Konuralp Journal of Mathematics, 7 (1) (2019) 208-216

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

On Nonlightlike Curves in Minkowski Space-time

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Abstract

In this paper, some special nonlightlike curves are investigated depending on characterizations of their Serret-Frenet frames in Minkowski space-time. Firstly, a great survey of nonlightlike curves in Minkowski space-time is stated. To generalize the results, the nonlightlike curves with the character of Serret-Frenet frame $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ are considered. So that, the necessary or/and sufficient conditions of a given nonlightlike curve to be a straight line, plane curve, helix and to lie on pseudohyperbolical space $H_0^3(r)$ and Lorentzian hypersphere $S_0^3(r)$ are stated both depending on curvature functions and character of Serret-Frenet frame of the curve, respectively.

Keywords: *Minkowski Space-time, Nonlightlike Curve, Serret-Frenet Frame*
2010 Mathematics Subject Classification: *14H50, 51B20.*

1. Introduction

In Euclidean space, Frenet frame is a moving reference frame of orthonormal vectors which are used to describe a curve locally at each point. It is the main tool in the differential geometric treatment of curves as it is far easier and more natural to describe local properties (e.g. curvature, torsion) in terms of a local reference system than using a global one like the Euclidean coordinates. For a regular curve $\gamma: I\subset\mathbb{R}\to\mathbb{E}^4$ with arclength parameter s and the Frenet vectors $\{T,N_1,N_2,N_3\}$, Frenet–Serret formulas can be given as

$$\frac{d}{ds}\begin{bmatrix} T\\ N_1\\ N_2\\ N_3\end{bmatrix}=\begin{bmatrix}0&\kappa&0&0\\ -\kappa&0&\tau&0\\ 0&-\tau&0&\sigma\\ 0&0&-\sigma&0\end{bmatrix}\begin{bmatrix} T\\ N_1\\ N_2\\ N_3\end{bmatrix}$$

where the functions κ, τ and σ are called the curvature, first and second torsion of the curve, [1], [2], [3]. Frenet-Serret formulas play an important role to describe the kinematic properties of a particle moving along on a regular curve or the geometric properties of the curve itself. Some special curves can be characterized as follows:

- $\kappa=0$ if and only if γ is a straight line;
- $\tau=0$ if and only if γ is a plane curve;
- $\sigma=0$ if and only if γ lies in a three dimensional subspace of $\mathbb{E}^4$;
- $\tau=0$ and $\kappa=$constant>0 if and only if γ is a circle;
- $\sigma=0$ and $\tau=c_2$, $\kappa=c_1$ where $c_1,c_2\in\mathbb{R}_0$ if and only if γ is a circular helix;
- $\sigma=c_3$, $\tau=c_2$ and $\kappa=c_1$ where $c_1,c_2,c_3\in\mathbb{R}_0$ if and only if

  $$\gamma(s)=\frac{1}{\alpha}\sin(\alpha s)V_1-\frac{1}{\alpha}\cos(\alpha s)V_2+\frac{1}{\beta}\sin(\beta s)V_3-\frac{1}{\beta}\cos(\beta s)V_4$$

  where $\alpha^2=\frac{\kappa^2+\tau^2+\sigma^2-\sqrt{(\kappa^2+\tau^2+\sigma^2)^2-4\kappa^2\sigma^2}}{2}$, $\beta^2=\frac{\kappa^2+\tau^2+\sigma^2+\sqrt{(\kappa^2+\tau^2+\sigma^2)^2-4\kappa^2\sigma^2}}{2}$, V_i are orthogonal constant vectors satisfying

  $\langle V_1,V_1\rangle=\langle V_2,V_2\rangle$ and $\langle V_3,V_3\rangle=\langle V_4,V_4\rangle$.

  The curve γ lies on a sphere with radius $\frac{1}{|\sigma|}$, [4].

The motion of the particle in Minkowski spaces is an another developing research area especially in physics and mathematics. This makes the curve theory in Minkowski spaces an interesting topic which helps to describe some kinematic properties of a moving particle. In the

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study [5], spacelike curves with nonnull frame vectors are investigated and a method to calculate Frenet apparatus of these curves is stated with a definition of vector product in Minkowski space-time. Then, the characterizations of curves lying on the pseudohyperbolic space H_0^3 are given and it is proved that there is no timelike or null curves lying on the pseudohyperbolic space H_0^3 in [6]. Then, the necessary and sufficient condition of a unit speed spacelike curve with nonzero curvature and torsion functions to lie on H_0^3 is stated in [7]. Finally, the parallel frames of nonlightlike curves is investigated in [8] by using the relations between Frenet-Serret frame of a given curve. Additionally, the parallel frames of nonlightlike curves for higher dimensions are discussed in [9].

