Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 399 – 408
DOI : 10.15672/hujms.470975

Research Article

The strong convergence of a proximal point algorithm in complete CAT(0) metric spaces

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Abstract

In this paper, we consider a proximal point algorithm for finding zeros of maximal monotone operators in complete CAT(0) spaces. First, a necessary and sufficient condition is presented for the zero set of the operator to be nonempty. Afterwards, we prove that, under suitable conditions, the proposed algorithm converges strongly to the metric projection of some point onto the zero set of the involving maximal monotone operator.

Mathematics Subject Classification (2010). 47H04, 47H05, 26B25, 26E25, 90C33

Keywords. maximal monotone operator, proximal point algorithm, Hadamard space, nonexpansive mapping, resolvent operator

1. Introduction

Monotone operator theory plays an important role in nonlinear and convex analysis. It has an essential role in convex analysis, optimization, variational inequalities, semigroup theory and evolution equations.
Let H be a real Hilbert space and $A : H \to 2^H$ be a maximal monotone operator. Consider the following set-valued problem:

$$\text{find } x \in X \text{ such that } 0 \in A(x).$$

The proximal point algorithm (PPA) is the most popular method for solving this problem, which was introduced by Martinet [25] and Rockafellar [26]. Rockafellar [26] showed the weak convergence of the sequence generated by the proximal point algorithm to a zero of the maximal monotone operator in Hilbert spaces. Güler's counterexample [17] showed that the sequence generated by the proximal point algorithm does not necessarily converge strongly even if the maximal monotone operator is the subdifierential of a convex, proper and lower semicontinuous function. For more information on convergence results of proximal point algorithm, we refer the reader to [6,12,17,26] and the references therein. In this paper, the proximal point algorithm has been considerd in complete CAT(0) spaces. Our results extend the previous results in Hilbert spaces as well as the recent results on Hadamard manifolds (see, for example, [1, 23] and the references therein) to complete CAT(0) spaces. Recently, Heydari and Ranjbar [18] studied the convergence of a PPA for a maximal monotone operator A. They assumed that the zero set of A is nonempty. In this

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Received: 16.10.2018; Accepted: 16.12.2018

paper, we significantly improve their results by giving a necessary and sufficient condition for the zero set of A to be nonempty. Under suitable conditions, the strong convergence of the iterative sequence generated by the proposed PPA, which is the generalization of the PPA presented by Rouhani and Moradi [13] to CAT(0) spaces, is established. Finally, we present some applications of our results.

2. Preliminaries

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0, d(x, y)] \to X$ such that $c(0) = x, c(d(x, y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. The metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be an uniquely geodesic space if there is exactly one geodesic joining x and y for each $x, y \in X$. A metric space (X, d) is said to be a CAT(0) space if it is a geodesic space and satisfies the following inequality:
CN − inequality: If $x, y_0, y_1, y_2 \in X$ such that $d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2)$, then

$$d^2(x, y_0) \leq \tfrac{1}{2}d^2(x, y_1) + \tfrac{1}{2}d^2(x, y_2) - \tfrac{1}{4}d^2(y_1, y_2).$$

(see more details in [7,9,16,19]. Some examples of CAT(0) spaces are pre-Hilbert spaces [7], $\mathbb{R}$-trees [21], Euclidean buildings [8], the complex Hilbert ball with a hyperbolic metric [15], Hadamard manifolds and many others. A complete CAT(0) space is called a Hadamard space. For all x and y belong to a CAT(0) space X, we write $(1 - t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$.
To proceed, let us first recall the following technical lemma.

