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Centro-Equiaffine Differential Invariants of Curve Families

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ABSTRACT

The generator set of all centro-equiaffine differential invariant rational functions field for arbitrary curves is obtained. By using these generators, the conditions of equivalence for two curve families are found. Then the relations between elements of generator set are investigated.

Keywords: *Differential invariant, parametric curve, equivalence.*

AMS Subject Classification (2010): *53A35 ; 53A55.*

1. Introduction

Invariant theory has been studied for along time on the theory of curves and surfaces. There are many papers on the invariant theory of curves in differential geometry. Also, there are many books on affine differential geometry [12], [18], [20]. In most of the studies, special invariants were considered such as arc length, curvature and torsion. The problem of equivalence has also been investigated.

The concept of affine geometry was introduced by Felix Klein in Erlangen Programme in 1872. According to this programme, affine geometry deals with the properties of curves and surfaces which are invariant under affine maps. Since that time, affine invariants of curves have been investigated. This paper is concerned with the basic theory of centro-equiaffine geometry of curves and related questions of centro-equiaffine invariants. We give the complete system of centro-equiaffine invariants for arbitrary r curves.

In [8] the problem of equivalence investigated for equiaffine curves and [14] it is solved for centro-affine curves. The first comprehensive treatment of affine geometry is given in the seminal work of Paukowitsch [13]. For further developments of subject, we refer the reader to [7], and more modern texts [1], [15], the commentaries [5], [10], [19] and survey papers [11], [3]. The fundamental theorem of curves in centro-affine geometry is obtained in [2]. A discussion of centro-affine plane and space curves can be found in [17]. A detailed discussion of curves in centro-affine geometry can be obtained in [2]. In [6] equiaffine invariants of 3-dimensional curves and in [4], [13] equiaffine curvatures of n-dimensional curves are investigated. Complete systems of global equiaffine invariants for space paths are obtained in [8]. The global $SL(n)$-equivalence of path in $\mathbb{R}^n$ is considered in [7] and [16].

The problem of equivalence has been already solved for a single curve and for two curves by Sağıroğlu for the group $SL(n,\mathbb{R})$ [16]. It is solved the equivalence problem for arbitrary r curves in this paper. Firstly, the generator system of $SL(n,\mathbb{R})$-differential invariants for arbitrary r parametric curves is obtained. Then it is given the conditions of equivalence of curve families in terms of generator invariants. It is observed that the generator invariants obtained are functionally independent, namely the generator invariant set is minimal.

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^n$ be n-dimensional Euclidean space. The set $SL(n,\mathbb{R}) = \{A = [a_{ij}] | i,j = 1,2,...,n \quad and \quad a_{ij} \in \mathbb{R}, which \quad detA = 1\}$ is a group in according to multiplication of matrix. The action of group $SL(n,\mathbb{R})$ on $\mathbb{R}^n$ is given by

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$$g.x = \begin{pmatrix} g_{11} & ... & g_{1n} \\ ... & ... & ... \\ g_{n1} & ... & g_{nn} \end{pmatrix} . \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} g_{11}x_1 + ... + g_{1n}x_n \\ \vdots \\ g_{n1}x_1 + ... + g_{nn}x_n \end{pmatrix}$$

for $g \in SL(n, \mathbb{R})$ and $x \in \mathbb{R}$.

Definition 1.1. A C^∞-function $x : I \to \mathbb{R}^n$ will be called a parametric curve or briefly a curve in $\mathbb{R}^n$.

Definition 1.2. Let $\{x_1, x_2, ..., x_r\}$ and $\{y_1, y_2, ..., y_r\}$ be two pairs of curve families. If $y_i = gx_i$, $i = 1, 2, ..., r$ for some $g \in SL(n, \mathbb{R})$, then these curve families will be called $SL(n, \mathbb{R})$-equivalent and denoted by $\{x_1, x_2, ..., x_r\} \overset{G}{\approx} \{y_1, y_2, ..., y_r\}$ for the group $G = SL(n, \mathbb{R})$.

Definition 1.3. Let $\{x_1, x_2, ..., x_r\}$ be a curve family in $\mathbb{R}^n$. The polynomial

$$P(x_1, x_2, ..., x_r) = P(x_1, x_2, ..., x_r, x_1', x_2', ..., x_r', ..., x_1^{(m)}, x_2^{(m)}, ..., x_r^{(m)})$$

for some natural number m will be called the differential polynomial of $x_1, x_2, ..., x_r$.