The present article is concern with the characterization of some special nonlightlike curves in Minkowski space-time. Firstly, a brief summary of Frenet frame of nonlightlike curves in Minkowski space-time. In addition, the necessary or/and sufficient conditions of nonlightlike curves to be a straight line and plane curve are given, respectively. Then the nonlightlike curves lying on a pseudohyperbolical space $H_0^3(r)$ and Lorentzian sphere $S_0^3(r)$ are investigated. Moreover, a way of finding the center and radius of the pseudohyperbolical space and Lorentzian sphere in terms of Frenet apparatus of the curve lying on $H_0^3(r)$ and $S_0^3(r)$, respectively. Furthermore, an example is given to show how this method works. Finally, Lorentzain circles and helices are came up for review.

2. Preliminaries

The Minkowski space-time is four dimensional Euclidean space provided with the Lorentzian inner product

$$\langle u,v\rangle_{\mathbb{L}} = -u_1v_1+u_2v_2+u_3v_3+u_4v_4$$

where $u=(u_1,u_1,u_3,u_4)$, $v=(v_1,v_2,v_3,v_4)$ and denoted by $\mathbb{E}_1^4$. Any vector u in $\mathbb{E}_1^4$ can be characterized as follows: the vector u is called spacelike, lightlike or timelike if $\langle u,u\rangle_{\mathbb{L}}>0$, $\langle u,u\rangle_{\mathbb{L}}=0$ or $\langle u,u\rangle_{L}<0$, respectively. The norm of the vector $u\in\mathbb{E}_1^4$ is defined by

$$\|u\| = \sqrt{|\langle u,u\rangle_{\mathbb{L}}|}.$$

A definition of a vector product in Minkowski space-time is introduced in [5] as follows:

Definition 2.1. *Let $u=(u_1,u_2,u_3,u_4)$, $v=(v_1,v_2,v_3,v_4)$ and $w=(w_1,w_2,w_3,w_4)$ be vectors in $\mathbb{E}_1^4$. The vector product in Minkowski space-time is defined with the determinant*

$$u\times v\times w = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}$$

where $\{e_1,e_2,e_3,e_4\}$ is the orthogonal basis of Minkowski space-time.

An arbitrary curve $\gamma: I\to\mathbb{E}_1^4$ is called spacelike, timelike or lightlike (null), if all of its velocity vectors $\gamma'(s)$ are spacelike, timelike or lightlike (null) for all $s\in I$, respectively. If $\|\gamma'(s)\|=1$, then γ is called unit speed curve. For any unit speed nonlightlike curve γ with Frenet-Serret frame $\{T,N_1,N_2,N_3\}$, Frenet-Serret formulas of the curve γ can be given as:

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa & 0 & 0 \\ -\varepsilon_1\kappa & 0 & \varepsilon_3\tau & 0 \\ 0 & -\varepsilon_2\tau & 0 & \varepsilon_4\sigma \\ 0 & 0 & -\varepsilon_3\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{2.1}$$

where $\varepsilon_1 = <T,T>_{\mathbb{L}}$ and $\varepsilon_i = <N_{i-1},N_{i-1}>_{\mathbb{L}}$ for $i=2,3,4$. Here, we call the value of $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ as the character of Frenet-Serret frame of the curve γ and κ, τ and σ as curvature, first and second torsion functions of the curve γ, respectively. Then the first vector field is defined by

$$T(s)=\gamma'(s).$$

The second vector field N_1 of the curve γ is defined as

$$N_1(s)=\frac{1}{\kappa(s)}T'(s)$$

where

$$\kappa(s)=\sqrt{|<\gamma''(s),\gamma''(s)>_{\mathbb{L}}|}>0$$

is a real valued function and called the curvature function of the curve γ. The third vector field N_2 of the curve γ is defined by

$$N_2(s)=\frac{1}{\|N_1'(s)+\varepsilon_1\kappa(s)T(s)\|}N_1'(s)+\varepsilon_1\kappa(s)T(s).$$

The first torsion function τ of the curve is defined by

$$\tau(s)=\|N_1'(s)+\varepsilon_1\kappa(s)T(s)\|.$$

Consider the vector product $T\times N_1\times N_2$, the fourth unit vector field is defined by

$$N_3(s)=\eta T(s)\times N_1(s)\times N_2(s)$$

where η is taken as -1 or $+1$ to make the determinant of the matrix $[T,N_1,N_2,N_3]$ as 1. The second torsion function σ of the curve is defined by

$$\sigma(s)=\|N_2'(s)+\varepsilon_2\tau(s)N_1(s)\|.$$

3. Some Special Nonlightlike Curves in Minkowski Space-time

Theorem 3.1. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed nonlightlike curve. Then γ is a straight line if and only if the curvature κ of the curve γ is zero.*