Lemma 2.1. ([11]) *Let (X, d) be a CAT(0) space. Then, for all $x, y, z \in X$ and all $t \in [0, 1]$:*
(1) $d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y)$,
(2) $d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z)$.
In addition, by using (1), we have

$$d(tx \oplus (1 - t)y, tx \oplus (1 - t)z) \leq (1 - t)d(y, z).$$

Lim [24] introduced a concept of convergence in complete CAT(0) spaces which is called $\triangle$-convergence:
let (x_n) be a bounded sequence in a complete CAT(0) space (X, d) and $x \in X$. Set $r(x, (x_n)) := \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of (x_n) is given by $r((x_n)) := \inf\{r(x, (x_n)) : x \in X\}$ and the asymptotic center of (x_n) is the set $A((x_n)) := \{x \in X : r(x, (x_n)) = r((x_n))\}$. It is known that in complete CAT(0) spaces, $A((x_n))$ consists of exactly one point [22]. A sequence (x_n) in the complete CAT(0) space (X, d) is said to be $\triangle$-convergent to $x \in X$ if $A((x_{n_k})) = \{x\}$ for every subsequence (x_{n_k}) of (x_n). The concept of $\triangle$-convergence has been studied by many authors (e.g. [11, 14]).
The concept of quasilinearization for CAT(0) space X has been introduced by Berg and Nikolaev [5]. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. Then the quasilinearization map $\langle .,. \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \tfrac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \quad (a, b, c, d \in X).$$

It can be easily verified that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b), \langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ are satisfied for all $a, b, c, d, e \in X$. Also, we can formally add compatible vectors, more precisely $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$, for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d), \quad (a, b, c, d \in X).$$

It is known ([5], Corollary 3) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.
The concept of dual space of a complete CAT(0) space X have been introduced by Ahmadi Kakavandi and Amini [3], based on Berg and Nikolaev's work [5], as follows.
Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (t \in \mathbb{R}, \;\; a, b, x \in X).$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b), (t \in \mathbb{R}, \;\; a, b \in X)$, where $L(\phi) = \sup\{\frac{\phi(x)-\phi(y)}{d(x,y)} : \; x, y \in X, \;\; x \neq y\}$ is the Lipschitz semi-norm for any function $\phi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R}\times X\times X$ is defined by

$$D((t, a, b),(s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \quad (t, s \in \mathbb{R}, \;\; a, b, c, d \in X).$$

For a Hadamard space (X, d), the pseudometric space $(\mathbb{R}\times X\times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$. It is obtained that $D((t, a, b),(s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x, y \in X$ ([3], Lemma 2.1). Then, D can impose an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : \;\; D((t, a, b),(s, c, d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : \;\; (t, a, b) \in \mathbb{R}\times X\times X\}$ is a metric space with metric $D([t\overrightarrow{ab}], [s\overrightarrow{cd}]) := D((t, a, b),(s, c, d))$, which is called the dual space of (X, d).
It is clear that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$. Fix $o \in X$, we write $0 = [\overrightarrow{oo}]$ as the zero of the dual space. In [3], it is shown that the dual of a closed and convex subset of Hilbert space H with nonempty interior is H and $t(b - a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R}, \;\; a, b \in H$. Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad (x^* = [t\overrightarrow{ab}] \in X, \;\; x, y \in X).$$

Also, we use the following notation:

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy} \rangle = \alpha\langle x^*, \overrightarrow{xy} \rangle + \beta\langle y^*, \overrightarrow{xy} \rangle, \quad (\alpha, \beta \in \mathbb{R}, \;\; x, y \in X, \;\; x^*, y^* \in X^*).$$

Introducing a dual for a CAT(0) space implies a concept of weak convergence with respect to the dual space which is named w-convergence in [3]. In [3], the authors also showed that w-convergence is stronger than $\triangle$-convergence.
Ahmadi Kakavandi in [2] presented an equivalent definition of w-convergence in complete CAT(0) spaces without using dual space, as follows:

Definition 2.2. ([2]) A sequence (x_n) in the complete CAT(0) space (X, d) is w-converges to $x \in X$ if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0$ for all $y \in X$.

The w-convergence is equivalent to the weak convergence in Hilbert space H. In fact, if $(. \;, .)$ is the inner product in Hilbert space H, then

$$2\langle \overrightarrow{xz}, \overrightarrow{xy} \rangle = d^2(x, y) + d^2(x, z) - d^2(z, y) = 2(x - z, x - y).$$

It must be noted that, any bounded sequence does not have a w-convergent subsequence. It is obvious that convergence with respect to the topology induced by the metric implies w-convergence. In [3], it has been shown that, w-convergence implies $\triangle$-convergence but the converse is not valid [2]. However Ahmadi Kakavandi [2] proved the following characterization of $\triangle$-convergence.

