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Soft hyperrings and their (fuzzy) isomorphism theorems

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Abstract

We introduce the notions of soft hyperrings, idealistic soft hyperrings, soft subhyperrings and soft hyperideals, and discuss some related properties. Moreover, we establish three (fuzzy) isomorphism theorems of soft hyperrings.

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1. Introduction

Soft set theory has been considered as an effective mathematical tool for modeling uncertainties [21]. Different from traditional mathematical tools for dealing with uncertainties, such as probability theory, fuzzy set theory [34] and rough set theory [25], soft set theory is free from the inadequacy of the parametrization tools of these theories [21]. Molodtsov demonstrated that soft set theory has potential applications in many directions, including function smoothness, Riemann integration, measurement theory, game theory and operations research [21]. Also, soft set theory has been applied to forecasting [32], decision making [6, 17, 39], association rules mining [13] and mobile cloud computing [31].

In theoretical aspect of soft sets, after Molodtsov's pioneer work [21], Maji et al. [20] gave further a detailed theoretical study on soft sets. Based on the analysis of several operations on soft sets introduced in [20], Ali et al. [3] proposed some new operations. In [4], Çağman and Enginoğlu defined the soft matrices, which are representative of soft

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sets, and Gong et al. [12] presented the bijective soft sets, which are special soft sets. As an extended concept of bijective soft sets, the exclusive disjunctive soft sets [33] were introduced. Furthermore, Jiang et al. [14] presented an extended soft set theory by using the concepts of description logics to act as the parameters of soft sets. Recently, many researchers studied the algebraic structures of soft sets. Aktaş and Çağman [2] defined soft groups and showed that fuzzy groups can be considered a special case of soft groups. Moreover, some basic properties of soft semirings [11] and soft rings [1] were introduced. Also, Sun et al. [28] presented the soft modules, and Li [19] analyzed the soft lattices. In addition, Jun et al. [15, 16, 26, 38] considered the applications of soft sets in BCK/BCI-algebras, BCH-algebras, WS-algebras and BL-algebras, and considered their related properties.

On the other hand, the theory of algebraic hyperstructures, introduced by Marty in 1934 [23], is a natural generalization of the theory of algebraic structures. It has been applied to many areas [5], such as probabilities, geometry, fuzzy sets, automata, cryptography, combinatorics, and artificial intelligence. Several books on hyperstructure theory have been published [5, 7, 29]. The book [7] was devoted especially to the study of hyperring theory and applications, in which several kinds of hyperrings were introduced and investigated. Krasner hyperring [18], which is a well known type of hyperring, has been studied by many authors. In what follows, by a hyperring we mean a Krasner hyperring. In [8], Davvaz and Salasi defined the notions of normal hyperideal, prime hyperideal, maximal hyperideal, and Jacobson radical of a hyperring and obtained some related results. Furthermore, Davvaz [9] established three isomorphism theorems of hyperrings, and derived the Jordan-Holder theorem for hyperrings. Moreover, Vougiouklis [30] considered the fundamental relation on a hyperring as the smallest equivalence relation so that the quotient is the fundamental ring. In [35], Zhan et al. applied fuzzy sets to hyperrings and introduced the concept of fuzzy hyperideals of a hyperring. By using the normal fuzzy hyperideals of a hyperring, Ma and Zhan [22] derived three fuzzy isomorphism theorems of hyperrings. Also, they considered isomorphism theorems and fuzzy isomorphism theorems of hypermodules [36, 37].

In this paper, we apply the notion of soft sets to hyperrings. Some related notions, such as soft hyperrings, idealistc soft hyperrings, soft subhyperrings, soft hyperideals, are defined, and several basic properties are investigated. Furthermore, we consider the isomorphism of soft hyperrings, and establish three (fuzzy) isomorphism theorems of soft hyperrings.

2. Preliminaries

In this section, we review some notions and results about hyperrings and soft sets. A hypergroupoid $(H, \circ)$ is a non-empty set H together with a hyperoperation $\circ$ defined on H, i.e., a mapping $H \times H \to \mathscr{P}^*(H)$, where $\mathscr{P}^*(H)$ is the set of all non-empty subsets of H. If $x \in H$ and A, B are subsets of H, then $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$. $(H, \circ)$ is called a hypergroup if for all $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and $x \circ H = H \circ x = H$ [27].

2.1. Definition. [18] A hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, i.e.,
- (a) for every $x, y, z \in R$, $(x + y) + z = x + (y + z)$;
- (b) for every $x, y \in R$, $x + y = y + x$;
- (c) there exists $0 \in R$ such that $0 + x = x$ for all $x \in R$;

(d) for every $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$ (we shall write $-x$ for x' and we call it the oposite of x);
(e) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$.
(3) The multiplication is distributive with respect to the hyperoperation $+$.

The following elementary facts follow easily from the axioms: $-(-x) = x$ and $-(x + y) = -x - y$ for all $x, y \in R$.

2.2. Example. [18] Let $(R, +, \cdot)$ be a ring and N a normal subgroup of its multiplicative semigroup. Then the multiplicative classes $\overline{x} = x \cdot N (x \in R)$ form a partition of R, and let $\overline{R} = R/N$ be the set of these classes. Define the hyperaddition and the multiplication on $\overline{R}$ by $\overline{x} \oplus \overline{y} = \{\overline{z} | z \in \overline{x} + \overline{y}\}$ and $\overline{x} \odot \overline{y} = \overline{x \cdot y}$. Then $(\overline{R}, \oplus, \odot)$ is a hyperring.

A non-empty subset S of a hyperring $(R, +, \cdot)$ is called a subhyperring of R if $(S, +, \cdot)$ itself is a hyperring. A subhyperring I of R is a left (right) hyperideal of R if $r \cdot a \in I (a \cdot r \in I)$ for all $r \in R$ and $a \in I$. A subhyperring I is called a hyperideal if I is both left and right hyperideal [9].

2.3. Lemma. [9] A non-empty subset I of a hyperring R is a left (right) hyperideal if and only if (1) $a, b \in I$ implies $a - b \subseteq I$; (2) $a \in I, r \in R$ imply $r \cdot a \in I (a \cdot r \in I)$.