Proof. Suppose that γ is a straight line. Then we can write

$$\gamma(s) = u + sv$$

where $u, v \in \mathbb{E}_1^4$ such that $< v, v >_{\mathbb{L}} = \varepsilon_1$. Since $\kappa(s) = \sqrt{|< \gamma''(s), \gamma''(s) >_{\mathbb{L}}|}$, we get $\kappa = 0$. Conversely, suppose that γ is a unit speed nonlightlike curve with $\kappa = 0$. Then we get

$$T' = \varepsilon_2 \kappa N_1 = 0.$$

This implies that the tangent vector field T of the curve γ is constant at all points. Thus γ is a straight line. □

Theorem 3.2. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed nonlightlike curve with $\kappa > 0$. If γ is a plane curve then the first torsion τ or second torsion σ of the curve γ is zero.*

Proof. Suppose that $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a unit speed nonlightlike plane curve with $\kappa > 0$. Then there exists a nonzero parallel vector field n such that

$$\langle \gamma(s) - \gamma(s_0), n \rangle_{\mathbb{L}} = 0$$

for any arbitrary $s_0 \in I$ and for all s. Differentiation yields

$$\langle \gamma', n \rangle_{\mathbb{L}} = 0$$

for all s. Thus we have

$$\langle T, n \rangle_{\mathbb{L}} = 0.$$

Differentiating again

$$\langle \varepsilon_2 \kappa N_1, n \rangle_{\mathbb{L}} = 0$$

by using the Frenet-Serret formulas in 2.1. Thus, n is orthogonal to both the vector fields T and N_1. We may write

$$n = f N_2 + g N_3$$

where f and g are functions of real parameter s. Since n is a parallel vector field by assumption, then we have

$$n' = (-\varepsilon_2 f \tau) N_1 + (f' - \varepsilon_3 g \sigma) N_2 + (g' + \varepsilon_4 \sigma) N_3 = 0$$

with the use of Frenet-Serret formulas in 2.1. This implies that

$$\begin{aligned} f\tau &= 0, \\ \varepsilon_3 f' - g\sigma &= 0, \\ \varepsilon_4 g' + \sigma &= 0. \end{aligned}$$

By the first equation above, we obtain two different cases: $\tau = 0$ and $f = 0$. The first case $\tau = 0$ is the one of desired results. For the other case $f = 0$, we see that

$$g\sigma = 0$$

by substituting $f = 0$ to the second equation above. Since n is a nonzero parallel vector field, then the function g should be nonzero which means $\sigma = 0$. This is the other desired result. □

Theorem 3.3. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed nonlightlike curve with $\kappa(s) > 0$, σ =constant. If first torsion τ of the curve γ is zero, then γ is a plane curve.*

Proof. Suppose that γ is a unit speed nonlightlike curve with $\kappa(s) > 0$, σ =constant and $\tau = 0$. We will give the proof in three different cases depending on the character of Frenet vector fields N_2 and N_3 of the curve γ.
Case 1: If both the vector fields N_2 and N_3 are spacelike, then we will consider the vector field

$$n = (c_1 \sin(\sigma s) - c_2 \cos(\sigma s)) N_2 + (c_1 \cos(\sigma s) + c_2 \sin(\sigma s)) N_3$$

where c_1 and c_2 are arbitrary real constants. By using Frenet-Serret formulas in 2.1, differentiation yields

$$\begin{aligned} n' &= (c_1 \sigma \cos(\sigma s) + c_2 \sigma \sin(\sigma s)) N_2 + (c_1 \sin(\sigma s) - c_2 \cos(\sigma s)) \sigma N_3 \\ &\quad + (-c_1 \sigma \sin(\sigma s) + c_2 \sigma \cos(\sigma s)) N_3 - (c_1 \cos(\sigma s) + c_2 \sin(\sigma s)) \sigma N_2 \\ &= 0. \end{aligned}$$

This implies that n is a parallel vector field. We assert that γ lies on the plane through $\gamma(s_0)$ orthogonal to the parallel vector field n for arbitrary $s_0 \in I$ and for all s. To prove this, consider the real valued function

$$f(s) = \langle \gamma(s) - \gamma(s_0), n \rangle_{\mathbb{L}}$$

for all s. Then we get

$$f'(s) = \langle \gamma'(s), n \rangle_{\mathbb{L}} = \langle T, n \rangle_{\mathbb{L}} = 0.$$

