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Hyperspaces of Superparacompact Spaces and Continuous Maps

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Abstract

In the present paper we establish that the space $\exp_\beta X$ of compact subsets of a Tychonoff space X is superparacompact iff X is so. Further, we prove the Tychonoff map $\exp_\beta f: \exp_\beta X \to \exp_\beta Y$ is superparacompact iff a given map $f: X \to Y$ is superparacompact.

1. Introduction

In the present paper under space we mean a topological T_1-space, under compact a Hausdorff compact space and under map a continuous map.

A collection ω of subsets of a set X is said [1] to be *star-countable* (respectively, *star-finite*) if each element of ω intersects at most a countable (respectively, finite) set of elements of ω. A collection ω of subsets of a set X *refines* a collection Ω of subsets of X if for each element $A \in \omega$ there is an element $B \in \Omega$ such that $A \subset B$. They also say that ω is *a refinement* of Ω.

A finite sequence of subsets $M_0, ..., M_s$ of a set X is [2] a *chain* connecting sets M_0 and M_s, if $M_{i-1} \cap M_i \neq \varnothing$ for $i = 1, ..., s$. A collection ω of subsets of a set X is said to be *connected* if for any pair of sets $M, M' \subset X$ there exists a chain in ω connecting sets M and M'. The maximal connected subcollections of ω are called *components* of ω. A star-finite open cover of a space X is said to be *a finite-component cover* if the number of elements of each component is finite. A space X is said to be *superparacompact* if every open cover of X has a finite-component cover which refines it.

Note that any compact space is superparacompact, and any superparacompact space is strongly paracompact. Infinite discrete space is superparacompact, but it is not compact. Real line is strongly paracompact, but it is not superparacompact.

For a collection $\omega = \{O_\alpha : \alpha \in A\}$ of subsets of a space X we suppose $[\omega] = [\omega]_X = \{[O_\alpha]_X : \alpha \in A\}$. For a space X, its some subspace W and a set $B \subset X \setminus W$ they say [2] that an open cover λ of the space W pricks out the set B in X if $B \cap (\cup[\lambda]_X) = \varnothing$.

The following criterion plays a key role in investigation the class of superparacompact spaces.

Theorem 1.1. *[3] A Tychonoff space X is superparacompact iff for every closed set F in βX lying in the growth $\beta X \setminus X$ there exists a finite-component cover λ of X pricking out F in βX (i. e. $F \cap (\cup[\lambda]_{\beta X}) = \varnothing$).*

D.Buhagiar and T.Miwa offered the following criterion of superparacompactness.

Theorem 1.2. *[4] A Tychonoff space X is superparacompact iff for every closed set F in perfect compactification bX lying in the growth $bX \setminus X$ there is a finite-component cover λ of X pricking out F in bX (i. e. $F \cap (\cup[\lambda]_{bX}) = \varnothing$).*

Let us recall a notion of the perfect compactification. For a topological space X and its subset A a set $Fr_X A = [A]_X \cap [X \setminus A]_X = [A]_X \setminus Int_X A$ is called [5] a boundary of A.

Let vX be a compact extension of a Tychonoff space X. If $H \subset X$ is an open set in X, then by $O(H)$ (or by $O_{vX}(H)$) we denote a maximal open set in vX satisfying $O_{vX}(H) \cap X = H$. It is easy to see that

$$O_{vX}(H) = \bigcup_{\substack{\Gamma \in \tau_{vX}, \\ \Gamma \cap X = H}} \Gamma,$$

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where $\tau_{\nu X}$ is the topology of the space νX.
A compactification νX of a Tychonoff space X is called *perfect with respect to an open set* H in X, if the equality $[Fr_X H]_{\nu X} = Fr_{\nu X} O_{\nu X}(X)$ holds. If νX is perfect with respect to every open set in X, then it is called *a perfect compactification* of the space X ([1], P. 232). A compactification νX of a space X is perfect iff for any two disjoint open sets U_1 and U_2 in X the equality $O(U_1 \bigcup U_2) = O(U_1) \bigcup O(U_2)$ is carried out. The Stone-Cĕch compactification βX of X is perfect. The equality $O(U_1 \bigcup U_2) = O(U_1) \bigcup O(U_2)$ is satisfied for every pair of open sets U_1 and U_2 in X iff X is normal, and the compactification νX coincides with the Stone-Cĕch compactification βX, i. e. $\nu X \cong \beta X$.
Let X be a space. By $\exp X$ we denote a set of all nonempty closed subsets of X. A family of sets of the view

