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Research Article

Mappings between the lattices of saturated submodules with respect to a prime ideal

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Abstract

Let $\mathfrak{S}_p({}_RM)$ be the lattice of all saturated submodules of an R-module M with respect to a prime ideal p of a commutative ring R. We examine the properties of the mappings $\eta : \mathfrak{S}_p({}_RR) \to \mathfrak{S}_p({}_RM)$ defined by $\eta(I) = S_p(IM)$ and $\theta : \mathfrak{S}_p({}_RM) \to \mathfrak{S}_p({}_RR)$ defined by $\theta(N) = (N : M)$, in particular considering when these mappings are lattice homomorphisms. It is proved that if M is a semisimple module or a projective module, then η is a lattice homomorphism. Also, if M is a faithful multiplication R-module, then η is a lattice epimorphism. In particular, if M is a finitely generated faithful multiplication R-module, then η is a lattice isomorphism and its inverse is θ. It is shown that if M is a distributive module over a semisimple ring R, then the lattice $\mathfrak{S}_p({}_RM)$ forms a Boolean algebra and η is a Boolean algebra homomorphism.

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1. Introduction

We assume throughout this paper that all rings are commutative with nonzero identity and all modules are unitary. Let R be a ring and M be an R-module. For any submodule N of M, we denote the annihilator of the R-module M/N by $(N : M)$, i.e., $(N : M) = \{r \in R \mid rM \subseteq N\}$.

It is well-known that the collection of all submodules of M forms a lattice with respect to the operations $\vee$ and $\wedge$ defined by

$$L \vee N = L + N \text{ and } L \wedge N = L \cap N.$$

Note that this lattice, denoted $\mathcal{L}({}_RM)$, is bounded with the least element (0) and greatest element M. Recently, P.F. Smith has studied several mappings between $\mathcal{L}({}_RR)$ and $\mathcal{L}({}_RM)$ [22–24]. For instance, in [22], he examined conditions under which the mappings $\lambda : \mathcal{L}({}_RR) \to \mathcal{L}({}_RM)$ defined by $\lambda(I) = IM$ and $\mu : \mathcal{L}({}_RM) \to \mathcal{L}({}_RR)$ defined by $\mu(N) = (N : M)$ are injective, surjective or lattice homomorphisms. An R-module M is called a λ-*module* (respectively μ-*module*), if λ (respectively μ) is a lattice homomorphism.

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The study of the mappings λ and μ continued in [23], considering when these mappings are complete lattice homomorphisms.

A proper submodule P of M is called a *prime submodule* if for $r \in R$ and $x \in M$, $rx \in P$ implies that $r \in (P : M)$ or $x \in P$ (see, for example, [2,6,18,19]). For a proper submodule N of an R-module M, the intersection of all prime submodules of M containing N is called the *radical* of N and denoted by $\operatorname{rad} N$; if there are no such prime submodules, $\operatorname{rad} N$ is M (see, for example, [11, 14, 17]). A submodule N of M is called a *radical submodule* if $\operatorname{rad} N = N$. The collection of all radical submodules of M which is denoted by $\mathcal{R}(_RM)$ forms a lattice with respect to the following operations:

$$L \vee N = \operatorname{rad}(L + N) \text{ and } L \wedge N = L \cap N.$$

Note that $\mathcal{R}(_RM)$ is a bounded lattice with the least element $\operatorname{rad}(0)$ and the greatest element M.

In [20], H.F. Moghimi and J.B. Harehdashti have studied the properties of the mappings $\rho : \mathcal{R}(_RR) \to \mathcal{R}(_RM)$ defined by $\rho(I) = \operatorname{rad}(IM)$ and $\sigma : \mathcal{L}(_RR) \to \mathcal{L}(_RM)$ defined by $\sigma(N) = (N : M)$, in particular considering when these mappings are lattice monomorphisms or epimorphisms. Later in [9], they investigated conditions under which these mappings are complete homomorphisms. Note that ρ is always a lattice homomorphism, but not necessarily a complete lattice homomorphism. An R-module M is called a *σ-module* if σ is a lattice homomorphism.

Let M be an R-module. For a prime ideal p of R and a submodule N of M, the set $S_p(N) = \{m \in M \mid cm \in N \text{ for some } c \in R \setminus p\}$ is called the *saturation* of N with respect to p. It is clear that $N \subseteq S_p(N)$. It is said that N is *saturated* with respect to p, if $N = S_p(N)$. It is easily seen that $S_p(N)$ is a saturated submodule of M (see [15, 16], for more details about saturation of submodules). The collection of all saturated submodules of an R-module M with respect to a fixed prime ideal p of R is a lattice with the following operations:

$$L \vee N = S_p(L + N) \quad \text{and} \quad L \wedge N = L \cap N.$$

We shall denote this lattice by $\mathfrak{S}_p(_RM)$, or by $\mathfrak{S}_p(M)$ if there is no ambiguity about R. Note that $\mathfrak{S}_p(M)$ is bounded, with the least element $S_p(0)$ and the greatest element M.