Lemma 2.3. ([2]) *A bounded sequence (x_n) in Hadamard space (X, d) is $\triangle$-convergent to $x \in X$ if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$ for all $y \in X$.*

3. Maximal monotone operators

Let X be a complete CAT(0) space with dual X^* and $A : X \to 2^{X^*}$ be a multi-valued operator with domain $D(A) := \{x \in X : \ A(x) \neq \emptyset\}$, range $R(A) := \cup_{x\in X} A(x)$, $A^{-1}(x^*) := \{x \in X : \ x^* \in A(x)\}$ and graph $gra(A) := \{(x, x^*) \in X \times X^* : \ x \in D(A), \ x^* \in A(x)\}$.
For the main results of this paper, we need the following definitions and useful propositions.

Definition 3.1. Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a multi-valued operator.

(i) A is said to be monotone if for all $x, y \in D(A), \ x^* \in A(x), \ y^* \in A(y)$,

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0.$$

(ii) A is said to be maximal monotone, if A is monotone and the graph of A is not properly contained in the graph of any other monotone operator, i.e. for any $(y, y^*) \in X \times X^*$, the inequality $\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$, for all $(x, x^*) \in gra(A)$, implies $y \in D(A)$ and $y^* \in A(y)$.

Definition 3.2. Let $\lambda > 0$. The resolvent and Yosida approximation of the multi-valued operator of $A : X \to 2^{X^*}$ of order λ are the multi-valued mappings $J_\lambda : X \to 2^X$ and $A_\lambda : X \to 2^{X^*}$ defined, respectively, by $J_\lambda(x) := \{z \in X : \ [\frac{1}{\lambda}\overrightarrow{zx}] \in A(z)\}$ and $A_\lambda(x) := \{[\frac{1}{\lambda}\overrightarrow{yx}] : \ y \in J_\lambda(x)\}$.

Definition 3.3. Let X be a Hadamard space with dual X^* and let $T : C \subset X \to X$ be a mapping. We say that T is firmly nonexpansive if $d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle$ for any $x, y \in X$.

By Cauchy-Schwarz inequality, it is clear that any firmly nonexpansive mapping T is nonexpansive.

Proposition 3.4. ([20]) *Let X be a Hadamard space with dual X^*. The mapping $T : C \subset X \to X$ is firmly nonexpansive if and only if*

$$\langle \overrightarrow{TxTy}, \overrightarrow{(Tx)x}\rangle + \langle \overrightarrow{TyTx}, \overrightarrow{(Ty)y}\rangle \leq 0, \quad \forall x, y \in C.$$

Proposition 3.5. ([10]) *Let C be a nonempty convex subset of a CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C(x)$ if and only if*

$$\langle \overrightarrow{xu}, \overrightarrow{yu}\rangle \leq 0, \quad \forall y \in C.$$

Proposition 3.6. ([20]) *Let X be a CAT(0) space and let $A : X \to 2^{X^*}$ be a multi-valued operator. Suppose that J_λ and A_λ are, respectively, resolvent and Yosida approximation of the operator A of order λ. Then*

(i) *For any $\lambda \geq 0$, $R(J_\lambda) \subset D(A)$, $F(J_\lambda) = A^{-1}(0) = A_\lambda^{-1}(0)$, where $R(J_\lambda)$ and $F(J_\lambda)$ are, respectively, the range and the fixed points set of J_λ,*
(ii) *If J_λ is single-valued, then A_λ is single-valued and $A_\lambda(x) \subset A(J_\lambda(x))$,*
(iii) *If A is monotone, then J_λ is single-valued and firmly nonexpansive mapping,*
(iv) *If A is monotone, then A_λ is a monotone operator,*
(v) *If A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda x, J_\mu x) \leq \frac{\mu-\lambda}{\mu+\lambda} d^2(x, J_\mu x)$, which implies that $d(x, J_\lambda(x)) \leq 2d(x, J_\mu(x))$.*

Remark 3.7. If A is a monotone operator on a CAT(0) space X, then, by parts (i) and (iii) of Proposition 3.6, $A^{-1}(0)$ is closed and convex.