$$O\langle U_1, ..., U_n\rangle = \{F \in \exp X : \ F \subset \bigcup_{i=1}^{n} U_n, F \cap U_1 \neq \varnothing, ..., F \cap U_n \neq \varnothing\}$$

forms a base of a topology on $\exp X$, where $U_1, \ldots, U_n$ are open nonempty sets in X. This topology is called *the Vietoris topology*. A space $\exp X$ equipped with Vietoris topology is called *hyperspace* of X. For a compact space X its hyperspace $\exp X$ is also a compact space (for details, see [6], [7], [8]).
Note for any space X it is well known that

$$[O\langle U_1, ..., U_n\rangle]_{\exp X} = O\left\langle [U_1]_X, ..., [U_n]_X\right\rangle.$$

Let $f : X \to Y$ be continuous map of compacts, $F \in \exp X$. We put

$$(\exp f)(F) = f(F).$$

This equality defines a map $\exp f : \exp X \to \exp Y$. For a continuous map f the map $\exp f$ is continuous. Really, it follows from the formula

$$(\exp f)^{-1} O\langle U_1, ..., U_m\rangle = O\langle f^{-1}(U_1), ..., f^{-1}(U_m)\rangle$$

what one can check directly. Note that if $f : X \to Y$ is an epimorphism, then $\exp f$ is also an epimorphism.
For a Tychonoff space X we put

$$\exp_\beta X = \{F \in \exp \beta X : F \subset X\}.$$

It is clear, that $\exp_\beta X \subset \exp X$. Consider the set $\exp_\beta X$ as a subspace of the space $\exp X$. For a Tychonoff spaces X the space $\exp_\beta X$ is also a Tychonoff space with respect to the induced topology.
For a continuous map $f : X \to Y$ of Tychonoff spaces we put

$$\exp_\beta f = \left.(\exp \beta f)\right|_{\exp_\beta X},$$

where $\beta f : \beta X \to \beta Y$ is the Stone-Cĕch compactification [5] of f (it is unique).
As it is well-known the action of functors on various categories of topological spaces and their continuous maps is one of the main problems of theory of covariant functors, in the present paper we investigate the action of the functor exp (the construction of taking of a hyperspace of a given space) on superparacompact spaces (section 2) and superparacompact maps (section 3).

2. Hyperspace of superparacompact spaces

It is well known that for a Tychonoff space X the set $\exp_\beta X$ is everywhere dense in $\exp \beta X$, i. e. $\exp \beta X$ is a compactification of the space $\exp_\beta X$. We claim $\exp \beta X$ is a perfect compactification of $\exp_\beta X$. At first we will prove the following technical statement.

Lemma 2.1. *Let γX be a compact extension of a space X and, V and W be disjoint open sets in γX. Let $V^X = X \cap V$ and $W^X = X \cap W$. Then the following equality is true:*

$$[X \setminus V^X]_{\gamma X} \cap [X \setminus W^X]_{\gamma X} = [X \setminus (V^X \cup W^X)]_{\gamma X}.$$

Proof. It is clear that $[X \setminus V^X]_{\gamma X} \cap [X \setminus W^X]_{\gamma X} \supset [X \setminus (V^X \cup W^X)]_{\gamma X}$. Let $x \in [X \setminus V^X]_{\gamma X} \cap [X \setminus W^X]_{\gamma X}$. Then each open neighbourhood Ox in γX of x intersects with the sets $X \setminus V^X$ and $X \setminus W^X$. Hence, $Ox \not\subset V^X$ and $Ox \not\subset W^X$. Therefore, since $V^X \cap W^X = \varnothing$, we have $Ox \not\subset V^X \cup W^X$, i. e. $Ox \cap X \setminus (V^X \cup W^X) \neq \varnothing$. By virtue of arbitrariness of the neighbourhood Ox we conclude that $x \in [X \setminus (V^X \cup W^X)]_{\gamma X}$. □

