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Positive Linear Operators Preserving τ and τ^2

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ABSTRACT. In the paper we introduce a general class of linear positive approximation processes defined on bounded and unbounded intervals designed using an appropriate function. Voronovskaya type theorems are given for these new constructions. Some examples including well known operators are presented.

Keywords: Generalized operators, Voronovskaya theorem.

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1. INTRODUCTION

In the theory of approximation by linear positive operators (l.p.o) Korovkin famous theorem has a crucial role to determine whether the corresponding sequence of l.p.o converges to the identity operator. However, Korovkin theorem for a sequence of l.p.o requires uniform convergence on an extended complete Tchebychev system, in special, the set of test functions $e_i(t) = t^i$, $i = 0, 1, 2$. In [6], to obtain better error estimation, J. P. King introduced and studied a generalization of the classical Bernstein operators. These operators preserve the test functions e_0 and e_2, while the classical Bernstein operators preserve the test functions e_0 and e_1. Starting from this approximation process King's idea has been successfully applied to several well known sequences of operators. In [5], the authors introduced the sequence of operators B_n^τ by

$$B_n^\tau(f;x) = \sum_{k=0}^{n} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right)\binom{n}{k}\tau^k(x)\left(1-\tau(x)\right)^{n-k}, \; x \in [0,1], \; n \in \mathbb{N},$$

which is a new form of well-known Bernstein operators, where $\tau \in C[0,1]$ is a strictly increasing function , $\tau(0) = 0$, $\tau(1) = 1$. Shape preserving and convergence properties as well as the asymptotic behavior and saturation for the sequence (B_n^τ) were deeply studied using the test functions $\{1, \tau, \tau^2\}$. Durrmeyer version of the operators B_n^τ was introduced and studied in [1]. A similar idea was used for the operators defined on unbounded intervals given in [2].
In this short note, we introduce linear positive operators defined on bounded and unbounded intervals that preserve the functions τ and τ^2 such that $\tau \in C[0,1]$ is strictly increasing, $\tau(0) = 0$, $\tau(1) = 1$ (for the operators defined on the unbounded interval, we consider the function $\rho \in C[0,\infty)$ such that $\rho(0) = 0$ and $\rho'(x) > 0$ for $x \in [0,\infty)$). Then, we give a Voronovskaya type theorem for our general operators. Some examples including very well known operators are also obtained.

2. GENERALIZED OPERATORS

Let $L_n : C[0,1] \to C[0,1]$ be a sequence of l.p.o such that $L_n e_0 = e_0$ and $L_n e_1 = e_1$.

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Let $\tau : [0,1] \to [m, M]$ be continuous such that $0 < m < M$, $\tau^{'}(x) > 0$ for $x \in [0,1]$, $\tau(0) = m$ and $\tau(1) = M$. For any $f \in C[0,1]$ consider the function $\frac{f\circ\tau^{-1}}{e_1}$ such that

$$\frac{f \circ \tau^{-1}}{e_1}(m + (M-m)t) = \frac{f\left(\tau^{-1}(m + (M-m)t)\right)}{m + (M-m)t}, \quad t \in [0,1].$$

For $x \in [0,1]$ and $f \in C[0,1]$ consider the operators

$$\left(V_n^L f\right)(x) = \tau(x) L_n\left(\frac{f \circ \tau^{-1}}{e_1}(m + (M-m)t); \frac{\tau(x) - m}{M - m}\right).$$

It is obvious that

$$V_n^L \tau(x) = \tau(x) \quad \text{and} \quad V_n^L \tau^2(x) = \tau^2(x).$$

2.1. **Examples.**

(1) Let $\tau(x) = x + 1$ and $L_n = B_n$, where (B_n) is the sequence of Bernstein operators. For $m = 1$ and $M = 2$,

$$V_n^B f(x) = (x+1) B_n\left(\frac{f(t)}{1+t}, x\right).$$

(2) Let $\tau(x) = e^{\mu x}$, $\mu > 0$ and $L_n = B_n$, where (B_n) is the sequence of Bernstein operators. For $m = 1$ and $M = e^{\mu}$,

$$V_n^B f(x) = e^{\mu x} B_n\left(\frac{f\left(\frac{1}{\mu}\log\left(1 + (e^{\mu} - 1)t\right)\right)}{1 + (e^{\mu} - 1)t}; \frac{e^{\mu x} - 1}{e^{\mu} - 1}\right).$$

Let $K_n : C[0,\infty) \to C[0,\infty)$ be a sequence of l.p.o such that $K_n e_0 = e_0$ and $K_n e_1 = e_1$.
Let $\rho : [0,\infty) \to [m,\infty)$ be continuous such that $m > 0$, $\rho^{'}(x) > 0$ for $x \in [0,\infty)$ and $\rho(0) = m$.
For $f \in C[0,\infty)$, consider the function $\frac{f\circ\rho^{-1}}{e_1}$ such that

$$\frac{f \circ \rho^{-1}}{e_1}(t + m) = \frac{f\left(\rho^{-1}(t+m)\right)}{t + m}, \quad t \in [0,\infty).$$