A subhyperring I of a hyperring R is normal if and only if $x + I - x \subseteq I$ for all $x \in R$. Let I be a normal hyperideal of a hyperring R, then for all $x, y \in R$, $(I + x) + (I + y) = I + x + y = I + z$ for all $z \in x + y$ and $I + x = I + y$ for all $y \in I + x$. If K and N are hyperideals of a hyperring R with N normal in R, then $K \cap N$ is a normal hyperideal of K, and N is a normal hyperideal of $K + N$ [9].

If I is a normal hyperideal of a hyperring R, then the relation I^* defined by $x \equiv y(mod\ I)$ if and only if $(x - y) \cap I \neq \emptyset$ is an equivalence relation [9]. Let $I^*[x]$ be the equivalence class of the element $x \in R$. Then $I + x = I^*[x]$ for all $x \in R$. On the set of all classes $R/I = \{I^*[x] \mid x \in R\}$, the hyperoperation $\oplus$ and the multiplication $\odot$ are defined by $I^*[x] \oplus I^*[y] = \{I^*[z] \mid z \in I^*[x] + I^*[y]\}$, and $I^*[x] \odot I^*[y] = I^*[x \cdot y]$, respectively. Then $(R/I, \oplus, \odot)$ is a hyperring. For all $I + x, I + y \in R/I$, we have $(I + x) \oplus (I + y) = \{I + z \mid z \in x + y\}$.

Let R_1 and R_2 be two hyperrings. A mapping φ from R_1 into R_2 is called a strong homomorphism if $\varphi(a + b) = \varphi(a) + \varphi(b)$, $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$, and $\varphi(0) = 0$, for all $a, b \in R_1$. A strong homomorphism φ is an isomorphism if φ is one to one and onto. If φ is a strong homomorphism from R_1 into R_2, then the kernel of φ is the set $ker\varphi = \{x \in R_1 \mid \varphi(x) = 0\}$. It is trivial that $ker\varphi$ is a hyperideal of R_1, but in general it is not normal in R_1 [9].

Let U be an initial universe set and E be a set of parameters. $\mathscr{P}(U)$ denotes the power set of U and $A \subseteq E$.

2.4. Definition. [21] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathscr{P}(U)$.

In fact, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). Please readers see the reference [20] for some examples.

2.5. Definition. [20] For two soft sets (F, A) and (G, B) over U, we say that (F, A) is a soft subset of (G, B), denoted by $(F, A)\widetilde{\subseteq}(G, B)$, if the following conditions hold: (1) $A \subseteq B$; (2) for all $e \in A$, $F(e) \subseteq G(e)$. Two soft sets (F, A) and (G, B) over U are called soft equal if $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$.

2.6. Definition. [3, 20] The extended intersection (or union) of two soft sets (F, A) and (G, B) over U is the soft set $(H, C) = (F, A) \cap_{\mathscr{E}} (G, B)$ (or $(F, A)\widetilde{\cup}(G, B)$), where $C = A \cup B$, and for all $e \in C$, if $e \in A - B$, $H(e) = F(e)$; if $e \in B - A$, $H(e) = G(e)$; if $e \in A \cap B$, $H(e) = F(e) \cap G(e)$ (or $F(e) \cup G(e)$).

2.7. Definition. [3] The restricted intersection (or restricted union) of two soft sets (F, A) and (G, B) over U such that $A \cap B \neq \varnothing$, is the soft set $(H, C) = (F, A) \cap_{\mathscr{R}} (G, B)$ (or $(F, A) \cup_{\mathscr{R}} (G, B)$), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$ (or $F(e) \cup G(e)$).

2.8. Definition. [20] If (F, A) and (G, B) are two soft sets over U, then "(F, A) AND (G, B) (or (F, A) OR (G, B))", denoted by $(F, A)\widetilde{\wedge}(G, B)$ (or $(F, A)\widetilde{\vee}(G, B)$), is defined as $(F, A)\widetilde{\wedge}(G, B)$ (or $(F, A)\widetilde{\vee}(G, B)$)$=(H, A \times B)$, where $H(x, y) = F(x) \cap G(y)$ (or $F(x) \cup G(y)$) for all $(x, y) \in A \times B$.

2.9. Definition. [11] Let (F, A) be a soft set. The set $\mathrm{Supp}(F, A) = \{x \in A \mid F(x) \neq \varnothing\}$ is called the support of the soft set (F, A). A soft set (F, A) is non-null if $\mathrm{Supp}(F, A) \neq \varnothing$.

3. (Idealistic) soft hyperrings

In what follows, R denotes a hyperring and A is a nonempty set. A set-valued function $F : A \to \mathscr{P}(R)$ can be defined as $F(x) = \{y \in R \mid (x, y) \in \rho\}$ for all $x \in A$, where ρ is an arbitrary binary relation between an element of A and an element of R, i.e., ρ is a subset of $A \times R$, then (F, A) is a soft set over R.

3.1. Definition. Let (F, A) be a non-null soft set over R. Then (F, A) is called an (idealistic) soft hyperring over R if $F(x)$ is a subhyperring (hyperideal) of R for all $x \in \mathrm{Supp}(F, A)$.

3.2. Example. Suppose that $R = \{0, 1, 2, 3\}$ and define the operations $+$ and $\cdot$ on R as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	0	0	0

Then $(R, +, \cdot)$ is a hyperring [10]. Let (F, A) be a soft set over R, where $A = R$ and $F : A \to \mathscr{P}(R)$ is a set-valued function defined by $F(x) = \{0\} \cup \{y \in R \mid x\rho y \Leftrightarrow x + y = \{2\}\}$ for all $x \in A$. Then $F(0) = \{0, 2\}$, $F(1) = \{0, 3\}$, $F(2) = \{0\}$ and $F(3) = \{0, 1\}$ are subhyperrings of R. Hence (F, A) is a soft hyperring over R.

Let $B = R$ and $G : B \to \mathscr{P}(R)$ be a set-valued function defined by $G(x) = \{0, 3\} \cup \{y \in R \mid x\rho' y \Leftrightarrow x + y \subseteq \{0, 3\}\}$ for all $x \in B$. Then $G(0) = G(3) = \{0, 3\}$ and $G(1) = G(2) = \{0, 1, 2, 3\}$ are hyperideals of R. Thus (G, B) is an idealistic soft hyperring over R.