Theorem 2.2. *For a Tychonoff space X the space $\exp \beta X$ is a perfect compactification of the space $\exp_\beta X$.*

Proof. It is enough to consider basic open sets. Let U_1 and U_2 be disjoint open sets in X. Since βX is perfect compactification of X we have $O_{\beta X}(U_1 \cup U_2) = O_{\beta X}(U_1) \cup O_{\beta X}(U_2)$. Consider open sets

$$O\langle U_i\rangle = \{F : F \in \exp_\beta X, F \subset U_i\}, \qquad i = 1,\ 2$$

in $\exp_\beta X$. It is clear, that $O\langle U_1\rangle \cap O\langle U_2\rangle = \varnothing$. We will show that

$$O_{\exp \beta X}(O\langle U_1\rangle \cup O\langle U_2\rangle) = O_{\exp \beta X}(O\langle U_1\rangle) \cup O_{\exp \beta X}(O\langle U_2\rangle).$$

The inclusion $\supset$ follows from the definition of the set $O(H)$ (see [1], P. 234). That is why it is enough to show the inverse inclusion. Let $\Phi \subset \beta X$ be a closed set such that $\Phi \notin O_{\exp \beta X}(O\langle U_1\rangle) \cup O_{\exp \beta X}(O\langle U_2\rangle)$. Then $\Phi \in \exp \beta X \setminus O_{\exp \beta X}(O\langle U_i\rangle)$, $i = 1,\ 2$. From [1] (see, P. 234) we have

$$\exp \beta X \setminus O_{\exp \beta X}(O\langle U_i\rangle) = [\exp_\beta X \setminus O\langle U_i\rangle]_{\exp \beta X}, \qquad i = 1,\ 2.$$

Hence $\Phi \in [\exp_\beta X \setminus O\langle U_i\rangle]_{\exp\beta X}$, $i = 1,\ 2$. Since $O\langle U_1\rangle \cap O\langle U_2\rangle = \varnothing$ by Lemma 2.1 we have

$$[\exp_\beta X \setminus O\langle U_1\rangle]_{\exp\beta X} \cap [\exp_\beta X \setminus O\langle U_2\rangle]_{\exp\beta X} = [\exp_\beta X \setminus O(\langle U_1\rangle \cup O\langle U_2\rangle)]_{\exp\beta X}.$$

Therefore, $\Phi \in [\exp_\beta X \setminus O_{\exp\beta X}(O\langle U_1\rangle \cup O\langle U_2\rangle)]_{\exp\beta X}$, what is equivalent $\Phi \in \exp\beta X \setminus O_{\exp\beta X}(\langle U_1\rangle \cup \langle U_2\rangle)$ (see [1], P. 234). In other words, $\Phi \notin O_{\exp\beta X}(\langle U_1\rangle \cup \langle U_2\rangle)$. Thus, we have established that inclusion $O_{\exp\beta X}(\langle U_1\rangle \cup \langle U_2\rangle) \subset O_{\exp\beta X}(O\langle U_1\rangle) \cup O_{\exp\beta X}(O\langle U_2\rangle)$ is also fair. □

Lemma 2.3. *Let $U_1, \ldots, U_n$; $V_1, \ldots, V_m$ be open subsets of a space X. Then $O\langle U_1, \ldots, U_n\rangle \cap O\langle V_1, \ldots, V_m\rangle \neq \varnothing$ iff for each $i \in \{1, \ldots, n\}$ and for each $j \in \{1, \ldots, m\}$ there exists, respectively $j(i) \in \{1, \ldots, m\}$ and $i(j) \in \{1, \ldots, n\}$, such that $U_i \cap V_{j(i)} \neq \varnothing$ and $U_{i(j)} \cap V_j \neq \varnothing$.*

