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Research Article

On *-differential identities in prime rings with involution

Shakir Ali*[1], Ali N.A. Koam[2], Moin A. Ansari[2]

[1] *Department of Mathematics, Faculty of Science, Aligarh Muslim University, Aligarh, India*
[2] *Department of Mathematics, Faculty of Science, Jazan University, Kingdom of Saudi Arabia*

Abstract

Let $\mathcal{R}$ be a ring. An additive map $x \mapsto x^*$ of $\mathcal{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in \mathcal{R}$. In this paper, we study the effect of involution "$*$" on prime rings that satisfying certain differential identities. The identities considered in this manuscript are new and interesting. As the applications, many known theorems can be either generalized or deduced. In particular, a classical theorem due to Herstein [A note on derivation II, Canad. Math. Bull., 1979] is deduced.

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1. Notations and introduction

In all that follows, unless specially stated, $\mathcal{R}$ always denotes an associative ring with centre $\mathcal{Z}(\mathcal{R})$. As usual the symbols $s \circ t$ and $[s, t]$ will denote the anti-commutator $st + ts$ and commutator $st - ts$, respectively. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if $nx = 0$ (where $x \in \mathcal{R}$) implies that $x = 0$. A ring $\mathcal{R}$ is called prime if $a\mathcal{R}b = (0)$ (where $a, b \in \mathcal{R}$) implies $a = 0$ or $b = 0$, and is called semiprime ring if $a\mathcal{R}a = (0)$ (where $a \in \mathcal{R}$) implies $a = 0$. An additive map $x \mapsto x^*$ of $\mathcal{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in \mathcal{R}$. A ring equipped with an involution is called a ring with involution or $*$-ring. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $\mathcal{R}$ will be denoted by $H(\mathcal{R})$ and $S(\mathcal{R})$, respectively. The involution is called the first kind if $\mathcal{Z}(\mathcal{R}) \subseteq H(\mathcal{R})$, otherwise it is said to be of the second kind. In the later case $S(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R}) \neq (0)$. Notice that x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in $\mathcal{R}$ are normal, then $\mathcal{R}$ is called a normal ring (see [15] for more details).

An additive mapping $\delta : \mathcal{R} \to \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $\delta(st) = \delta(s)t + s\delta(t)$ for all $s, t \in \mathcal{R}$. A derivation δ is said to be inner if there exists $a \in \mathcal{R}$ such that $\delta(s) = as - sa$ for all $s \in \mathcal{R}$. Over the last some decades, several authors have investigated the relationship between the commutativity of the ring $\mathcal{R}$ and certain special types of maps like derivations and automorphisms of $\mathcal{R}$. The criteria to discuss the commutativity of prime rings via

*Corresponding Author.
Email addresses: shakir.ali.mm@amu.ac.in (S. Ali), akoum@jazanu.edu.sa (A.N.A. Koam), maansari@jazanu.edu.sa (M.A. Ansari)
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derivations has been given for the first time by Posner [19]. In fact, he proved that the existence of a nonzero centralizing derivation(*i.e.*, $\delta(x)x - x\delta(x) \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{Z}(\mathcal{R})$ on a prime ring forces the ring to be commutative. Since then many algebraists established the commutativity of prime and semiprime rings via derivations and automorphisms that satisfying certain differential identities (see [1,3,6–10,13,14,17,18] and references therein).

In this paper, our intent is to continue to investigate and discuss the commutativity of prime rings with involution '$*$" satisfying certain $*$- differential identities. In fact, our results generalize and unify several well known and classical theorems proved in [4], [12], and [16].

2. Preliminaries

We shall do a great deal of calculation with commutators and anti-commutators, routinely using the following basic identities:

For all $s, t, w \in \mathcal{R}$;

$$[st, w] = s[t, w] + [s, w]t \text{ and } [s, tw] = t[s, w] + [s, t]w$$
$$so(tw) = (sot)w - t[s, w] = t(sow) + [s, t]w$$
$$(st)ow = s(tow) - [s, w]t = (sow)t + s[t, w].$$

We start our investigation with some known facts and results about rings which will be used frequently throughout the discussions.