Clearly, every idealistic soft hyperring over R is a soft hyperring over R, but the converse is not true in general. In Example 3.2, the (F,A) is not an idealistic soft hyperring over R since $\{0,1\}$ and $\{0,2\}$ are not hyperideals of R.

(F,A) is an (idealistic) soft hyperring over R and $B \subseteq A$. From the Definition 3.1, we have that $(F|_B, B)$ is an (idealistic) soft hyperring over R when it is non-null. Next, we give an example to show that (F,A) is not an (idealistic) soft hyperring over R, but there exists a subset B of A such that $(F|_B, B)$ is an (idealistic) soft hyperring over R.

3.3. Example. Let $R = \{0,1,2,3\}$ be a set with the hyperoperation $+$ and the multiplication $\cdot$ defined as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	$\{0,1\}$	3	$\{2,3\}$
2	2	3	0	1
3	3	$\{2,3\}$	1	$\{0,1\}$

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

It follows that $(R,+,\cdot)$ is a hyperring [24]. Let (F,A) be the soft set over R where $A = R$ and $F : A \to \mathscr{P}(R)$ is a set-valued function defined by $F(x) = \{y \in R \mid x\rho y \Leftrightarrow x + y \subseteq \{1,3\}\}$ for all $x \in A$. Then $F(1) = F(3) = \{0,2\}$ is a hyperideal of R, but $F(0) = F(2) = \{1,3\}$ is not a hyperideal of R, and also is not a subhyperring of R since $1 + 3 = \{2,3\} \nsubseteq \{1,3\}$. Therefore, (F,A) is not an idealistic soft hyperring over R, and also is not a soft hyperring over R. However, if we take $B = \{1,3\} \subseteq A$, then $(F|_B, B)$ is an idealistic soft hyperring over R. Also, it is a soft hyperring over R.

3.4. Theorem. Let (F,A) and (G,B) be two (idealistic) soft hyperrings over R, then

(1) $(F,A) \cap_{\mathscr{E}} (G,B)$ is an (idealistic) soft hyperring over R if it is non-null;
(2) if $A \cap B \neq \varnothing$, then $(F,A) \cap_{\mathscr{R}} (G,B)$ is an (idealistic) soft hyperring over R whenever it is non-null;
(3) if $A \cap B = \varnothing$, then $(F,A)\widetilde{\cup}(G,B)$ is an (idealistic) soft hyperring over R;
(4) $(F,A)\widetilde{\wedge}(G,B)$ is an (idealistic) soft hyperring over R.

Proof. We only prove (1), and the proofs of (2)-(4) are similar. By Definition 2.6, we have $(H,C) = (F,A) \cap_{\mathscr{E}} (G,B)$. For all $x \in \mathrm{Supp}(H,C)$, if $x \in A - B$, because (F,A) is an (idealistic) soft hyperring over R, we have $H(x) = F(x)$ is a subhyperring (hyperideal) of R; if $x \in B - A$, because (G,B) is an (idealistic) soft hyperring over R, $H(x) = G(x)$ is a subhyperring (hyperideal) of R; if $x \in A \cap B$, $H(x) = F(x) \cap G(x)$ is a subhyperring (hyperideal) of R, since the intersection of any two subhyperrings (hyperideals) of R is also a subhyperring (hyperideal) of R. Therefore, $(H,C) = (F,A) \cap_{\mathscr{E}} (G,B)$ is an (idealistic) soft hyperring over R.

If A and B are not disjoint, Theorem 3.4(3) is not true in general.

3.5. Example. Consider the hyperring R defined in Example 3.3. Let $A = R$ and $F : A \to \mathscr{P}(R)$ be a set-valued function defined by $F(x) = \{0,1\} \cup \{y \in R \mid x\rho y \Leftrightarrow x + y \subseteq \{2,3\}\}$ for all $x \in A$. Then $F(0) = F(1) = \{0,1,2,3\}$ and $F(2) = F(3) = \{0,1\}$ are hyperideals of R. Thus (F,A) is an idealistic soft hyperring over R.

Let $B = \{0,2\}$ and $G : B \to \mathscr{P}(R)$ be the set-valued function defined by $G(x) = \{y \in R \mid x\rho' y \Leftrightarrow x + y \subseteq \{0,2\}\}$ for all $x \in B$. Then $G(0) = G(2) = \{0,2\}$ is a hyperideal of R. Hence, (G,B) is an idealistic soft hyperring over R. However, $(H,C) = (F,A)\widetilde{\cup}(G,B)$ is not an idealistic soft hyperring over R and also is not a soft hyperring, since $H(2) = F(2) \cup G(2) = \{0,1,2\}$ is not a subhyperring of R for

$1 + 2 = \{3\} \nsubseteq H(2)$.

3.6. Corollary. Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of (idealistic) soft hyperrings over R, where Λ is an index set, then

(1) $(\cap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i)$ is an (idealistic) soft hyperring over R if it is non-null;
(2) if $\cap_{i\in\Lambda} A_i \neq \varnothing$, then $(\cap_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i)$ is an (idealistic) soft hyperring over R whenever it is non-null;
(3) if $A_i \cap A_j = \varnothing$ for all $i, j \in \Lambda$ and $i \neq j$, then $\widetilde{\cup}_{i\in\Lambda}(F_i, A_i)$ is an (idealistic) soft hyperring over R;
(4) $\widetilde{\wedge}_{i\in\Lambda}(F_i, A_i)$ is an (idealistic) soft hyperring over R.

3.7. Definition. Let (F, A) be an idealistic soft hyperring over R, then (F, A) is called an identity idealistic soft hyperring over R if $F(x) = \{0\}$ for all $x \in A$; (F, A) is called an absolute idealistic soft hyperring over R if $F(x) = R$ for all $x \in A$.

3.8. Example. Consider the hyperring R defined in Example 3.3. Let $A = R$ and $F : A \to \mathscr{P}(R)$ be the set-valued function defined by $F(x) = \{y \in R \mid x\rho y \Leftrightarrow x+y = \{x\}\}$ for all $x \in A$. Then $F(0) = F(1) = F(2) = F(3) = \{0\}$ and so (F, A) is an identity idealistic soft hyperring over R.

