A CLASS OF THIRD-ORDER BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITION AT RESONANCE

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Abstract. In this paper, we consider third-order boundary value problem with, Dirichlet, Neumann and integral conditions at resonance case, where the kernel's dimension of the ordinary differential operator is equal to one and the ordinary differential equation which can be written as the abstract equation $Lu = Nu$, called semilinear form, where L is a linear Fredholm operator of index zero, and N is a nonlinear operator. First, we prove a priori estimates, and then we use Mawhin's coincidence degree theory to deduce the existence of solutions. One important ingredient to be able to apply this abstract results (Mawhin's coincidence degree theory) is proving the Fredholm property of the operator L. An example is also presented to illustrate the effectiveness of the main results.

1. Introduction

In this paper, we consider the following nonlinear third-order boundary value problem

$$u''' (t) = f (t, u (t), u' (t), u'' (t)), \ t \in (0, 1), \tag{1.1}$$

$$u (0) = u' (0) = 0, \ u (1) = \frac{3}{\eta^3} \int_0^\eta u (t) \, dt, \ \eta \in (0, 1), \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, and $\eta \in (0, 1)$. We say that the boundary value problem $(1.1) - (1.2)$ is a resonance problem if the linear equation $Lu = u''' = 0$, with the boundary value conditions (1.2) has non-trivial solution i.e. $dimKerL \geq 1$.

The theory of the boundary value problems with integral boundary conditions arises in different areas of applied mathematics and physics. For example, heat

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conduction, chemical engineering, underground water flow, thermo-elasticity and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. Recently, several authors have studied nonlocal boundary value problems at resonance and non-resonance for second-order, third-order and higher-order (in particular, third-order) ordinary differential equations, for instance see [2, 3, 5, 6, 8, 12, 14, 16, 18, 19, 21] and the references therein. However, to our knowledge the corresponding results for third-order with integral boundary conditions are rarely seen [1, 7, 9, 10, 11, 15, 17, 20, 22] and the references therein. In the most papers mentioned above, the coincidence degree theory of Mawhin was applied to establish existence theorem.

Inspired and motivation by works mentioned above, in the present article, we use the coincidence degree theory of Mawhin [13] to discuss the existence of solution for third-order nonlocal boundary value problem $(1.1) - (1.2)$ at resonance case, and establish an existence theorem. The paper is organized as follows. In Section 2 we give the background information from coincidence degree theory, we also define appropriate mappings and projectors that will be used in the sequel. We state and prove our main result in Section 3, and we give an example to illustrate Theorem 3.1.

2. Preliminaries

We first recall some notations and an abstract existence result (Mawhin 1979). Let Y, Z be two real Banach spaces and let $L : domL \subset Y \to Z$ be a linear operator which is Fredholm map of index zero (that is, ImL, the image of L, $KerL$, the kernel of L is finite dimensional with the same dimension as the Z/ImL), and $P : Y \to Y$, $Q : Z \to Z$ be continuous projections such that $ImP = KerL$, $KerQ = ImL$ and $Y = KerL \oplus KerP$, $Z = ImL \oplus ImQ$. It follows that $L \mid_{domL \cap KerP} \to ImL$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $domL \cap \Omega \neq \phi$, the map $N : Y \to Z$ is said to be $L - compact$ on $\overline{\Omega}$ if the map $QN : \overline{\Omega} \to Z$ is bounded and $K_P (I - Q) N : \overline{\Omega} \to Y$ is compact.

We will formulate the boundary value problem $(1.1) - (1.2)$ as $Lu = Nu$ where L and N are appropriate operators. To obtain our existence results we use the following fixed point theorem of Mawhin.

Theorem 2.1. *(See* [13]*) Let L be a Fredholm operator of index zero and N be $L - compact$ on $\overline{\Omega}$. Assume that the following conditions are satisfied:*
(i) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0, 1)$.
(ii) $Nu \notin ImL$ for every $u \in KerL \cap \partial\Omega$.
(iii) $deg (QN \mid_{KerL}, \Omega \cap KerL, 0) \neq 0$,
where $Q : Z \to Z$ is a projection as above with $ImL = KerQ$.
Then the abstract equation $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$.