But obviously, $f(s_0) = 0$. So that $f = 0$ which shows that γ lies entirely in the plane orthogonal to the parallel vector field n. Other two cases can be proved similarly by different choice of the vector field n.
Case 2: If the vector field N_2 is timelike, then we will consider the vector field

$$n = (-c_1 e^{\sigma s} + c_2 e^{-\sigma s}) N_2 + (c_1 e^{\sigma s} + c_2 e^{-\sigma s}) N_3$$

where c_1 and c_2 are arbitrary real constants.
Case 3: If the vector field N_3 is timelike, then we will consider the vector field

$$n = (c_1 e^{\sigma s} - c_2 e^{-\sigma s}) N_2 + (c_1 e^{\sigma s} + c_2 e^{-\sigma s}) N_3$$

where c_1 and c_2 are arbitrary real constants. □

Theorem 3.4. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed nonlightlike curve with $\kappa > 0$. If second torsion σ of the curve γ is zero, then γ is a plane curve.*

Proof. Assume that γ is a nonlighlike plane curve with $\kappa > 0$ and $\sigma = 0$. That is

$$N_3' = -\varepsilon_3 \sigma N_2 = 0.$$

This means that the vector field N_3 of the curve γ is constant at all points. The proof can be done similar to the proof of above theorem by choosing $n = N_3$ and considering the real valued function

$$f(s) = \langle \gamma(s) - \gamma(s_0), N_3(s) \rangle_{\mathbb{L}}$$

for all s. □

Definition 3.5. *The pseudohyperbolic space with center P_0 and radius $r \in \mathbb{R}^+$ is the hyperquadratic*

$$H_0^3(r) = \{x \in \mathbb{E}_1^4 : \langle x - P_0, x - P_0 \rangle_{\mathbb{L}} = -r^2\}.$$

The Lorentzian hypersphere with center P_0 and radius $r \in \mathbb{R}^+$ is the hyperquadratic

$$S_0^3(r) = \{x \in \mathbb{E}_1^4 : \langle x - P_0, x - P_0 \rangle_{\mathbb{L}} = r^2\}.$$

In the study [6], it is proved that there is no timelike or null curves lying on the pseudohyperbolical space $H_0^3(r)$. Only spacelike curves exist that lie on the pseudohyperbolical space $H_0^3(r)$.

Theorem 3.6. *[6] Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with the character of Frenet-Serret frame $(1,1,1,-1)$ with nonzero curvature and torsion functions. The curve γ lies on a $H_0^3(r)$ if and only if the following equality is satisfied*

$$\frac{1}{\kappa^2} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 - \frac{1}{\sigma^2}\left[\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' - \frac{\tau}{\kappa}\right]^2 = -r^2$$

for some $r \in \mathbb{R}^+$.

The following theorem gives the necessary and sufficient condition of a unit speed spacelike curve to lie on $H_0^3(r)$ with the character of Frenet-Serret frame $(1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$.

Theorem 3.7. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with the character of Frenet-Serret frame $(1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and nonzero curvature, torsion functions. The curve γ lies on a $H_0^3(r)$ if and only if then the following equality is satisfied*

$$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_3 \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 + \varepsilon_4 \frac{1}{\sigma^2}\left[\varepsilon_4\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' - \frac{\tau}{\kappa}\right]^2 = -r^2$$

for some constant $r \in \mathbb{R}^+$.

Proof. Suppose that $\{T, N_1, N_2, N_3\}$ is the Serret-Frenet frame field of the unit speed spacelike curve γ with the character of $(1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and nonzero curvature and torsion functions. Suppose that the curve γ lies on $H_0^3(r)$ with center P_0. Then we have

$$\langle \gamma - P_0, \gamma - P_0 \rangle_{\mathbb{L}} = -r^2.$$

By differentiating this relation, we get

$$\langle T, \gamma - P_0 \rangle_{\mathbb{L}} = 0. \tag{3.1}$$

Thus, we may write

$$\gamma - P_0 = aN_1 + bN_2 + cN_3$$

for some a, b and c such that

$$-r^2 = \varepsilon_2 a^2 + \varepsilon_3 b^2 + \varepsilon_4 c^2.$$

If we differentiate the Equation 3.1, then we get

$$\langle T', \gamma - P_0\rangle_{\mathbb{L}} + \langle T, T\rangle_{\mathbb{L}} = 0.$$

By using the Frenet-Serret formulas in 2.1, we see that

$$\langle N_1, \gamma - P_0\rangle_{\mathbb{L}} = -\varepsilon_2 \frac{1}{\kappa}. \tag{3.2}$$

Thus, we get

$$a = \varepsilon_2 \langle \gamma - P_0, N_1\rangle_{\mathbb{L}} = -\frac{1}{\kappa}.$$