Proposition 3.8. ([18]) *Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a multi-valued maximal monotone operator. Suppose that $(x_n, x_n^*) \in gra(A)$ for all $n \in \mathbb{N}$ such that (x_n) is a bounded sequence in X which is w-convergent to $x \in X$ and $(x_n^*) \subset X^*$ converges to $x^* \in X^*$ in the metric D, then $x^* \in A(x)$.*

4. Proximal point algorithm

One of the most important problems in monotone operator theory is finding a zero of a maximal monotone operator. This problem can be formulated in Hadamard spaces as follows:

$$\text{find } x \in X \text{ such that } 0 \in A(x),$$

where $A : X \to 2^{X^*}$ is a monotone operator on the Hadamard space X and 0 is the zero of the dual space X^*. We say that A satisfies the range condition if, for every $\lambda > 0$, $D(J_\lambda) = X$. It is well known that if A is a maximal monotone operator on a Hilbert space H, then $R(I+\lambda A) = H$ for all $\lambda > 0$, where I is the identity operator. Thus every maximal monotone operator A on a Hilbert space satisfies the range condition. Also as it has been shown in [19] that, if A is a maximal monotone operator on a Hadamard manifold, then A satisfies the range condition. We say that a Hadamard space X satisfies the condition Q if every bounded sequence in X has a subsequence that is w-convergent. Let $A : X \to 2^{X^*}$ be a multi-valued maximal monotone operator on the Hadamard space X with dual X^* that satisfies the range condition. Suppose that (λ_n) is a sequence of positive real numbers, (α_n) is a sequence in $]0,1[$ and $u \in X$ to be fixed. The proximal point algorithm for maximal monotone operator A in Hadamard space X, is the iterative sequence generated by

$$\frac{1}{\lambda_n}\overrightarrow{x_{n+1}(\alpha_n u \oplus (1-\alpha_n)x_n)} \in Ax_{n+1}, x_0 \in X, \tag{4.1}$$

which, by the definition of the resolvent operator, is equivalent to

$$x_{n+1} = J_{\lambda_n}(\alpha_n u \oplus (1-\alpha_n)x_n), x_0 \in X. \tag{4.2}$$

Note that the range condition of A and part (iii) of Proposition 3.3 guarantee the existence and well definedness of the sequence (x_n) in (4.1) or (4.2).
The inexact version of (4.2) can be formulated as follow

$$z_{n+1} = J_{\lambda_n}(\alpha_n u \oplus (1-\alpha_n)y_n), d(z_n, y_n) \le e_n, y_0 \in X, \tag{4.3}$$

where (e_n) is a sequence in $]0,\infty[$.

Lemma 4.1. *Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a multi-valued monotone operator which satisfies the range condition. Assume that the sequences $(x_n), (y_n)$ are generated by the algorithms (4.2) and (4.3), respectively, and that $\Sigma_{n=1}^{\infty} e_n < +\infty$. Then,*

(1) *the sequence (x_n) is bounded, if and only if the sequence (y_n) is bounded,*
(2) *if $A^{-1}(0) \neq \emptyset$ then, the sequence (x_n) converges strongly to $P_{A^{-1}(0)}u$, if and only if the sequence (y_n) converges strongly to $P_{A^{-1}(0)}u$.*

Proof. (1): For all $n \ge 0$,

$$\begin{aligned} d(x_n, y_n) \le & d(x_n, z_n) + d(z_n, y_n), \\ = & d\Big(J_{\lambda_{n-1}}(\alpha_{n-1}u \oplus (1-\alpha_{n-1})x_{n-1}), \\ & J_{\lambda_{n-1}}(\alpha_{n-1}u \oplus (1-\alpha_{n-1})y_{n-1})\Big) + d(z_n, y_n), \\ & \le (1-\alpha_{n-1})d(x_{n-1}, y_{n-1}) + e_n \\ & \le d(x_{n-1}, y_{n-1}) + e_n \\ & \vdots \\ & \le d(x_0, y_0) + \Sigma_{i=1}^{n} e_i < \infty. \end{aligned}$$

Thus the sequence (x_n) is bounded, if and only if the sequence (y_n) is bounded.
(2): For all $n \geq 0$, from (1) we have,

$$d(x_n, y_n) \leq (1-\alpha_{n-1})d(x_{n-1}, y_{n-1}) + e_n.$$

By letting $n \to \infty$ and by using (1), the sequence (x_n) converges strongly to $P_{A^{-1}(0)}u$, if and only if the sequence (y_n) converges strongly to $P_{A^{-1}(0)}u$. □

The proof of our main result is based on the following useful lemma. This lemma has been proved for convex functions and we refer the reader to [4] for its proof. But the following lemma extends this result for monotone operators.