In the following, we shall use the classical spaces $C[0, 1]$, $C^1[0, 1]$, $C^2[0, 1]$ and $L^1[0, 1]$. For $u \in C^2[0, 1]$, we use the norm $\|u\| = \max \{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$ where $\|u\|_\infty = \max_{t \in [0,1]} |u(t)|$ and denote the norm in $L^1[0, 1]$ by $\|\cdot\|_1$.

We will use the Sobolev space $W^{3,1}(0, 1)$ which is defined by

$W^{3,1}(0, 1) = \left\{ u : [0, 1] \to \mathbb{R} : u, u', u'' \text{are absolutely continuous on } [0, 1] \text{ with } u''' \in L^1[0, 1] \right\}.$

Let $Y = C^2[0,1]$, $Z = L^1[0,1]$, define the linear operator $L : domL \subset Y \to Z$ by

$$Lu = u''', \quad u \in domL,$$

where

$$domL = \left\{ u \in W^{3,1}(0,1) : u(0) = u'(0) = 0, \ u(1) = \frac{3}{\eta^3} \int_0^\eta u(t)\, dt \right\},$$

and define $N : Y \to Z$ by

$$Nu(t) = f(t, u(t), u'(t), u''(t)), \ t \in (0,1).$$

Then the boundary value problem $(1.1) - (1.2)$ can be written as $Lu = Nu$.

3. Existence results

We will assume that the following conditions hold and in all this paper let us set

$$Rf(s, u(s), u'(s), u''(s)) = \int_0^1 (1-s)^2 f(s, u(s), u'(s), u''(s))\, ds$$

$$-\frac{1}{\eta^3} \int_0^\eta (\eta - s)^3 f(s, u(s), u'(s), u''(s))\, ds.$$

(H_1) There exist functions $\alpha, \beta, \gamma, r \in L^1[0,1]$, such that for $(u, v, w) \in \mathbb{R}^3$, $t \in [0,1]$, it holds

$$|f(t, u, v, w)| \le \alpha(t)|u| + \beta(t)|v| + \gamma(t)|w| + r(t). \tag{3.1}$$

(H_2) There exists a constant $M > 0$ such that for $u \in domL$, if $|u''(t)| > M$ for all $t \in [0,1]$, then

$$Rf(s, u(s), u'(s), u''(s)) \ne 0. \tag{3.2}$$

(H_3) There exists a constant $M^* > 0$ such that for any $u(t) = \frac{b}{2}t^2 \in KerL$ with $\left|\frac{b}{2}\right| > M^*$, either

$$\frac{b}{2}[Rf(s, u(s), u'(s), u''(s))] < 0, \tag{3.3}$$

or else

$$\frac{b}{2}[Rf(s, u(s), u'(s), u''(s))] > 0. \tag{3.4}$$

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function, assume that conditions $(H_1) - (H_3)$ hold and that*

$$\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 < \frac{1}{2}. \tag{3.5}$$

Then the boundary value problem $(1.1) - (1.2)$ has at least one solution in $C^2[0,1]$.

For the Proof of Theorem 3.1 we shall apply Theorem 2.1 and the following Lemmas. Before we state our lemmas, we say that L is a Fredholm operator of index zero, that is, ImL is closed and $\dim KerL = co \dim ImL$. This implies that there exist a continuous projections $P : Y \to Y$ and $Q : Z \to Z$ such that $ImP = KerL$ and $KerQ = ImL$. For this purpose, we must define P by (3.1) (see later), the linear continuous projector operator Q by

$$Qy(t) = \frac{1}{C} \left[\int_0^1 (1-s)^2 y(s) \, ds - \frac{1}{\eta^3} \int_0^\eta (\eta - s)^3 y(s) \, ds \right] t^2,$$

