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Research Article

# On initial value problem of random fractional differential equation with impulses

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**Abstract**

In this paper, we prove the existence and uniqueness of solution for random fractional differential equation with impulses via Banach fixed point theorem and Schauder fixed point theorem. Moreover, the continuous dependence of the solution on the initial data is investigated.

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**Keywords.** random fractional differential equation with impulses, second-order stochastic processes, mean square continuous solution

## 1. Introduction

Differential equations of fractional-order have been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, environmental, physics and economics. In the current years, fractional calculus and fractional differential equations have undergone expanded study as a considerable interest both in mathematics and in applications. One of the recently influential works on the subject of fractional differential equation is the monograph of Kilbas et al. [7], Lakshmikantham et al. [9], Miller et al. [13] and the papers of Bayour et al. [1], Mei et al. [12], Zou et al. [20].

Random differential equations and random integral equations were introduced as good models in various branches of science and engineering since random coefficients and uncertainties have been taken into consideration (see [2, 8, 15, 16]). Recently, the issue of fractional calculus and random differential equation has emerged as the significant subject and this new theory becomes very attractive to many scientists. Therefore, this theory has been developed in theoretical directions, and a wide number of applications of this theory have been considered (see [10, 11, 14, 19]). Lupulescu et al. [10, 11] proved the existence and uniqueness of solutions for random fractional differential equations (RFDEs) under Carathéodory condition, and the existence results of extremal random solutions of the RFDEs are studied. In [17, 18], Vu et al. discussed the existence and uniqueness of solutions of RFDEs with delay. Zhang and Sun [14] proved the existence and uniqueness

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of solution for random impulsive differential equations by applying random Banach fixed point theorem and Schaefer's type random fixed point theorem. Yang and Wang [19] established a framework to study impulsive fractional sample path associated with impulsive fractional $L^p$-problem, and the existence, Ulam-Hyers-Rassias stability of solution of a class of non-instantaneous impulsive fractional-order implicit differential equations with random effects were investigated.

Hafiz et al. [5,6] considered the stochastic Abel integral equations of the first and second kind using a concept of the stochastic m.s. fractional integration for mean square ( m.s.) continuous second-order stochastic processes, and authors also studied the m.s. Riemann-Liouville fractional integration of m.s. integrable stochastic processes, the m.s. fractional derivative in the sense of Riemann-Liouville and Caputo. El-Sayed [4] defined the Caputo-via Riemann-Liouville fractional-order operator for the second order stochastic processes, studied some equivalent properties for these fractional-order operators and some equivalent Cauchy type problems. The existence of mild solution of the nonlinear fractional-order stochastic differential equations also is proved.

Based on the motivation stated in the work of Hafiz et al. [5,6] and El-Sayed [4]. In this paper, we study the existence and uniqueness of solution for random fractional differential equation with impulses via Banach fixed point theorem and Schauder fixed point theorem. The continuous dependence of the solution on the initial data also is investigated.

The rest of the paper is organized as follows: In Section 2, we give some basic theorems, definitions and notations which are used throughout this paper . In Section 3, we investigate the existence and uniqueness of solution for random fractional differential equation with impulses and the continuous dependence of the solution on the initial data.

## 2. Preliminaries

In this section, we introduce some basic theorems, definitions and notations which are used throughout this paper. These results can be found in the papers [3–6].

Let $(\Omega, F, \mathbb{P})$ be complete probability space. Let $X(t,\omega) = \{X(t), t \in J = [0,T], \omega \in \Omega\}$, $T > 0$, be a second-order stochastic process, i.e., $E(X^2(t)) := \int_\Omega X^2 d\mathbb{P} < \infty$. Let $L_2(\Omega)$ is the Banach space of random variables $X : \Omega \to \mathbb{R}$ such that $E(X^2(t)) < \infty$. Let $C := C(J, L_2(\Omega))$ be the class of all second-order stochastic processes which are m.s. Riemann integrable on $J$, i.e.,

$$\int_J E(X^2(t))dt < \infty.$$

In $C(J, L_2(\Omega))$, we denote the Banach space of all continuous functions from $J \times \Omega$ into $\mathbb{R}$ with the norm

$$\|X\|_C = \max_{t\in J} \|X(t)\|_2, \text{ where } \|X(t)\|_2 = \left(E(X^2(t))\right)^{1/2}.$$

**Theorem 2.1** (see [5, 6]). *Let $\alpha \in (0,1]$ and $X \in C(J, L_2(\Omega))$. The stochastic m.s. fractional integral $I^\alpha_{0+}X(t)$ is defined by*

$$I^\alpha_{0+}X(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} X(s)ds.$$

**Theorem 2.2** (see [5, 6]). *Let $\alpha, \beta \in (0,1]$. If $X \in C(J, L_2(\Omega))$, then $I^\alpha_{0+}X(t)$ exists in m.s. sense as a second-order m.s. continuous second-order process $I^\alpha_{0+}X(t) \in C(J, L_2(\Omega))$ with the following properties*

i) $I^\alpha_{0+} : C(J, L_2(\Omega)) \to C(J, L_2(\Omega))$;
ii) $I^\alpha_{0+} I^\beta_{0+} X(t) = I^\beta_{0+} I^\alpha_{0+} X(t) = I^{\alpha+\beta}_{0+} X(t)$;
iii) $\lim_{\alpha\to 0} I^\alpha_{0+}X(t) = I^0_{0+}X(t) = X(t)$.

