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Research Article

Quartic B-spline Differential Quadrature Method for Solving the Extended Fisher-Kolmogorov Equation

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Abstract

Some numerical solutions of the extended Fisher-Kolmogorov(EFK) equation have been obtained via quartic B-spline differential quadrature method(DQM). 2nd order weighting coefficients are obtained directly by quartic B-splines. Since the 4th order derivatives of quartic B-splines do not exist, the 4th order weighting coefficients have been obtained by matrix multiplication approach. After the discretization of the eFK equation via DQM, ordinary differential equation systems have been obtained and strong stability preserving Runge-Kutta method has been used for time integration. To be able to control the accuracy of the method three test problems have been solved and error norms L_2 and L_∞ are calculated.

Keywords: Partial Differential Equations, Differential Quadrature Method, Runge-Kutta method, B-splines.

Extended Fisher-Kolmogorov Denklemini Çözmek İçin Kuartik B-Spline Diferansiyel Kuadratur Metot

Öz

Extended Fisher-Kolmogorov (EFK) denkleminin bazı çözümleri kuartik B-spline diferansiyel quadrature metot (DQM) ile elde edildi. İkinci mertebeden ağırlık katsayıları kuartik B-spline fonksiyonlar ile direkt olarak elde edildi. Kuartik B-spline fonksiyonların dördüncü mertebeden türevleri mevcut olmadığından, dördüncü mertebeden ağırlık katsayıları matris çarpımı yaklaşımı ile elde edildi. EFK denklemi DQM ile ayrıklaştırıldıktan sonra adi diferansiyel denklem sistemi elde edildi ve kararlılığı güçlü bir şekilde koruyan Runge-Kutta metot ile zamana bağlı integre edildi. Metodun tamlığını kontrol etmek için üç adet test problemi çözüldü ve L_2 ile L_∞ hata normları hesaplandı.

Anahtar Kelimeler: Kısmi diferansiyel denklemler, Diferansiyel Quadrature Metot, Runge-Kutta metot, B-spline fonksiyonlar.

1. Introduction

The extended Fisher-Kolmogorov (EFK) equation has been investigated via quartic B-spline Differential Quadrature Method (QAB-DQM). The EFK equation is given in the following format

$$u_t - u_{2x} + \varphi u_{4x} + f(u) = 0, \qquad (1)$$
$$a \le x \le b, \;\; t \ge 0,$$

where $f(u) = u^3 - u$ and $\varphi > 0$.

Coullet et al. (1987) and van Saarlos (1987); van Saarlos (1988); van Saarlos (1989) and Dee and van Saarlos (1988) introduced the EFK equation given in the Eq.(1) for the value of the $\varphi \neq 0$. When the value of the $\varphi = 0$ the Eq.(1) turns into the standard Fisher-Kolmogorov (FK) equation.EFK equation has been used in many applications in various scientific areas such as model of a phase transition in a binary system near the Lipschitz point (Hornreich et al.,1975; Zimmerman, 1991), propagation of domain walls in liquid crystals (Zhu,1982), spatiotemporal chaos (Coullet et al.,1987) and pattern formation in bistable systems (Dee and van Saarlos, 1988). EFK equation (1) has been solved with various methods by many researchers (Danumjaya and Pani, 2005; Mittal and

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Arora,2010; Mittal and Dahiya, 2016). We investigated numerical solutions of EFK equation via QAB-DQM. DQM was firstly supposed by Bellman et al. (1972). DQM has had wide application areas because of its using of considerably less number of nodal points and simplicity. DQM is a method in which partial derivative of a function in terms of a coordinate direction is expressed as a linear combination of all the values of the function at all nodal points along that direction (Shu,2000). Several authors have recently developed various types of DQM by using different base functions (Striz et al., 1995; Shu and Xue, 1997; Shu and Wu, 2007; Korkmaz and Dağ, 2011; Başhan et al.,2015; Başhan et al.,2016; Başhan et al.,2017; Başhan, 2018; Başhan et al.,2018).