Let R be a ring, p a fixed prime ideal of R and M an R-module. Now consider the mappings $\eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M)$ defined by

$$\eta(I) = S_p(IM),$$

for every saturated ideal I of R, and $\theta : \mathfrak{S}_p(M) \to \mathfrak{S}_p(R)$ defined by

$$\theta(N) = (N : M),$$

for every saturated submodule N of M. It will be convenient for us to call the module M an η-module (resp. a θ-module) in case the above mapping η (resp. θ) is a lattice homomorphism.

In this paper, we investigate conditions under which η and θ are lattice homomorphisms, in particular considering when η and θ are Boolean algebra homomorphisms. It is shown that modules over Prüfer domains (Corollary 2.4), projective modules (Corollary 2.6) and semisimple R-modules (Corollary 2.7) are three classes of η-modules. It is proved that if M is a faithful multiplication R-module, then η is a lattice epimorphism, and in particular $\mathfrak{S}_p(M)$ is isomorphic to a quotient of $\mathfrak{S}_p(R)$ (Theorem 2.8) for all prime ideals p of R. It is shown that a finitely generated module M is a θ-module if and only if it is a multiplication module (Corollary 2.11). In particular, every cyclic R-module is a θ-module (Corollary 2.10). Moreover, if M is a finitely generated faithful multiplication R-module then η and θ are lattice isomorphisms (Corollary 2.17).

An R-module M is called *distributive* if $\mathcal{L}(_RM)$ is a distributive lattice (see, for example,

[8]). A ring R is called *arithmetical* if it is a distributive R-module. We say that an R-module M is $\mathfrak{S}$-*distributive* with respect to a prime ideal p of R if $\mathfrak{S}_p(M)$ is a distributive lattice. It is proved that an R-module M is distributive if and only if it is $\mathfrak{S}$-distributive with respect to any prime ideal of R (Corollary 3.4). In particular, every multiplication module over an arithmetical ring R is $\mathfrak{S}$-distributive with respect to any prime ideal of R (Corollary 3.5). It is shown that if M is a distributive module over a semisimple ring R, then $\mathfrak{S}_p(M)$ forms a Boolean algebra (Theorem 3.7) and η is a Boolean algebra homomorphism (Theorem 3.13). In particular, if M is a multiplication module over a semisimple ring R, then η is a Boolean algebra epimorphism (Corollary 3.14).

2. η-modules and θ-modules

We start with a lemma which collects some facts about saturation of submodules.

Lemma 2.1. *Let R be a ring, p a prime ideal of R and M an R-module. Then*

(1) $S_p(L \cap N) = S_p(L) \cap S_p(N)$ *for all submodules L and N of M;*

(2) $S_p(S_p(IM) + S_p(JM)) = S_p(S_p(I+J)M) = S_p(IM + JM)$ *for all ideals I and J of R.*

Proof. (1) Clear.
(2) Since $IM \subseteq (I+J)M \subseteq S_p(I+J)M$, we conclude that $S_p(IM) \subseteq S_p(S_p(I+J)M)$. Similarly, $S_p(JM) \subseteq S_p(S_p(I+J)M)$. Therefore, we have $S_p(IM)+S_p(JM) \subseteq S_p(S_p(I+J)M)$. Hence we have $S_p(S_p(IM)+S_p(JM)) \subseteq S_p(S_p(I+J)M)$. Now, let $x \in S_p(S_p(I+J)M)$. Then there exists $c \in R \setminus p$ such that $cx \in S_p(I+J)M$. Therefore $cx = \sum_{i=1}^{k} r_i x_i$ for some $r_i \in S_p(I+J)$ and $x_i \in M$ $(1 \le i \le k)$. Thus there are $c_i \in R \setminus p$ $(1 \le i \le k)$ such that $c_i r_i \in I+J$, and so $c_1 \dots c_k cx \in (I+J)M$. It follows that $x \in S_p((I+J)M)$. Hence we have $S_p(S_p(I+J)M) \subseteq S_p(IM+JM)$. It is also clear that $S_p(IM+JM) \subseteq S_p(S_p(IM)+S_p(JM))$. □

Theorem 2.2. *Let R be a ring, p a prime ideal of R and M an R-module. Then the following statements are equivalent:*

(1) *M is an η-module over R;*

(2) $S_p((I \cap J)M) = S_p(IM) \cap S_p(JM)$ *for all ideals I and J of R;*

(3) $(I_p \cap J_p)M_p = I_pM_p \cap J_pM_p$ *for all ideals I and J of R;*

(4) *M_p is a λ-module over R_p.*

Proof. (1) ⇒ (2) By definition.
(2) ⇒ (1) Let $I, J \in \mathfrak{S}_p(R)$. By the assumption, $\eta(I \wedge J) = \eta(I) \wedge \eta(J)$.
By using Lemma 2.1, we have

$$\begin{aligned}\eta(I \vee J) &= S_p((I \vee J)M) = S_p(S_p(I+J)M)\\ &= S_p(S_p(IM) + S_p(JM))\\ &= S_p(IM) \vee S_p(JM)\\ &= \eta(I) \vee \eta(J).\end{aligned}$$