Similarly, if we differentiate the Equation 3.2, then we get

$$b = \varepsilon_3 \langle \gamma - P_0, N_2\rangle_{\mathbb{L}} = -\varepsilon_2 \frac{1}{\tau}(\frac{1}{\kappa})'.$$

By differentiating the equation

$$\langle \gamma - P_0, N_2\rangle_{\mathbb{L}} = \varepsilon_4 \frac{1}{\tau}(\frac{1}{\kappa})',$$

we obtain

$$c = \varepsilon_4 \langle \gamma - P_0, N_3\rangle_{\mathbb{L}} = \frac{1}{\sigma}[\varepsilon_4(\frac{1}{\tau}(\frac{1}{\kappa})')' - \frac{\tau}{\kappa}].$$

That is

$$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_3(\frac{1}{\tau}(\frac{1}{\kappa})')^2 + \varepsilon_4 \frac{1}{\sigma^2}[\varepsilon_4(\frac{1}{\tau}(\frac{1}{\kappa})')' - \frac{\tau}{\kappa}]^2 = -r^2.$$

□

Theorem 3.8. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with the character of Frenet-Serret frame $(1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and nonzero constant curvature, torsion functions. If the inequality*

$$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_4 \frac{\tau^2}{\kappa^2\sigma^2} < 0$$

is satisfied then the curve γ lies on $H_0^3(r)$ with center

$$P_0 = \gamma + \frac{1}{\kappa}N_1 + \frac{\tau}{\kappa\sigma}N_3.$$

Proof. Suppose that γ is a given unit speed spacelike curve character of Frenet-Serret frame $(1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and nonzero constant curvature, torsion functions such that

$$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_4 \frac{\tau^2}{\kappa^2\sigma^2} < 0.$$

Then we get

$$\begin{aligned}\langle \gamma - P_0, \gamma - P_0\rangle_{\mathbb{L}} &= \left\langle -\frac{1}{\kappa}N_1 - \frac{\tau}{\kappa\sigma}N_3, -\frac{1}{\kappa}N_1 - \frac{\tau}{\kappa\sigma}N_3 \right\rangle_{\mathbb{L}} \\ &= \varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_4 \frac{\tau^2}{\kappa^2\sigma^2}\end{aligned}$$

which is equal to a negative constant. Thus, there exists $r \in \mathbb{R}^+$ such that

$$\langle \gamma - P_0, \gamma - P_0\rangle_{\mathbb{L}} = -r^2.$$

If the center of pseudohyperbolical space P_0 is taken as

$$P_0 = \gamma + \frac{1}{\kappa}N_1 + \frac{\tau}{\kappa\sigma}N_3$$

then we get

$$P_0' = T + \frac{1}{\kappa}(-\kappa T + \varepsilon_3 \tau N_2) + \frac{\tau}{\kappa\sigma}(-\varepsilon_3 \sigma N_2)$$
$$= T - T + \varepsilon_3 \frac{\tau}{\kappa} N_2 - \varepsilon_3 \frac{\tau}{\kappa} N_2 = 0.$$

So that the center P_0 is a constant.

$$\langle \gamma - P_0, \gamma - P_0 \rangle_{\mathbb{L}} = \left\langle -\varepsilon_1 \frac{1}{\kappa} N_1 - \varepsilon_1 \frac{\tau}{\kappa\sigma} N_3, -\varepsilon_1 \frac{1}{\kappa} N_1 - \varepsilon_1 \frac{\tau}{\kappa\sigma} N_3 \right\rangle_{\mathbb{L}}$$
$$= \varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_4 \frac{\tau^2}{\kappa^2\sigma^2}$$

is also a constant. This means that the curve γ lies on pseudohyperbolical space. □

Theorem 3.9. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed nonlightlike curve with the character of Frenet-Serret frame $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and nonzero curvature, torsion functions. The curve γ lies on Lorentzian hypersphere $S_0^3(r)$ if and only if then the following equality is satisfied*

$$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_3 (\frac{1}{\tau}(\frac{1}{\kappa})')^2 + \varepsilon_4 \frac{1}{\sigma^2}[\varepsilon_4(\frac{1}{\tau}(\frac{1}{\kappa})')' - \varepsilon_1 \frac{\tau}{\kappa}]^2 = r^2$$

for some constant $r \in \mathbb{R}^+$.