where $\frac{1}{C} = \frac{60}{2-\eta^3}$ and the linear operator $K_P : ImL \to domL \cap KerP$ by

$$K_P y(t) = \frac{1}{2} \int_0^t (t-s)^2 y(s) \, ds, \quad \forall y \in ImL.$$

Lemma 3.2. (i) *The operator* $L : domL \subset Y \to Z$ *is a Fredholm operator of index zero.*
(ii) *For every* $y \in ImL$, *we have*

$$\|K_P y\| \leq \|y\|_1.$$

Proof. First, we prove (i). It is clear that

$$KerL = \{u \in domL : Lu = 0\},$$

$$= \{u \in domL : u''' = 0\},$$

$$= \left\{u \in domL : u(t) = \frac{b}{2}t^2, \ b \in \mathbb{R}\right\} \simeq \mathbb{R}.$$

Now, we show that

$$ImL = \left\{y \in Z : \int_0^1 (1-s)^2 y(s) \, ds - \frac{1}{\eta^3} \int_0^\eta (\eta - s)^3 y(s) \, ds = 0\right\}. \qquad (3.6)$$

In fact

$$u''' = y, \qquad (3.7)$$

has a solution $u(t)$ that satisfies the boundary value conditions (1.2), if and only if

$$\int_0^1 (1-s)^2 y(s) \, ds - \frac{1}{\eta^3} \int_0^\eta (\eta - s)^3 y(s) \, ds = 0. \qquad (3.8)$$

From (3.7), we have

$$u(t) = u(0) + u'(0) t + u''(0) \frac{t^2}{2} + \frac{1}{2} \int_0^t (t-s)^2 y(s) \, ds.$$

Thus from the condition $u(0) = u'(0) = 0$, we have

$$u(t) = u''(0) \frac{t^2}{2} + \frac{1}{2} \int_0^t (t-s)^2 y(s) \, ds.$$

According to $u\left(1\right) = \frac{3}{\eta^3}\int_0^\eta u\left(t\right)dt$, we have

$$\frac{1}{2}u''\left(0\right) + \frac{1}{2}\int_0^1\left(1-s\right)^2 y\left(s\right)ds = \frac{3}{\eta^3}\left[\int_0^\eta u''\left(0\right)\frac{t^2}{2}dt + \frac{1}{2}\int_0^\eta\int_0^t\left(t-s\right)^2 y\left(s\right)dsdt\right],$$

i.e.

$$\int_0^1\left(1-s\right)^2 y\left(s\right)ds - \frac{1}{\eta^3}\int_0^\eta\left(\eta-s\right)^3 y\left(s\right)ds = 0.$$

Hence

$$ImL = \left\{y \in Z \; : \; \int_0^1\left(1-s\right)^2 y\left(s\right)ds - \frac{1}{\eta^3}\int_0^\eta\left(\eta-s\right)^3 y\left(s\right)ds = 0\right\}.$$

On the other hand, if (3.8) holds, setting

$$u\left(t\right) = \frac{b}{2}t^2 + \frac{1}{2}\int_0^t\left(t-s\right)^2 y\left(s\right)ds,$$

where b is an arbitrary constant, then $u\left(t\right)$ is a solution of (3.7). Hence (3.6) holds. For simplicity of notation in the definition of the projector operator Q, we set

$$Ry = \int_0^1\left(1-s\right)^2 y\left(s\right)ds - \frac{1}{\eta^3}\int_0^\eta\left(\eta-s\right)^3 y\left(s\right)ds.$$

Let $C = \int_0^1\left(1-t\right)^2 t^2 dt - \frac{1}{\eta^3}\int_0^\eta\left(\eta-t\right)^3 t^2 dt \neq 0$, $t \in \left(0,1\right]$. By simple calculation, we get $C = \frac{2-\eta^3}{60}$.