Let $C^1(J, L_2(\Omega))$ be a second-order stochastic process which is m.s. differentiable with m.s. continuous derivative.

**Definition 2.3** (see [15]). A second-order stochastic processes $X(t)$, $t \in J$, has a mean square derivative or m.s. derivative $\frac{d}{dt}X(t)$ at $t \in J$ if

$$\lim_{h\to 0}\left\|\frac{X(t+h)-X(t)}{h} - \frac{d}{dt}X(t)\right\|_2 = 0.$$

**Definition 2.4** (see [5, 6]). The Caputo fractional derivative of order $\alpha \in (0,1]$ of the stochastic process $X$, denoted by $D^{\alpha}_{0+}X(t)$ is defined by

$$D^{\alpha}_{0+}X(t) = I^{1-\alpha}\frac{d}{dt}X(t),$$

where $\frac{d}{dt}X(t)$ denotes the m.s. differentiation of $X(t)$.

**Theorem 2.5** (see [5, 6]). *Let $\alpha \in (0,1]$. If $X$ is m.s. differentiable with m.s integrable second-order derivative, then*

i) $\lim_{\alpha\to 1} D^{\alpha}_{0+}X(t) = \frac{d}{dt}X(t)$;
ii) $\lim_{\alpha\to 0} D^{\alpha}_{0+}X(t) = X(t) - X(0)$;
iii) $I^{\alpha}_{0+}D^{\alpha}_{0+}X(t) = X(t) - X(0)$;
iv) $D^{\alpha}_{0+}I^{\alpha}_{0+}X(t) = X(t)$.

## 3. Main results

We consider the following random fractional-order differential equation with impulses (RFDEIs):

$$\begin{cases} \frac{d}{dt}X(t) = F(t, X(t), D^{\alpha}X(t)), \ t \in J', \\ \Delta X(t_k) = I_k(X(t_k)), \ k = 1, 2, \ldots, m, \\ X(0) = X_0, \end{cases} \tag{3.1}$$

where $D^{\alpha}$ is m.s. Caputo fractional derivative of order $\alpha \in (0,1]$, $J' = [0,t_1]\cup(t_1,t_2]\cup\ldots\cup(t_m,T]$, $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$, $F \in C(J' \times L_2(\Omega)\times L_2(\Omega), L_2(\Omega))$, $X_0$ is a random variable with $E(X_0)^2 < \infty$, $I_k \in C(L_2(\Omega), L_2(\Omega))$ and $\Delta X(t_k) = X(t_k^+) - X(t_k^-)$,

$$X(t_k^-) = \lim_{h\to 0^-} X(t_k + h), \quad X(t_k^+) = \lim_{h\to 0^+} X(t_k + h)$$

represent the right and left limits of $X(t)$ at $t = t_k$ respectively, and they satisfy $X(t_k^-) = X(t_k)$ for $k = 1, \ldots, m$.

Now, we denote $PC(J) = \{X : J \to L_2(\Omega) \mid X \in C(J', L_2(\Omega)),\ X(t_k^+)$ and $X(t_k^-)$ exist and $X(t_k^-) = X(t_k),\ k = 1, \ldots, m\}$.

Let $\frac{d}{dt}X(t) = Y(t)$ for any $t \in J'$. Consider the following random impulsive differential equation:

$$\begin{cases} \frac{d}{dt}X(t) = Y(t), \\ \Delta X(t_k) = I_k(X(t_k)), \ k = 1, 2, \ldots, m, \\ X(0) = X_0. \end{cases} \tag{3.2}$$

**Lemma 3.1.** *Assume that $Y \in C^1(J, L_2(\Omega))$. A function $X \in PC(J)$ is a solution of the problem (3.2) if and only if $X$ satisfies the following impulsive integral equation (IIE)*

$$X(t) = \begin{cases} X_0 + \int_0^t Y(s)ds, & \text{if } t \in [0, t_1], \\ \quad \vdots & \\ X_0 + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} Y(s)ds + \int_{t_k}^{t} Y(s)ds + \sum_{i=1}^{k} I_i(X(t_i)), & \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{3.3}$$

*where $k = 1, 2, \ldots, m$.*

***Proof.*** Assume that $X$ satisfies the problem (3.2). For $t \in [0, t_1]$ and using Theorem 5.1.1 (see [15, pp. 118]), we have

$$X(t) = X(0) + \int_0^t Y(s)ds.$$

In view of $X(t_1^+) = I_1(X(t_1)) + X(t_1^-)$, we obtain

$$X(t_1^+) = I_1(X(t_1)) + X_0 + \int_0^{t_1} Y(s)ds.$$

For $t \in (t_1, t_2]$, we get

$$\begin{aligned} X(t) &= X(t_1^+) + \int_{t_1}^{t} Y(s)ds \\ &= I_1(X(t_1)) + X_0 + \int_0^{t_1} Y(s)ds + \int_{t_1}^{t} Y(s)ds. \end{aligned}$$