Proof. Assume that for every $i \in \{1, \ldots, n\}$ there exists $j(i) \in \{1, \ldots, m\}$ such that $U_i \cap V_{j(i)} \neq \varnothing$ and for every $j \in \{1, \ldots, m\}$ there exists $i(j) \in \{1, \ldots, n\}$ such that $U_{i(j)} \cap V_j \neq \varnothing$. For any pair $(i,\ j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ for which $U_i \cap V_j \neq \varnothing$, choose a point $x_{ij} \in U_i \cap V_j$ and make a closed set F consisting of these points. Then $F \subset \bigcup\limits_{i=1}^{n} U_i$ and $F \subset \bigcup\limits_{j=1}^{m} V_j$. Besides, $F \cap U_i \neq \varnothing$, $i = 1, \ldots, n$, and $F \cap V_j \neq \varnothing$, $j = 1, \ldots, m$. Therefore, $F \in O\langle U_1, \ldots, U_n\rangle \cap O\langle V_1, \ldots, V_m\rangle$.
Suppose there exists $i_0 \in \{1, \ldots, n\}$ such that $U_{i_0} \cap V_j = \varnothing$ for all $j \in \{1, \ldots, m\}$. Then $U_{i_0} \cap \bigcup\limits_{j=1}^{m} V_j = \varnothing$ and for each $F \in O\langle U_1,\ ...,\ U_n\rangle$ we have $F \not\subset \bigcup\limits_{j=1}^{m} V_j$. Hence, $F \notin O\langle V_1,\ ...,\ V_m\rangle$. Similarly, every $\Gamma \in O\langle V_1,\ ...,\ V_m\rangle$ lies in $\bigcup\limits_{j=1}^{m} V_j$ what implies $\Gamma \cap U_{i_0} = \varnothing$. From here $\Gamma \notin O\langle U_1,\ ...,\ U_n\rangle$. Thus, $O\langle U_1,\ ...,\ U_n\rangle \cap O\langle V_1,\ ...,\ V_m\rangle = \varnothing$. □

Lemma 2.4. *Let υ be a finite-component cover of a Tychonoff space X. Then the family $\exp_\beta \upsilon = \{O\langle U_1, \ldots, U_n\rangle : U_i \in \upsilon,\ i = 1, \ldots, n;\ n \in \mathbb{N}\}$ is a finite-component cover of the space $\exp_\beta X$.*

Proof. Let $O\langle G_1, \ldots, G_k\rangle$ be an element of $\exp_\beta \upsilon$. Each $G_i \in \upsilon$ intersects with finite elements of υ. Let $|\{\alpha :\ G_i \cap U_\alpha \neq \varnothing,\ U_\alpha \in \upsilon\}| = n_i$, $i = 1,\ 2, \ldots, k$. Denote $\gamma = \{G_i \cap U_j :\ G_i \cap U_j \neq \varnothing,\ i = 1,\ 2, \ldots, k,\ U_j \in \upsilon\}$. Then $|\gamma| \leq n_1 \cdot \ldots \cdot n_k$. Therefore, the set $O\langle G_1, \ldots, G_k\rangle$ crosses not more then $\prod\limits_{i=1}^{k} n_i$ elements of $\exp_\beta \upsilon$. It means that the collection $\exp_\beta \upsilon$ is star-finite.
Let $F \in \exp_\beta X$. There is a subfamily $\upsilon_F \subset \upsilon$ such that $F \subset \bigcup\limits_{U \in \upsilon_F} U$. From a cover $\{F \cap U :\ U \in \upsilon_F,\ F \cap U \neq \varnothing\}$ of the compact F it is possible to allocate a finite subcover $\{F \cap U_i :\ i = 1, \ldots, m\}$. We have $F \in O\langle U_1, \ldots, U_m\rangle$. So, the family $\exp_\beta \upsilon$ is a cover of $\exp_\beta X$. On the other hand by the definition of Vietoris topology the cover $\exp_\beta \upsilon$ is open. Thus, $\exp_\beta \upsilon$ is a star-finite open cover of $\exp_\beta X$.
We will show now that all components of the $\exp_\beta \upsilon$ are finite.
Let $M = O\langle G_1, \ldots, G_s\rangle$ and $M' = O\langle G'_1, \ldots, G'_t\rangle$ be arbitrary elements of $\exp_\beta \upsilon$. Further, let $\gamma_{G_i G'_j} = \{U_l^{ij} :\ l = 1,\ 2, \ldots, n_{ij}\}$ be the maximal chain of υ connecting G_i and G'_j, $i = 1,\ 2, \ldots, s$; $j = 1,\ 2, \ldots, t$. By definition these sets satisfy the following properties:

(1) $U_1^{ij} = G_i, \qquad i = 1, \ldots, s;\ j = 1, \ldots, t;$
(2) $U_{n_{ij}}^{ij} = G'_j, \qquad i = 1, \ldots, s;\ j = 1, \ldots, t;$
(3) $U_l^{ij} \cap U_{l+1}^{ij} \neq \varnothing, \qquad l = 1, \ldots, n_{ij} - 1;\ i = 1, \ldots, s;\ j = 1, \ldots, t.$

If $s < t$ we have $O\langle G_1, \ldots, G_s\rangle = O\langle U_1^{1j}, \ldots, U_1^{sj}, U_1^{i_1(s+1)}, \ldots, U_1^{i_{t-s}t}\rangle$, where $j = 1, \ldots, t$ and $i_1, \ldots, i_{t-s} \in \{1, \ldots, s\}$. Further, $O\langle G'_1, \ldots, G'_t\rangle = O\langle U_{n_{i1}}^{i1}, \ldots, U_{n_{it}}^{it}\rangle$, $i = 1, \ldots, s$. Thus, the cover $\exp_\beta \upsilon$ has a chain connecting the given sets $M = O\langle G_1, \ldots, G_s\rangle$ and $M' = O\langle G'_1, \ldots, G'_t\rangle$. The case $s > t$ is analogously.
Now using Lemma 2.1 and calculating directly we find that each maximal chain of $\exp_\beta \upsilon$ connecting the sets $M = O\langle G_1, \ldots, G_s\rangle$ and $M' = O\langle G'_1, \ldots, G'_t\rangle$ has no more than $\prod\limits_{\substack{i=1, \\ j=1}}^{\substack{t \\ s}} n_{ij}$ elements. Thus, all components of $\exp_\beta \upsilon$ is finite. □

Theorem 2.5. *For a Tychonoff space X its hyperspace $\exp_\beta X$ is superparacompact iff X is superparacompact.*

Proof. As the superparacompactness is inherited to the closed subsets [2], the superparacompactness of $\exp_\beta X$ implies superparacompactness of the closed subset $X \subset \exp_\beta X$.
Let Ω be an open cover of $\exp_\beta X$. For each element $G \in \Omega$ there exists $O_G\langle U_1, \ldots, U_n\rangle$ such that $O_G\langle U_1, \ldots, U_n\rangle \subset G$, where $U_1, \ldots, U_n$ are open sets in X. We can choose sets $G \in \Omega$ so that a collection of sets $O_G\langle U_1, \ldots, U_n\rangle$ forms a cover of $\exp_\beta X$, what we denote by Ω'. It is easy to see that a collection $\omega' = \bigcup\limits_{O_G\langle U_1, \ldots, U_n\rangle \in \Omega'} \{U_1, \ldots, U_n\}$ is an open cover of X. There exists a finite-component cover ω of X which refines ω'. Then by Lemma 2.4 the collection

$$\exp_\beta \omega = \{O\langle V_1, \ldots, V_k\rangle :\ V_i \in \omega,\ i = 1, \ldots, n;\ n \in \mathbb{N}\}$$