Lemma 4.2. *Let X be a Hadamard space with dual X^* that satisfies the condition Q and $A : X \to 2^{X^*}$ be a multi-valued maximal monotone operator which satisfies the range condition and let $A^{-1}(0) \neq \emptyset$. Suppose that $u \in X$. Then $\lim_{t\to\infty} J_t u = p$, where $p = P_{A^{-1}(0)}u$.*

Proof. By using the Proposition 3.4 and the fact that the resolvent operator is firmly nonexpansive, for all $y \in X$ we have,

$$\langle \overrightarrow{J_t u J_t y}, \overrightarrow{(J_t u)u} \rangle + \langle \overrightarrow{J_t y J_t u}, \overrightarrow{(J_t y)y} \rangle \leq 0.$$

Hence for every $q \in A^{-1}(0)$,

$$\langle \overrightarrow{J_t u J_t q}, \overrightarrow{(J_t u)u} \rangle + \langle \overrightarrow{J_t q J_t u}, \overrightarrow{(J_t q)q} \rangle \leq 0.$$

Part (i) of Proposition 3.6 follows

$$\langle \overrightarrow{(J_t u)q}, \overrightarrow{(J_t u)u} \rangle + \langle \overrightarrow{q(J_t u)}, \overrightarrow{qq} \rangle \leq 0,$$

which implies that

$$\langle \overrightarrow{(J_t u)q}, \overrightarrow{(J_t u)u} \rangle \leq 0.$$

So

$$\langle \overrightarrow{(J_t u)u}, \overrightarrow{(J_t u)u} \rangle + \langle \overrightarrow{uq}, \overrightarrow{(J_t u)u} \rangle \leq 0.$$

By Cauchy-Schwarz inequality

$$d^2(J_t u, u) \leq \langle \overrightarrow{qu}, \overrightarrow{(J_t u)u} \rangle \leq d(u,q)d(J_t u, u),$$

and hence

$$d(J_t u, u) \leq d(u, q). \tag{4.4}$$

In particular, $(J_t u)$ is bounded. By the condition Q, there exists a sequence of $(J_t u)$ that is w-convergent. Suppose that (t_n) is a sequence in $]0, \infty[$ such that $\lim_{n\to\infty} t_n = \infty$ and $(J_{t_n} u)$ is w-convergent to some $z \in X$.
Since $\{d(J_{t_n} u, u)\}$ is bounded, $[\frac{1}{t_n}\overrightarrow{(J_{t_n} u)u}] \in A(J_{t_n} u)$ and $\lim_{n\to\infty} t_n = \infty$, then

$$\lim_{n\to\infty} D([\tfrac{1}{t_n}\overrightarrow{(J_{t_n} u)u}], 0) = \lim_{n\to\infty} \tfrac{1}{t_n} d(J_{t_n} u, u) = 0.$$

By using the Proposition 3.8,

$$z \in A^{-1}(0). \tag{4.5}$$

Since (J_{t_n}) is w-convergent, for every $y \in X$ we heve

$$\limsup_{n\to\infty} \langle \overrightarrow{z(J_{t_n} u)}, \overrightarrow{zy} \rangle = 0,$$

which implies that

$$\limsup_{n\to\infty} \langle \overrightarrow{z(J_{t_n} u)}, \overrightarrow{zu} \rangle = 0.$$

Thus

$$\limsup_{n\to\infty}[d^2(z,u)+d^2(J_{t_n}u,z)-d^2(J_{t_n}u,u)]=0. \tag{4.6}$$

Also by (4.4) and (4.5),

$$d^2(z,u)-d^2(J_{t_n}u,u)\geq 0, \tag{4.7}$$

Therefore by using the inequalities (4.6) and (4.7),

$$\limsup_{n\to\infty} d^2(J_{t_n}u,z)=0.$$

Thus, $(J_{t_n}u)$ converges strongly to z. On the other hand by (4.4),

$$d(u,z)\leq d(u,q),$$

for all $q\in A^{-1}(0)$, which implies that, $z=p=P_{A^{-1}(0)}u$. This completes the proof. □

In the following theorem, we give a necessary and sufficient condition for the zero set of A to be nonempty.