Similary, from $X(t_2^+) = I_2(X(t_2)) + X(t_2^-)$, we get for $t \in (t_2, t_3]$

$$\begin{aligned} X(t) &= X(t_2^+) + \int_{t_2}^{t} Y(s)ds \\ &= I_1(X(t_1)) + I_2(X(t_2)) + X_0 + \int_0^{t_1} Y(s)ds + \int_{t_1}^{t_2} Y(s)ds + \int_{t_2}^{t} Y(s)ds. \end{aligned}$$

Repeating the above process, for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ we infer that

$$X(t) = X_0 + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} Y(s)ds + \int_{t_k}^{t} Y(s)ds + \sum_{i=1}^{k} I_i(X(t_i)).$$

Conversely, assume that $X$ satisfies the problem (3.3). Then, we use Theorem 5.1.1 (see [15, pp. 118]) to the subintervals $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ to complete the proof. □

**Remark 3.2.** In view of $\frac{d}{dt}X(t) = Y(t)$ for any $t \in J'$, we have

$$Y(t) = F(t, X(t), D^\alpha X(t)) = F(t, X(t), I^{1-\alpha}\frac{d}{dt}X(t)) = F(t, X(t), I^{1-\alpha}Y(t))$$

and from IIE (3.2), we obtain

$$Y(t) = \begin{cases} F\Big(t, X_0 + \int_0^t Y(s)ds, I^{1-\alpha}Y(t)\Big), & \text{if } t \in [0, t_1], \\ F\Big(t, X_0 + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} Y(s)ds + \int_{t_k}^{t} Y(s)ds + \sum_{i=1}^{k} I_i(X(t_i)), I^{1-\alpha}Y(t)\Big), & \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{3.4}$$

where $k = 1, 2, \ldots, m$.

**Remark 3.3.** The solution $X$ of the problem (3.1) can be represented by IIE (3.3), where $Y$ is the solution of the IIE (3.4).

Assume that $F \in C(J' \times L_2(\Omega) \times L_2(\Omega), L_2(\Omega))$ and $I_k \in C(L_2(\Omega), L_2(\Omega)), k = 1, 2, \dots, m$ satisfy the following assumptions:

(A1) there exist $L_1, L_2 > 0$ such that

$$\|F(t, X_1, Y_1) - F(t, X_2, Y_2)\|_2 \leq L_1 \|X_1 - X_2\|_2 + L_2 \|Y_1 - Y_2\|_2 ,$$

for $t \in J$ and $X_1, X_2, Y_1, Y_2 \in L_2(\Omega)$;

(A2) there exists $L_3 > 0$ satisfied $mL_3 \in (0, 1)$ such that

$$\|I_k(X) - I_k(Y)\|_2 \leq L_3 \|X - Y\|_2 ,$$

for $k = 1, 2, \dots, m$ and $X, Y \in L_2(\Omega)$.

First, we prove the existence and uniqueness of solution for the problem (3.1) based on Banach fixed point theorem.

**Theorem 3.4.** *Assume that the assumptions (A1)-(A2) hold and $X_0 \in L_2(\Omega)$. If*

$$(2 + \frac{mL_3}{1 - mL_3})L_1 T + \frac{L_2 T^{1-\alpha}}{\Gamma(2-\alpha)} < 1,$$

*then the RFDEI (3.1) has a unique solution on $J$.*

***Proof.*** Let $X_0 \in L_2(\Omega)$. Define an operator $Q$ on $PC(J)$ by

$$(QY)(t) = \begin{cases} F\Big(t, X_0 + \int_0^t Y(s)ds, I^{1-\alpha}Y(t)\Big), & \text{if } t \in [0, t_1], \\ F\Big(t, X_0 + \sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k} Y(s)ds + \int_{t_k}^t Y(s)ds + \sum\limits_{0<t_k<t} I_k(X(t_k)), I^{1-\alpha}Y(t)\Big), & \text{if } t \in (t_k, t_{k+1}], \end{cases}$$

where $k = 1, 2, \dots, m$. Then it is clear that $Q : PC(J) \to PC(J)$. Now, we show that the operator $Q$ is contraction. Using the assumptions (A1)-(A2) and $Y, \widetilde{Y} \in L_2(\Omega)$, we have for $t \in [0, t_1]$

$$\begin{aligned} &\|(QY)(t) - (Q\widetilde{Y})(t)\|_2 \\ &\quad = \Big\|F\Big(t, X_0 + \int_0^t Y(s)ds, I^{1-\alpha}Y(t)\Big) - F\Big(t, X_0 + \int_0^t \widetilde{Y}(s)ds, I^{1-\alpha}\widetilde{Y}(t)\Big)\Big\|_2 \\ &\quad \leq L_1 \Big\|X_0 + \int_0^t Y(s)ds - X_0 - \int_0^t \widetilde{Y}(s)ds\Big\|_2 + L_2 \|I^{1-\alpha}Y(t) - I^{1-\alpha}\widetilde{Y}(t)\|_2 \\ &\quad \leq L_1 \int_0^t \|Y(s) - \widetilde{Y}(s)\|_2 ds + \frac{L_2}{\Gamma(1-\alpha)} \int_0^t |(t-s)^{-\alpha}| \|Y(s) - \widetilde{Y}(s)\|_2 ds \\ &\quad \leq \Big(L_1 T + \frac{L_2 T^{1-\alpha}}{\Gamma(2-\alpha)}\Big) \|Y - \widetilde{Y}\|_C < \|Y - \widetilde{Y}\|_C. \end{aligned}$$