The derivation of $P(x_1, x_2, ..., x_r)$ will be denoted by P' and this derivation is obtained as follows:

$$x_\tau^{(0)} = x_\tau,\ (x_\tau^{(m-1)})' = x_\tau^{(m)},\ \tau = 1, 2, ..., r.$$

Definition 1.4. Let P_1 and P_2 be two differential polynomials. Then the function

$$f < x_1, x_2, ..., x_r >= \frac{P_1(x_1, x_2, ..., x_r)}{P_2(x_1, x_2, ..., x_r)},\ P_2(x_1, x_2, ..., x_r) \neq 0$$

will be called a differential rational function. If

$$f < gx_1, gx_2, ..., gx_r >= f < x_1, x_2, ..., x_r >$$

for some $g \in SL(n, \mathbb{R})$, the differential rational function f is called centro-equiaffine invariant differential rational function. Centro-equiaffine differential polynomial is defined by the same way.

There no exists centro-equiaffine differential polynomial except constant. But there exists the centro-equiaffine differential rational function different from constant.

Remark 1.1. Let $\{x_1, x_2, ..., x_r\} \overset{G}{\approx} \{y_1, y_2, ..., y_r\}$. So for some $g \in SL(n, \mathbb{R})$ we get $y_i = gx_i$, $i = 1, 2, ..., r$. Then for all differential invariant rational function f, since

$$f < y_1, y_2, ..., y_r >= f < gx_1, gx_2, ..., gx_r >= f < x_1, x_2, ..., x_r >$$

we obtain $f < y_1, y_2, ..., y_r >= f < x_1, x_2, ..., x_r >$. But the reverse is not true.

The set of all differential rational functions will be denoted by $R < x_1, x_2, ..., x_r >$. It is a differential field and $\mathbb{R}$-algebra. Let G be the group $SL(n, \mathbb{R})$. The set of all centro-equiaffine invariant differential rational functions will be denoted by $R < x_1, x_2, ..., x_r >^G$. $R < x_1, x_2, ..., x_r >^G$ is a differential subfield and subalgebra of $R < x_1, x_2, ..., x_r >$.

Definition 1.5. Let $f_1, f_2, ..., f_k \in R < x_1, x_2, ..., x_r >^G$. If the differential field and algebra generated by these functions is equal to $R < x_1, x_2, ..., x_r >^G$ then these functions will be called the generator set of $R < x_1, x_2, ..., x_r >^G$.

2. Centro-Equiaffine Invariants of Arbitrary Curves

Let $x_1, x_2, ..., x_n \in \mathbb{R}^n$. The determinant $\begin{vmatrix} x_{11} & ... & x_{n1} \\ ... & ... & ... \\ x_{1n} & ... & x_{nn} \end{vmatrix}$ will be denoted by $[x_1 ... x_n]$. In here, k.column of this determinant is consist of the components of x_k, which are $x_{k1}, x_{k2}, ..., x_{kn}$.

Lemma 2.1. *Let $x_0, x_1, ..., x_n, y_2, ..., y_n$ be vectors in $\mathbb{R}^n$. Then the following equality holds:*

$$[x_1 x_2 ... x_n][x_0 y_2 ... y_n] - [x_0 x_2 ... x_n][x_1 y_2 ... y_n] - ... - [x_1 x_2 ... x_0][x_n y_2 ... y_n] = 0 \tag{2.1}$$

Proof. [7]. □

Definition 2.1. A curve x in $\mathbb{R}^n$ will be called $SL(n,\mathbb{R})$-regular (briefly regular) if $[xx'...x^{(n-1)}] \neq 0$. Hence for all $t \in I$, $[x(t)x'(t)...x^{(n-1)}(t)] \neq 0$.

Let G be the group $SL(n,\mathbb{R})$.