Theorem 4.3. *Let X be a Hadamard space with dual X^* that satisfies the condition Q and $A: X\to 2^{X^*}$ be a multi-valued maximal monotone operator which satisfies the range condition. If (x_n) and (y_n) are generated by the algorithms (4.2) and (4.3), respectively, such that, $\alpha_n\to 1$, $\lambda_n\to\infty$ and $\Sigma_{n=1}^{\infty}e_n<\infty$, then*
  (1) *$A^{-1}(0)\neq\emptyset$ if and only if (x_n) is bounded,*
  (2) *$A^{-1}(0)\neq\emptyset$ if and only if (y_n) is bounded.*

Proof. (1): Assume that $A^{-1}(0)\neq\emptyset$ and $q\in A^{-1}(0)$. From (4.2), and the fact that the resolvent operator is nonexpansive, for all $m\geq 0$ we have:

$$\begin{aligned} d(x_{m+1},q)&=d(x_{m+1},J_{\lambda_m}q)\\ &\leq d(\alpha_m u\oplus(1-\alpha_m)x_m,q)\\ &\leq \alpha_m d(u,q)+(1-\alpha_m)d(x_m,q). \end{aligned}$$

By the above inequality and using the induction, for all $m\geq 0$

$$d(x_{m+1},q)\leq\max\{d(u,q),d(x_1,q)\}.$$

This inequality shows that (x_n) is bounded.
Conversely, assume that (x_n) is bounded. By the condition Q, there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k+1}) is w-convergent to $x\in X$.
From (4.1), we get:

$$\frac{1}{\lambda_{n_k}}\overrightarrow{x_{n_k+1}(\alpha_{n_k}u\oplus(1-\alpha_{n_k})x_{n_k})}\in A(x_{n_k+1}),x_0\in X.$$

Since (x_{n_k}) is bounded and $\lambda_{n_k}\to\infty$, we have

$$\begin{aligned}\lim_{k\to\infty} D([\frac{1}{\lambda_{n_k}}\overrightarrow{(x_{n_k+1}(\alpha_{n_k}u\oplus(1-\alpha_{n_k})x_{n_k})}],0)&=\lim_{k\to\infty}\frac{1}{\lambda_{n_k}}d(x_{n_k+1},(\alpha_{n_k}u\oplus(1-\alpha_{n_k})x_{n_k}))\\ &=0.\end{aligned}$$

By using the Proposition 3.8, we have $x\in A^{-1}(0)$, and this completes the proof of (1).
(2): The result follows by using the Lemma 4.1 and (1). □

The following theorem extends the previous results given by Heydari and Ranjbar [18] who assumed that the zero set of A is nonempty.

Theorem 4.4. *Let X be a Hadamard space with dual X^* that satisfies the condition Q and $A: X\to 2^{X^*}$ be a multi-valued maximal monotone operator which satisfies the range condition. If (x_n) and (y_n) are generated by the algorithms (4.2) and (4.3), respectively, such that, $\alpha_n\to 1$, $\lambda_n\to\infty$ and $\Sigma_{n=1}^{\infty}e_n<\infty$, then*

(1) (x_n) *converges strongly to* $P_{A^{-1}(0)}u$ *if and only if* $A^{-1}(0) \neq \emptyset$,
(2) (y_n) *converges strongly to* $P_{A^{-1}(0)}u$ *if and only if* $A^{-1}(0) \neq \emptyset$.