Fact 2.1 ([2, Lemma 2.1]). *Let $\mathcal{R}$ be a prime ring with involution "$*$" of second kind such that char$(\mathcal{R}) \neq 2$. If $\mathcal{R}$ is normal i.e., $[x, x^*] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Fact 2.2. *The center of a prime ring is free from zero divisors.*

Fact 2.3. *Let $\mathcal{R}$ be a 2-torsion free ring with involution "$*$" . Then every $x \in \mathcal{R}$ can be uniquely represented as $2x = h + k$, where $h \in H(\mathcal{R})$ and $k \in S(\mathcal{R})$.*

In view of the Fact 2.1 and Theorem 2.4 of [2], we have the following.

Fact 2.4. *Let $\mathcal{R}$ be a prime ring with involution "$*$" of second kind such that char$(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $[\delta(x), x^*] = 0$ for all $x \in \mathcal{R}$. Then, $\mathcal{R}$ is a commutative integral domain.*

Lemma 2.5. *Let $\mathcal{R}$ be a prime ring with involution "$*$" of the second kind such that char$(\mathcal{R}) \neq 2$. Let δ be a derivation of $\mathcal{R}$ such that $\delta(h) = 0$ for all $h \in H(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$. Then $\delta(x) = 0$ for all $x \in \mathcal{Z}(\mathcal{R})$.*

Proof. Suppose that we have $\delta(h) = 0$ for all $h \in S(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$. Substituting k^2 (where $k \in S(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$) for h and using the fact that $\delta(k) \in \mathcal{Z}$, we obtain $2\delta(k)k = 0$ for all $k \in S(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$. This implies that $\delta(k)k = 0$ for all $k \in S(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$. Application of the Fact 2.2 yields $\delta(k) = 0$ for all $k \in S(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$. In view of the Fact 2.3, we conclude that $2\delta(x) = \delta(2x) = \delta(h + k) = \delta(h) + \delta(k) = 0$ and hence $\delta(x) = 0$ for all $x \in \mathcal{Z}(\mathcal{R})$. □

Lemma 2.6. *Let $\mathcal{R}$ be a prime ring with involution "$*$" of second kind such that char$(\mathcal{R}) \neq 2$. If $x \circ x^* = 0$ for all $x \in \mathcal{R}$ or $xx^* = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is a commutative integral domain.*

Proof. First we assume that $x \circ x^* = 0$ for all $x \in \mathcal{R}$. Direct linearization of the above relation gives

$$xy^* + yx^* + x^*y + y^*x = 0 \tag{2.1}$$

for all $x, y \in \mathcal{R}$. Substituting sy for y (where $s \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$ in (2.1), we get

$$-sxy^* + syx^* + sx^*y - sy^*x = 0 \tag{2.2}$$

for all $x, y \in \mathcal{R}$ and $s \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$. Multiplying (2.1) by s and then combining with the obtained relation, we arrive at $s(yx^* + x^*y) = 0$ for all $x, y \in \mathcal{R}$ and $s \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$. Invoking the primeness of $\mathcal{R}$, we get $yx^* + x^*y = 0$ for all $x, y \in \mathcal{R}$. This implies that $x \circ y = 0$ for all $x, y \in \mathcal{R}$. Replacing x by xz and using the second anti-commutator identity, we get $y[x, z] = 0$ for all $x, y, z \in \mathcal{R}$. The primeness of $\mathcal{R}$ furnishes the required result. On the other hand, we consider the case $xx^* = 0$ for all $x \in \mathcal{R}$. This implies that $x \circ x^* = 0$ for all $x \in \mathcal{R}$ and therefore the result follows by above discussion. Hence, $\mathcal{R}$ is commutative. This proves the lemma. □

3. The results

In [16], Herstein proved that a prime ring $\mathcal{R}$ of characteristic different from two with a nonzero derivation δ satisfying the differential identity $[\delta(x), \delta(y)] = 0$ for all $x, y \in \mathcal{R}$, must be commutative. Further, Daif [11] showed that for a 2-torsion free semiprime ring $\mathcal{R}$ admitting a derivation δ such that $[\delta(x), \delta(y)] = 0$ for all $x, y \in I$, where I is a nonzero ideal of $\mathcal{R}$ and δ is nonzero on I, then $\mathcal{R}$ contains a nonzero central ideal. Further, this result was extended by first author together with Dar in [[12], Theorem 3.1] for prime rings with involution. In fact, they proved that if $\mathcal{R}$ is prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$ and satisfying the $*$-differential identity $[\delta(x), \delta(x^*)] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ must be commutative. This result motivated us to prove the following theorem.