Now, we need to show that the operator Q is projector. From $Qy\left(t\right) = \frac{1}{C}\cdot\left(Ry\right)\cdot t^2$, we have

$$\left(Q^2 y\right)\left(t\right) = \left(Q\left(Qy\right)\right)\left(t\right),$$

$$= \frac{1}{C}\left(\frac{1}{C}Ry\right)\left(\int_0^1\left(1-t\right)^2 t^2 ds - \frac{1}{\eta^3}\int_0^\eta\left(\eta-t\right)^3 t^2 ds\right)t^2,$$

$$= \frac{1}{C}\left(Ry\right)t^2$$

$$= \left(Qy\right)\left(t\right),$$

which implies that the operator Q is a projector. Furthermore, $ImL = KerQ$. In order, to show $Z = ImL \oplus ImQ$, it remains to shows two following steps.

Step 1. For $y \in Z$, let $y = \left(y - Qy\right) + Qy$, since $Q\left(y - Qy\right) = Qy - Q^2 y = 0$, we know $\left(y - Qy\right) \in KerQ = ImL$ and $Qy \in ImQ$. Thus

$$Z = ImL + ImQ.$$

Step 2. Let $y \in ImL \cap ImQ$. Since $y \in ImQ$, then there exists $\rho \in \mathbb{R}$ such that $y\left(t\right) = \rho t^2$, $t \in \left[0,1\right]$. Since $y \in ImL = KerQ$, then

$$0 = \rho\left(Ry\right)\left(t\right) = \rho\left(\int_0^1\left(1-t\right)^2 t^2 ds - \frac{1}{\eta^3}\int_0^\eta\left(\eta-t\right)^3 t^2 ds\right) = \rho C.$$

Since $C \neq 0$, then $\rho = 0$, so we have $y(t) = 0$, $t \in [0,1]$, which implies

$$ImL \cap ImQ = \{0\}.$$

As consequence of Step 1 and Step 2, we deduce that

$$Z = ImL \oplus ImQ,$$

and so

$$\dim KerL = co \dim ImL = \dim ImQ = 1.$$

Thus L is Fredholm operator of index zero.

We are now ready to give the other projector employed in the proof of (ii). Define $P : Y \to Y$ by

$$(Pu)(t) = u''(0) \frac{t^2}{2}. \tag{3.9}$$

Note that $KerP = \left\{ u \in Y : u''(0) \frac{t^2}{2} = 0 \right\} = \{ u \in Y : u''(0) = 0 \}$ and $ImP = KerL$.

Similarly, we shall prove that the operator P is projector and $Y = KerP \oplus KerL$. Fistly, since $(Pu)''(t) = u''(0)$, then $(P^2u)(t) = Pu(t)$, $t \in [0,1]$.

Secondly, for all $u \in Y$ and $t \in [0,1]$, we have

$$u(t) = (u(t) - Pu(t)) + Pu(t)$$

$$= \left(u(t) - u''(0) \frac{t^2}{2} \right) + u''(0) \frac{t^2}{2},$$

that is $Y = KerP + KerL$. By simple calculation we can get $KerL \cap KerP = \{0\}$. Then $Y = KerP \oplus KerL$.

Before, to estimate the supremum norm of the generalized inverse operator K_P. It remains to prove that the operator K_P is the generalized inverse of L. In fact, if $y \in ImL$, then

$$(LK_P)y(t) = [(K_P y)(t)]''' = y(t).$$

And for $u \in domL \cap KerP$, we know

$$(K_P L)u(t) = (K_P)u'''(t) = \frac{1}{2} \int_0^t (t-s)^2 u'''(s)\,ds = u(t) - u(0) - u'(0)t - u''(0) \frac{t^2}{2},$$

in view of $u \in domL \cap KerP$, $u(0) = u'(0) = 0$ and $Pu = 0$, it follows that

$$(K_P L)u(t) = u(t).$$

This shows that $K_P = (L \mid_{domL \cap KerP})^{-1}$.