Proof. (1): Assume that $A^{-1}(0) \neq \emptyset$ and $P_{A^{-1}(0)}u = p$. For all $n \geq 1$,

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_{n+1}, J_{\lambda_n}u) + d(J_{\lambda_n}u, p) \\ &\leq d(\alpha_n u \oplus (1-\alpha_n)x_n, u) + d(J_{\lambda_n}u, p) \\ &\leq (1-\alpha_n)d(x_n, u) + d(J_{\lambda_n}u, p). \end{aligned}$$

where the second inequality follows from (4.2) and the fact that the resolvent operator is nonexpansive. Now the result follows immediately by letting $n \to \infty$ in the above inequality and by using Lemma 4.2.
Conversely, if (x_n) be converges, then by using the Lemma 4.1, we conclude that $A^{-1}(0) \neq \emptyset$ and this completes the proof of (1).
(2): The result follows by using the Lemma 4.1 and (1). □

The following example shows that the condition $\lambda_n \to \infty$, is necessary in Theorem 4.4.

Example 4.5. Let $A : \mathbb{R} \to \mathbb{R}$ be defined by $A(x) = x$. Let $u = 1$, $x_0 = 0$, $\alpha_n = \frac{n}{n+1}$ and $\lambda_n \in (0, \infty)$, for all $n \geq 0$ be such that $\lambda_n = \lambda$, for some $\lambda \in (0, \infty)$. Obviously $A^{-1}(0) = \{0\}$, and

$$x_{n+1} = J_{\lambda_n}(\alpha_n u + (1-\alpha_n)x_n),$$

which implies that

$$x_{n+1} = \frac{\frac{n}{n+1} + \frac{1}{n+1}x_n}{1+\lambda}.$$

Since (x_n) is bounded $(0 \leq x_n \leq \frac{4}{1+\lambda})$, by above equality, we have

$$\lim_{n\to\infty} x_n = \frac{1}{1+\lambda} \notin A^{-1}(0).$$

5. Application to minimization problem

One of the most important examples of maximal monotone operators are subdifferentials of convex, proper and lower semicontinuous functions. Let (X, d) be a Hadamard space and $f : X \to (-\infty, \infty]$ be a convex, proper and lower semicontinuous function. We know from [3] that the subdifferential $A = \partial f$ of f is a maximal monotone operator which satisfies the range condition and the zero set of A coincides with the set of minimizers of f.

Theorem 5.1. *Let X be a Hadamard space with dual X^* that satisfies the condition Q and $f : X \to (-\infty, \infty]$ be a convex, proper and lower semicontinuous function. If $A = \partial f$ and (x_n) generated by (4.2) with $A = \partial f$, such that $\alpha_n \to 1$ and $\lambda_n \to \infty$, then:*

(1) *$argminf \neq \emptyset$ if and only if (x_n) is bounded,*
(2) *if $argminf \neq \emptyset$ then (x_n) converges strongly to $P_{A^{-1}(0)}u$, the metric projection of u onto $A^{-1}(0) = argminf$.*

Proof. The result (1) follows from Theorem 4.3 and The result (2) follows from Theorem 4.4 . □

6. Application to fixed point problem

Let (X, d) be a Hadamard space and $T : X \to X$ is a nonexpansive mapping, i.e $d(Tx, Ty) \leq d(x, y)$ for each $x, y \in X$. We know from [20] that the operator $A(z) = [\overrightarrow{Tzz}]$ is a monotone operator in X and the zero set of A coincides with the set of fixed points of T.

Theorem 6.1. *Let X be a Hadamard space with dual X^* that satisfies the condition Q and $T : X \to X$ be a nonexpansive mapping. If (x_n) generated by (4.2) with $A(z) = [\overrightarrow{Tzz}]$, such that A satisfies the range condition, $\alpha_n \to 1$ and $\lambda_n \to \infty$, then:*

(1) *the set of fixed points of T is nonempty if and only if (x_n) is bounded,*
(2) *if the set of fixed points of T is nonempty then (x_n) converges strongly to $P_{A^{-1}(0)}u$, the metric projection of u onto the set of fixed points of T.*

Proof. The result (1) follows from Theorem 4.3 and The result (2) follows from Theorem 4.4 . □

Acknowledgment. The authors are grateful to the referees for valuable suggestions leading to the improvement of the paper.

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