2. Quartic B-spline DQM

Let us take the uniform grid distribution $a = x_1 < x_2 < \cdots < x_N = b$ of the finite domain $[a, b]$ into account. Assuming that any given function $f(x)$ is smooth enough throughout the solution domain of problem, its derivatives in terms of x at a nodal point x_i can best be approached by a linear combination of all the functional values over the solution domain, that is,

$$f_x^{(r)}(x_i) = \left.\frac{d^{(r)}f}{dx^{(r)}}\right|_{x_i} = \sum_{j=1}^{N} w_{ij}^{(r)} f(x_j), \quad i = 1, 2, \dots, N, \quad r = 1, 2, \dots, N-1 \qquad (2)$$

where r represents the order of derivative, $w_{ij}^{(r)}$'s denote the weighting coefficients of the r^{th} order derivative approximation, and N represents the number of nodal points in the given solution domain. Here, the index j indicates the fact that $w_{ij}^{(r)}$ is the corresponding weighting coefficient of the value of the function $f(x_j)$.

Let $Q_m(x)$ be the quartic B-splines with nodal points at the points x_i where the uniformly distributed N grid points are chosen as $a = x_1 < x_2 < \dots < x_N = b$ on the real axis. Then, the quartic B-splines $\{Q_{-1}, Q_0, \dots, Q_{N+1}\}$ constitute a basis for all functions described throughout $[a, b]$. The quartic B-splines $Q_m(x)$ are described by the relationships:

$$(x) = \frac{1}{h^4}\begin{cases} (x - x_{m-2})^4 & , [x_{m-2}, x_{m-1}] \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4 & , [x_{m-1}, x_m] \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4 + 10(x - x_m)^4 & , [x_m, x_{m+1}] \\ (x_{m+3} - x)^4 - 5(x_{m+2} - x)^4 & , [x_{m+1}, x_{m+2}] \\ (x_{m+3} - x)^4 & , [x_{m+1}, x_{m+2}] \\ 0 & , otherwise \end{cases}$$

where $h = x_m - x_{m-1}$ for all m (Prenter,1975).

Table 1. The value of quartic B-splines and derivatives functions at the nodal points.

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
Q_m	0	1	11	11	1	0
Q'_m	0	4/h	12/h	-12/h	-4/h	0
Q''_m	0	$12/h^2$	$-12/h^2$	$-12/h^2$	$12/h^2$	0
Q'''_m	0	$24/h^3$	$-72/h^3$	$72/h^3$	$-24/h^3$	0

Substitution of each quartic B-spline function into the DQM equation (2) for a fixed x_i and $r = 2$ gives

$$\frac{d^{(2)}Q_m(x_i)}{dx^{(2)}} = \sum_{j=m-1}^{m+2} w_{ij}^{(2)} Q_m(x_j). \qquad (4)$$

After using the value of Q_m at the first nodal point x_1, we obtained linear equation system (4) which contains N+3 equations and at the same time N+6 unknowns given in the matrix form as below

$$\begin{bmatrix} 1 & 11 & 11 & 1 & & & & & \\ & 1 & 11 & 11 & 1 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & 1 & 11 & 11 & 1 & & \\ & & & & 1 & 11 & 11 & 1 & \end{bmatrix} \begin{bmatrix} w_{1-2}^{(2)} \\ w_{1-1}^{(2)} \\ \vdots \\ w_{1N+2}^{(2)} \\ w_{1N+3}^{(2)} \end{bmatrix} = \begin{bmatrix} 12/h^2 \\ -12/h^2 \\ -12/h^2 \\ 12/h^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5}$$

As it can be seen, this unsolvable equation system (5) needs three additional equations. By using

$$\frac{d^{(3)}Q_{-1}(x_1)}{dx^{(3)}} = \sum_{j=-2}^{1} w_{1j}^{(2)}\, Q'_{-1}(x_j),$$

$$\frac{d^{(3)}Q_{N}(x_1)}{dx^{(3)}} = \sum_{j=N-1}^{N+2} w_{1j}^{(2)}\, Q'_{N}(x_j),$$

$$\frac{d^{(3)}Q_{N+1}(x_1)}{dx^{(3)}} = \sum_{j=N}^{N+3} w_{1j}^{(2)}\, Q'_{N+1}(x_j),$$