Similary, using the assumptions (A1)-(A2) and for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $Y, \widetilde{Y} \in L_2(\Omega)$, we obtain

$$\begin{aligned}
&\|(QY)(t) - (Q\widetilde{Y})(t)\|_2 \\
&\quad \leq L_1 \Big\| \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} Y(s)ds + \int_{t_m}^{t} Y(s)ds + \sum_{k=1}^{m} I_k(X(t_k)) - \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \widetilde{Y}(s)ds \\
&\qquad - \int_{t_m}^{t} \widetilde{Y}(s)ds - \sum_{k=1}^{m} I_k(\widetilde{X}(t_k)) \Big\|_2 + L_2 \|I^{1-\alpha}Y(t) - I^{1-\alpha}\widetilde{Y}(t)\|_2 \\
&\quad \leq L_1 \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|Y(s) - \widetilde{Y}(s)\|_2 ds + L_1 \int_{t_m}^{t} \|Y(s) - \widetilde{Y}(s)\|_2 ds \\
&\qquad + L_1 \sum_{k=1}^{m} \|I_k(X(t_k)) - I_k(\widetilde{X}(t_k))\|_2 + L_2 \|I^{1-\alpha}Y(t) - I^{1-\alpha}\widetilde{Y}(t)\|_2 \\
&\quad \leq L_1 \sum_{k=1}^{m} (t_k - t_{k-1}) \|Y - \widetilde{Y}\|_C + L_1(t - t_m)\|Y - \widetilde{Y}\|_C \\
&\qquad + \frac{mL_1L_3T}{1 - mL_3} \|Y - \widetilde{Y}\|_C + \frac{L_2T^{1-\alpha}}{\Gamma(2-\alpha)} \|Y - \widetilde{Y}\|_C \\
&\quad \leq \left( (2 + \frac{mL_3}{1 - mL_3}) L_1 T + \frac{L_2T^{1-\alpha}}{\Gamma(2-\alpha)} \right) \|Y - \widetilde{Y}\|_C .
\end{aligned}$$

Since the condition $(2 + \frac{mL_3}{1-mL_3})L_1T + \dfrac{L_2T^{1-\alpha}}{\Gamma(2-\alpha)} < 1$, we get $\|QY - Q\widetilde{Y})\|_C \leq \|Y - \widetilde{Y}\|_C$, $\forall Y, \widetilde{Y} \in L_2(\Omega)$. Therefore, the operator $Q$ is contraction. As a consequence the Banach fixed point theorem, we conclude that there exists a unique fixed point which is a unique solution of IIE (3.4) on $J$. Using Remark 3.3, then the RFDEI (3.1) has a unique solution on $J$.The proof is completed. □

**Remark 3.5.** From the assumption (A1), we have for $t \in J$ and $X, Y \in L_2(\Omega)$

$$\begin{aligned}
\|F(t, X, Y)\|_2 &\leq \|F(t, X, Y) - F(t, 0, 0)\|_2 + \|F(t, 0, 0)\|_2 \\
&\leq L_1 \|X\|_2 + L_2 \|Y\|_2 + \|F(t, 0, 0)\|_2 \leq L(1 + \|X\|_C + \|Y\|_C),
\end{aligned}$$

where $L = \max\{L_1, L_2, \sup_{t\in J} \|F(t, 0, 0)\|_2\}$. Moverover, from the assumption (A2), we obtain

$$\|I_k(X)\|_2 \leq \|I_k(X) - I_k(0)\|_2 + \|I_k(0)\|_2 \leq L_3 \|X\|_2 + \|I_k(0)\|_2 \leq C(1 + \|X\|_C),$$

where $C = \max\{L_3, \|X(0)\|_2\}$ for $k = 1, 2, \ldots$ and $X \in L_2(\Omega)$.

In the sequel, we show the existence of solution of IIE (3.4) via Schauder fixed point theorem. To this purpose, let $B(X_0, \rho)$ be a closed ball with center $X_0$ and radius $\rho$, i.e., $B(X_0, \rho) := \{X \in PC(J, L_2(\Omega)) \mid \|X - X_0\|_2 \leq \rho\}$.

**Theorem 3.6.** *Assume that the assumptions (A1)-(A2) hold and $X_0 \in L_2(\Omega)$. If*

$$\frac{L}{\rho}\left(1 + mC + \|X_0\|_2 + mC\|X\|_2 + (m+1)T\|Y\|_2 + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|Y\|_2\right) < 1,$$

*then the RFDEI (3.1) has a unique solution on $J$.*

***Proof.*** We prove that the operator $Q$ satisfies the conditions of Schauder's fixed point theorem. To this purpose, we consider the operator $Q : B(X_0, \rho) \to B(X_0, \rho)$ defined as in the proof of Theorem 3.4. The proof is given in three steps as follows.

