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Notes on near-ring ideals with (σ, τ)-derivation

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Abstract

In the present paper, we extend some well known results concerning derivations of prime near-rings in [4], [5] and [13] to $(\sigma, \tau)-$derivations and semigroup ideals of prime near-rings.

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1. Introduction

An additively written group $(N, +)$ equipped with a binary operation $. : N \times N \to N, (x, y) \to xy$, such that $x(yz) = (xy)z$ and $x(y + z) = xy + xz$ for all $x, y, z \in N$ is called a left near-ring. A near-ring N is called zero symmetric if $0x = 0$ for all $x \in N$ (recall that left distributive yields $x0 = 0$). An element x of N is said to be distributive if $(y + z)\,x = yx + zx$ for all $x, y, z \in N$. In what follows all near-rings are zero symmetric left near-rings. A near-ring N is said to be $3-$prime if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. For any $x, y \in N$, as usual $[x, y] = xy - yx$ and $xoy = xy + yx$ will denote the well-known Lie and Jordan products respectively, while the symbol (x, y) will denote the additive commutator $x + y - x - y$. Given an element a of N, we put $C(a) = \{x \in N \mid ax = xa\}$. The set $Z = \{x \in N \mid yx = xy \text{ for all } y \in N\}$ is called multiplicative center of N. A nonempty subset U of N will be said a semigroup right ideal (resp. a semigroup left ideal) if $UN \subseteq U$ (resp. $NU \subseteq U$) and U is both a semigroup right ideal and a semigroup left ideal, it will be called a semigroup ideal. An additive mapping $d : N \to N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently, as noted in [13, Proposition 1], if $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. An element $x \in N$ for which $d(x) = 0$ is called constant. Following [8], an additive mapping d of N is called $(\sigma, \tau)-$derivation if there exist automorphisms $\sigma, \tau : N \to N$ such that

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$d(xy) = \tau(x)d(y) + d(x)\sigma(y)$ for all $x, y \in N$ or equivalently, as noted in [8, Lemma 1], if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in N$. Of course a $(1,1)-$derivation where 1 is the identity map on N is a derivation.

As well known that derivations or $(\sigma, \tau)-$derivations are important both in algebra and ring theory. These topics have many implications such as generalizations of Lie algebra, differantial and homological algebra. Some researchers have studied on these topics.(see [6], [7], [10] and [12]). Since E. C. Posner published his paper [11] in 1957, many authors have investigated properties of derivations of prime and semiprime rings. In view of these results it is natural to look for comparable results on near-rings. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [3], but thus far only few papers on this subject in near-rings have been published (see refererences for a partial bibliography).

In the present paper, we shall attempt to generalize some known results for derivations to $(\sigma, \tau)-$derivations and semigroup ideals of a left prime near-ring N. In Theorem 3.3, we extend [13, Theorem 1]. Theorem 3.7 is a generalization of [4, Lemma 3.2] to $(\sigma, \tau)-$derivation and semigroup ideals of N. Finally, it is shown that under appropriatiate additional hypothesis near-ring N must be a commutative ring.

2. Preliminaries

We begin with the following known results.

2.1. Lemma. [3, Lemma 3]*Let N be a prime near-ring.*

(i) *If $z \in Z \setminus \{0\}$, then z is not a zero divisor.*

(ii) *If Z contains a nonzero element z for which $z + z \in Z$, then $(N, +)$ is abelian.*

(iii) *Let d be a nonzero derivation on N. Then $xd(N) = (0)$ implies $x = 0$, and $d(N)x = (0)$ implies $x = 0$.*

(iv) *If N is $2-$torsion free and d is a derivation on N such that $d^2 = 0$, then $d = 0$.*

2.2. Lemma. [4, Lemma 1.3]*Let N be a $3-$prime near-ring and d be a nonzero derivation on N.*

(i) *If U is a nonzero semigroup right ideal (resp. semigroup left ideal) and $x \in N$ such that $Ux = (0)$ (resp. $xU = (0)$), then $x = 0$.*