additional equations, three unknown terms $w_{1-2}^{(2)}, w_{1N+2}^{(2)}$ and $w_{1N+3}^{(2)}$ are eliminated. So, the equation system has N+3 equations and N+3 unknowns given in the matrix for as below

$$\begin{bmatrix} 8 & 14 & 2 & & & & & \\ 1 & 11 & 11 & 1 & & & & \\ & 1 & 11 & 11 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 11 & 11 & 1 \\ & & & & & 2 & 14 & 8 \\ & & & & & & 30 & 42 \end{bmatrix} \begin{bmatrix} w_{1-1}^{(2)} \\ w_{10}^{(2)} \\ \vdots \\ w_{1N-1}^{(2)} \\ w_{1N}^{(2)} \\ w_{1N+1}^{(2)} \end{bmatrix} = \begin{bmatrix} 18/h^2 \\ -12/h^2 \\ -12/h^2 \\ 12/h^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6}$$

By the same process, for the x_m nodal points, $2 \le m \le N-1$ we obtained

$$\begin{bmatrix} 8 & 14 & 2 & & & & & \\ 1 & 11 & 11 & 1 & & & & \\ & 1 & 11 & 11 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 11 & 11 & 1 \\ & & & & & 2 & 14 & 8 \\ & & & & & & 30 & 42 \end{bmatrix} \begin{bmatrix} w_{m-1}^{(2)} \\ \vdots \\ w_{mm-2}^{(2)} \\ w_{mm-1}^{(2)} \\ w_{mm}^{(2)} \\ w_{mm+1}^{(2)} \\ \vdots \\ w_{mN+1}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 12/h^2 \\ -12/h^2 \\ -12/h^2 \\ 12/h^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7}$$

equation system. And for the final nodal point x_N we obtained

$$\begin{bmatrix} 8 & 14 & 2 & & & & & \\ 1 & 11 & 11 & 1 & & & & \\ & 1 & 11 & 11 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 11 & 11 & 1 \\ & & & & & 2 & 14 & 8 \\ & & & & & & 30 & 42 \end{bmatrix} \begin{bmatrix} w_{N-1}^{(2)} \\ \vdots \\ w_{NN-3}^{(2)} \\ w_{NN-2}^{(2)} \\ w_{NN-1}^{(2)} \\ w_{NN}^{(2)} \\ w_{NN+1}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 12/h^2 \\ -12/h^2 \\ -30/h^2 \\ 6/h^2 \end{bmatrix} \tag{8}$$

equation system (8). By using Thomas algorithm, equation systems (6)-(8) have been solved easily.

Then to get the weighting coefficients of the 4th order derivatives, we used 2nd order weighting coefficients. By using the matrix multiplication approach, 4th order weighting coefficients have been calculated as shown below [12]:

$$[A^{(4)}] = [A^{(2)}][A^{(2)}],$$

where $[A^{(2)}]$ and $[A^{(4)}]$ represent the weighting coefficients' matrices of the 2nd order and 4th order derivatives, respectively.

3. Numerical discretization

The EFK equation of the form

$$u_t - u_{2x} + \varphi u_{4x} + u^3 - u = 0 \quad (9)$$

has the boundary conditions

$U(a,t) = \mu_1(t),\ U(b,t) = \mu_2(t),$
$a \le x \le b,\ \ t \ge 0,$
and the initial condition
$U(x,0) = u_0(x), a \le x \le b.$
Eq. (9) is rewritten as

$$u_t = u_{2x} - \varphi u_{4x} - u^3 + u = 0. \quad (10)$$

Next, the differential quadrature derivative approaches of the 2nd order and 4th order have been substituted for u_{2x} and u_{4x} in Eq. (10)

$$\frac{dU(x_i)}{dt} = \sum_{j=1}^{N} w_{ij}^{(2)}\, U(x_j,t) - \varphi \sum_{j=1}^{N} w_{ij}^{(4)}\, U(x_j,t) - U^3(x_i,t) + U(x_i,t),$$
i=1,2,...,N (11)

and ordinary differential equation (ODE) (11) was obtained. After that, the ODE represented by Eq. (11) was integrated with respect to time. Here, we have chosen strong stability-preserving Runge-Kutta43 method Ketcheson (2010) thanks to its many advantages like accuracy, efficiency and simplicity.