Lastly, we estimate the supremum norm of the generalized inverse operator K_P. From the definition of K_P, it follows that

$$\|K_P y\|_\infty \leq \frac{1}{2} \int_0^1 (1-s)^2 |y(s)|\,ds \leq \int_0^1 |y(s)|\,ds = \|y\|_1.$$

From $(K_P y)'(t) = \int_0^t (t-s) y(s)\,ds$, we obtain

$$\|(K_P y)'\|_\infty \leq \int_0^1 (1-s)|y(s)|\,ds \leq \int_0^1 |y(s)|\,ds = \|y\|_1,$$

and from $(K_P y)''(t) = \int_0^t y(s)\, ds$, we obtain

$$\left\| (K_P y)'' \right\|_\infty \leq \int\limits_0^1 |y(s)|\, ds = \|y\|_1 \,,$$

then

$$\|K_P y\| \leq \|y\|_1 \,. \tag{3.10}$$

$\square$

Lemma 3.3. *Let* $\Omega_1 = \{u \in domL \setminus KerL : \ Lu = \lambda Nu, \ for \ some \ \lambda \in [0,1]\}$. *Then* Ω_1 *is bounded.*

Proof. Suppose that $u \in \Omega_1$, and $Lu = \lambda Nu$. Thus $\lambda \neq 0$ and $QNu = 0$, so it yields

$$Rf(s, u(s), u'(s), u''(s)) = 0.$$

Thus, by condition (H_2), there exists $t_1 \in [0,1]$, such that $|u''(t_1)| \leq M$. In view of

$$u''(0) = u''(t_1) - \int\limits_0^{t_1} u'''(t)\, dt,$$

then, we have

$$|u''(0)| \leq M + \int\limits_0^1 |u'''(s)|\, ds = M + \|u'''\|_1 = M + \|Lu\|_1 \leq M + \|Nu\|_1 \,. \tag{3.11}$$

Again for $u \in \Omega_1$, then $(I-P)u \in domL \cap KerP = ImK_P$ and $LPu = 0, 0 < \lambda < 1$ and $Nu = \frac{1}{\lambda} Lu \in ImL$, thus from Lemma 3.2, we know

$$\|(I-P)u\| = \|K_P L(I-P)u\| \leq \|L(I-P)u\|_1 = \|Lu\|_1 \leq \|Nu\|_1 \,. \tag{3.12}$$

From (3.11),(3.12) and $\|Pu\| = |u''(0)|$, we have

$$\|u\| \leq \|Pu\| + \|(I-P)u\| = |u''(0)| + \|(I-P)u\| \leq M + 2\|Nu\|_1 \,. \tag{3.13}$$

From (3.1) and (3.13), we obtain

$$\|u\| \leq 2\left[\|\alpha\|_1 \|u\|_\infty + \|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2} \right]. \tag{3.14}$$

Thus, from $\|u\|_\infty \leq \|u\|$ and (3.14), we have

$$\|u\|_\infty \leq \frac{2}{1 - 2\|\alpha\|_1} \left[\|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2} \right]. \tag{3.15}$$

From $\|u'\|_\infty \leq \|u\|$, (3.14) and (3.15), we have

$$\|u'\|_\infty \leq \|u\| \,,$$

$$\|u'\|_\infty \leq 2\left[1 + \frac{2\|\alpha\|_1}{1 - 2\|\alpha\|_1} \right] \left[\|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2} \right],$$

$$= \frac{2}{1 - 2\|\alpha\|_1} \left[\|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2} \right],$$

i.e

$$\|u'\|_\infty \left[\frac{1 - 2\|\alpha\|_1 - 2\|\beta\|_1}{1 - 2\|\alpha\|_1} \right] \leq \frac{2}{1 - 2\|\alpha\|_1} \left[\|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2} \right].$$