Theorem 2.1. *Let $x_1, x_2, ..., x_r$ be a curve family in $\mathbb{R}^n$ such that x_1 is regular. Then the generator set of $R < x_1, x_2, ..., x_r >^G$ is*

$$\begin{array}{l} [x_1{x_1}'...{x_1}^{(n-1)}], [x_1{x_1}'...{x_1}^{(i-1)}{x_1}^{(n)}{x_1}^{(i+1)}...{x_1}^{(n-1)}], \quad i = 0, 1, ..., n-1 \\ [x_1{x_1}'...{x_1}^{(i-1)}x_K{x_1}^{(i+1)}...{x_1}^{(n-1)}], \quad i = 0, 1, ..., n-1, K = 2, 3, ..., r. \end{array} \tag{2.2}$$

Proof. For the group $G = SL(n,\mathbb{R})$, the generator set of $R < x_1, x_2, ..., x_r >^G$ is

$$[x_1...x_n], [x_1...x_{i-1}x_\tau x_{i+1}...x_n], i = 1, 2, ..., n, \tau \in \Delta/\{1, ..., n\}$$

where Δ is an index set [20]. Let us take $x_1, x_2, ..., x_r, x_1', x_2', ..., x_r', x_1^{(K)}, x_2^{(K)}, ..., x_r^{(K)}, ...$ instead of the vectors x_τ. Then the generator set of $R < x_1, x_2, ..., x_r, x_1', x_2', ..., x_r', x_1^{(K)}, x_2^{(K)}, ..., x_r^{(K)}, ... >^G = R < U >^G$ is

$$[x_1{x_1}'...{x_1}^{(n-1)}], [x_1{x_1}'...{x_1}^{(i-1)}{x_1}^{(s)}{x_1}^{(i+1)}...{x_1}^{(n-1)}], s \geq n$$

$$[x_1{x_1}'...{x_1}^{(i-1)}{x_K}^{(\tau)}{x_1}^{(i+1)}...{x_1}^{(n-1)}], \tau \geq 0, K = 2, 3, ..., r.$$

We know that $[x_1{x_1}'...{x_1}^{(n-1)}]' = [x_1...{x_1}^{(n-2)}{x_1}^{(n)}]$.

Firstly, we want to show that $[x_1{x_1}'...{x_1}^{(i-1)}{x_1}^{(s)}{x_1}^{(i+1)}...{x_1}^{(n-1)}]$, $s \geq n$ is generated by $[x_1{x_1}'...{x_1}^{(i-1)}{x_1}^{(n)}{x_1}^{(i+1)}...{x_1}^{(n-1)}]$, $i = 0, 1, ..., n-2$. Let $s = n$. Then $R < U >^G$ is generated by (2.2).

Let $s > n$. By induction hypothesis, for $s-1$ let the set (2.2) be the generator set. Therefore $[x_1{x_1}'...{x_1}^{(i-1)}{x_1}^{(s-1)}{x_1}^{(i+1)}...{x_1}^{(n-1)}]$ is generated by (2.2). We get

$$[x_1...{x_1}^{(i-1)}{x_1}^{(s)}{x_1}^{(i+1)}...{x_1}^{(n-1)}] = [x_1...{x_1}^{(i-1)}{x_1}^{(s-1)}{x_1}^{(i+1)}...{x_1}^{(n-1)}]' -$$

$$[x_1...{x_1}^{(i-2)}{x_1}^{(i)}{x_1}^{(s-1)}{x_1}^{(i+1)}...{x_1}^{(n-1)}] - [x_1...{x_1}^{(i-1)}{x_1}^{(s-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}{x_1}^{(n)}].$$

In this equality, except of the last determinant, the others is generated by the set (2.2) and in according to induction hypothesis. In Lemma 2.1, if we take $x_1 = x_1$, $x_2 = x_1'$, ..., $x_n = {x_1}^{(n-1)}$, $x_0 = {x_1}^{(n)}$, $y_2 = x_1$, ..., $y_{i+1} = {x_1}^{(i-1)}$, $y_{i+2} = {x_1}^{(s-1)}$, $y_{i+3} = {x_1}^{(i+1)}$, ..., $y_n = {x_1}^{(n-2)}$ and eliminate the zero terms it is obtained that

$$[x_1{x_1}'...{x_1}^{(n-1)}].[{x_1}^{(n)}x_1{x_1}'...{x_1}^{(i-1)}{x_1}^{(s-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}] +$$

$$[x_1...{x_1}^{(i-1)}{x_1}^{(n)}{x_1}^{(i+1)}...{x_1}^{(n-1)}].[{x_1}^{(i)}x_1...{x_1}^{(i-1)}{x_1}^{(s-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}] +$$

$$[x_1...{x_1}^{(n-2)}{x_1}^{(n)}].[{x_1}^{(n-1)}x_1...{x_1}^{(n-2)}] = 0.$$

So the term $[{x_1}^{(n)}x_1{x_1}'...{x_1}^{(i-1)}{x_1}^{(s-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}]$ generated by the set (2.2).