Let $B = R$ and $G : B \to \mathscr{P}(R)$ be the set-valued function defined by $G(x) = \{y \in R \mid x\rho' y \Leftrightarrow x + y \subseteq R\}$ for all $x \in B$. Then $G(x) = R$ for all $x \in B$ and so (G, B) is an absolute idealistic soft hyperring over R.

3.9. Theorem. Let φ be a strong homomorphism from hyperring R_1 to hyperring R_2. If (F, A) is a soft hyperring over R_1, then $(\varphi(F), A)$ is a soft hyperring over R_2; if φ is onto and (F, A) is an idealistic soft hyperring over R_1, then $(\varphi(F), A)$ is an idealistic soft hyperring over R_2, where $\varphi(F)(x) = \varphi(F(x))$ for all $x \in A$.

Proof. Clearly, $\mathrm{Supp}(\varphi(F), A) = \mathrm{Supp}(F, A)$. For all $x \in \mathrm{Supp}(\varphi(F), A)$, $\varphi(F)(x) = \varphi(F(x))$. Since (F, A) is a soft hyperring over R_1, it follows that $F(x)$ is a subhyperring of R_1, so $\varphi(F(x))$ is also a subhyperring of R_2. Hence, $(\varphi(F), A)$ is a soft hyperring over R_2. Moreover, for every $x \in \mathrm{Supp}(\varphi(F), A)$, because $F(x)$ is a hyperideal of R_1 and φ is onto, we have that $\varphi(F)(x) = \varphi(F(x))$ is a hyperideal of R_2. Therefore, $(\varphi(F), A)$ is an idealistic soft hyperring over R_2.

3.10. Theorem. Let φ be a strong homomorphism from hyperring R_1 to hyperring R_2, and (F, A) be an idealistic soft hyperring over R_1. If $F(x) = ker\varphi$ for all $x \in A$, then $(\varphi(F), A)$ is an identity idealistic soft hyperring over R_2. If φ is onto and (F, A) is an absolute idealistic soft hyperring over R_1, then $(\varphi(F), A)$ is an absolute idealistic soft hyperring over R_2.

Proof. It is straightforward.

3.11. Definition. Let (F, A) and (G, B) be two soft hyperrings over R. Then (G, B) is called a soft subhyperring (hyperideal) of (F, A) if $B \subseteq A$, and $G(x)$ is a subhyperring (hyperideal) of $F(x)$ for all $x \in \mathrm{Supp}(G, B)$.

3.12. Example. Consider the hyperring R given in Example 3.2. Let $A = R$ and $F : A \to \mathscr{P}(R)$ be the set-valued function defined by $F(x) = \{0, 2\} \cup \{y \in R \mid x\rho y \Leftrightarrow x + y \subseteq \{1, 3\}\}$ for all $x \in A$. Then $F(0) = F(2) = \{0, 1, 2, 3\}$, and $F(1) = F(3) = \{0, 2\}$ are subhyperrings of R. Therefore, (F, A) is a soft hyperring over R.

Let $B = \{1, 2, 3\} \subseteq A$ and $G : B \to \mathscr{P}(R)$ be the set-valued function defined by $G(x) = \{0\} \cup \{y \in R \mid x\rho' y \Leftrightarrow x + y = \{1\}\}$ for all $x \in B$. Then $G(1) = \{0\}$, $G(2) = \{0, 3\}$ and $G(3) = \{0, 2\}$ are hyperideals of $F(1)$, $F(2)$ and $F(3)$, respectively, so (G, B) is a soft hyperideal of (F, A).

3.13. Theorem. Let (F, A) and (G, B) be soft hyperrings over R. For all $x \in \mathrm{Supp}(G, B)$, if $B \subseteq A$ and $G(x) \subseteq F(x)$, then (G, B) is a soft subhyperring of (F, A). Furthermore, if (G, B) is an idealistic soft hyperring over R, then (G, B) is a soft hyperideal of (F, A).

Proof. Straightforward.

3.14. Theorem. Let (F, A) be a soft hyperring over R, and $(G_i, B_i)_{i\in\Lambda}$ be a non-empty family of soft subhyperrings (hyperideals) of (F, A), where Λ is an index set, then

(1) $(\cap_{\mathscr{E}})_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal) of (F, A) if it is non-null;
(2) if $\cap_{i\in\Lambda} B_i \neq \varnothing$, then $(\cap_{\mathscr{R}})_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal) of (F, A) whenever it is non-null;
(3) if $B_i \cap B_j = \varnothing$ for all $i, j \in \Lambda$ and $i \neq j$, then $\widetilde{\cup}_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal)of (F, A);
(4) $\widetilde{\wedge}_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal) of the soft hyperring $\widetilde{\wedge}_{i\in\Lambda}(F, A)$ if it is non-null.

Proof. We only prove (1), and the proofs of (2)-(4) are similar. By Definition 2.6, we have $(H, C) = (\cap_{\mathscr{E}})_{i\in\Lambda}(G_i, B_i)$, where $C = \bigcup_{i\in\Lambda} B_i$, $H(x) = \bigcap_{i\in\Lambda(x)} G_i(x)$ and $\Lambda(x) = \{i \in \Lambda | x \in B_i\}$, for all $x \in C$. Since $(G_i, B_i)_{i\in\Lambda}$ be a non-empty family of soft subhyperrings (hyperideals) of (F, A), we have that $C = \bigcup_{i\in\Lambda} B_i \subseteq A$, and $H(x) = \bigcap_{i\in\Lambda(x)} G_i(x)$ is a subhyperring (hyperideal) of $F(x)$, for all $x \in \mathrm{Supp}(H, C)$. Therefore, $(H, C) = (\cap_{\mathscr{E}})_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal) of (F, A).

3.15. Theorem. Let φ be a strong homomorphism from hyperring R_1 to hyperring R_2. If (F, A) is a soft hyperring over R_1, and (G, B) is a soft subhyperring (hyperideal) of (F, A), then $(\varphi(G), B)$ is a soft subhyperring (hyperideal) of $(\varphi(F), A)$.

Proof. From Theorem 3.9, we have that $(\varphi(F), A)$ and $(\varphi(G), B)$ are soft hyperrings over R_2. Clearly, $\mathrm{Supp}(\varphi(G), B) = \mathrm{Supp}(G, B)$. It follows that $B \subseteq A$ and $G(x)$ is a subhyperring of $F(x)$ for all $x \in \mathrm{Supp}(G, B)$, because (G, B) is a soft subhyperring of (F, A). So $\varphi(G)(x) \subseteq \varphi(F)(x)$ for all $x \in \mathrm{Supp}(\varphi(G), B)$. According to Theorem 3.13, $(\varphi(G), B)$ is a soft subhyperring of $(\varphi(F), A)$.