is a finite-component cover of $\exp_\beta X$ and it refines Ω. □

3. Superparacompactness of the map $\exp_\beta f$

For a continuous map $f:(X,\tau_X)\to(Y,\tau_Y)$ and $O\in\tau_Y$ a preimage $f^{-1}O$ is called *a tube* (above O). Remind, a continuous map $f:\ X\to Y$ is called [2] a T_0-*map*, if for each pair of distinct points $x,x^{'}\in X$, such that $f(x)=f(x^{'})$, at least one of these points has an open neighbourhood in X which does not contain another point. A continuous map $f:\ X\to Y$ is called *totally regular*, if for each point $x\in X$ and every closed set F in X not containing x there exists an open neighbourhood O of $f(x)$ such that in the tube $f^{-1}O$ the sets $\{x\}$ and F are functional separable. Totally regular T_0-map is said to be *a Tychonoff map*.
Obviously, each continuous map $f:\ X\to Y$ of a Tychonoff space X into a topological space Y is a Tychonoff map. In this case owing to the set $\exp_\beta X$ is a Tychonoff space concerning to Vietoris topology for every Tychonoff space X, the map $\exp_\beta f:\ \exp_\beta X\to\exp_\beta Y$ is a Tychonoff map.
A continuous, closed map $f:X\to Y$ is said to be *compact* if the preimage $f^{-1}y$ of each point $y\in Y$ is compact. A continuous map $f:X\to Y$ is compact iff for each point $y\in Y$ and every cover ω of the fibre $f^{-1}y$, consisting of open sets in X, there is an open neighbourhood O of y in Y such that the tube $f^{-1}O$ can be covered with a finite subfamily of ω.
A compact map $bf:\ b_fX\to Y$ is said to be *a compactification* of a continuous map $f:X\to Y$ if X is everywhere dense in b_fX and $bf|_X=f$. On the set of all compactifications of the map f it is possible to introduce a partial order: for the compactifications $b_1f:\ b_{1f}X\to Y$ and $b_2f:\ b_{2f}X\to Y$ of f we put $b_1f\le b_2f$ if there is a natural map of $b_{2f}X$ onto $b_{1f}X$. B. A. Pasynkov showed that for each Tychonoff map $f:X\to Y$ there exists its maximal compactification $g:Z\to Y$, which he denoted by βf, and the space Z where this maximal compactification defines by β_fX. To within homeomorphism for a given Tychonoff map f its maximal compactification βf is unique.

Remark 3.1. *Note that the maps b_1f, b_2f, βf are compactifications of the map f. The spaces $b_{1f}X$, b_{2f}, β_fX are some extensions of X but they are not obliged to be compactifications.*

A Tychonoff map $f:X\to Y$ is said to be *superparacompact*, if for every closed set F in β_fX lying in the growth $\beta_fX\setminus X$ there exists a finite-component cover λ of X pricking out F in β_fX (i. e. $F\cap(\cup[\lambda]_{\beta_fX})=\varnothing$) [3].
It is easy to see that one can define superparacompactness of a map as follows: a map $f:X\to Y$ is superparacompact if for each $y\in Y$ and every open cover Υ of $f^{-1}y$ in X there exists an open neighbourhood O of y in Y such that Υ has a finite-component cover υ of $f^{-1}O$ in X which refines Υ.

Definition 3.2. *A compactification $bf:b_fX\to Y$ of a Tychonoff map $f:X\to Y$ is said to be perfect compactification of f if for each point $y\in Y$ and for every disjoint open sets U_1 and U_2 in X there exists an open neighbourhood $O\subset Y$ of y such that the equality*

$$O_{b_fX}(U_1\cup U_2)\cap bf^{-1}O=\Big(O_{b_fX}(U_1)\cup O_{b_fX}(U_2)\Big)\cap bf^{-1}O$$

holds.

Let $f:\ X\to Y$ be a continuous map of a Tychonoff space X into a space Y. It is well known there exists a compactification vX of X such that f has a continuous extension $vf:\ vX\to Y$ on vX. It is clear, vf is a perfect compactification of f.
The following result is an analog of Theorem 1.2 for a case of maps.

Theorem 3.3. *Let $bf:\ b_fX\to Y$ be a perfect compactification of a Tychonoff map $f:X\ \to Y$. The map f is superparacompact iff for every closed set F in b_fX lying in the growth $b_fX\setminus X$ there exists a finite-component cover λ of X pricking out the set F in b_fX.*

Proof. The proof is carried out similar to the proof of Theorem 1.1 П from [2]. □

Evidently a restriction $f|_\Phi:\ \Phi\to Y$ of a superparacompact map $f:\ X\to Y$ on the closed subset $\Phi\subset X$ is a superparacompact map.
The following result is a variant of Theorem 2.2 for a case of maps.

Theorem 3.4. *Let $f:\ X\to Y$ be a Tychonoff map. Then the map $\exp_\beta\beta f:\exp_\beta\beta_fX\to\exp_\beta Y$ is a perfect compactification of $\exp_\beta f:\exp_\beta X\ \to\exp_\beta Y$.*

Proof. The proof is similar to the proof of Theorem 2.2. Here the equality

$$(\exp_\beta\beta f)^{-1}O\langle U_1,...,U_m\rangle=O\langle\beta f^{-1}(U_1),...,\beta f^{-1}(U_m)\rangle$$

is used. □

The following statement is the main result of this section.