Theorem 3.1. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ_1 and δ_2 be derivations of $\mathcal{R}$ such that at least one of them is nonzero and satisfying the identity $[\delta_1(x), \delta_1(x^*)] + \delta_2(x \circ x^*) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Proof. We are given that $\delta_1,\ \delta_2 : \mathcal{R} \to \mathcal{R}$ derivations such that

$$[\delta_1(x), \delta_1(x^*)] + \delta_2(x \circ x^*) = 0 \tag{3.1}$$

for all $x \in \mathcal{R}$. We divide the proof in three cases.

Case (i): First we assume that $\delta_1 \neq 0$ and $\delta_2 = 0$. Then, the relation (3.1) reduces to

$$[\delta_1(x), \delta_1(x^*)] = 0 \tag{3.2}$$

for all $x \in \mathcal{R}$. Henceforth, the proof follows by [[12], Theorem 3.1]. But the proof of Theorem 3.1. given in [12] is very complicated and technical. Therefore, we present here short and elegant proof that may be considered as an alternative and brief proof of Theorem 3.1. given in [12]. Polarizing the equation (3.2), we obtain

$$[\delta_1(x), \delta_1(y^*)] + [\delta_1(y), \delta_1(x^*)] = 0 \tag{3.3}$$

for all $x, y \in \mathcal{R}$. Substituting yh for y (where $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$) in (3.3) and using the fact that $\delta_1(h) \in \mathcal{Z}(\mathcal{R})$, we arrive at

$$\{[\delta_1(x), \delta_1(y^*)] + [\delta_1(y), \delta_1(x^*)]\}h + \{[\delta_1(x), y^*] + [y, \delta_1(x^*)]\}\delta_1(h) = 0 \tag{3.4}$$

for all $x, y \in \mathcal{R}$. In view of (3.3), the above relation reduces to

$$\{[\delta_1(x), y^*] + [y, \delta_1(x^*)]\}\delta_1(h) = 0 \tag{3.5}$$

for all $x, y \in \mathcal{R}$ and $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$. The primeness of $\mathcal{R}$ yields that either $\delta_1(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$ or $[\delta_1(x), y^*] + [y, \delta_1(x^*)] = 0$ for all $x, y \in \mathcal{R}$. If $\delta_1(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$, then by the Fact 2.5 we conclude that $\delta_1(x) = 0$ for all $x \in \mathcal{Z}(\mathcal{R})$. Substituting ky for y(where $k \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$) in (3.3) and then combining it with (3.3), we obtain $[\delta_1(y), \delta_1(x^*)] = 0$ for all $x, y \in \mathcal{R}$. This implies that $[\delta_1(x^2), \delta_1(x)] = 0$ for all $x \in \mathcal{R}$. In view of [[5], Theorem 3.1], we conclude the result. Finally, we have the remaining case $[\delta_1(x), y^*] + [y, \delta_1(x^*)] = 0$ for all $x, y \in \mathcal{R}$. Replace y by ys(where $s \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$) in the last expression to get $s[\delta_1(x), y^*] - s[y, \delta_1(x^*)] = 0$ for all $x, y \in \mathcal{R}$ and $s \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$.

Multiplying the above relation by s from left side and combine with last relation, we find that $2s[\delta_1(x), x^*] = 0$ for all $x \in \mathcal{R}$. Since $\mathcal{R}$ is prime ring and char $(\mathcal{R}) \neq 2$, so the last identity reduces to $[\delta_1(x), x^*] = 0$ for all $x \in \mathcal{R}$. In view of the Fact 2.4, we conclude the required conclusion. Hence, $\mathcal{R}$ is commutative.