Now, for all $x \in \mathrm{Supp}(\varphi(G), B)$, $r' \in \varphi(F)(x)$, $a' \in \varphi(G)(x)$, there exists $r \in F(x)$, $a \in G(x)$ such that $\varphi(r) = r'$, $\varphi(a) = a'$. Because $G(x)$ is a hyperideal of $F(x)$, we have that $r' \cdot a' = \varphi(r) \cdot \varphi(a) = \varphi(r \cdot a) \in \varphi(G(x)) = \varphi(G)(x)$ and $a' \cdot r' = \varphi(a) \cdot \varphi(r) = \varphi(a \cdot r) \in \varphi(G(x)) = \varphi(G)(x)$. It follows that $\varphi(G)(x)$ is a hyperideal of $\varphi(F)(x)$ for all $x \in \mathrm{Supp}(\varphi(G), B)$. Therefore, $(\varphi(G), B)$ is a soft hyperideal of $(\varphi(F), A)$.

4. Isomorphism theorems of soft hyperrings

In this section, we consider the isomorphism theorems of soft hyperrings. First, we give the notions of soft homomorphism, soft monomorphism, soft epimorphism, and soft isomorphism.

4.1. Definition. Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2, respectively, and $\varphi : R_1 \to R_2$ and $\psi : A \to B$ be two mappings. If φ is a strong homomorphism and for all $x \in A$, $\varphi(F(x)) = G(\psi(x))$, then (φ, ψ) is called a soft homomorphism, and (F, A) is soft homomorphic to (G, B), denoted by $(F, A) \sim (G, B)$. If φ is a monomorphism (resp. epimorphism, isomorphism) and ψ is a injective (resp. surjective, bijective) mapping, then (φ, ψ) is called a soft monomorphism (resp. epimorphism, isomorphism), and (F, A) is soft monomorphic (resp. epimorphic, isomorphic) to (G, B). $(F, A) \simeq (G, B)$ is used to denote that (F, A) is soft isomorphic to (G, B).

4.2. Theorem. Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2, respectively, and (F, A) be soft epimorphic to (G, B). If (F, A) is an idealistic soft hyperring over R_1, then (G, B) is an idealistic soft hyperring over R_2.

Proof. Suppose that (φ, ψ) is a soft epimorphism from (F, A) to (G, B). For every $x \in \mathrm{Supp}(F, A)$, $F(x)$ is a hyperideal of R_1, by Theorem 3.9, we have that $\varphi(F(x))$ a hyperideal of R_2. For every $y \in \mathrm{Supp}(G, B)$, there exists $x \in A$ such that $\psi(x) = y$, so $G(y) = G(\psi(x)) = \varphi(F(x))$ is a hyperideal of R_2. It follows that (G, B) is an idealistic soft hyperring over R_2.

In what follows, we say that $(F/I, A)$ is a soft hyperring over R/I, which means $(F/I)(x) = F(x)/I$ for all $x \in A$, $I \subseteq F(x)$ for all $x \in \mathrm{Supp}(F, A)$, and $(F/I)(x) = \varnothing$ for $x \in A-$ $\mathrm{Supp}(F, A)$, where (F, A) is a soft hyperring over R, and I is a normal hyperideal of R.

4.3. Theorem. Let I be a normal hyperideal of R, and (F, A) be a soft hyperring over R, then (F, A) is soft epimorphic to $(F/I, A)$.

Proof. Since $I \subseteq F(x)$ for all $x \in \mathrm{Supp}(F, A)$, it follows that $F(x)/I$ is a subhyperring of R/I. So $(F/I, A)$ is a soft hyperring over R/I. Define $\varphi : R \to R/I$ by $\varphi(x) = I^*[x]$, for all $x \in R$, then φ is an epimorphism. Define $\psi : A \to A$ by $\psi(x) = x$ for all $x \in A$. Clearly, ψ is surjective. For all $x \in A$, $\varphi(F(x)) = F(x)/I = F(\psi(x))/I$. Therefore, (φ, ψ) is a soft epimorphism, and (F, A) is soft epimorphic to $(F/I, A)$.

4.4. Theorem. (*First Isomorphism Theorem*) Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2, respectively. If (φ, ψ) is a soft epimorphism from (F, A) to (G, B) with kernel I such that I is a normal hyperideal of R_1, then $(F/I, A) \simeq (\varphi(F), A)$. Moreover, if ψ is bijective, then $(F/I, A) \simeq (G, B)$.

Proof. Clearly, $(F/I, A)$ and $(\varphi(F), A)$ are soft hyperrings over R_1/I and R_2, respectively. We define $\varphi' : R_1/I \to R_2$ by $\varphi'(I^*[x]) = \varphi(x)$, for all $x \in R_1$. According to the first isomorphism theorem of hyperrings, φ' is an isomorphism. Define $\psi' : A \to A$ by $\psi'(x) = x$ for all $x \in A$, then ψ' is bijective. Also $\varphi'(F(x)/I) = \varphi(F(x)) = \varphi(F(\psi'(x)))$ for all $x \in A$. It follows that (φ', ψ') is a soft isomorphism, and $(F/I, A) \simeq (\varphi(F), A)$. Moreover, since φ' is an isomorphism, ψ is bijective and for all $x \in A$, $\varphi'(F(x)/I) = \varphi(F(x)) = G(\psi(x))$. So (φ', ψ) is a soft isomorphism, and $(F/I, A) \simeq (G, B)$.

4.5. Theorem. (*Second Isomorphism Theorem*) Let I and K be hyperideals of R, with I normal in R. If (F, A) is a soft hyperring of K, then $(F/(I \cap K), A) \simeq ((I + F)/I, A)$.