4. Numerical Examples and Results

Here, we obtained the approximate solutions of the EFK by the QAB-DQM. The accuracy and efficiency of the new method have been controlled by calculating the error norms L_2 and L_∞, respectively:

$$L_2 = \|u - U\|_2 \cong \sqrt{h \sum_{j=1}^{N} \left|U_j^{exact} - (U_N)_j\right|^2},$$
$$L_\infty = \|u - U\|_\infty \cong \max_j \left|U_j^{exact} - (U_N)_j\right|, \quad j = 1, 2, ..., N-1.$$

Since the analytical solution of EFK equation does not exist, newly obtained numerical solution was compared with those solutions obtained when nodal point number was taken as N = 160 instead of exact solution.

4.1. Test problem 1

The first test problem has the initial condition as follows:

$$U(x,0) = -sin(\pi x) \quad (12)$$

having the boundary conditions

$$U(-4,t) = U(4,t) = 0, \quad (13)$$

at the domain $-4 \le x \le 4$.
We fixed the number of nodal points as N = 81 and time increment as $\Delta t = 0.0001$ when $\varphi = 0$, $\varphi = 0.0001$ and $\varphi = 0.1$, respectively. Numerical simulations are shown in Figures $1-3$.

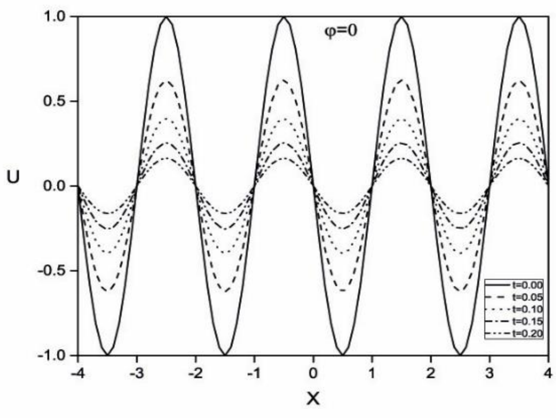

Figure 1. Numerical simulations of Test problem 1 for $\varphi = 0$.

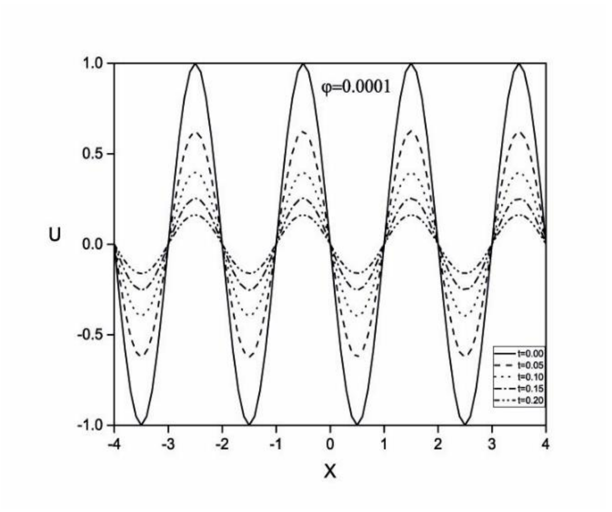

Figure 2. Numerical simulations of Test problem 1 for $\varphi = 0.0001$.

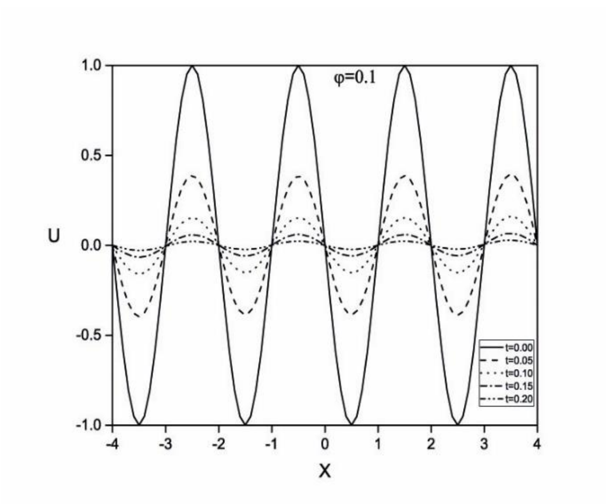

Figure 3. Numerical simulations of Test problem 1 for $\varphi = 0.1$.