**Step 1**. The operator $Q$ is a m.s. continuous on $PC(J)$. Indeed, let $\{Y_n\}$ be a sequences such that $Y_n \to Y$ as $n \to \infty$, respectively. For $t \in [0, t_1]$, we obtain

$$\begin{aligned}
&\|(QY_n)(t) - (QY)(t)\|_2 \\
&\quad = \Big\|F\Big(t, X_0 + \int_0^t Y_n(s)ds, I^{1-\alpha}Y_n(t)\Big) - F\Big(t, X_0 + \int_0^t Y(s)ds, I^{1-\alpha}Y(t)\Big)\Big\|_2 \\
&\quad \le L_1\Big\|X_0 + \int_0^t Y_n(s)ds - X_0 - \int_0^t Y(s)ds\Big\|_2 + L_2\|I^{1-\alpha}Y_n(t) - I^{1-\alpha}Y(t)\|_2 \\
&\quad \le L_1 \int_0^t \|Y_n(s) - Y(s)\|_2 ds + \frac{L_2}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\|Y_n(s) - Y(s)\|_2 ds \\
&\quad \le \Big(L_1 T + \frac{L_2 T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\|Y_n - Y\|_C. \qquad (3.5)
\end{aligned}$$

Similarly, for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\|(QY_n)(t) - (QY)(t)\|_2 \\
&\quad \le L_1 \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|Y_n(s) - Y(s)\|_2 ds + L_1 \int_{t_m}^{t} \|Y_n(s) - Y(s)\|_2 ds \\
&\qquad + L_1 \sum_{k=1}^{m} \|I_k(X_n(t_k)) - I_k(X(t_k))\|_2 + L_2\|I^{1-\alpha}Y_n(t) - I^{1-\alpha}Y(t)\|_2 \\
&\quad \le L_1 \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|Y_n(s) - Y(s)\|_2 ds + L_1 \int_{t_m}^{t} \|Y_n(s) - Y(s)\|_2 ds \\
&\qquad + L_1 L_3 \sum_{k=1}^{m} \|X_n(t_k) - X(t_k)\|_2 + \frac{L_2}{\Gamma(1-\alpha)} \int_0^t |(t-s)^{-\alpha}|\|Y_n(s) - Y(s)\|_2 ds \\
&\quad \le \Big(2L_1 T + \frac{mL_1L_3T}{1 - mL_3} + \frac{L_2 T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\|Y_n - Y\|_C. \qquad (3.6)
\end{aligned}$$

Since $F$ and $I_k$ are m.s. continuous on $PC(\rho)$ for $k = 1, 2, \ldots, m$, we infer that

$$\|QY_n - QY\|_2 \to 0 \text{ as } n \to \infty.$$

**Step 2**. The operator $Q(B(X_0, \rho))$ is m.s. bounded. Indeed, for $t \in [0, t_1]$, $X \in PC(J)$ and using Remark 3.5, we have

$$\begin{aligned}
\|(QY)(t)\|_2 &= \Big\|F\Big(t, X_0 + \int_0^t Y(s)ds, I^{1-\alpha}Y(t)\Big)\Big\|_2 \\
&\le L\Big(1 + \Big\|X_0 + \int_0^t Y(s)ds\Big\|_2 + \|I^{1-\alpha}Y(t)\|_2\Big) \\
&\le L\Big(1 + \|X_0\|_2 + \int_0^t \|Y(s)\|_2 ds + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\|Y(s)\|_2 ds\Big) \\
&\le L\Big(1 + \|X_0\|_2 + T\|Y\|_C + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|Y\|_C\Big) < \rho. \qquad (3.7)
\end{aligned}$$

Similarly, using Remark 3.5, for $X \in PC(J)$ and $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, one obtain

$$
\begin{aligned}
&\|(QY)(t)\|_2 \\
&\quad \le L\Big(\|X_0\|_2 + \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|Y(s)\|_2 ds + \int_{t_m}^{t} \|Y(s)\|_2 ds + \sum_{k=1}^{m} \|I_k(X(t_k))\|_2 + \|I^{1-\alpha}Y(t)\|_2\Big) \\
&\quad \le L\Big(1 + \|X_0\|_2 + \sum_{k=1}^{m}(t_k - t_{k-1})\|Y\|_C + (t - t_m)\|Y\|_C + mC(1 + \|X\|_C) + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|Y\|_C\Big) \\
&\quad \le L\Big(1 + mC + \|X_0\|_C + mC\|X\|_C + (m+1)T\|Y\|_C + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|Y\|_C\Big) < \rho. \qquad (3.8)
\end{aligned}
$$

Combining the inequalities (3.7) and (3.8), we infer that $\|QY\|_2 \le \rho$.

**Step 3**. The operator $Q$ is m.s. equicontinuous. Let $Y \in PC(J)$ and for $\tau_1, \tau_2 \in [0, t_1]$, $\tau_1 < \tau_2$, we have