For $x \in [0,\infty)$, consider the operators

$$\left(U_n^K f\right)(x) = \rho(x) K_n\left(\frac{f\left(\rho^{-1}(m+t)\right)}{m+t}; \rho(x) - m\right).$$

It is obvious that

$$\left(U_n^K \rho\right)(x) = \rho(x) \quad \text{and} \quad \left(U_n^K \rho^2\right)(x) = \rho^2(x).$$

2.2. **Examples.**

(1) Let $\rho(x) = e^{\mu x} + 1$, $x \geq 0$, $\mu > 0$ and $K_n = S_n$, where (S_n) is the sequence of Szász-Mirakyan operators. For $m = 2$, $\rho^{-1} : [2,\infty) \to [0,\infty)$, $\rho^{-1}(x) = \frac{1}{\mu}\log(x-1)$ and $x \in [2,\infty)$,

$$\left(U_n^S f\right)(x) = (e^{\mu x} + 1) S_n\left(\frac{f\left(\frac{1}{\mu}\log(1+t)\right)}{2+t}, e^{\mu x} - 1\right).$$

(2) Let $\rho(x) = e^{\mu x}$ and $K_n = T_n$, where (T_n) is the sequence of Baskakov operators. For $x \geq 0$, $m = 1$ and $\mu > 0$,

$$\left(U_n^B f\right)(x) = e^{\mu x} T_n \left(\frac{f\left(\frac{1}{\mu}\log(1+t)\right)}{1+t}; e^{\mu x} - 1 \right).$$

3. Transferring the Voronovskaya Result

Theorem 3.1. *Let $f \in C[0,1]$ with $f''(t)$ finite at any $t \in [0,1]$. Suppose that $L_n e_0 = e_0$, $L_n e_1 = e_1$ and*

$$V_n f(x) = \tau(x) L_n \left(\frac{f \circ \tau^{-1}}{e_1} (m + (M-m)t); \frac{\tau(x) - m}{M - m} \right).$$

If there exists $\alpha \in C[0,1]$ such that

$$\lim_{n\to\infty} n \left(L_n f(t) - f(t)\right) = \alpha(t) f''(t),$$

then we have

$$\begin{aligned} &\lim_{n\to\infty} n \left(V_n f(x) - f(x)\right) \\ = &\frac{(M-m)^2 \alpha\left(\frac{\tau(x)-m}{M-m}\right)}{\tau^2(x)\left(\tau'(x)\right)^3} \Big[\tau'(x)\tau^2(x) f''(x) - \tau(x)\left(\tau(x)\tau''(x) + 2\left(\tau'(x)\right)^2\right) f'(x) \\ + &2\left(\tau'(x)\right)^3 f(x)\Big]. \end{aligned}$$

Proof. We have

$$\begin{aligned} &n \left(V_n f(x) - f(x)\right) \\ = &n\tau(x) \left[L_n \left(\frac{f \circ \tau^{-1}}{e_1} (m + (M-m)t); \frac{\tau(x) - m}{M - m} \right) - \frac{f(x)}{\tau(x)} \right], \\ = &n\tau(x) \left[L_n \left(\frac{f \circ \tau^{-1}}{e_1} (m + (M-m)t); \frac{\tau(x) - m}{M - m} \right) - \frac{f \circ \tau^{-1}}{e_1} (m + (M-m)t) \Big|_{\left(\frac{\tau(x)-m}{M-m}\right)} \right]. \end{aligned}$$

Thus we have from the hypothesis that

$$\lim_{n\to\infty} n \left(V_n f(x) - f(x)\right) = \tau(x)\alpha\left(\frac{\tau(x)-m}{M-m}\right) \frac{d^2}{du^2} \left(\frac{f\left(\tau^{-1}(m + (M-m)u)\right)}{m + (M-m)u} \right) \Bigg|_{u = \frac{\tau(x)-m}{M-m}},$$

with $u = \frac{\tau(x)-m}{M-m}$ and $\frac{dx}{du} = \frac{M-m}{\tau'(x)}$.