Case (ii): Now we assume that $\delta_1 = 0$ and $\delta_2 \neq 0$ Then, the relation (3.1) reduces to

$$\delta_2(x \circ x^*) = 0 \tag{3.6}$$

for all $x \in \mathcal{R}$. Substituting $x + y$ for x in (3.6), we obtain

$$\delta_2(x \circ y^*) + \delta_2(y \circ x^*) = 0 \tag{3.7}$$

for all $x, y \in \mathcal{R}$. Replacing x by hx (where $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$) in (3.7) and using the anti-commutator identities, we get

$$\delta_2(h(x \circ y^*)) + \delta_2(h(y \circ x^*)) = 0 \tag{3.8}$$

for all $x, y \in \mathcal{R}$ and $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$. Since $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$, so $\delta_2(h) \in \mathcal{Z}(\mathcal{R})$ and consequently equation (3.8) gives

$$\delta_2(h)\{(x \circ y^*) + (y \circ x^*)\} + h\{\delta_2((x \circ y^*) + \delta_2((y \circ x^*)\} = 0 \tag{3.9}$$

for all $x, y \in \mathcal{R}$ and $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R}))$. Application of relation (3.7) yields

$$\delta_2(h)\{(x \circ y^*) + (y \circ x^*)\} = 0 \tag{3.10}$$

for all $x, y \in \mathcal{R}$ and $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$. Taking $x = y$ in 3.10) and using the fact that char $\mathcal{R} \neq 2$, we obtain $\delta_2(h)(x \circ x^*) = 0$ for all $x \in \mathcal{R}$. Invoking the primeness of $\mathcal{R}$, it follows that either $x \circ x^* = 0$ for all $x \in \mathcal{R}$ or $\delta_2(h) = 0$. In case $\delta_2(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$, application of the Fact 2.5 implies that $\delta_2(x) = 0$ for all $x \in \mathcal{Z}$. Substituting ky for y(where $k \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$) in (3.7) and then combining it with (3.7), we find that $\delta_1(x \circ y) = 0$ for all $x, y \in \mathcal{R}$. This implies that $\mathcal{R}$ is commutative. In consequence, we have $x \circ x^* = 0$ for all $x \in \mathcal{R}$. Hence, Lemma 2.6 yields the required result.

Case (iii): Finally, we assume that both δ_1 and δ_2 are nonzero. Interchanging the role of x and x^* in equation (3.1) and using the fact that $[x, x^*] = -[x^*, x]$ and $x \circ x^* = x^* \circ x$, gives

$$-[\delta_1(x^*), \delta_1(x)] + \delta_2(x^* \circ x) = 0 \tag{3.11}$$

for all $x \in \mathcal{R}$. This implies that

$$[\delta_1(x), \delta_1(x^*)] - \delta_2(x \circ x^*) = 0 \tag{3.12}$$

for all $x \in \mathcal{R}$. Combining (3.11) and (3.14) and using the fact that $char(\mathcal{R}) \neq 2$, we get

$$[\delta_1(x), \delta_1(x^*)] = 0 \tag{3.13}$$

for all $x \in \mathcal{R}$. Therefore, the result follows by Case (i). Hence, $\mathcal{R}$ is a commutative Integral domain. This completes the proof of the theorem. $\square$

Using a similar technique with necessary variations, we can prove the following result.

Theorem 3.2. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ_1 and δ_2 be derivations of $\mathcal{R}$ such that at least one of them is nonzero and satisfying the identity $[\delta_1(x), \delta_1(x^*)] - \delta_2(x \circ x^*) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Theorem 3.3. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ_1 and δ_2 be derivations of $\mathcal{R}$ such that at least one of them is nonzero and satisfying the identity $\delta_1(x) \circ \delta_1(x^*) + \delta_2([x, x^*]) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Proof. By the assumption, we have

$$\delta_1(x) \circ \delta_1(x^*) + \delta_2([x, x^*]) = 0 \tag{3.14}$$

for all $x \in \mathcal{R}$. Substituting x^* for x in (3.14) and using the fact that $x \circ x^* = x^* \circ x$, we obtain

$$\delta_1(x) \circ \delta_1(x^*) - \delta_2([x, x^*]) = 0 \tag{3.15}$$

for all $x \in \mathcal{R}$. From relations (3.14) and (3.15), we conclude that

$$\delta_1(x) \circ \delta_1(x^*) = 0 \tag{3.16}$$

for all $x \in \mathcal{R}$. Henceforward, the result is follows by [[12], Theorem 3.3]. This proves the theorem. □