$$
\begin{aligned}
&\|(QY)(\tau_2) - (QY)(\tau_1)\|_2 \\
&\quad = \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_2} Y(s)ds, I^{1-\alpha}Y(t)\|_{t=\tau_2}\Big) - F\Big(\tau_1, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\quad \le \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_2} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_2}\Big) - F\Big(\tau_2, X_0 + \int_0^{\tau_2} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\qquad + \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_2} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big) - F\Big(\tau_1, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\quad \le \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_2} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_2}\Big) - F\Big(\tau_2, X_0 + \int_0^{\tau_2} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\qquad + \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_2} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big) - F\Big(\tau_2, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\qquad + \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big) - F\Big(\tau_1, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\quad \le L_1\Big\|X_0 + \int_0^{\tau_2} Y(s)ds - X_0 - \int_0^{\tau_1} Y(s)ds\Big\|_2 + L_2\|I^{1-\alpha}Y(t)|_{t=\tau_2} - I^{1-\alpha}Y(t)|_{t=\tau_1}\|_2 \\
&\qquad + \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big) - F\Big(\tau_1, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\quad \le L_1 \int_{\tau_1}^{\tau_2} \|Y(s)\|_2 ds + \frac{L_2}{\Gamma(1-\alpha)}\Big\|\int_0^{\tau_2}(\tau_2 - s)^{-\alpha}Y(s)ds - \int_0^{\tau_1}(\tau_1 - s)^{-\alpha}Y(s)ds\Big\|_2 \\
&\qquad + \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big) - F\Big(\tau_1, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\quad \le L_1\|Y\|_C(\tau_2 - \tau_1) + \frac{L_2}{\Gamma(1-\alpha)}\|Y\|_C\Big|\int_0^{\tau_2}(\tau_2 - s)^{-\alpha}ds - \int_0^{\tau_1}(\tau_1 - s)^{-\alpha}ds\Big| \\
&\qquad + \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big) - F\Big(\tau_1, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\quad \le L_1\|Y\|_C(\tau_2 - \tau_1) + \frac{L_2\|Y\|_C}{\Gamma(1-\alpha)}\Big|\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{-\alpha}ds + \int_0^{\tau_1}\big[(\tau_2 - s)^{-\alpha} - (\tau_1 - s)^{-\alpha}\big]ds\Big| \\
&\qquad + \Big\|F\Big(\tau_2, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big) - F\Big(\tau_1, X_0 + \int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)|_{t=\tau_1}\Big)\Big\|_2 \\
&\quad \le L_1\|Y\|_C(\tau_2 - \tau_1) + \frac{L_2}{\Gamma(2-\alpha)}\Big(2(\tau_2 - \tau_1)^{1-\alpha} + \tau_2^{1-\alpha} - \tau_1^{1-\alpha}\Big)\|Y\|_C
\end{aligned}
$$

$$+\Big\|F\Big(\tau_2, X_0+\int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)\big|_{t=\tau_1}\Big)-F\Big(\tau_1, X_0+\int_0^{\tau_1} Y(s)ds, I^{1-\alpha}Y(t)\big|_{t=\tau_1}\Big)\Big\|_2.$$

Performing the same calculations as above, for $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, $\tau_1 < \tau_2$, $k = 1, 2, \ldots, m$ and $X \in PC(J)$, we obtain

$$\begin{aligned}
&\|(QY)(\tau_2)-(QY)(\tau_1)\|_2\\
&= \Big\|F\Big(\tau_2, X_0+\sum_{0<t_k<\tau_2}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_2} Y(s)ds+\sum_{0<t_k<\tau_2} I_k(X(t_k)), I^{1-\alpha}Y(t)\big|_{t=\tau_2}\Big)\\
&\quad -F\Big(\tau_1, X_0+\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_1} Y(s)ds+\sum_{0<t_k<\tau_1} I_k(X(t_k)), I^{1-\alpha}Y(t)\big|_{t=\tau_1}\Big)\Big\|_2\\
&\le L_1\Big\|\sum_{0<t_k<\tau_2}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_2} Y(s)ds+\sum_{0<t_k<\tau_2} I_k(X(t_k))-\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds\\
&\quad -\int_{t_k}^{\tau_1} Y(s)ds-\sum_{0<t_k<\tau_1} I_k(X(t_k))\Big\|_2+L_2\|I^{1-\alpha}Y(t)\big|_{t=\tau_2}-I^{1-\alpha}Y(t)\big|_{t=\tau_1}\|_2\\
&\quad +\Big\|F\Big(\tau_2, X_0+\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_1} Y(s)ds+\sum_{0<t_k<\tau_1} I_k(X(t_k)), I^{1-\alpha}Y(t)\big|_{t=\tau_1}\\
&\quad -F\Big(\tau_1, X_0+\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_1} Y(s)ds+\sum_{0<t_k<\tau_1} I_k(X(t_k)), I^{1-\alpha}Y(t)\big|_{t=\tau_1}\Big)\Big)\Big\|_2\\
&\le L_1\Big\|\sum_{0<t_k<\tau_2}\int_{t_{k-1}}^{t_k} Y(s)ds-\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds\Big\|_2+L_1\Big\|\int_{t_k}^{\tau_2} Y(s)ds-\int_{t_k}^{\tau_1} Y(s)ds\Big\|_2\\
&\quad +L_1\Big\|\sum_{0<t_k<\tau_2} I_k(X(t_k))-\sum_{0<t_k<\tau_1} I_k(X(t_k))\Big\|_2+L_2\|I^{1-\alpha}Y(t)\big|_{t=\tau_2}-I^{1-\alpha}Y(t)\big|_{t=\tau_1}\|_2\\
&\quad +\Big\|F\Big(\tau_2, X_0+\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_1} Y(s)ds+\sum_{0<t_k<\tau_1} I_k(X(t_k)), I^{1-\alpha}Y(t)\big|_{t=\tau_1}\\
&\quad -F\Big(\tau_1, X_0+\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_1} Y(s)ds+\sum_{0<t_k<\tau_1} I_k(X(t_k)), I^{1-\alpha}Y(t)\big|_{t=\tau_1}\Big)\Big)\Big\|_2\\
&\le L_1\|Y\|_C\sum_{0<t_k<\tau_2-\tau_1}(t_k-t_{k-1})+L_1\|Y\|_C(\tau_2-\tau_1)+L_1\sum_{0<t_k<\tau_2-\tau_1}\|I_k(X(t_k))\|_2\\
&\quad +\frac{L_2}{\Gamma(2-\alpha)}\Big(2\big(\tau_2-\tau_1\big)^{1-\alpha}+\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\Big)\|Y\|_C\\
&\quad +\Big\|F\Big(\tau_2, X_0+\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_1} Y(s)ds+\sum_{0<t_k<\tau_1} I_k(X(t_k)), I^{1-\alpha}Y(t)\big|_{t=\tau_1}\\
&\quad -F\Big(\tau_1, X_0+\sum_{0<t_k<\tau_1}\int_{t_{k-1}}^{t_k} Y(s)ds+\int_{t_k}^{\tau_1} Y(s)ds+\sum_{0<t_k<\tau_1} I_k(X(t_k)), I^{1-\alpha}Y(t)\big|_{t=\tau_1}\Big)\Big)\Big\|_2,
\end{aligned}$$