(ii) *If U is a nonzero semigroup right ideal or semigroup left ideal, then $d(U) \neq (0)$.*

(iii) *If U is a nonzero semigroup right ideal and $x \in N$ which centralizes U, then $x \in Z$.*

2.3. Lemma. [4, Lemma 1.4]*Let N be a $3-$prime near-ring and U be a nonzero semigroup ideal of N. Let d be a nonzero derivation on N.*

(i) *If $x, y \in N$ and $xUy = (0)$, then $x = 0$ or $y = 0$.*

(ii) *If $x \in N$ and $d(U)x = (0)$, then $x = 0$.*

(iii) *If $x \in N$ and $xd(U) = (0)$, then $x = 0$.*

2.4. Lemma. [4, Theorem 2.1]*Let N be a $3-$prime near-ring and U be a nonzero semigroup right ideal or a nonzero semigroup left ideal of N. If N admits a nonzero derivation d for which $d(U) \subset Z$, then N is a commutative ring.*

2.5. Lemma. [4, Lemma 3.2]*Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. Let d be a nonzero derivation on N such that $d^2(U) \neq (0)$. If $a \in N$ and $[a, d(U)] = (0)$, then $a \in Z$.*

2.6. Lemma. [5, Lemma 1.8]*Let N be a $3-$prime near-ring with $2N \neq (0)$, and U a nonzero semigroup ideal. If d is a derivation on N such that $d^2(U) = (0)$, then $d = 0$.*

2.7. Lemma. [5, Lemma 2.4]*Let N be an arbitrary near-ring. Let S and T be nonempty subsets of N such that $st = -ts$ for all $s \in S$ and $t \in T$. If $a, b \in S$ and c is an element of T for which $-c \in T$, then $(ab)\,c = c\,(ab)$.*

2.8. Lemma. [8, Lemma 1]*Let N be a $3-$prime near-ring and d be a (σ, τ)-derivation on N. Then $d\,(xy) = d\,(x)\,\sigma\,(y) + \tau\,(x)\,d\,(y)$, for all $x, y \in N$.*

2.9. Lemma. [9, Lemma 4]*Let N be a $3-$prime near-ring, d a (σ, τ)-derivation of N and U a nonzero semigroup right ideal (resp. semigroup left ideal). If $d\,(U) = (0)$, then $d = 0$.*

2.10. Lemma. [9, Theorem 1]*Let N be a 3-prime near-ring, d a nonzero (σ, τ)-derivation of N and U a nonzero semigroup right ideal of N. If $d\,(U) \subset Z$, then N is a commutative ring.*

2.11. Lemma. [9, Theorem 3]*Let N be a 3-prime near-ring, d a nonzero (σ, τ)-derivation of N such that $\sigma d = d\sigma, \tau d = d\tau$ and U a nonzero semigroup ideal of N. If $d^2\,(U) = (0)$, then $d = 0$.*

2.12. Lemma. [1, Lemma 2.2]*Let d be a (σ, τ)-derivation on the near-ring N. Then N satisfies the following partial distributive laws:*
(i) $(\tau\,(x)\,d\,(y) + d\,(x)\,\sigma\,(y))\,z = \tau\,(x)\,d\,(y)\,z + d\,(x)\,\sigma\,(y)\,z$, *for all* $x, y, z \in N$.
(ii) $(d\,(x)\,\sigma\,(y) + \tau\,(x)\,d\,(y))\,z = d\,(x)\,\sigma\,(y)\,z + \tau\,(x)\,d\,(y)\,z$ *for all* $x, y, z \in N$.