Proof. Suppose that $\{T, N_1, N_2, N_3\}$ is the Serret-Frenet frame field of the unit speed spacelike curve γ with the character of $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and nonzero curvature and torsion functions. Assume that the curve γ lies on $S_0^3(r)$ with center P_0. Then we obtain

$$\langle \gamma - P_0, \gamma - P_0 \rangle_{\mathbb{L}} = r^2.$$

If we differentiate this relation, then we get

$$\langle T, \gamma - P_0 \rangle_{\mathbb{L}} = 0. \tag{3.3}$$

Thus, we may write

$$\gamma - P_0 = aN_1 + bN_2 + cN_3$$

for some a, b and c such that

$$r^2 = \varepsilon_2 a^2 + \varepsilon_3 b^2 + \varepsilon_4 c^2.$$

Similar to the proof of Theorem 3.7, we obtain

$$a = -\varepsilon_1 \frac{1}{\kappa},\ b = \varepsilon_3 \varepsilon_4 \frac{1}{\tau}(\frac{1}{\kappa})' \text{ and } c = \frac{1}{\sigma}[\varepsilon_4(\frac{1}{\tau}(\frac{1}{\kappa})')' - \varepsilon_1 \frac{\tau}{\kappa}].$$

Thus, we get

$$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_3 (\frac{1}{\tau}(\frac{1}{\kappa})')^2 + \varepsilon_4 \frac{1}{\sigma^2}[\varepsilon_4(\frac{1}{\tau}(\frac{1}{\kappa})')' - \varepsilon_1 \frac{\tau}{\kappa}]^2 = r^2.$$

□

Theorem 3.10. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed nonlightlike curve with the character of Frenet-Serret frame $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and nonzero constant curvature, torsion functions. If the inequality*

$$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_4 \frac{\tau^2}{\kappa^2\sigma^2} > 0$$

is satisfied then the curve γ lies on $S_0^3(r)$ with center

$$P_0 = \gamma + \varepsilon_1 \frac{1}{\kappa} N_1 + \varepsilon_1 \frac{\tau}{\kappa\sigma} N_3.$$

Example 3.11. *Consider the unit speed timelike curve*

$$\gamma(s) = (\sqrt{2}\sinh s, \sqrt{2}\cosh s, \sin s, \cos s).$$

Let $\{T, N_1, N_2, N_3\}$ denote the Frenet frame of the curve γ. We find the Frenet apparatus of the curve γ as follows:

$$T = (\sqrt{2}\cosh s, \sqrt{2}\sinh s, \cos s, -\sin s),$$

$$N_1 = (\tfrac{\sqrt{2}}{\sqrt{3}}\sinh s, \tfrac{\sqrt{2}}{\sqrt{3}}\cosh s, -\tfrac{1}{\sqrt{3}}\sin s, -\tfrac{1}{\sqrt{3}}\cos s),$$

$$N_2 = (-\cosh s, -\sinh s, -\sqrt{2}\cos s, \sqrt{2}\sin s),$$

$$N_3 = (-\tfrac{1}{\sqrt{3}}\sinh s, -\tfrac{1}{\sqrt{3}}\cosh s, -\tfrac{\sqrt{2}}{\sqrt{3}}\sin s, -\tfrac{\sqrt{2}}{\sqrt{3}}\cos s),$$

and

$$\kappa=\sqrt{3},\ \tau=\frac{2\sqrt{2}}{\sqrt{3}},\ \sigma=-\frac{1}{\sqrt{3}}.$$

By above corollary, this curve lies on a Lorentzian sphere with the center

$$\begin{aligned}P_0&=\gamma-\frac{1}{\kappa}N_1-\frac{\tau}{\kappa\sigma}N_3\\&=(\sqrt{2}\sinh s,\sqrt{2}\cosh s,\sin s,\cos s)-\frac{1}{\sqrt{3}}(\tfrac{\sqrt{2}}{\sqrt{3}}\sinh s,\tfrac{\sqrt{2}}{\sqrt{3}}\cosh s,-\tfrac{1}{\sqrt{3}}\sin s,-\tfrac{1}{\sqrt{3}}\cos s)\\&+\frac{2\sqrt{2}}{\sqrt{3}}(-\tfrac{1}{\sqrt{3}}\sinh s,-\tfrac{1}{\sqrt{3}}\cosh s,-\tfrac{\sqrt{2}}{\sqrt{3}}\sin s,-\tfrac{\sqrt{2}}{\sqrt{3}}\cos s)\\&=(0,0,0,0)\end{aligned}$$

and radius

$$r=\sqrt{\frac{1}{\kappa^2}+\frac{\tau^2}{\kappa^2\sigma^2}}=\sqrt{3}.$$

Theorem 3.12. *Let $\gamma:I\subset\mathbb{R}\to\mathbb{E}_1^4$ be a unit speed nonlightlike curve with nonzero constant curvature function κ, first torsion function σ and the character of Frenet-Serret frame $(\varepsilon_1,1,\varepsilon_3,\varepsilon_4)$. If the first torsion τ is zero, then the curve γ is a part of a Lorentzian circle with radius $\frac{1}{\kappa}$.*