Theorem 3.4. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that char$(\mathcal{R}) \neq 2$. Let δ_1 and δ_2 be derivations of $\mathcal{R}$ such that at least one of them is nonzero and satisfying the identity $\delta_1(x) \circ \delta_1(x^*) - \delta_2([x, x^*]) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Theorem 3.5. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that char$(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $\delta([x, x^*]) + [\delta(x), \delta(x^*)] = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Proof. By the assumption, we have

$$\delta([x, x^*]) + [\delta(x), \delta(x^*)] = 0 \tag{3.17}$$

for all $x \in \mathcal{R}$. Polarizing the relation (3.17), we obtain

$$\delta([x, y^*]) + \delta([y, x^*]) + [\delta(x), \delta(y^*)] + [\delta(y), \delta(x^*)] = 0 \tag{3.18}$$

for all $x, y \in \mathcal{R}$. Substituting yh for y (where $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$) in (3.18) and using the fact $\delta(h) \in \mathcal{Z}(\mathcal{R})$ (where $h \in \mathcal{Z}(\mathcal{R}) \cap H(\mathcal{R})$), we arrive at

$$\{[x, y^*] + [y, x^*]\}\delta(h) + h\{\delta([x, y^*)] + \delta([y, x^*])\} + \delta(h)[\delta(x), y^*]$$
$$+h[\delta(x), \delta(y^*)] + [\delta(y), \delta(x^*)]h + [y, \delta(x^*)]\delta(h) = 0$$

for all $x, y \in \mathcal{R}$. This implies that

$$\{[x, y^*] + [y, x^*]\}\delta(h) + \{\delta([x, y^*)] + \delta([y, x^*]) + [\delta(x), \delta(y^*)] + [\delta(y), \delta(x^*)]\}h$$
$$+\{[\delta(x), y^*] + [y, \delta(x^*)]\}\delta(h) = 0$$

Application of the relation (3.18) yields

$$\{[x, y^*] + [y, x^*]\}\delta(h) + \{[\delta(x), y^*] + [y, \delta(x^*)]\}\delta(h) = 0 \tag{3.19}$$

for all $x \in \mathcal{R}$. Since $\delta(h) \in \mathcal{Z}(\mathcal{R})$, so the above expression can be written as

$$\{[x, y^*] + [y, x^*] + [\delta(x), y^*] + [y, \delta(x^*)]\}\delta(h) = 0 \tag{3.20}$$

for all $x \in \mathcal{R}$. The primeness of $\mathcal{R}$ yields that either $\delta(h) = 0$ or $[x, y^*] + [y, x^*] + [\delta(x), y^*] + [y, \delta(x^*)]) = 0$ for all $x, y \in \mathcal{R}$. If $\delta(h) = 0$, then $\delta(x) = 0$ for all $x \in \mathcal{Z}(\mathcal{R})$ by the Fact 2.5. Replacing y by ky (where $s \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$) in (3.18) and combining it with the obtained result, we get $\delta[(x, y]) + [\delta(x), \delta(y)] = 0$ for all $x, y \in \mathcal{R}$. Substituting x^2 for y in the last relation, we find that $[\delta(x), \delta(x^2)] = 0$ for all $x \in \mathcal{R}$. Hence, $\mathcal{R}$ is commutative by Theorem 3.1 of [5]. On the other hand, we have

$$[x, y^*] + [y, x^*] + [\delta(x), y^*] + [y, \delta(x^*)] = 0 \tag{3.21}$$

for all $x, y \in \mathcal{R}$. Replace x by xs (where $s \in \mathcal{Z}(\mathcal{R}) \cap S((\mathcal{R}))$ in (3.21) to get

$$[x, y^*]s - [y, x^*]s + [\delta(x), y^*]s + [x, y^*]\delta(s) - [y, x^*]\delta(s) - [y, \delta(x^*)]s = 0 \tag{3.22}$$

for all $x, y \in \mathcal{R}$ and $s \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$. Multiplying by s to (3.21) from right and combining with (3.22), we arrive at

$$2[x, y^*]s + 2[\delta(x), y^*]s + 2[x, y^*]\delta(s) = 0 \tag{3.23}$$

for all $x, y \in \mathcal{R}$. Taking $y = x^*$ in (3.23) and using the fact that $char(\mathcal{R}) \neq 2$, we obtain $[\delta(x), x]s = 0$ for all $x \in \mathcal{R}$ and $s \in \mathcal{Z}(\mathcal{R}) \cap S(\mathcal{R})$. The primeness of $\mathcal{R}$ gives that $[\delta(x), x] = 0$ for all $x \in \mathcal{R}$. Since $\delta \neq 0$, Posner's theorem [19] yields the desired conclusion. This proves the theorem completely. □

Theorem 3.6. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $\delta(x \circ x^*) + \delta(x) \circ \delta(x^*) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

In view of above discussions, results, and as the applications of the main theorems, we obtain the following corollaries.