3. The Main Results

3.1. Theorem. *Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. If d_1 is a nonzero (σ, τ)-derivation and d_2 a nonzero derivation of N such that $d_1\,(x)\,\sigma\,(d_2\,(y)) = -\tau\,(d_2\,(x))\,d_1\,(y)$ for all $x, y \in U$, then $(N, +)$ is abelian.*

Proof. Writing yr, $y \in U$, $r \in N$ instead of y, we have

$$d_1\,(x)\,\sigma\,(d_2\,(y)\,r + yd_2\,(r)) = -\tau\,(d_2\,(x))\,(\tau\,(y)\,d_1\,(r) + d_1\,(y)\,\sigma\,(r))$$

and so

$$d_1\,(x)\,\sigma\,(d_2\,(y))\sigma(r) + d_1\,(x)\,\sigma(y)\sigma(d_2\,(r)) = -\tau\,(d_2\,(x))\,d_1\,(y)\,\sigma\,(r) - \tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(r)$$

Using the hypothesis, we get

$$d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r)) = -\tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(r)\,, \text{ for all } x, y \in U, r \in N. \tag{3.1}$$

Replacing r by $r + t$, $t \in N$ in (3.1), we get

$$\begin{aligned} &d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r)) + d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(t)) \\ &= -\tau\,(d_2\,(x))\,\tau\,(y)\,(d_1\,(r) + d_1\,(t)). \end{aligned}$$

Using $-(a + b) = (-b) + (-a)$, for all $a, b \in N$, we have

$$\begin{aligned} &d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r)) + d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(t)) \\ &= -\tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(t) - \tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(r) \end{aligned}$$

and so

$$d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r)) + d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(t)) + \tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(r) + \tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(t) = 0.$$

Using the (3.1) and $(r, t) = r + t - r - t$ in the last equation, we arrive at

$$d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r, t)) = 0, \text{ for all } x, y \in U, r, t \in N.$$

That is

$$\sigma^{-1}(d_1\,(x))Ud_2\,(r, t) = (0), \text{ for all } x \in U, r, t \in N. \tag{3.2}$$

By Lemma 2.3 (i), we get $d_1(U) = (0)$ or $d_2(r,t) = 0$, for all $r,t \in N$. If $d_1(U) = (0)$, then $d_1 = 0$ by Lemma 2.9. This is a contradiction. So that $d_2(r,t) = 0$ for all $r,t \in N$. For any $w \in N$, we have $d_2(wr, wt) = 0$. Hence we obtain that $d_2(w)(r,t) = 0$, for all $w,r,t \in N$. From Lemma 2.1 (iii) and $d_2 \neq 0$, we get $(r,t) = 0$, for all $r,t \in N$. Thus the proof is completed. □

3.2. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring and U a nonzero semigroup ideal of N. If d_1 is a (σ,τ)-derivation and d_2 a derivation of N such that $d_1(x)\sigma(d_2(y)) = -\tau(d_2(x))d_1(y)$ for all $x,y \in U$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. Assume that $d_2 \neq 0$. Using the same method as in the proof of Theorem 3.1, we have

$$d_1(x)\sigma(y)\sigma(d_2(r)) = -\tau(d_2(x))\tau(y)d_1(r), \text{ for all } x,y,r \in U. \tag{3.3}$$

Replacing y by $yd_2(z)$ in (3.3), we have

$$d_1(x)\sigma(y)\sigma(d_2(z))\sigma(d_2(r)) = -\tau(d_2(x))\tau(y)\tau(d_2(z))d_1(r), \text{ for all } x,y,r,z \in U. \tag{3.4}$$

Using (3.3) in (3.4), we arrive at

$$-\tau(d_2(x))\tau(y)d_1(z)\sigma(d_2(r)) = -\tau(d_2(x))\tau(y)\tau(d_2(z))d_1(r)$$

and so

$$\tau(d_2(x))\tau(y)(d_1(z)\sigma(d_2(r)) - \tau(d_2(z))d_1(r)) = 0, \text{ for all } x,y,r,z \in U.$$

Since τ is an automorphism of N, we get

$$d_2(x)U\tau^{-1}(d_1(z)\sigma(d_2(r)) - \tau(d_2(z))d_1(r)) = (0), \text{ for all } x,r,z \in U. \tag{3.5}$$