It is obvious that

$$\begin{aligned} \frac{d}{du} \left(\frac{f\left(\tau^{-1}(m + (M-m)u)\right)}{m + (M-m)u} \right) \Bigg|_{u = \frac{\tau(x)-m}{M-m}} &= \frac{d}{du} \left(\frac{f \circ \tau^{-1}}{e_1} (m + (M-m))(u) \right) \Bigg|_{u = \frac{\tau(x)-m}{M-m}}, \\ &= \frac{dx}{du} \frac{d}{dx} \left(\frac{f(x)}{\tau(x)} \right), \\ &= \frac{M-m}{\tau'(x)} \frac{d}{dx} \left(\frac{f(x)}{\tau(x)} \right), \\ &= (M-m) \frac{f'(x)\tau(x) - \tau'(x) f(x)}{\tau'(x)\tau^2(x)}, \end{aligned}$$

and

$$\begin{aligned}
&\frac{d^2}{du^2}\left(\frac{f\left(\tau^{-1}\left(m+(M-m)\,u\right)\right)}{m+(M-m)\,u}\right)\Bigg|_{u=\frac{\tau(x)-m}{M-m}} \\
=\ & \frac{M-m}{\tau'(x)}\frac{d}{dx}\left((M-m)\,\frac{f'(x)\,\tau(x)-f(x)\,\tau'(x)}{\tau'(x)\,\tau^2(x)}\right), \\
=\ & \frac{(M-m)^2}{\left(\tau(x)\,\tau'(x)\right)^3}\left[\tau'(x)\,\tau^2(x)\,f''(x)-\tau(x)\left(\tau(x)\,\tau''(x)+2\left(\tau'(x)\right)^2\right)f'(x)\right. \\
+\ & \left. 2\left(\tau'(x)\right)^3 f(x)\right].
\end{aligned}$$

Hence we have the desired result. □

Corollary 3.1. *Let $\tau(x)=e^{\mu x}$ and $L_n=B_n$, where (B_n) is the sequence of Bernstein operators. For $m=1$ and $M=e^{\mu}$, we get*

$$\lim_{n\to\infty} n\left(V_n f(x)-f(x)\right)=\frac{\left(e^{\mu x}-1\right)\left(e^{\mu}-e^{\mu x}\right)}{2\mu^2 e^{2\mu x}}\left(f''(x)-3\mu f'(x)+2\mu^2 f(x)\right).$$

Corollary 3.2. *Let $\tau(x)=x+1$ and $L_n=B_n$, where (B_n) is the sequence of Bernstein operators. For $m=1$ and $M=2$, we obtain*

$$\lim_{n\to\infty} n\left(V_n f(x)-f(x)\right)=\frac{x(1-x)}{2}\left(f''(x)-\frac{2}{x+1}f'(x)+\frac{2}{(x+1)^2}f(x)\right).$$

Theorem 3.2. *Let $f\in C[0,\infty)$ with $f''(t)$ finite, $t\in[0,\infty)$. Suppose that $K_n e_0=e_0$, $K_n e_1=e_1$ and*

$$U_n f(x)=\rho(x)\,K_n\left(\frac{f\circ\rho^{-1}}{e_1}(m+t)\,;\rho(x)-m\right).$$

If there exists $\gamma\in C[0,\infty)$ such that

$$\lim_{n\to\infty} n\left(K_n f(t)-f(t)\right)=\gamma(t)\,f''(t),$$

then we have

$$\begin{aligned}
&\lim_{n\to\infty} n\left(U_n f(x)-f(x)\right) \\
=\ & \frac{\gamma\left(\rho(x)-m\right)}{\rho^2(x)\left(\rho'(x)\right)^3}\left[\rho'(x)\,\rho^2(x)\,f''(x)-\rho(x)\left(\rho(x)\,\rho''(x)+2\left(\rho'(x)\right)^2\right)f'(x)\right. \\
+\ & \left. 2\left(\rho'(x)\right)^3 f(x)\right].
\end{aligned}$$

Proof. The proof of this theorem is similar to that of Theorem 1. □

Corollary 3.3. *Let $\rho(x)=e^{\mu x}+1$, $x\geq 0$, $\mu>0$ and $K_n=S_n$, where (S_n) is the sequence of Szász-Mirakyan operators. For $m=2$, we have*

$$\lim_{n\to\infty} n\left(U_n f(x)-f(x)\right)=\frac{e^{\mu x}-1}{2\mu^2 e^{2\mu x}}\left(f''(x)-\mu\frac{3e^{\mu x}+1}{e^{\mu x}+1}f'(x)+2\mu^2\frac{e^{2\mu x}}{\left(e^{\mu x}+1\right)^2}f(x)\right).$$

Corollary 3.4. *Let $\rho(x) = e^{\mu x}$, and $K_n = T_n$, where (T_n) is the sequence of Baskakov operators. For $x \geq 0$, $\mu > 0$ and $m = 1$, we get*

$$\lim_{n\to\infty} n\left(U_n f(x) - f(x)\right) = \frac{e^{\mu x}-1}{2\mu^2 e^{\mu x}}\left(f''(x) - 3\mu f'(x) + 2\mu^2 f(x)\right).$$

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