Similarly, $[x_1...{x_1}^{(i-1)}{x_K}^{(\tau)}{x_1}^{(i+1)}...{x_1}^{(n-1)}]$, $\tau \geq 0, K = 2, 3, ..., r$ is obtained by induction on τ. For $\tau = 0$, $[x_1...{x_1}^{(i-1)}x_K{x_1}^{(i+1)}...{x_1}^{(n-1)}]$ is the generator. For $\tau = n-1$, let $[x_1...{x_1}^{(i-1)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-1)}]$ generated by the set (2.2) in according to induction hypothesis. Let us show that this is true for $\tau = n$.

$$\begin{array}{c} [x_1...{x_1}^{(i-1)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-1)}]' = \\ [x_1...{x_1}^{(i-2)}{x_1}^{(i)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-1)}] + \\ [x_1...{x_1}^{(i-1)}{x_K}^{(n)}{x_1}^{(i+1)}...{x_1}^{(n-1)}] + \\ [x_1...{x_1}^{(i-1)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}{x_1}^{(n)}]. \end{array} \tag{2.3}$$

In (2.3), we want to show that the determinant $[x_1...{x_1}^{(i-1)}{x_K}^{(n)}{x_1}^{(i+1)}...{x_1}^{(n-1)}]$ is generated by the set 2.2. Except the determinant

$$[x_1...{x_1}^{(i-1)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}{x_1}^{(n)}]$$

other determinants in (2.3) are generated by the set (2.2) and the induction hypothesis. For the last determinant, we use Lemma 2.1. If we take $x_1 = x_1$, $x_2 = x_1'$, ..., $x_n = {x_1}^{(n-1)}$, $x_0 = {x_1}^{(n)}$, $y_2 = x_1$, ..., $y_{i+1} = {x_1}^{(i-1)}$, $y_{i+2} = {x_K}^{(n-1)}$, $y_{i+3} = {x_1}^{(i+1)}$, ..., $y_n = {x_1}^{(n-2)}$ and eliminate the zero terms, it is obtained that

$$[x_1{x_1}'...{x_1}^{(n-1)}].[{x_1}^{(n)}x_1{x_1}'...{x_1}^{(i-1)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}]+$$
$$[x_1...{x_1}^{(i-1)}{x_1}^{(n)}{x_1}^{(i+1)}...{x_1}^{(n-1)}].[{x_1}^{(i)}x_1...{x_1}^{(i-1)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}]+$$
$$[x_1...{x_1}^{(n-2)}{x_1}^{(n)}].[{x_1}^{(n-1)}x_1...{x_1}^{(i-1)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}]=0.$$

So the term $[x_1...{x_1}^{(i-1)}{x_K}^{(n-1)}{x_1}^{(i+1)}...{x_1}^{(n-2)}{x_1}^{(n)}]$ generated by the set (2.2). By the induction hypothesis, the set (2.2) is generator set of $R<U>^G$. □

Theorem 2.2. *Let $G=SL(n,\mathbb{R})$ and $\{x_1,x_2,...,x_r\}$ and $\{y_1,y_2,...,y_r\}$ be two curve families such that x_1 and y_1 are regular. If for $i=0,1,...,n-1$ and $K=2,3,...,r$*

$$\begin{aligned}[x_1{x_1}'...{x_1}^{(n-1)}] &= [y_1{y_1}'...{y_1}^{(n-1)}]\\ [x_1{x_1}'...{x_1}^{(i-1)}{x_1}^{(n)}{x_1}^{(i+1)}...{x_1}^{(n-1)}] &= [y_1{y_1}'...{y_1}^{(i-1)}{y_1}^{(n)}{y_1}^{(i+1)}...{y_1}^{(n-1)}]\\ [x_1{x_1}'...{x_1}^{(i-1)}x_K{x_1}^{(i+1)}...{x_1}^{(n-1)}] &= [y_1{y_1}'...{y_1}^{(i-1)}y_K{y_1}^{(i+1)}...{y_1}^{(n-1)}]\end{aligned}$$

then $\{x_1,x_2,...,x_r\}\overset{G}{\approx}\{y_1,y_2,...,y_r\}$.