By Lemma 2.3 (i), we get $d_2(x) = 0$ or $d_1(z)\sigma(d_2(r)) = \tau(d_2(z))d_1(r)$, for all $x,r,z \in U$. The first case contradicts $U \neq (0)$ by Lemma 2.2 (ii). So we must have $d_1(z)\sigma(d_2(r)) = \tau(d_2(z))d_1(r)$, for all $r,z \in U$. Hence we obtain that $2d_1(z)\sigma(d_2(r)) = 0$ by the hypothesis. Since N is $2-$torsion free, we have $\sigma^{-1}(d_1(z))d_2(r) = 0$, for all $r,z \in U$. Hence $d_1(U) = (0)$ by Lemma 2.3 (iii) . This gives us $d_1 = 0$ by Lemma 2.9. This completes the proof. □

3.3. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring and U a nonzero semigroup ideal of N. If d_1 is a (σ,τ)-derivation and d_2 a derivation of N such that d_1d_2 acts as a (σ,τ)-derivation on U, then $d_1 = 0$ or $d_2 = 0$.*

Proof. By calculating $d_1d_2(xy)$ in two different ways, we see that

$$d_1d_2(xy) = d_1d_2(x)\sigma(y) + \tau(x)d_1d_2(y)$$

and

$$d_1d_2(xy) = d_1d_2(x)\sigma(y) + \tau(d_2(x))d_1(y) + d_1(x)\sigma(d_2(y)) + \tau(x)d_1d_2(y).$$

Equating these two expressions for $d_1d_2(xy)$, we obtain that

$$d_1(x)\sigma(d_2(y)) = -\tau(d_2(x))d_1(y) \text{ for all } x,y \in U.$$

Then $d_1 = 0$ or $d_2 = 0$ by Theorem 3.2. □

3.4. Theorem. *Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. If d is a (σ,τ)-derivation of N such that $d(x)\sigma(y) = \tau(x)d(y)$ for all $x,y \in U$, then $d = 0$.*

Proof. Assume that

(3.6) $\quad d(x)\sigma(y) = \tau(x)d(y), \text{ for all } x, y \in U.$

Replacing y by $yz, z \in U$ in (3.6), we have

$$d(x)\sigma(y)\sigma(z) = \tau(x)(d(y)\sigma(z) + \tau(y)d(z))$$

and so

$$d(x)\sigma(y)\sigma(z) = \tau(x)d(y)\sigma(z) + \tau(x)\tau(y)d(z), \text{ for all } x, y, z \in U.$$

Applying (3.6) in this equation, we get $\tau(x)\tau(y)d(z) = 0$, for all $x, y, z \in U$. Hence

$$xU\tau^{-1}(d(z)) = (0), \text{ for all } x, z \in U.$$

By Lemma 2.3 (i) and $U \neq (0)$, we get $d(z) = 0$, for all $z \in U$, and so $d = 0$ by Lemma 2.9. □

In [4], Bell and Argaç studied commutativity in $3-$prime near-rings with a nonzero derivation d for which $d(xy) = d(yx)$ for all x, y in some nonzero one sided ideal. Ashraf and Ali showed this result for (σ, σ)-derivation on N in [2]. Now, we continue this study for a (σ, τ)-derivation d and a semigroup ideal U of near-rings without any restriction on U.

