda, f(z)] = (0)$ for any $z \in Z(R)$. □

2.16. Theorem. Let R be a semiprime ring with $Z(R) \neq (0)$, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x \circ y) \pm (F(x) \circ F(y)) = 0$ for all $x, y \in \lambda$, then $\lambda[\lambda, f(Z)] = (0)$.

Proof. By hypothesis, we have

(2.87) $\quad F(x \circ y) \pm F(x) \circ F(y) = 0$ for all $x, y \in \lambda$.

Since $Z(R) \neq (0)$. Let $z \in Z(R)$. Replacing y by yz in (2.87), we have

(2.88)
$$\begin{aligned} &0 = F(x \circ yz) \pm F(x) \circ F(yz) = F((x \circ y)z) \pm (F(x) \circ y)z + (F(x) \circ y)f(z) \\ &-y[F(x), f(z)] = (x \circ y)f(z) \pm ((F(x) \circ y)f(z) - y[F(x), f(z)]) \text{ for all } x, y \in \lambda. \end{aligned}$$

Using (2.87) in the above relation, we get

(2.89) $\quad (x \circ y)f(z) \mp [F(x), y]f(z) = 0$ for all $x, y \in \lambda$.

Substituting ry for y in (2.89), we obtain

(2.90) $\quad r(x \circ y)f(z) + [x,r]yf(z) \mp r[F(x), y]f(z) + [F(x), r]yf(z) = 0$ for all $x, y \in \lambda$.

Left multiplying (2.89) by r and then subtracting from (2.90), we get

(2.91) $\quad [x,r]yf(z) \mp [F(x),r]yf(z) = 0$ for all $x, y \in \lambda$.

Arguing in the similar manner as in the proof of Theorem 2.15, we get the result. □

2.17. Corollary. Let R be a semiprime ring with $Z(R) \neq (0)$ and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x,y]) \pm [F(x), F(y)] = 0$ or $F(x \circ y) \pm (F(x) \circ F(y)) = 0$ for all $x, y \in R$, then $f : Z(R) \to Z(R)$.

2.18. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(xy) \pm [x,y] \in Z(R)$ for all $x, y \in \lambda$, then $\lambda \subseteq Z(R)$ for all $x \in \lambda$ and $F(xy) \in Z(R)$ for all $x, y \in \lambda$.

Proof. By hypothesis, we have

(2.92) $\quad F(xy) \pm [x,y] = G(xy) \mp yx \in Z(R)$

for all $x, y \in \lambda$. By [10, Theorem 2.11], we obtain that $x[x,\lambda] \subseteq Z(R)$ for all $x \in \lambda$. Replacing y with xy in (2.92) and then using the fact $x[x,\lambda] \subseteq Z(R)$ for all $x \in \lambda$, we get $F(x^2y) \in Z(R)$ for all $x, y \in \lambda$. Now we put $x = x^2$ in (2.92) and then obtain

(2.93) $\quad F(x^2y) \pm x[x,y] \pm [x,y]x \in Z(R)$ for all $x, y \in \lambda$.

This implies $[x,y]x \in Z(R)$ for all $x, y \in \lambda$. Therefore we can write that $x[y,x] - [y,x]x \in Z(R)$ for all $x \in \lambda$, that gives $[y,x]_3 = [[[y,x],x],x] = 0$ for all $x, y \in \lambda$. Then by [14, Theorem 2], we get $\lambda \subseteq Z(R)$. Thus our hypothesis reduces to $F(xy) \in Z(R)$ for all $x, y \in \lambda$. □

2.19. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(xy) \pm (x \circ y) \in Z(R)$ for all $x, y \in \lambda$, then $\lambda \subseteq Z(R)$ and $F(xy) \in Z(R)$ for all $x, y \in \lambda$.

Proof. By hypothesis, we have

(2.94) $\quad F(xy) \pm (x \circ y) = G(xy) \pm yx \in Z(R)$

for all $x, y \in \lambda$. By [10, Theorem 2.11], we obtain that $x[x,\lambda] \subseteq Z(R)$ for all $x \in \lambda$. Now replacing y with xy in (2.94) and then using the fact $x[x,\lambda] \subseteq Z(R)$ for all $x \in \lambda$, we get $F(x^2y) \pm 2xyx \in Z(R)$ for all $x, y \in \lambda$. Now we put $x = x^2$ in (2.94) and then obtain

(2.95) $\quad F(x^2y) \pm (x^2 \circ y) \in Z(R)$

that is

(2.96) $\quad F(x^2y) \pm (2xyx + x[x,y] + [y,x]x) \in Z(R)$ for all $x, y \in \lambda$.

This implies $[x,y]x \in Z(R)$ for all $x, y \in \lambda$. Therefore we can write that $x[y,x] - [y,x]x \in Z(R)$ for all $x \in \lambda$, which gives $[y,x]_3 = [[[y,x],x],x] = 0$ for all $x, y \in \lambda$. Then by [14, Theorem 2], we get $\lambda \subseteq Z(R)$. Thus our hypothesis gives $F(xy) \in Z(R)$ for all $x, y \in \lambda$. □

2.20. Corollary. Let R be a semiprime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If

(1) $F(xy) \pm [x,y] \in Z(R)$ for all $x, y \in R$;
(2) $F(xy) \pm (x \circ y) \in Z(R)$ for all $x, y \in R$;

then R is commutative.

3. Examples

The following examples demonstrate that the restrictions in the hypothesis of the results are not superfluous.

3.1. Example. Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers. Since $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$, so R is not semiprime ring. We define maps $F, f : R \to R$, by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then F is a multiplicative (generalized)-derivation associated with the map f.

It is very easy to verify that R satisfies (i) $F(x)F(y) \pm [x, y] \in Z(R)$; (ii) $F(x)F(y) \pm (x \circ y) \in Z(R)$, (iii) $F(xy) \pm [x, y] \in Z(R)$; (iv) $F(xy) \pm (x \circ y) \in Z(R)$; Since R is not commutative, the hypothesis of semiprimeness in Corollary 2.3 and Corollary 2.20 can not be omitted.

3.2. Example. Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$. Note that R is not a semiprime ring. Define maps $F, f : R \to R$ by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^2 & a^2 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Then it is verified that F is a multiplicative (generalized)-derivation associated with the map f. It is easy to see that $F([x, y]) \pm [x, y] \in Z(R)$ and $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x, y \in R$. But neither f is commuting on R nor $f : Z(R) \to Z(R)$. Hence R to be semiprime in the hypothesis of Corollary 2.8 is essential.

3.3. Example. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{S} \right\}$, where S is any ring. Note that R is not a semiprime ring. Define maps F and $f : R \to R$ by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then F is a multiplicative generalized derivation associated with the map f. It is easy to see that (i) $[F(x), y] \pm [F(y), x] \in Z(R)$ and (ii)$F([x, y]) \pm [F(x), y] = 0$ or $F(x \circ y) \pm (F(x) \circ y) = 0$ for all $x, y \in R$. But neither f is commuting nor $f : Z(R) \to Z(R)$. Hence R to be semiprime in the hypothesis of Corollary 2.11 and Corollary 2.14 are essential.

Moreover, it satisfies $F([x, y]) \pm [F(x), F(y)] = 0$ or $F(x \circ y) \pm (F(x) \circ F(y)) = 0$ for all $x, y \in R$. But f does not map $Z(R)$ to $Z(R)$. Hence R to be semiprime in the hypothesis of Corollary 2.17 is essential.

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