Proof. Assume that γ is a unit speed nonlightlike curve with nonzero constant κ, σ and $\tau=0$. As a result of Theorem 3.3, γ is a plane curve. If the center of the Lorentzian circle P_0 is taken as

$$P_0=\gamma+\varepsilon_1\frac{1}{\kappa}N_1$$

then we get

$$P_0'=T+\varepsilon_1\frac{1}{\kappa}(-\varepsilon_1\kappa T)=T-T=0.$$

Hence, the center P_0 of the circle is a constant. Moreover, we have

$$\langle\gamma-P_0,\gamma-P_0\rangle_{\mathbb{L}}=\left\langle-\varepsilon_1\frac{1}{\kappa}N_1,-\varepsilon_1\frac{1}{\kappa}N_1\right\rangle_{\mathbb{L}}=\frac{1}{\kappa^2}.$$

□

In the study [10], it is stated that a regular unit speed curve in Euclidean four space $\mathbb{E}^4$ with nonzero curvature and torsion functions is a helix if and only if

$$\frac{\kappa^2}{\tau^2}+[\frac{1}{\sigma}(\frac{\kappa}{\tau})']^2$$

is a constant. On the other hand, spacelike helices are investigated in Minkowski space-time in [11]. Moreover, the differential equations of characterizations of spacelike helices are also found in [11]. Now, the following theorem gives the necessary and sufficient condition of unit speed nonlightlike curve with nonzero curvature, torsion functions and the character of Frenet-Serret frame $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ to be a helix in Minkowski space-time.

Theorem 3.13. *Let $\gamma:I\subset\mathbb{R}\to\mathbb{E}_1^4$ be a unit speed nonlightlike curve with nonzero curvature, torsion functions and the character of Frenet-Serret frame $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$. The curve γ is a helix if and only if*

$$\begin{aligned}(\frac{\kappa}{\tau})'(1-\varepsilon_2\varepsilon_3)&=0,\\ \varepsilon_4\frac{\kappa\sigma}{\tau}+\varepsilon_3(\frac{1}{\sigma}(\frac{\kappa}{\tau})')'&=0.\end{aligned}$$

Proof. Suppose that $\gamma:I\subset\mathbb{R}\to\mathbb{E}_1^4$ is a given unit speed nonlightlike curve with nonzero curvature, torsion functions and the character of Frenet-Serret frame $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$. Assume that γ is a helix. Then there exists a constant vector U such that

$$\langle T,U\rangle_{\mathbb{L}}=c \tag{3.4}$$

where c is a real constant $\langle U,U\rangle_{\mathbb{L}}=\varepsilon$. Here, we may choose ε as 1, -1 or 0. Differentiating the Equation 3.4 with respect to s and using the Frenet-Serret formula in 2.1, we get

$$\langle N_1,U\rangle_{\mathbb{L}}=0. \tag{3.5}$$

Therefore U is in the subspace $Span\{T,N_2,N_3\}$ and can be written as follows:

$$U=\varepsilon_1cT+gN_2+hN_3.$$

Differentiating Equation 3.5 gives that

$$g=\varepsilon_3\langle N_2,U\rangle_{\mathbb{L}}=\varepsilon_1 c\frac{\kappa}{\tau}. \tag{3.6}$$

If we differentiate Equation 3.6, then we obtain

$$h=\varepsilon_4\langle N_3,U\rangle_{\mathbb{L}}=\varepsilon_1\varepsilon_2\frac{c}{\sigma}(\frac{\kappa}{\tau})'.$$

That is

$$U=\varepsilon_1 c(T+\frac{\kappa}{\tau}N_2+\varepsilon_2\frac{1}{\sigma}(\frac{\kappa}{\tau})'N_3).$$

Since U is a constant vector field, then we obtain

$$U'=\varepsilon_1 c[(\frac{\kappa}{\tau})'(1-\varepsilon_2\varepsilon_3)N_2+(\varepsilon_4\frac{\kappa\sigma}{\tau}+\varepsilon_3(\frac{1}{\sigma}(\frac{\kappa}{\tau})')')N_3]=0$$

This implies

$$(\frac{\kappa}{\tau})'(1-\varepsilon_2\varepsilon_3)=0,$$

$$\varepsilon_4\frac{\kappa\sigma}{\tau}+\varepsilon_3(\frac{1}{\sigma}(\frac{\kappa}{\tau})')'=0.$$