3.5. Theorem. *Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. If d is a (σ, τ)-derivation of N such that $d([x, y]) = 0$ for all $x, y \in U$, then N is a commutative ring.*

Proof. In the view of our hypothesis, we have

(3.7) $\quad d([x, y]) = 0, \text{ for all } x, y \in U.$

Replacing y by xy in (3.7), we get

$$\begin{aligned} 0 &= d([x, xy]) = d(x[x, y]) \\ &= d(x)\sigma([x, y]) + \tau(x)d([x, y]), \end{aligned}$$

Using (3.7) in this equation, we have

$$d(x)\sigma([x, y]) = 0, \text{ for all } x, y \in U.$$

That is

(3.8) $\quad d(x)\sigma(x)\sigma(y) = d(x)\sigma(y)\sigma(x), \text{ for all } x, y \in U.$

Writing yr, $r \in N$ instead of y in (3.8), we get

$$d(x)\sigma(x)\sigma(y)\sigma(r) = d(x)\sigma(y)\sigma(r)\sigma(x), \text{ for all } x, y \in U.$$

Using (3.8) in this equation, we arrive at

$$d(x)\sigma(y)\sigma(x)\sigma(r) = d(x)\sigma(y)\sigma(r)\sigma(x)$$

and so

$$d(x)\sigma(y)\sigma([x, r]) = 0.$$

That is

$$\sigma^{-1}(d(x))U[x, r] = (0), \text{ for all } x \in U, r \in N.$$

This yields that for each fixed $x \in U$ either $d(x) = 0$ or $x \in Z$ by Lemma 2.3 (i). But $x \in Z$ also implies that $d(x) \in Z$. Therefore, for any cases we find that $d(x) \in Z$, for any $x \in U$. By Lemma 2.10, we obtain that N is a commutative ring. This completes proof of our theorem. □

3.6. Theorem. *Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. If d is a (σ,τ)-derivation of N such that $d(xoy) = 0$ for all $x, y \in U$, then N is a commutative ring.*

Proof. Replacing x by yx in the hypothesis, we get

$$\begin{aligned} 0 &= d(yxoy) = d(y(xoy)) \\ &= d(y)\sigma(xoy) + \tau(y)d(xoy). \end{aligned}$$

Using the hypothesis, we find that

$$d(y)\sigma(xoy) = 0.$$

That is

$$d(y)\sigma(x)\sigma(y) = -d(y)\sigma(y)\sigma(x), \text{ for all } x, y \in U. \tag{3.9}$$

Taking $xr, r \in N$ instead of x in (3.9) and using (3.9), we obtain

$$\sigma^{-1}(d(y))U[r, y] = (0), \text{ for all } y \in U, r \in N.$$

Now using the same arguments in the last paragraph of the proof of Theorem 3.5, we get the required result. □

3.7. Theorem. *Let N be a $3-$prime near-ring and U be a nonzero semigroup ideal of N. If $a \in N, d$ is a nonzero (σ,τ)-derivation on N such that $\sigma d = d\sigma, \tau d = d\tau$ and $[a, d(U)] = (0)$, then $a \in Z$.*

Proof. Note that $d(U) \subseteq C(a)$ by the hypothesis. Assume that $\tau(y) \in C(a)$. Then it is obvious that $\tau(y)d(x), d(x) \in C(a)$. Also, we get $ad(yx) = d(yx)a$ by the hypothesis. That is

$$a(\tau(y)d(x) + d(y)\sigma(x)) = (\tau(y)d(x) + d(y)\sigma(x))a, \text{ for all } x \in U.$$

We can apply Lemma 2.12 (i) to get

$$a\tau(y)d(x) + ad(y)\sigma(x) = \tau(y)d(x)a + d(y)\sigma(x)a, \text{ for all } x \in U.$$

Since $\tau(y)d(x) \in C(a)$, we get

$$ad(y)\sigma(x) = d(y)\sigma(x)a, \text{ for all } x \in U. \tag{3.10}$$

Thus we obtain $d(y)\sigma(x) \in C(a)$, for all $x \in U$. That is

$$d(y)\sigma(U) \subseteq C(a), \text{ for all } \tau(y) \in C(a). \tag{3.11}$$

On the other hand, if $d^2(U) = (0)$, then $d = 0$ by Lemma 2.11. Since $d \neq 0$, we can choose any $z \in U$ such that $d^2(z) \neq 0$. Let $\tau(y) = d(z)$. Since $d(y)\sigma(x) \in C(a)$, for all $x \in U, \tau(y) \in C(a)$, we have $d(y)\sigma(xr) = d(y)\sigma(x)\sigma(r) \in C(a)$ for all $x \in U, r \in N$. Thus

$$ad(y)\sigma(x)\sigma(r) = d(y)\sigma(x)\sigma(r)a, \text{ for all } x \in U, r \in N.$$