so $\|(QY)(\tau_2)-(QY)(\tau_1)\|_2 \to 0$ as $\tau_2 \to \tau_1$ for any $Y \in L_2(\Omega)$. Applying the Arzelá-Ascoli theorem together with the results of Steps 1 to 3, we conclude the operator $Q$ is m.s. equicontinuous on $PC(J)$.

As a consequence of Schauder's fixed point theorem, we deduce that the operator $Q$ has a fixed point which is a solution of IIE (3.4). Using Remark 3.3, the RFDEI (3.1) has a solution on $J$. The proof is complete. □

In the sequel, we prove the continuous dependence of solution with respect to initial condition of the problem (3.1). Consider the following two problems.

$$\begin{cases} \frac{d}{dt}X(t) = F\big(t, X(t), D^\alpha X(t)\big),\ t \in J', \\ \Delta X(t_k) = I_k(X(t_k)),\ k = 1, 2, \dots, m, \\ X(0) = X_0 \end{cases} \tag{3.9}$$

and

$$\begin{cases} \frac{d}{dt}X(t) = F\big(t, X(t), D^\alpha X(t)\big),\ t \in J', \\ \Delta X(t_k) = I_k(X(t_k)),\ k = 1, 2, \dots, m, \\ X(0) = \widetilde{X}_0. \end{cases} \tag{3.10}$$

**Definition 3.7.** The solution of the problem (3.1) is said to depend continuously on the initial conditions $X_0$ if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\|X_0 - \widetilde{X}_0\|_2 \le \delta(\epsilon)$ implies that $\|X - \widetilde{X}\|_C < \epsilon$.

**Theorem 3.8.** *Assume that the conditions in Theorem 3.4 are satisfied. Then, the solution of the problem (3.1) depends continuously on the initial data.*

***Proof.*** From Lemma 3.1 and 3.3, then the problems (3.9) and (3.10) are transformed to the following IIEs:

$$X(t) = \begin{cases} X_0 + \int_0^t Y(s)ds, & \text{if } t \in [0, t_1], \\ \quad\vdots & \\ X_0 + \sum\limits_{i=1}^k \int_{t_{i-1}}^{t_i} Y(s)ds + \int_{t_k}^t Y(s)ds + \sum\limits_{i=1}^k I_i(X(t_i)), & \text{if } t \in (t_{k-1}, t_k], \end{cases} \tag{3.11}$$

where $Y$ is the solution of the IIE

$$Y(t) = \begin{cases} F\Big(t, X_0 + \int_0^t Y(s)ds, I^{1-\alpha}Y(t)\Big), & \text{if } t \in [0, t_1], \\ F\Big(t, X_0 + \sum\limits_{i=1}^k \int_{t_{i-1}}^{t_i} Y(s)ds + \int_{t_k}^t Y(s)ds + \sum\limits_{i=1}^k I_i(X(t_i)), I^{1-\alpha}Y(t)\Big), & \text{if } t \in (t_{k-1}, t_k], \end{cases}$$

and

$$\widetilde{X}(t) = \begin{cases} \widetilde{X}_0 + \int_0^t \widetilde{Y}(s)ds, & \text{if } t \in [0, t_1], \\ \quad\vdots & \\ \widetilde{X}_0 + \sum\limits_{i=1}^k \int_{t_{i-1}}^{t_i} \widetilde{Y}(s)ds + \int_{t_k}^t \widetilde{Y}(s)ds + \sum\limits_{i=1}^k I_i(\widetilde{X}(t_i)), & \text{if } t \in (t_{k-1}, t_k], \end{cases} \tag{3.12}$$

where $\widetilde{Y}$ is the solution of the IIE

$$\widetilde{Y}(t) = \begin{cases} F\Big(t, \widetilde{X}_0 + \int_0^t \widetilde{Y}(s)ds, I^{1-\alpha}\widetilde{Y}(t)\Big), & \text{if } t \in [0, t_1], \\ F\Big(t, \widetilde{X}_0 + \sum\limits_{i=1}^k \int_{t_{i-1}}^{t_i} \widetilde{Y}(s)ds + \int_{t_k}^t \widetilde{Y}(s)ds + \sum\limits_{i=1}^k I_i(\widetilde{X}(t_i)), I^{1-\alpha}\widetilde{Y}(t)\Big), & \text{if } t \in (t_{k-1}, t_k]. \end{cases}$$