Proof. Clearly, $(F/(I \cap K), A)$ and $((I + F)/I, A)$ are soft hyperring over $(K/(I \cap K)$ and $(I+K)/I$, respectively. $\varphi : K \to (I+K)/I$ is defined by $\varphi(x) = I^*[x]$ for all $x \in K$. Then φ is an epimorphism. $\psi : A \to A$ is defined by $\psi(x) = x$ for all $x \in A$. Then ψ is bijective. For all $x \in A$, we have $\varphi(F(x)) = \{I^*[a] \mid a \in F(x)\} = (I + F(x))/I = (I + F(\psi(x)))/I$. For $\{I^*[a] \mid a \in F(x)\} = (I + F(x))/I$, the proof is showed as follows.

Clearly, $\{I^*[a] \mid a \in F(x)\} \subseteq (I + F(x))/I$. For any $I^*[b] \in (I + F(x))/I$, where $b \in I + F(x)$, which implies that there exist $i \in I$ and $k \in F(x)$ such that $b \in i + k$, so $I^*[b] = I + b = I + i + k = I + k = I^*[k] \in \{I^*[a] \mid a \in F(x)\}$.

Therefore, (φ, ψ) is a soft epimorphism from (F, A) to $((I+F)/I, A)$. Since $I \cap K$ is a normal hyperideal of K, if $ker\varphi = I \cap K$, then, by Theorem 4.4, $(F/(I \cap K), A) \simeq ((I + F)/I, A)$. For any $x \in K$, $x \in ker\varphi \Leftrightarrow \varphi(x) = I^*[0] = I \Leftrightarrow I^*[x] = I + x = I \Leftrightarrow x \in I$ (since $x \in K$)$\Leftrightarrow x \in I \cap K$.

4.6. Theorem. (*Third Isomorphism Theorem*) Let K and I be normal hyperideals of R such that $I \subseteq K$. If (F, A) is a soft hyperring over R, and $K \subseteq F(x)$ for all $x \in \mathrm{Supp}(F, A)$, then $((F/I)/(K/I), A) \simeq (F/K, A)$.

Proof. We have that K/I is a normal hyperideal of R/I, because K and I are normal hyperideals of R with $I \subseteq K$. Thus, $(R/I)/(K/I)$ is defined. Since $F(x)$ is a subhyperring of R and $I \subseteq K \subseteq F(x)$ for all $x \in \mathrm{Supp}(F, A)$, $(F(x)/I)/(K/I)$ is defined and is a subhyperring of $(R/I)/(K/I)$. Clearly, $\mathrm{Supp}((F/I)/(K/I), A)= \mathrm{Supp}(F, A)$. It follows that $((F/I)/(K/I), A)$ is a soft hyperring over $(R/I)/(K/I)$. Also, it is easy to obtain that $(F/I, A)$ and $(F/K, A)$ are soft hyperrings over R/I and R/K, respectively. $\varphi : R/I \to R/K$, defined by $\varphi(I^*[x]) = K^*[x]$, is an epimorphism, and $\psi : A \to A$, defined by $\psi(x) = x$ for all $x \in A$, is bijective. Moreover, for all $x \in A$, $\varphi(F(x)/I) = F(x)/K = F(\psi(x))/K$. So (φ, ψ) is a soft epimorphism from $(F/I, A)$ to $(F/K, A)$. By Theorem 4.4, if $kerf = K/I$, then $((F/I)/(K/I), A) \simeq (F/K, A)$. For any $I^*[x] \in R/I$, $I^*[x] \in kerf \Leftrightarrow f(I^*[x]) = K^*[0] = K \Leftrightarrow K^*[x] = K + x = K \Leftrightarrow x \in K \Leftrightarrow I^*[x] \in K/I$.

5. Fuzzy isomorphism theorems of soft hyperrings

In this scetion, we eatablish three fuzzy isomorphism theorems of soft hyperrings. Firstly, we review some related results about fuzzy hyperideal of hyperrings [22, 35].

A fuzzy set μ of a hyperring R is called a fuzzy hyperideal of R if the following conditions hold: (1) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y} \mu(z)$ for all $x, y \in R$; (2) $\mu(x) \leq \mu(-x)$ for all $x \in R$; (3) $\max\{\mu(x), \mu(y)\} \leq \mu(xy)$ for all $x, y \in R$. A fuzzy hyperideal μ of R is called normal if $\mu(y) \leq \inf_{\alpha \in x+y-x} \mu(\alpha)$ for all $x, y \in R$.

Let μ be a normal fuzzy hyperideal of R. Define the relation on R: $x \equiv y(mod\ \mu)$ if and only if there exists $\alpha \in x - y$ such that $\mu(\alpha) = \mu(0)$, denoted by $x\mu^*y$, and μ^* is an equivalence relation. If $x\mu^*y$, then $\mu(x) = \mu(y)$. Let $\mu^*[x]$ be the equivalence class containing $x \in R$, and R/μ be the set of all equivalence classes, i.e., $R/\mu = \{\mu^*[x] \mid x \in R\}$. Define operations $\oplus$ and $\odot$ in R/μ by $\mu^*[x] \oplus \mu^*[y] = \{\mu^*[z] \mid z \in \mu^*[x] + \mu^*[y]\}$, and $\mu^*[x] \odot \mu^*[y] = \mu^*[x \cdot y]$, respectively. Then $(R/\mu, \oplus, \odot)$ is a hyperring.

Let I be a normal hyperideal of R, and μ be a normal fuzzy hyperideal of R. If μ is restricted to I, then μ is a normal fuzzy hyperideal of I, and I/μ is a normal hyperideal of R/μ. If μ and ν are normal fuzzy hyperideals of R, then $\mu \cap \nu$ is normal fuzzy hyperideals of R.

If X and Y are two non-empty sets, $\varphi : X \to Y$ is a mapping, and μ and ν are the fuzzy sets of X and Y, respectively, then the image $\varphi(\mu)$ of μ is the fuzzy subset of Y defined as follows: for all $y \in Y$, if $\varphi^{-1}(y) \neq \emptyset$, $\varphi(\mu)(y) = \sup\limits_{x \in \varphi^{-1}(y)} \{\mu(x)\}$; otherwise, $\varphi(\mu)(y) = 0$. The inverse image $\varphi^{-1}(\nu)$ of ν is the fuzzy subset of X defined by $\varphi^{-1}(\nu)(x) = \nu(\varphi(x))$ for all $x \in X$.