Using the equation (3.10), we have

$$d(y)\sigma(x)a\sigma(r) = d(y)\sigma(x)\sigma(r)a,$$

and so

$$d(y)\sigma(x)[a, \sigma(r)] = 0, \text{ for all } x \in U, r \in N.$$

That is

$$\sigma^{-1}(d(y))U[\sigma^{-1}(a), r]) = (0), \text{ for all } r \in N.$$

This yields $d(y)=0$ or $a\in Z$ by Lemma 2.3 (i). If $d(y)=0$, then $d(\tau^{-1}(d(z))=0$. Using $\tau d=d\tau$, we have $d^2(z)=0$. But this contradicts $d^2(z)\neq 0$. Thus we must have $a\in Z$. This completes the proof. □

As immediate corollaries of Theorem 3.7 and Lemma 2.10 we give the following theorem.

3.8. Theorem. *Let N be a $3-$prime near-ring and U be a nonzero semigroup ideal of N. If $a\in N, d$ is a nonzero (σ,τ)-derivation on N such that $\sigma d=d\sigma, \tau d=d\tau$ and $[d(U),d(U)]=(0)$, then N is a commutative ring.*

3.9. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring and U be a nonzero semigroup ideal of N. If d_1 is a derivation and d_2 is a (σ,τ)-derivation on N such that $d_2\tau=\tau d_2, d_2\sigma=\sigma d_2$ and $d_1(x)d_2(y)=-d_2(y)d_1(x)$, for all $x,y\in U$, then $d_1=0$ or $d_2=0$.*

Proof. Assume that $d_1\neq 0$ and $d_2\neq 0$. By the hypothesis, we have

(3.12) $\quad d_1(x)d_2(y)=-d_2(y)d_1(x)$, for all $x,y\in U$.

We may assume $d_1^2(U)\neq(0)\neq d_2^2(U^2)$ by Lemma 2.6 and Lemma 2.11. Let $w\in d_2(U^2)$. It is easy to see $w,-w\in d_2(U)$. If we take $T=d_2(U), S=d_1(U)$, then $[uv, d_2(U^2)]=(0)$, for all $u,v\in d_1(U)$ by Lemma 2.7. Thus $uv\in Z$ for all $u,v\in d_1(U)$ by Theorem 3.7. Also we have $d_1(x)d_1(y)\in Z$, for all $x,y\in U$. It follows that

$$d_1(x)d_1(x)d_1(y)=d_1(x)d_1(y)d_1(x).$$

Multipliying this equation by $d_1(y)$ from right hand, we have

$$d_1(x)d_1(x)d_1(y)d_1(y)=d_1(x)d_1(y)d_1(x)d_1(y).$$

Using $d_1(x)d_1(y), d_1(y)d_1(x)\in Z$ respectively in the last equation, we find that