For $t \in [0, t_1]$ and $Y, \widetilde{Y} \in L_2(\Omega)$, we have

$$\begin{aligned} \big\|Y(t) - \widetilde{Y}(t)\big\|_2 &= \Big\|F\Big(t, X_0 + \int_0^t Y(s)ds, I^{1-\alpha}Y(t)\Big) - F\Big(t, \widetilde{X}_0 + \int_0^t \widetilde{Y}(s)ds, I^{1-\alpha}\widetilde{Y}(t)\Big)\Big\|_2 \\ &\le L_1\big\|X_0 - \widetilde{X}_0\big\|_2 + L_1 \int_0^t \big\|Y(s) - \widetilde{Y}(s)\big\|_2 ds + \frac{L_2}{\Gamma(1-\alpha)} \int_0^t |(t-s)^{-\alpha}| \big\|Y(s) - \widetilde{Y}(s)\big\|_2 ds \\ &\le L_1\big\|X_0 - \widetilde{X}_0\big\|_2 + \Big(L_1 T + \frac{L_2 T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\big\|Y - \widetilde{Y}\big\|_C. \end{aligned} \tag{3.13}$$

Similarly, for $t \in (t_k, t_{k+1}]$ and $Y, \widetilde{Y} \in L_2(\Omega)$, we obtain

$$\begin{aligned}
\|Y(t) - \widetilde{Y}(t)\|_2 &= \Big\| F\Big(t, X_0 + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} Y(s)ds + \int_{t_k}^{t} Y(s)ds + \sum_{i=1}^{k} I_i(X(t_i)), I^{1-\alpha}Y(t)\Big) \\
&\quad - F\Big(t, \widetilde{X}_0 + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \widetilde{Y}(s)ds + \int_{t_k}^{t} \widetilde{Y}(s)ds + \sum_{i=1}^{k} I_i(\widetilde{X}(t_i)), I^{1-\alpha}\widetilde{Y}(t)\Big) \Big\|_2 \\
&\le L_1\|X_0 - \widetilde{X}_0\|_2 + L_1 \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|Y(s) - \widetilde{Y}(s)\|_2 ds + \int_{t_k}^{t} \|Y(s) - \widetilde{Y}(s)\|_2 ds \\
&\quad + L_1 \sum_{i=1}^{k} \|I_i(X(t_i)) - I_i(\widetilde{X}(t_i))\|_2 + \frac{L_2}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \|Y(s) - \widetilde{Y}(s)\|_2 ds \\
&\le L_1\|X_0 - \widetilde{X}_0\|_2 + \Big(2L_1T + \frac{mL_1L_3T}{1 - mL_3} + \frac{L_2T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\|Y - \widetilde{Y}\|_C,
\end{aligned}$$

then

$$\Big(1 - 2L_1T - \frac{mL_1L_3T}{1 - mL_3} - \frac{L_2T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\|Y - \widetilde{Y}\|_C \le L_1\|X_0 - \widetilde{X}_0\|_2$$

or

$$\|Y - \widetilde{Y}\|_C \le K\|X_0 - \widetilde{X}_0\|_2, \tag{3.14}$$

where $K = L_1\Big(1 - 2L_1T - \frac{mL_1L_3T}{1 - mL_3} - \frac{L_2T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)^{-1}$.

For $t \in [0, t_1]$ and from inequality (3.13), we get the following estimate:

$$\begin{aligned}
\|X - \widetilde{X}\|_C &\le \|X_0 - \widetilde{X}_0\|_2 + T\|Y - \widetilde{Y}\|_C \\
&\le (1 + TK)\|X_0 - \widetilde{X}_0\|_2 \le \delta(1 + TK) \le \epsilon_1,
\end{aligned} \tag{3.15}$$

where $\epsilon_1 = \delta(1 + TK)$.

Similarly, for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, and from inequality (3.14), we have the following estimate

$$\begin{aligned}
\|X(t) - \widetilde{X}(t)\|_2 &\le \|X_0 - \widetilde{X}_0\|_2 + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|Y(s) - \widetilde{Y}(s)\|_2 ds \\
&\quad + \int_{t_k}^{t} \|Y(s) - \widetilde{Y}(s)\|_2 ds + \sum_{i=1}^{k} \|I_i(X(t_i)) - I_i(\widetilde{X}(t_i))\|_2 \\
&\le \|X_0 - \widetilde{X}_0\|_2 + 2T\|Y - \widetilde{Y}\|_C + mL_3\|X - \widetilde{X}\|_C.
\end{aligned}$$

From the inequality above, we infer that

$$\|X - \widetilde{X}\|_C \le \frac{1 + 2TK}{1 - mL_3}\|X_0 - \widetilde{X}_0\|_2 \le \delta\frac{1 + 2TK}{1 - mL_3} \le \epsilon_2,$$

where $\epsilon_2 = \frac{1 + 2TK}{1 - mL_3}$.

Choosing $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ and by Definition 3.7, we conclude that the solution of the problem (3.1) depends continuously on the initial data $X_0$. The proof is completed. □

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