Therefore

$$\|u'\|_\infty \leq \frac{2}{1 - 2\|\alpha\|_1 - 2\|\beta\|_1} \left[\|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right]. \qquad (3.16)$$

Again, from $\|u''\|_\infty \leq \|u\|$, (3.14), (3.15) and (3.16), we have

$$\|u''\|_\infty \leq \left[2\|\beta\|_1 + \frac{4\|\beta\|_1 \|\alpha\|_1}{1 - 2\|\alpha\|_1}\right]\|u'\|_\infty + \left[\frac{4\|\alpha\|_1}{1 - 2\|\alpha\|_1} + 2\right]\left[\|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right],$$

$$\leq \left[\frac{4\|\beta\|_1}{(1 - 2\|\alpha\|_1 - 2\|\beta\|_1)(1 - 2\|\alpha\|_1)} + \frac{2}{1 - 2\|\alpha\|_1}\right]\left[\|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right],$$

$$= \frac{2}{(1 - 2\|\alpha\|_1 - 2\|\beta\|_1)}\left[\|\gamma\|_1 \|u''\|_\infty + \|r\|_1 + \frac{M}{2}\right],$$

i.e

$$\|u''\|_\infty \leq \left[\frac{2\left(\|r\|_1 + \frac{M}{2}\right)}{1 - 2\|\alpha\|_1 - 2\|\beta\|_1 - 2\|\gamma\|_1}\right], \qquad (3.17)$$

thus, from (3.17), there exists $M_1 > 0$ such that

$$\|u''\|_\infty \leq M_1, \qquad (3.18)$$

therefore, from (3.18) and (3.16), there exists $M_2 > 0$, such that

$$\|u'\|_\infty \leq M_2, \qquad (3.19)$$

hence, from (3.19) and (3.15), there exists $M_3 > 0$, such that

$$\|u\|_\infty \leq M_3. \qquad (3.20)$$

Consequently

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\} \leq \max\{M_1, M_2, M_3\}.$$

Again, from (3.1), (3.18), (3.19) and (3.20), we have

$$\|u'''\|_1 = \|Lu\|_1 \leq \|Nu\|_1 \leq \|\alpha\|_1 M_3 + \|\beta\|_1 M_2 + \|\gamma\|_1 M_1 + \|r\|_1.$$

So, Ω_1 is bounded. $\qquad \qquad \square$

Lemma 3.4. *The set* $\Omega_2 = \{u \in KerL : Nu \in ImL\}$ *is bounded.*

Proof. Let $u \in \Omega_2$, then $u \in KerL = \{u \in domL : u = \frac{b}{2}t^2, b \in \mathbb{R}, t \in [0,1]\}$. Also, since $KerQ = ImL$, then $QNu = 0$, therefore

$$Rf\left(s, \frac{b}{2}s^2, bs, b\right) = 0.$$

From condition (H_2), $\|u\|_\infty = \left|\frac{b}{2}\right| \leq M$, so $\|u\| \leq M$, thus Ω_2 is bounded. $\qquad \square$

Before we define the set Ω_3, we must state our isomorphism, $J : KerL \to ImQ$. Let

$$J\left(\frac{b}{2}t^2\right) = \frac{b}{2}t^2, \forall b \in \mathbb{R}, t \in [0,1],$$

and define

$$\Omega_3 = \{u \in KerL : -\lambda Ju + (1 - \lambda)QNu = 0, \lambda \in [0,1]\}.$$

Lemma 3.5. *If the first part of condition (H_3) holds, then*

$$\left(\frac{b}{2}\right)\left(\frac{60}{2-\eta^3}\right)\left[Rf\left(s,\frac{b}{2}s^2,bs,b\right)ds\right] < 0, \tag{3.21}$$

for all $\left|\frac{b}{2}\right| > M^*$ and Ω_3 is bounded.