Let R_1 and R_2 be two hyperrings, and $\varphi : R_1 \to R_2$ be a strong homomorphism. If μ and ν are (normal) fuzzy hyperideals of R_1 and R_2, respectively, then (1) $\varphi(\mu)$ is a (normal) fuzzy hyperideal of R_2; (2) if φ is onto, then $\varphi^{-1}(\nu)$ is a (normal) fuzzy hyperideal of R_1. If μ and ν are normal fuzzy hyperideals of R_1 and R_2, respectively, then (1) if φ is onto, then $\varphi(\varphi^{-1}(\nu)) = \nu$; (2) if μ is a constant on $ker\varphi$, then $\varphi^{-1}(\varphi(\mu)) = \mu$. Let μ be a normal fuzzy hyperideal of R, then $R_\mu = \{x \in R \mid \mu(x) = \mu(0)\}$ is a normal hyperideal of R.

5.1. Theorem. (*First Fuzzy Isomorphism Theorem*) Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2, respectively. If (φ, ψ) is a soft epimorphism from (F, A) to (G, B) and μ is a normal fuzzy hyperideal of R_1 with $(R_1)_\mu \supseteq ker\varphi$, then $(F/\mu, A) \simeq (\varphi(F)/\varphi(\mu), A)$, where $(F/\mu)(x) = F(x)/\mu$ for all $x \in A$. Moreover, if ψ is bijective, then $(F/\mu, A) \simeq (G/\varphi(\mu), B)$.

Proof. We obtain that $(F/\mu, A)$ is a soft hyperring over R_1/μ, since (F, A) is soft hyperring over R_1, and μ is a normal fuzzy hyperideal of R_1. For all $x \in \text{Supp}(F, A)$, $\varphi(F(x)) = G(\psi(x))$ is a subhyperring of R_2, so $(\varphi(F)/\varphi(\mu), A)$ is a soft hyperring over $R_2/\varphi(\mu)$. $\varphi' : R_1/\mu \to R_2/\varphi(\mu)$ defined by $\varphi'(\mu^*[x]) = \varphi(\mu)^*[\varphi(x)]$, for all $x \in R_1$, is an isomorphism, by the first fuzzy isomorphism theorem of hyperrings. $\psi' : A \to A$ defined by $\psi'(x) = x$ for all $x \in A$, is bijective. Moreover, $\varphi'(F(x)/\mu) = \{\varphi(\mu)^*[a] \mid a \in \varphi(F(x))\} = \varphi(F(x))/\varphi(\mu) = \varphi(F(\psi'(x)))/\varphi(\mu)$, for all $x \in A$. It follows that (φ', ψ') is a soft isomorphism, and $(F/\mu, A) \simeq (\varphi(F)/\varphi(\mu), A)$.

Moreover, for all $x \in A$, we have that $\varphi'(F(x)/\mu) = \{\varphi(\mu)^*[a] \mid a \in \varphi(F(x))\} = \varphi(F(x))/\varphi(\mu) = G(\psi(x))/\varphi(\mu)$. φ' is an isomorphism, and ψ is bijective. It follows that (φ', ψ) is a soft isomorphism, and $(F/\mu, A) \simeq (G/\varphi(\mu), B)$.

5.2. Theorem. Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2 respectively. If (φ, ψ) is a soft epimorphism from (F, A) to (G, B) and ν is a normal fuzzy hyperideal of R_2, then $(F/\varphi^{-1}(\nu), A) \simeq (\varphi(F)/\nu, A)$. Moreover, if ψ is bijective, then $(F/\varphi^{-1}(\nu), A) \simeq (G/\nu, B)$.

Proof. Since ν is a normal fuzzy hyperideal of R_2 and φ is an epimorphism, we have that $\varphi(\varphi^{-1}(\nu)) = \nu$ and $\varphi^{-1}(\nu)$ is a normal fuzzy hyperideal of R_1. Thus, $(F/\varphi^{-1}(\nu), A)$ and $(\varphi(F)/\nu, A)$ are soft hyperrings over hyperrings $R_1/\varphi^{-1}(\nu)$ and R_2/ν, respectively. For any $x \in ker\varphi$, $\varphi(x) = \varphi(0)$. It follows that $\nu(\varphi(x)) = \nu(\varphi(0))$, i.e., $\varphi^{-1}(\nu)(x) = \varphi^{-1}(\nu)(0)$, which implies that $x \in (R_1)_{\varphi^{-1}(\nu)}$. So $(R_1)_{\varphi^{-1}(\nu)} \supseteq ker\varphi$. By Theorem 5.1, we have $(F/\varphi^{-1}(\nu), A) \simeq (\varphi(F)/\nu, A)$. Furthermore, if ψ is bijective, then we have $(F/\varphi^{-1}(\nu), A) \simeq (G/\nu, B)$.

5.3. Theorem. (*Second Fuzzy Isomorphism Theorem*) Let (F, A) be a soft hyperring over R. If μ and ν are two normal fuzzy hyperideals with $\mu(0) = \nu(0)$, then $(F_\mu/(\mu \cap \nu), A) \simeq ((F_\mu + F_\nu)/\nu, A)$.

Proof. We have that $\mu \cap \nu$ and ν are normal fuzzy hyperideal of R_μ and $R_\mu + R_\nu$, respectively. It follows that $R_\mu/(\mu \cap \nu)$ and $(R_\mu + R_\nu)/\nu$ are hyperrings. Since (F, A) is a soft hyperring over R, we can obtain easily that $(F_\mu/(\mu \cap \nu), A)$ and $((F_\mu + F_\nu)/\nu, A)$ are soft hyperrings over $R_\mu/(\mu \cap \nu)$ and $(R_\mu + R_\nu)/\nu$, respectively. $\varphi : R_\mu/(\mu \cap \nu) \to (R_\mu + R_\nu)/\nu$ is defined by $\varphi((\mu\cap\nu)^*[x]) = \nu^*[x]$ for all $x \in R_\mu$. If $(\mu\cap\nu)^*[x] = (\mu\cap\nu)^*[y]$, then $(\mu\cap\nu)(x) = (\mu\cap\nu)(y)$, i.e., $\min\{(\mu(x), \nu(x)\} = \min\{(\mu(y), \nu(y)\}$. Because $x, y \in R_\mu$ and $\mu(0) = \nu(0)$, we have $\mu(x) = \mu(0) = \nu(0)$ and $\mu(y) = \mu(0) = \nu(0)$. So $\nu(x) = \nu(y)$. It follows that $\nu^*(x) = \nu^*(y)$. Thus, φ is well-defined. Moreover, we have