Corollary 3.7. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ_1 and δ_2 be derivations of $\mathcal{R}$ such that at least one of them is nonzero and satisfying the identity $[\delta_1(x), \delta_1(y)] \pm \delta_2(x \circ y) = 0$ for all $x, y \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Corollary 3.8. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ_1 and δ_2 be derivations of $\mathcal{R}$ such that at least one of them is nonzero and satisfying the identity $\delta_1(x) \circ \delta_1(y) \pm \delta_2([x, y]) = 0$ for all $x, y \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Corollary 3.9 ([16], Theorem)**.** *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $[\delta(x), \delta(y)] = 0$ for all $x, y \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.*

The next corollary is the $*$-version of Herstein classical theorem proved in [16].

Corollary 3.10 ([12], Theorem 3.1)**.** *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $[\delta(x), \delta(x^*)] = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.*

Corollary 3.11 ([12], Theorem 3.2)**.** *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $\delta(x) \circ \delta(x^*) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.*

Corollary 3.12 ([4], Theorem 2.2)**.** *Let $\mathcal{R}$ be a prime ring with involution "$*$" of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $\delta([x, x^*]) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.*

Corollary 3.13 ([4], Theorem 2.3)**.** *Let $\mathcal{R}$ be a prime ring with involution "$*$" of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $\delta(x \circ x^*) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.*

Corollary 3.14. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $\delta(xx^*) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Proof. We are given that δ is a nonzero derivation of $\mathcal{R}$ such that $\delta(xx^*) = 0$ for all $x \in \mathcal{R}$. For any $x \in \mathcal{R}$, x^* also is an element of $\mathcal{R}$. Substitution x^* for x in the given assertion, we obtain $\delta(x^*x) = 0$ for all $x \in \mathcal{R}$. This implies that $\delta(x \circ x^*) = 0$ for all $x \in \mathcal{R}$. Hence $\mathcal{R}$ is commutative by Corollary 3.13. □

Corollary 3.15. *Let $\mathcal{R}$ be a prime ring with involution " $*$ " of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $\delta(xx^*) + \delta(x)\delta(x^*) = 0$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain.*

Proof. By the assumption, we have $\delta(xx^*)+\delta(x)\delta(x^*)=0$ for all $x\in\mathcal{R}$. Replace x by x^* in the last expression to get $\delta(x^*x)+\delta(x^*)\delta(x)=0$ for all $x\in\mathcal{R}$. By combining the last two relations, we obtain $\delta([x,x^*]+[\delta(x),\delta(x^*)]=0$ for all $x\in\mathcal{R}$. Hence, $\mathcal{R}$ is commutative by Theorem 2.5. This proves the corollary. □

4. Some examples

The first example shows that the restriction of the second kind involution in Theorems 3.1 and 3.5 is not superfluous.

Example 4.1. Let $R=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} |a,b,c,d\in\mathcal{Z}\right\}$. Obviously, $\mathcal{R}$ is prime ring. Define the maps $\delta_1,\delta_2,\ ^*:R\longrightarrow R$ such that $\begin{pmatrix} a & b\\ c & d\end{pmatrix}^*=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$, $\delta_1\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 0 & -b\\ c & 0\end{pmatrix}$ Then, it is straightforward to check that δ_1 is a derivation of $\mathcal{R}$. It is easy to see that $Z(\mathcal{R})=\left\{\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} |a\in\mathcal{Z}\right\}$. Then, $x^*=x$ for all $x\in\mathcal{R}$ and hence $Z(\mathcal{R})\subseteq H(\mathcal{R})$, which shows that the involution * is of the first kind. Moreover, for $\delta_1=\delta_2$ the following conditions: (i) $[\delta_1(x),\delta_1(x^*)]+\delta_2(x\circ x^*)=0$ for all $x\in\mathcal{R}$, (ii) $\delta([x,x^*])+[\delta(x),\delta(x^*)]=0$ for all $x\in\mathcal{R}$ are satisfied. However, $\mathcal{R}$ is not commutative.