Proof. Since x_1 and y_1 are regular, we get $[x_1{x_1}'...{x_1}^{(n-1)}]\neq 0$ and $[y_1{y_1}'...{y_1}^{(n-1)}]\neq 0$. Let us take the matrices

$$A_{x_1}=\begin{pmatrix}x_{11}(t) & ... & {x_{11}}^{(n-1)}(t)\\ ... & ... & ...\\ x_{1n}(t) & ... & {x_{1n}}^{(n-1)}(t)\end{pmatrix} \text{ and } A'_{x_1}=\begin{pmatrix}{x_{11}}'(t) & ... & {x_{11}}^{(n)}(t)\\ ... & ... & ...\\ {x_{1n}}'(t) & ... & {x_{1n}}^{(n)}(t)\end{pmatrix}.$$

Since $[x_1{x_1}'...{x_1}^{(n-1)}]\neq 0$, there exists matrix inverse of A_{x_1}. Take the matrix $A_{x_1}^{-1}.A'_{x_1}=C$. Then $A'_{x_1}=A_{x_1}.C$. So the matrix C has the form

$$C=\begin{pmatrix}0 & ... & 0 & c_{1n}\\ 1 & ... & 0 & c_{2n}\\ ... & ... & ... & ...\\ 0 & ... & 1 & c_{nn}\end{pmatrix}$$

where

$$c_{1n}=\frac{[x^{(n)}{x_1}'...{x_1}^{(n-1)}]}{[x_1{x_1}'...{x_1}^{(n-1)}]}, c_{2n}=\frac{[x_1{x_1}^{(n)}...{x_1}^{(n-1)}]}{[x_1{x_1}'...{x_1}^{(n-1)}]}, ..., c_{nn}=\frac{[x_1{x_1}'...{x_1}^{(n-2)}{x_1}^{(n)}]}{[x_1{x_1}'...{x_1}^{(n-1)}]}.$$

From conditions of the theorem, it is obtained that $A_{x_1}^{-1}.A'_{x_1}=A_{y_1}^{-1}.A'_{y_1}$. So we have that

$$\begin{aligned}(A_{y_1}.A_{x_1}^{-1})' = A'_{y_1}.A_{x_1}^{-1}+A_{y_1}.(A_{x_1}^{-1})' &= A'_{y_1}.A_{x_1}^{-1}+A_{y_1}.(-A_{x_1}^{-1}.A'_{x_1}.A_{x_1}^{-1})\\ &= A_{y_1}.(A_{y_1}^{-1}.A'_{y_1}-A_{x_1}^{-1}.A'_{x_1}).A_{x_1}^{-1}=0\end{aligned}$$

Therefore $A_{y_1}.A_{x_1}^{-1}=g$, g is constant. And we get $A_{y_1}=gA_{x_1}$. So $detA_{y_1}=det(gA_{x_1})$ and since $[x_1{x_1}'...{x_1}^{(n-1)}]=[y_1{y_1}'...{y_1}^{(n-1)}]$, then it is obtain that $g\in SL(n,\mathbb{R})$. If we write this equality obviously, we have that $y_1(t)=gx_1(t), \forall t\in I$.

Let us take the matrix

$$D_{x_K}=\begin{pmatrix}x_{11}(t) & ... & {x_{11}}^{(n-2)}(t) & x_{K1}(t)\\ x_{12}(t) & ... & {x_{12}}^{(n-2)}(t) & x_{K2}(t)\\ ... & ... & ... & ...\\ x_{1n}(t) & ... & {x_{1n}}^{(n-2)}(t) & x_{Kn}(t)\end{pmatrix}$$

Let take $A_{x_1}^{-1}.D_{x_K}=H=[h_{ij}]$, $i,j=1,2,...,n$. Let us find the elements of this matrix. We have that $D_{x_K}=A_{x_1}.H$. Then similarly we get that

$$H=\begin{pmatrix}1 & ... & 0 & h_{1n}\\ 0 & ... & 0 & h_{2n}\\ ... & ... & ... & ...\\ 0 & ... & 0 & h_{nn}\end{pmatrix}$$

where

$$h_{1n}=\frac{[x_K{x_1}'...{x_1}^{(n-1)}]}{[x_1{x_1}'...{x_1}^{(n-1)}]}, h_{2n}=\frac{[x_1x_K...{x_1}^{(n-1)}]}{[x_1{x_1}'...{x_1}^{(n-1)}]}, ..., h_{nn}=\frac{[x_2{x_1}'...{x_1}^{(n-2)}x_K]}{[x_1{x_1}'...{x_1}^{(n-1)}]}.$$