Proof. Suppose that $u = \frac{b_0}{2}t^2 \in \Omega_3$. Then we obtain

$$\lambda\left(\frac{b_0}{2}\right) = (1-\lambda)\left(\frac{60}{2-\eta^3}\right)\left(Rf\left(s,b_0\frac{s^2}{2},b_0s,b_0\right)\right).$$

If $\lambda = 1$, then $b_0 = 0$, which gives Ω_3 bounded.
Otherwise, if $\lambda \neq 1$, there exist $M^* > 0$ such that $\left|\frac{b_0}{2}\right| > M^*$. Then in view of (3.21), we have

$$\lambda\left(\frac{b_0}{2}\right)^2 = (1-\lambda)\frac{b_0}{2}\left(\frac{60}{2-\eta^3}\right)\left(Rf\left(s,b_0\frac{s^2}{2},b_0s,b_0\right)\right) < 0,$$

which contradicts the fact that $\lambda\left(\frac{b_0}{2}\right)^2 \geq 0$. Then $|u| = \left|\frac{b_0}{2}t^2\right| \leq \left|\frac{b_0}{2}\right| \leq M^*$, we obtain $\|u\| \leq M^*$. Hence $\Omega_3 \subset \{u \in KerL : \|u\| \leq M^*\}$ is bounded.
If $\lambda = 0$, it yields

$$Rf\left(s,\frac{b_0}{2}s^2,b_0s,b_0\right) = 0.$$

Taking condition (H_2) into account, we obtain $\|u\| = \left|\frac{b}{2}\right| \leq M^*$. $\qquad\square$

Now, define Ω_3 by

$$\Omega_3 = \{u \in KerL : \lambda Ju + (1-\lambda)QNu = 0, \ \lambda \in [0,1]\}$$

Lemma 3.6. *If the second part of (H_3) holds, then*

$$\left(\frac{b}{2}\right)\left(\frac{60}{2-\eta^3}\right)\left[Rf\left(s,\frac{b}{2}s^2,bs,b\right)\right] > 0, \tag{3.22}$$

for all $\left|\frac{b}{2}\right| > M^*$ and Ω_3 is bounded.

Proof. A similar argument as above shows that Ω_3 is bounded. $\qquad\square$

The Proof of Theorem 3.1 is now an easy consequence of the above lemmas and Theorem 2.1.

Proof. of Theorem 3.1.
Let Ω to be an open bounded subset of Y such that $\overset{3}{\underset{i=1}{\cup}}\overline{\Omega}_i \subset \Omega$. By using the fact that u''' is bounded and the Arzela-Ascoli Theorem, we can prove that $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact, thus N is $L-compact$ on $\overline{\Omega}$. Then by Lemmas 3.3 and 3.4, we have
(i) $Lu \neq \lambda Nu$ for every $(u,\lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0,1)$.
(ii) $Nu \notin ImL$ for every $u \in KerL \cap \partial\Omega$.
(iii) Let $H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu = 0, \ \lambda \in [0,1]$.
According to Lemmas 3.5 and 3.6, we know that $H(u,\lambda) \neq 0$ for every $u \in KerL \cap \partial\Omega$. Thus, by the homotopy property of degree, we obtain

$$deg(QN\mid_{KerL}, \Omega \cap KerL, 0) = deg(H(\cdot,0), \Omega \cap KerL, 0),$$
$$= deg(H(\cdot,1), \Omega \cap KerL, 0),$$

$$= deg\left(\pm J, \Omega \cap KerL, 0\right) \neq 0.$$

Then, by Theorem 2.1, $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$, so the boundary value problem $(1.1) - (1.2)$ has at least one solution in $C^2[0, 1]$. The proof is complete. □

We construct an example to illustrate the applicability of the results presented.