Theorem 3.5. *The Tychonoff map $\exp_\beta f:\ \exp_\beta X\to\exp_\beta Y$ is superparacompact iff a map $f:\ X\to Y$ is superparacompact.*

Proof. Let $\exp_\beta f:\ \exp_\beta X\to\exp_\beta Y$ be a superparacompact map. It implies that $f:\ X\to Y$ is a superparacompact map since $X\cong\exp_1X$ is closed set in $\exp_\beta X$.
Let now $f:\ X\to Y$ be a superparacompact map. Consider arbitrary $\Gamma\in\exp_\beta Y$ and an open cover Ω of $(\exp_\beta f)^{-1}(\Gamma)=\{F\in\exp_\beta X:f(F)=\Gamma\}$ in $\exp_\beta X$. For each element $G\in\Omega$ there exists $O_G\langle U_1,\ldots,U_n\rangle$ such that $O_G\langle U_1,\ldots,U_n\rangle\subset G$, where $U_1,\ldots,U_n$ are open sets in X. We can choose sets $G\in\Omega$ so that a collection of sets $O_G\langle U_1,\ldots,\ U_n\rangle$ forms a cover of $(\exp_\beta f)^{-1}(\Gamma)$, what we denote by $\Omega^{'}$. It is easy to see that a collection $\omega^{'}=\bigcup\limits_{O_G\langle U_1,\ldots,U_n\rangle\in\Omega^{'}}\{U_1,\ldots,\ U_n\}$ is an open cover of $f^{-1}\Gamma$ in X. For each $y\in\Gamma$ there exists an open neighbourhood O_y of y in Y such that the collection $\omega_y=\{U\cap f^{-1}O_y:\ U\in\omega^{'}\}$ is an open cover of $f^{-1}y$ in X and ω_y has a finite-component cover $\omega_y^{'}$ of $f^{-1}O_y$ in X which refines ω_y. Gather such O_y and construct an open cover $\{O_y:y\in\Gamma\}$ of Γ in Y. Since $\Gamma\in\exp_\beta Y$ by construction of hyperspace, Γ is a compact subset of Y. Consequently, there exists a finite open subcover $\gamma=\{O_{y_1},\ldots,O_{y_n}\}$ in Y, which covers Γ. Put

$\omega = \bigcup_{O_{y_i} \in \gamma} \omega'_{y_i}$. Then ω is an open cover of $f^{-1}\left(\bigcup_{U \in \omega} U\right)$ in X. By the construction ω is a finite-component cover, and it refines ω'. Hence, $\exp_\beta \omega$ is a finite-component cover of $(\exp_\beta f)^{-1} O\langle O_{y_1}, \dots, O_{y_n}\rangle = \langle f^{-1}O_{y_1}, \dots, f^{-1}O_{y_n}\rangle$ in $\exp_\beta X$ and it refines Ω.

So, for each $\Gamma \in \exp_\beta Y$ and every open cover Ω of $(\exp_\beta f)^{-1}\Gamma$ in $\exp_\beta X$ there exists an open neighbourhood $O\langle O_{y_1}, \dots, O_{y_n}\rangle$ of Γ in $\exp_\beta Y$ such that Ω has a finite-component cover $\exp_\beta \omega$ of $(\exp_\beta f)^{-1} O\langle O_{y_1}, \dots, O_{y_n}\rangle$ in $\exp_\beta X$ which refines Ω. Thus, the map $\exp_\beta f : \exp_\beta X \to \exp_\beta Y$ is superparacompact. □

Corollary 3.6. *Let $f : X \to Y$ be a superparacompact map and Φ be a closed set in $\exp_\beta \beta_f X$ such that $\Phi \subset \exp_\beta \beta_f X \setminus \exp_\beta X$. Then there exists a finite-component cover Ω of $\exp_\beta X$ pricking out Φ in $\exp_\beta \beta_f X$ (i. e. $\Phi \cap (\cup[\Omega]_{\exp_\beta \beta_f X}) = \varnothing$).*

Corollary 3.7. *The functor $\exp_\beta$ lifts onto category of superparacompact spaces and their continuous maps.*

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