Similarly, we can find the matrix $A_{y_1}^{-1}.D_{y_K}$. From conditions of the theorem, we have that $A_{x_1}^{-1}.D_{x_K} = A_{y_1}^{-1}.D_{y_K}$. We know that $A_{y_1} = gA_{x_1}$. Therefore we get

$$A_{x_1}^{-1}.D_{x_K} = (gA_{x_1})^{-1}.D_{y_K} = A_{x_1}^{-1}.g^{-1}.D_{y_K}$$

and then

$$D_{x_K} = g^{-1}.D_{y_K} \Longrightarrow D_{y_K} = g.D_{x_K}.$$

Then we get $y_K(t) = gx_K(t)$, $\forall t \in I, K = 2, 3, ..., r$. So for the same $g \in SL(n, \mathbb{R})$, it is obtained that $y_1(t) = gx_1(t)$ and $y_K(t) = gx_K(t)$. Therefore we get $y_K(t) = gx_K(t)$, $\forall t \in I, K = 1, 2, ..., r$. Hence $\{x_1, x_2, ..., x_r\} \overset{G}{\approx} \{y_1, y_2, ..., y_r\}$. □

Theorem 2.3. *Let $G = SL(n, \mathbb{R})$ and $f_1(t), f_2(t), ..., f_n(t)$, $f_n(t) \neq 0$ and $f_{Ki}(t)$ $(i = 1, 2, ..., n-1, K = 2, 3, ..., r)$ be C^∞-functions on I. Then there exist curves $x_1, x_2, ..., x_r$ where x_1 is regular such that*

$$\begin{aligned}
[x_1 ... {x_1}^{(i-1)} {x_1}^{(n)} {x_1}^{(i+1)} ... {x_1}^{(n-1)}] &= f_i(t), i = 0, 1, ..., n-1 \\
[x_1 {x_1}' ... {x_1}^{(n-1)}] &= f_n(t) \\
[x_1 ... {x_1}^{(i-1)} x_K {x_1}^{(i+1)} ... {x_1}^{(n-1)}] &= f_{Ki}(t), i = 0, 1, ..., n-1, K = 2, 3, ..., r.
\end{aligned}$$

Proof. Including an x_1 unknown,

$$\begin{aligned}
\frac{[x_1 ... {x_1}^{(i-1)} {x_1}^{(n)} {x_1}^{(i+1)} ... {x_1}^{(n-1)}]}{[x_1 {x_1}' ... {x_1}^{(n-1)}]} &= \frac{f_i(t)}{f_n(t)} = g_i(t), i = 0, 1, ..., n-1 \\
\frac{[x_1 ... {x_1}^{(i-1)} x_K {x_1}^{(i+1)} ... {x_1}^{(n-1)}]}{[x_1 {x_1}' ... {x_1}^{(n-1)}]} &= \frac{f_{Ki}(t)}{f_n(t)} = g_{Ki}(t), i = 0, ..., n-1, K = 2, ..., r.
\end{aligned}$$

We take the matrix multiplication $A_{x_1}^{-1}.A'_{x_1} = B$ such that $A'_{x_1} = A_{x_1}.B$. In here, matrix B has the form

$$B = \begin{pmatrix} 0 & ... & 0 & g_1(t) \\ 1 & ... & 0 & g_2(t) \\ ... & ... & ... & ... \\ 0 & ... & 1 & g_n(t) \end{pmatrix}.$$

Then we have the following differential equation system from this multiplication:

$$\begin{aligned}
x_{11} g_1(t) + {x_{11}}' g_2(t) + ... + {x_{11}}^{(n-1)} g_n(t) &= {x_{11}}^{(n)} \\
x_{12} g_1(t) + {x_{12}}' g_2(t) + ... + {x_{12}}^{(n-1)} g_n(t) &= {x_{12}}^{(n)} \\
&\vdots \\
x_{1n} g_1(t) + {x_{1n}}' g_2(t) + ... + {x_{1n}}^{(n-1)} g_n(t) &= {x_{1n}}^{(n)}
\end{aligned}$$

Let we take $x_{1i} = y$, $i = 1, 2, ..., n$. So we can write the above differential equation system as $g_1(t)y + g_2(t)y' + ... + g_n(t)y^{(n-1)} - y^{(n)} = 0$.