The next example demonstrates that Theorem 3.5 cannot be extended for semiprime rings.

Example 4.2. Let $\mathcal{R}$ be a ring with involution "$*$" same as in Example 4.1. Next, let $\mathcal{C}$ be the field of complex numbers with the conjugation involution. Consider the set $\mathcal{L}=\mathcal{R}\times\mathcal{C}$. Then, it is obvious to see that $(\mathcal{L},\sigma)$ a semiprime ring with involution * of the second kind, where $\sigma(r,z)=(r^*,\bar{z})$ for all $(r,z)\in\mathcal{R}\times\mathcal{C}$. Define a derivation $\delta:\mathcal{L}\longrightarrow\mathcal{L}$ by $\delta(r,z)=(\delta_1(r),0)$ for all $(r,z)\in\mathcal{R}\times\mathcal{C}$ (where δ_1 is a derivation on $\mathcal{R}$). Then, it is straightforward to check that δ is a derivation of $\mathcal{R}\times\mathcal{C}$ satisfying the conditions of the mentioned theorems, but $\mathcal{R}$ is not commutative. Hence, in Theorem 3.5, the hypothesis of primeness is crucial.

Remark 4.3. At the end, let us also point out that we do not know yet whether Theorems 3.1, 3.3, and 3.5 true for automorphisms of semi(prime)rings. Hence, these are open problems for automorphisms of semi(prime)rings.

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References

[1] S. Ali and H. Alhazmi, *Some commutativity theorems in prime rings with involution and derivations*, J. Adv. Math. Comput. Sci. **24** (5), 1–6, 2017.

[2] S. Ali and N.A. Dar, *On $*$-centralizing mappings in rings with involution*, Georgian Math. J. **1**, 25–28, 2014.

[3] S. Ali and S. Huang, *On derivations in semiprime rings*, Algebr. Represent. Theory **15** (6), 1023–1033, 2012.

[4] S. Ali, N.A. Dar, and M. Asci, *On derivations and commutativity of prime rings with involution*, Georgian Math. J. **23** (1), 9–14, 2016.

[5] S. Ali, M.S. Khan, and M. Al-Shomrani, *Generalization of Herstein theorem and its applications to range inclusion problems*, J. Egyptian Math. Soc. **22**, 322–326, 2014.

[6] N. Argac, *On prime and semiprime rings with derivations*, Algebra Colloq. **13** (3), 371–380, 2006.

[7] M. Ashraf and M.A. Siddeeque, *On $*-$ n-derivations in prime rings with involution*, Georgian Math. J. **21** (1), 9–18, 2014.

[8] M. Ashraf and N. Rehman, *On commutativity of rings with derivations*, Results Math. **42** (1-2), 3–8, 2002.

[9] H.E. Bell, *On the commutativity of prime rings with derivation*, Quaest. Math. **22**, 329-333, 1991.

[10] H.E. Bell and M.N. Daif, *On derivations and commutativity in prime rings*, Acta Math. Hungar. **66**, 337–343, 1995.

[11] M.N. Daif, *Commutativity results for semiprime rings with derivation*, Int. J. Math. Math. Sci. **21** (3), 471–474, 1998.

[12] N.A. Dar and S. Ali, *On $*$-commuting mappings and derivations in rings with involution*, Turk. J. Math. **40**, 884–894, 2016.

[13] V.De. Filippis, *On derivation and commutativity in prime rings*, Int. J. Math. Math. Sci. 69-72, 3859–3865, 2004.

[14] A. Fosner and J. Vukman, *Some results concerning additive mappings and derivations on semiprime rings*, Pul. Math. Debrecen, **78** (3-4), 575–581, 2011.

[15] I.N. Herstein, *Rings with Involution*, University of Chicago Press, Chicago, 1976.

[16] I.N. Herstein, *A note on derivation II*, Canad. Math. Bull. **22**, 509–511, 1979.

[17] J. Mayne, *Centralizing automorphisms of prime rings*, Canad. Math. Bull. **19**, 113–117, 1976.

[18] L. Oukhtite, *Posner's second theorem for Jordan ideals in ring with involution*, Expo. Math. **4** (29), 415–419, 2011.

[19] E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8**, 1093–1100, 1957.