Conversely, suppose that above relations are satisfied. If we chose vector U as

$$U=\varepsilon_1 c(T+\frac{\kappa}{\tau}N_2+\varepsilon_2\frac{1}{\sigma}(\frac{\kappa}{\tau})'N_3)$$

where c is an arbitrary real constant, then we see that

$\langle T,U\rangle_{\mathbb{L}}=c$ and $U'=0$

which means that γ is a helix. □

Theorem 3.14. *Let $\gamma: I\subset\mathbb{R}\to\mathbb{E}_1^4$ be a unit speed timelike curve with nonzero curvature, torsion functions. The curve γ is a helix if and only if*

$$\frac{\kappa\sigma}{\tau}+(\frac{1}{\sigma}(\frac{\kappa}{\tau})')'=0.$$

4. Conclusion

For a unit speed nonlightlike curve $\gamma: I\subset\mathbb{R}\to\mathbb{E}_1^4$ with the Frenet vectors $\{T,N_1,N_2,N_3\}$ and the character of Frenet-Serret frame $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$, Frenet–Serret formulas can be given as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix}=\begin{bmatrix} 0 & \varepsilon_2\kappa & 0 & 0 \\ -\varepsilon_1\kappa & 0 & \varepsilon_3\tau & 0 \\ 0 & -\varepsilon_2\tau & 0 & \varepsilon_4\sigma \\ 0 & 0 & -\varepsilon_3\sigma & 0 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

where the functions κ, τ and σ are the curvature, first and second torsion of the curve, respectively. Some special curves can be characterized as follows:

- $\kappa=0$ if and only if γ is a straight line;
- If γ is a plane curve with $\kappa>0$ then $\tau=0$ or $\sigma=0$;
- If $\kappa>0$, $\sigma=$constant and $\tau=0$ then γ is a plane curve;
- If $\kappa>0$, $\sigma=0$ then γ is a plane curve;
- The spacelike curve γ with nonzero κ, τ and σ is lying on $H_0^3(r)$ if and only if

  $$\varepsilon_2\frac{1}{\kappa^2}+\varepsilon_3(\frac{1}{\tau}(\frac{1}{\kappa})')^2+\varepsilon_4\frac{1}{\sigma^2}[\varepsilon_4(\frac{1}{\tau}(\frac{1}{\kappa})')'-\frac{\tau}{\kappa}]^2=-r^2$$

  for some constant $r\in\mathbb{R}^+$;
- If $\kappa=c_1$, $\tau=c_2$, $\sigma=c_3$ where $c_1,c_2,c_3\in\mathbb{R}_0$ and

  $$\varepsilon_2\frac{1}{\kappa^2}+\varepsilon_4\frac{\tau^2}{\kappa^2\sigma^2}<0$$

  then spacelike curve γ lies on a pseudohyperbolical space with center

  $$P_0=\gamma+\frac{1}{\kappa}N_1+\frac{\tau}{\kappa\sigma}N_3;$$

- The curve γ with nonzero κ, τ and σ is lying on $S_0^3(r)$ if and only if

  $$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_3 (\frac{1}{\tau}(\frac{1}{\kappa})')^2 + \varepsilon_4 \frac{1}{\sigma^2}[\varepsilon_4(\frac{1}{\tau}(\frac{1}{\kappa})')' - \varepsilon_1 \frac{\tau}{\kappa}]^2 = r^2$$

  for some constant $r \in \mathbb{R}^+$;
- If $\kappa = c_1$, $\tau = c_2$, $\sigma = c_3$ where $c_1, c_2, c_3 \in \mathbb{R}_0$ and

  $$\varepsilon_2 \frac{1}{\kappa^2} + \varepsilon_4 \frac{\tau^2}{\kappa^2 \sigma^2} > 0$$

  then γ lies on a Lorentzian sphere with center

  $$P_0 = \gamma + \varepsilon_1 \frac{1}{\kappa} N_1 + \varepsilon_1 \frac{\tau}{\kappa\sigma} N_3;$$
- If $\kappa = c_1$, $\tau = 0$, $\sigma = c_2$ where $c_1, c_2 \in \mathbb{R}_0$, then γ is a part of circle with center

  $$P_0 = \gamma + \varepsilon_1 \frac{1}{\kappa} N_1$$

  and radius

  $$r = \frac{1}{\kappa};$$
- Let $\kappa \neq 0$, $\tau \neq 0$ and $\sigma \neq 0$. γ is a helix if and only if the followings are satisfied

  $$(\frac{\kappa}{\tau})'(1 - \varepsilon_2 \varepsilon_3) = 0,$$

  $$\varepsilon_4 \frac{\kappa\sigma}{\tau} + \varepsilon_3 (\frac{1}{\sigma}(\frac{\kappa}{\tau})')' = 0.$$

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