As it can obviously be seen from Figure 1 and Figure 2 that the behaviors of solutions for $\varphi = 0$ and $\varphi = 0.0001$ are similar to each other. Except for $\varphi = 0.1$ simulations given in Figure 3 that solutions decline to 0 very rapidly because of stabilizing behaviour of EFK. The calculated values of the error norms L_2 and L_∞ are illustrated in Table 2. As it is obviously seen from the results given in Table 2 that by the increasing of the nodal point numbers the error norms L_2 and L_∞ got decreased and also both results are in concordance with each other.

Table 2. L_2 and L_∞ error norms at t=0.2.

N	Present(QAB-DQM)	
	L_2	L_∞
20	0.01620	0.01313
40	0.00891	0.00794
80	0.00292	0.00276

4.2. Test problem 2

The initial condition of the second test problem is given below:

$$U(x,0) = 10^{-3}exp(-x^2) \quad (14)$$

with boundary conditions

$$U(-4,t) = U(4,t) = 1, \quad (15)$$

at the domain $-4 \leq x \leq 4$.

The simulations are running up from $t = 0.25$ to $t = 4.50$ are given in Figure 4. It is clear from Figure 4 that the approximate solution of U declines as time increases and eventually it comes near close to the value 1.

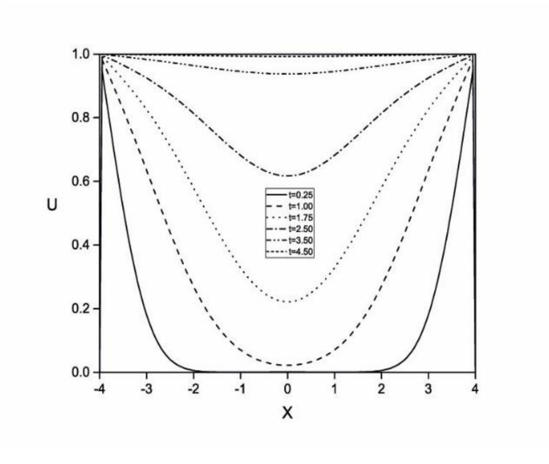

Figure 4. Numerical simulations of test problem 2 for $\varphi = 0.0001$.

4.3. Test problem 3

Finally, the third test problem has initial condition

$$U(x,0) = -10^{-3} exp(-x^2) \quad (16)$$

and boundary conditions

$$U(-4,t) = U(4,t) = -1, \quad (17)$$

at the domain $-4 \leq x \leq 4$.

The simulations running up from t = 0.25 to t = 4.50 are given in Figure 5. Figure 5 shows, the approximate solution of U declines when time increases and eventually it comes near close to the value -1.

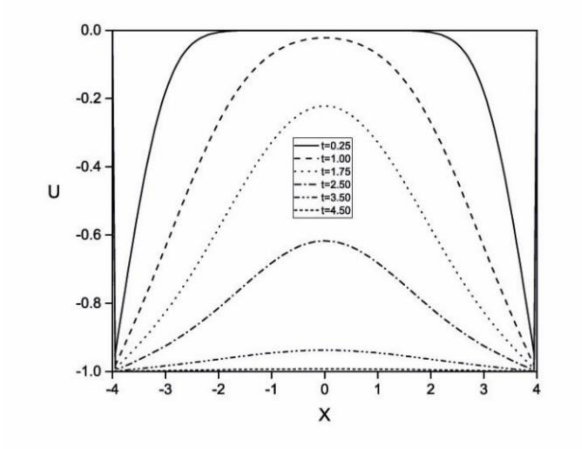

Figure 5. Numerical simulations of Test problem 3 for $\varphi = 0.0001$.

5. Discussion

In this study, we have implemented QAB-DQM for numerical solution of EFK equation. To obtain the 2nd order weighting coefficients, we solved the equation systems by Thomas algorithm and then to obtain the 4th order weighting coefficients we used the matrix multiplication approach. The accuracy of the present method has been tested by determining the error norms L_2 and L_∞. The results obtained here are in concordance to each other.

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