It is known that the theory of differential equations, there exist one solution of this differential equation. Let $x_1(t) = (y_1, y_2, ..., y_n)$ be the solution. Then the curve $x_1(t)$ satisfies the conditions of the theorem.

Similarly, take the matrices D_{x_K} and A_{x_1}. Let $A_{x_1}^{-1}.D_{x_K} = C$. So $D_{x_K} = A_{x_1}.C$. Then we get the matrix C as:

$$C = \begin{pmatrix} 1 & 0 & ... & 0 & g_{K0}(t) \\ 0 & 1 & ... & 0 & g_{K1}(t) \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & g_{Kn-2}(t) \\ 0 & 0 & ... & 0 & g_{Kn-1}(t) \end{pmatrix}.$$

Since $D_{x_K} = A_{x_1}.C$, we have the following differential equation system:

$$\begin{aligned}
x_{K1} &= x_{11} g_{K0}(t) + {x_{11}}' g_{K1}(t) + ... + {x_{11}}^{(n-1)} g_{Kn-1}(t) \\
x_{K2} &= x_{12} g_{K0}(t) + {x_{12}}' g_{K1}(t) + ... + {x_{12}}^{(n-1)} g_{Kn-1}(t) \\
&\vdots \\
x_{Kn} &= x_{1n} g_{K0}(t) + {x_{1n}}' g_{K1}(t) + ... + {x_{1n}}^{(n-1)} g_{Kn-1}(t)
\end{aligned}$$

So we get the curves $x_K = \begin{pmatrix} x_{K1} \\ x_{K2} \\ \vdots \\ x_{Kn} \end{pmatrix}$, $K = 2, 3, ..., r$.

It is obtained that $\frac{[x_1 x_1' ... x_1^{(n-1)}]'}{[x_1 x_1' ... x_1^{(n-1)}]} = \frac{f_n'(t)}{f_n(t)}$. Let $[x_1 x_1' ... x_1^{(n-1)}] = p(t)$. Then we get $\frac{p'(t)}{p(t)} = \frac{f_n'(t)}{f_n(t)}$.

So we get for some $\lambda \in \mathbb{R}/\{0\}$, $f_n(t) = \lambda p(t)$. Then it can be found a matrix h such that $deth \neq 0$. Let $y(t)$ be the curve $hx_1(t)$ and $y_K(t)$ be the curves $hx_K(t)$, $K = 2, 3, ..., r$. Then $y(t)$ and $y_K(t)$ are curves which provide the conditions of the theorem. Really, since

$$[y ... y^{(n-1)}] = [(hx_1) ... (hx_1)^{(n-1)}] = [hx_1 ... hx_1^{(n-1)}] = deth.[x_1 ... x_1^{(n-1)}] = f_n(t) \neq 0$$

then $y(t)$ is regular.

$$\frac{[y ... y^{(i-1)} y^{(n)} y^{(i+1)} ... y^{(n-1)}]}{[y ... y^{(n-1)}]} = \frac{deth.[x_1 ... x_1^{(i-1)} x_1^{(n)} x_1^{(i+1)} ... x_1^{(n-1)}]}{deth.[x_1 x_1' ... x_1^{(n-1)}]} = \frac{f_i(t)}{f_n(t)}$$

$$\frac{[y ... y^{(i-1)} y_K y^{(i+1)} ... y^{(n-1)}]}{[y ... y^{(n-1)}]} = \frac{deth.[x_1 ... x_1^{(i-1)} x_K x_1^{(i+1)} ... x_1^{(n-1)}]}{deth.[x_1 x_1' ... x_1^{(n-1)}]} = \frac{f_{Ki}(t)}{f_n(t)}$$

and since $[y ... y^{(n-1)}] = f_n(t)$, then for $i = 0, 1, ..., n-1$, $K = 2, 3, ..., r$ we get

$$[y ... y^{(i-1)} y^{(n)} y^{(i+1)} ... y^{(n-1)}] = f_i(t)$$

$$[y ... y^{(i-1)} y_K y^{(i+1)} ... y^{(n-1)}] = f_{Ki}(t).$$

Hence curves $y(t)$ and $y_K(t)$, $K = 2, 3, ..., r$ satisfy conditions of the theorem. □

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