$$d_1(x)d_1(y)d_1(x)d_1(y)=d_1(y)d_1(x)d_1(x)d_1(y).$$

Again using $d_1(y)d_1(x), d_1(x)d_1(y)\in Z$ respectively, we arrive at

$$d_1(y)d_1(x)d_1(x)d_1(y)=d_1(y)d_1(x)d_1(y)d_1(x)$$

and so

$$d_1(y)d_1(x)(d_1(x)d_1(y)-d_1(y)d_1(x))=0.$$

Since $d_1(y)d_1(x)$ is central, Lemma 2.1 (i) shows that for any $x,y\in U$, either $d_1(x)d_1(y)-d_1(y)d_1(x)=0$ or $d_1(y)d_1(x)=0$. Hence we get $d_1(x)d_1(y)=d_1(y)d_1(x)=0$, for all $x,y\in U$. That is $[d_1(U),d_1(U)]=(0)$. By Lemma 2.5, we get $d_1^2(U)=0$ or $d_1(U)\subset Z$. In the first case, we find that $d_1=0$ by Lemma 2.6. In the second case, we have N is a commutative ring by Lemma 2.4. But this fact that together with (3.12) shows that $2d_2(y)d_1(x)=0$ for all $x,y\in U$, i.e. $d_2(U)d_1(U)=(0)$. Therefore we get $d_1=0$ or $d_2=0$ from Lemma 2.1 (iii) and Lemma 2.9. Thus we must have $d_1=0$ or $d_2=0$. □

3.10. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring and U be a nonzero semigroup ideal of N. If d_1 is a nonzero derivation and d_2 is a nonzero (σ,τ)-derivation on N such that $d_2\tau=\tau d_2, d_2\sigma=\sigma d_2$.*

(i) If $d_1(x)d_2(y)+d_2(y)d_1(x)\in Z$, for all $x,y\in U$ and at least one of $d_1(U)\cap Z$ and $d_2(U)\cap Z$ is nonzero, then N is a commutative ring.

(ii) If $xd_2(y)+d_2(y)x\in Z$, for all $x,y\in U$, and $U\cap Z\neq(0)$, then N is a commutative ring.

Proof. (i) Let $d_1(U)\cap Z\neq(0)$. Let $x\in U$ such that $d_1(x)\in Z\setminus\{0\}$ and $y\in U$. Then $d_1(x)d_2(y)+d_2(y)d_1(x)=2d_1(x)d_2(y)\in Z$. Since N is a $2-$torsion free $3-$prime near-ring and $d_1(x)\in Z\setminus\{0\}$, we have $d_2(U)\subseteq Z$. Hence N is a commutative ring by Lemma 2.10.

(ii) Let $x\in U\cap Z$ and $y\in U$. Then, $xd_2(y)+d_2(y)x=2xd_2(y)\in Z$. Thus, $d_2(U)\subseteq Z$. As in the proof of (i), we get N is a commutative ring. □

3.11. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring, U be a nonzero semigroup ideal of N which is closed under addition. Suppose that N has nonzero derivation d_1 and nonzero (σ,τ)-derivation d_2 such that $d_1(x)d_2(y)+d_2(y)d_1(x)\in Z$ for all $x,y\in U$ and $d_1(U)\cap Z\neq(0)$ or $d_2(U)\cap Z\neq(0)$. Then N is a commutative ring.*

Proof. By Theorem 3.9, we cannot have $d_1(x)d_2(y)+d_2(y)d_1(x)=0$ for all $x,y\in U^2$. Since $d_1\left(U^2\right)\subseteq U$, there exist $x_0,y_0\in U^2$ such that $u_0=d_1(x_0)d_2(y_0)+d_2(y_0)d_1(x_0)$ is a nonzero central element in U. If either of $d_1(u_0)$ and $d_2(u_0)$ is nonzero, our conclusion follows from Theorem 3.10. On the other hand, if $d_1(u_0)=d_2(u_0)=0$, then $d_1(u_0x)d_2(u_0y)+d_2(u_0y)d_1(u_0x)=u_0d_1(x)\tau(u_0)d_2(y)+\tau(u_0)d_2(y)u_0d_1(x)$. Using $u_0\in Z,\tau(u_0)\in Z$ in the last equation, we get

$$u_0\tau(u_0)\left(d_1(x)d_2(y)+d_2(y)d_1(x)\right)\in Z.$$

Since $0\neq u_0\tau(u_0)\in Z$, we obtain that $d_1(x)d_2(y)+d_2(y)d_1(x)\in Z$. Hence N is a commutative ring by Theorem 3.10. □

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