Example 3.1. *Consider the following boundary value problem*

$$u'''(t) = f\left(t, u\left(t\right), u'\left(t\right), u''\left(t\right)\right), \ t \in (0, 1), \tag{3.23}$$

$$u\left(0\right) = u'\left(0\right) = 0, \ u\left(1\right) = \frac{3}{\eta^3} \int_0^\eta u\left(t\right) dt, \ \eta \in (0, 1), \tag{3.24}$$

where

$$f\left(t, u\left(t\right), u'\left(t\right), u''\left(t\right)\right) = \frac{1}{3}u''\left(t\right) + \frac{1}{6}\left(1 - t\right)\left(1 - \cos\left(u'(t)\right)\sin\left(u(t)\right)\right), \ t \in (0, 1).$$

Here we have

$$\left|\frac{1}{3}u''\left(t\right) + \frac{1}{6}\left(1 - t\right)\left(1 - \cos\left(u'(t)\right)\sin\left(u(t)\right)\right)\right| \leq \frac{1}{3} + \frac{1}{3}\left|u''\left(t\right)\right|,$$

that is

$$\alpha\left(t\right) = 0, \ \beta\left(t\right) = 0, \ \gamma\left(t\right) = \frac{1}{3} \ and \ r\left(t\right) = \frac{1}{3}.$$

So, condition (H_1) is satisfied, which gives

$$\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 = \frac{1}{3} < \frac{1}{2}.$$

Set

$$I = Rf\left(s, u\left(s\right), u'\left(s\right), u''\left(s\right)\right)$$

$$= \int_0^1 \left(1 - s\right)^2 f\left(s, u\left(s\right), u'\left(s\right), u''\left(s\right)\right) ds - \frac{1}{\eta^3} \int_0^\eta \left(\eta - s\right)^3 f\left(s, u\left(s\right), u'\left(s\right), u''\left(s\right)\right) ds.$$

If $u''(t) < -M = -10$, then

$$f\left(t, u\left(t\right), u'\left(t\right), u''\left(t\right)\right) < \frac{1}{3}\left(1 - M\right) = -3 < 0.$$

In this case, we have $I < 0$, because

$$\int_0^1 \left(1 - s\right)^2 f\left(s, u\left(s\right), u'\left(s\right), u''\left(s\right)\right) ds < \int_0^\eta \left(1 - \frac{s}{\eta}\right)^3 f\left(s, u\left(s\right), u'\left(s\right), u''\left(s\right)\right) ds.$$

If $u''(t) > M = 10$, then

$$f\left(t, u\left(t\right), u'\left(t\right), u''\left(t\right)\right) > \frac{1}{3}\left(1 + M\right) = \frac{11}{3} > 0.$$

Hence, $I > 0$, because

$$\int_0^1 \left(1 - s\right)^2 f\left(s, u\left(s\right), u'\left(s\right), u''\left(s\right)\right) ds > \int_0^\eta \left(1 - \frac{s}{\eta}\right)^3 f\left(s, u\left(s\right), u'\left(s\right), u''\left(s\right)\right) ds.$$

Therefore, the condition (H_2) *is satisfied.*
If $\frac{b}{2} < -M^* = -5$ *and* $u(t) = \frac{b}{2}t^2$ *, then*

$$f(t, u(t), u'(t), u''(t)) < \frac{1}{3}(1 + b) = -3 < 0.$$

In this case, we have $I < 0$, *because*

$$\int_0^1 (1 - s)^2 f(s, u(s), u'(s), u''(s)) \, ds < \int_0^\eta \left(1 - \frac{s}{\eta}\right)^3 f(s, u(s), u'(s), u''(s)) \, ds.$$

Hence

$$\frac{b}{2} \left[\int_0^1 (1 - s)^2 f(s, u(s), u'(s), u''(s)) \, ds - \frac{1}{\eta^3} \int_0^\eta (\eta - s)^3 f(s, u(s), u'(s), u''(s)) \, ds\right] > 0.$$

Therefore $\frac{b}{2}I > 0$. *So condition* (H_3) *is satisfied.*
Thus, all the conditions of Theorem 3.1 *are satisfied, which implies that the boundary value problem* (3.23) − (3.24) *has at least one solution* $u \in C^2[0, 1]$.

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