$$\varphi((\mu \cap \nu)^*[x] \oplus (\mu \cap \nu)^*[y]) = \varphi(\{(\mu \cap \nu)^*[z] \mid z \in (\mu \cap \nu)^*[x] + (\mu \cap \nu)^*[y]\})$$
$$= \{\nu^*[z] \mid z \in (\mu \cap \nu)^*[x] + (\mu \cap \nu)^*[y]\} = \nu^*((\mu \cap \nu)^*[x]) \oplus \nu^*((\mu \cap \nu)^*[y])$$
$$= \varphi((\mu \cap \nu)^*[x]) \oplus \varphi((\mu \cap \nu)^*[y]),$$

$$\varphi((\mu \cap \nu)^*[x] \odot (\mu \cap \nu)^*[y]) = \varphi((\mu \cap \nu)^*[x \cdot y]) = \nu^*[x \cdot y]$$
$$= \nu^*[x] \odot \nu^*[y] = \varphi((\mu \cap \nu)^*[x]) \odot \varphi((\mu \cap \nu)^*[y])$$

and $\varphi((\mu \cap \nu)^*[0]) = \nu^*[0] = 0$. Consequently, φ is a homomorphism.

If $(\mu\cap\nu)^*[x] \neq (\mu\cap\nu)^*[y]$, we have $(\mu\cap\nu)(x) \neq (\mu\cap\nu)(y)$. It follows that $\nu(x) \neq \nu(y)$, so $\nu^*[x] \neq \nu^*[y]$. Hence, φ is a monomorphism. For any $\nu^*[x] \in (R_\mu + R_\nu)/\nu$, where $x \in R_\mu + R_\nu$, which implies that there exist $a \in R_\mu$ and $b \in R_\nu$ such that $x \in a + b$, there is $\alpha \in x - a \subseteq a + b - a \subseteq R_\nu$, i.e., $\nu(\alpha) = \nu(0)$. Hence we have $\nu^*[x] = \nu^*[a]$. So $\varphi((\mu \cap \nu)^*[a]) = \nu^*[x]$, and φ is an epimorphism. Thus, φ is an isomorphism.

$\psi : A \to A$ defined by $\psi(x) = x$ for all $x \in A$, is bijective. For all $x \in A$, $\varphi(F_\mu(x)/(\mu\cap \nu)) = F_\mu(x)/\nu = (F_\mu + F_\nu)(x)/\nu = (F_\mu + F_\nu)(\psi(x))/\nu$. The proof of $F_\mu(x)/\nu = (F_\mu + F_\nu)(x)/\nu$ is showed as follows.

Clearly, $F_\mu(x)/\nu \subseteq (F_\mu + F_\nu)(x)/\nu$. For all $\nu^*[a] \in (F_\mu + F_\nu)(x)/\nu$, where $a \in (F_\mu + F_\nu)(x)$, which implies that there exist $m \in F_\mu(x)$ and $n \in F_\nu(x)$ such that $a \in m + n$, there is $\alpha \in a - m \subseteq m + n - m \subseteq F_\nu(x)$, i.e., $\nu(\alpha) = \nu(0)$. It follows that $\nu^*[a] = \nu^*[m] \in F_\mu(x)/\nu$.

Therefore, (φ, ψ) is a soft isomorphism and $(F_\mu/\mu \cap \nu, A) \simeq ((F_\mu + F_\nu)/\nu, A)$.

5.4. Theorem. (*Third Fuzzy Isomorphism Theorem*) Let (F, A) be a soft hyperring over a hyperring R. If μ and ν are two normal fuzzy hyperideals with $\nu \leq \mu$, $\mu(0) = \nu(0)$ and $F_\mu(x) = R_\mu$ for all $x \in$Supp(F, A), then $((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$.

Proof. We can easily deduce that R_μ/ν is a normal hyperideal of R/ν. Because (F, A) be a soft hyperring over R, we have that $(F/\nu, A)$, $((F/\nu)/(F_\mu/\nu), A)$ and $(F/\mu, A)$ are soft hyperrings over R/ν, $(R/\nu)/(R_\mu/\nu)$ and R/μ, respectively. $\varphi : R/\nu \to R/\mu$ is defined by $\varphi(\nu^*[x]) = \mu^*[x]$ for all $x \in R$. If $\nu^*[x] = \nu^*[y]$ for all $x, y \in R$, then there exists $\alpha \in x - y$ such that $\nu(\alpha) = \nu(0)$. Because $\nu \leq \mu$ and $\mu(0) = \nu(0)$, we get $\mu(\alpha) \geq \nu(\alpha) = \nu(0) = \mu(0)$, which implies that $\mu(\alpha) = \mu(0)$. So we have $\mu^*[x] = \mu^*[y]$. Thus, φ is well-defined. Clearly, φ is an epimorphism. $\psi : A \to A$ defined by $g(x) = x$ for all $x \in A$, is bijective. For all $x \in A$, $\varphi(F(x)/\nu) = F(x)/\mu = F(\psi(x))/\mu$. Hence, (φ, ψ) is a soft epimorphism from $(F/\nu, A)$ to $(F/\mu, A)$. Moreover, $ker\varphi = \{\nu^*[x] \in R/\nu \mid \varphi(\nu^*[x]) = \mu^*[0]\} = \{\nu^*[x] \in R/\nu \mid \mu^*[x] = \mu^*[0]\} = \{\nu^*[x] \in R/\nu \mid \mu(x) = \mu(0)\} = \{\nu^*[x] \in R/\nu \mid x \in R_\mu\} = R_\mu/\nu$. By Theorem 4.4, we have $((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$.

6. Conclusions

In this paper, we define soft hyperrings, idealistic soft hyperrings, soft subhyperrings and soft hyperideals, and introduce homomorphism and isomorphism of soft hyperrings. Furthermore, we generalize three (fuzzy) isomorphism theorems of hyperrings to three (fuzzy) isomorphism theorems of soft hyperrings. Based on these results, we will apply the notion of soft sets to other algebraic hyperstructures, and consider some applications of soft hyperrings in decision making problems.

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