(2) ⇒ (3) Let $z \in I_pM_p \cap J_pM_p$. Then $z = \sum_{i=1}^{k} a_i x_i / s_i = \sum_{i=1}^{k} b_i y_i / t_i$ for some $a_i \in I$, $b_i \in J$, $x_i, y_i \in M$, $s_i, t_i \in R \setminus p$. Hence we have $s_1 \dots s_k t_1 \dots t_k z \in IM \cap JM$ which follows that $z \in S_p(IM) \cap S_p(JM)$. Therefore by (2), $z \in S_p((I \cap J)M)$. Thus $cz \in (I \cap J)M$ for some $c \in R \setminus p$, and so $z \in (I_p \cap J_p)M_p$ as desired. The reverse inclusion is clear.
(3) ⇒ (2) Let $x \in S_p(IM) \cap S_p(JM)$. Then $cx \in IM$ and $dx \in JM$ for some $c, d \in R \setminus p$. Therefore $cx = \sum_{i=1}^{k} c_i x_i$ and $dx = \sum_{j=1}^{k} d_j x'_j$ for some $c_i \in I$, $d_j \in J$ and $x_i, x'_j \in M$ $(1 \le i, j \le k)$. Thus $c_1 dx = \sum_{j=1}^{k} c_1 d_j x'_j$ and hence $c_1 dx \in (I \cap J)M$ such that $c_1 d \in R \setminus p$. Thus $x \in S_p((I \cap J)M)$. The reverse inclusion is clear.
(3) ⇔ (4) Follows from [22, Lemma 2.1 (ii)]. □

Let R be a domain with the field of fractions K. A non-zero ideal I of R is called *invertible* provided $I^{-1}I = R$ where $I^{-1} = \{k \in K : kI \subseteq R\}$. A domain R is called *Prüfer* if every non-zero finitely generated ideal of R is invertible (see, for more details, [13]).

Corollary 2.3. *Let R be a domain, p a prime ideal of R and M an R-module. Then the following statements are equivalent:*

(1) *R_p is Prüfer;*
(2) *Every R_p-module is a λ-module;*
(3) *Every R-module is an η-module.*

Proof. (1) $\Leftrightarrow$ (2) By [22, Theorem 2.3].
(2) $\Leftrightarrow$ (3) By Theorem 2.2. □

Corollary 2.4. *Let R be any Prüfer domain. Then every R-module is an η-module.*

Proof. Let R be a Prüfer domain and p be a prime ideal of R. Then by [13, Theorem 6.6], R_p is a valuation ring. Thus by [22, Proposition 2.4], every R_p-module is a λ-module and hence by Corollary 2.3, every R-module is an η-module. □

Theorem 2.5. *Let R be any ring. Then*

(1) *Every direct summand of an η-module is an η-module.*
(2) *Every direct sum of λ-modules is an η-module.*

Proof. (1) Let K be a direct summand of an η-module M. Let I and J be any ideals of R and p be a prime ideal of R. Then by Lemma 2.1 (1) and Theorem 2.2, we have

$$\begin{aligned} S_p(IK) \cap S_p(JK) &= S_p(K \cap IM) \cap S_p(K \cap JM) \\ &= S_p(K) \cap S_p(IM) \cap S_p(JM) \\ &= S_p(K) \cap S_p((I \cap J)M) \\ &= S_p(K \cap (I \cap J)M) \\ &= S_p((I \cap J)K). \end{aligned}$$

Thus by Theorem 2.2, K is an η-module.
(2) Let M_i $(i \in \mathfrak{I})$ be any collection of λ-modules and let $M = \oplus_{i\in\mathfrak{I}} M_i$. Given any ideals I and J of R, by [22, Lemma 2.1], we have

$$\begin{aligned} S_p(IM) \cap S_p(JM) &= S_p(\oplus_{i\in\mathfrak{I}} IM_i) \cap S_p(\oplus_{i\in\mathfrak{I}} JM_i) \\ &= S_p(\oplus_{i\in\mathfrak{I}} IM_i \cap \oplus_{i\in\mathfrak{I}} JM_i) \\ &= S_p(\oplus_{i\in\mathfrak{I}} (IM_i \cap JM_i)) \\ &= S_p(\oplus_{i\in\mathfrak{I}} (I \cap J)M_i) \\ &= S_p((I \cap J)M). \end{aligned}$$

Thus by Theorem 2.2, M is an η-module. □

Corollary 2.6. *For any ring R, every projective R-module is an η-module.*

Proof. By [22, Lemma 2.1], every ring R is a λ-module. Thus by [10, Theorem IV.2.1] and Theorem 2.5(2), every free R-module is an η-module, and therefore by [10, Theorem IV.3.4] and Theorem 2.5(1), every projective R-module is an η-module. □

Corollary 2.7. *For any ring R, every semisimple R-module is an η-module.*

Proof. Clearly every simple module is a λ-module. Since any semisimple module is a direct sum of a family of simple submodules, the result follows from Theorem 2.5(2). □

An R-module M is called a *multiplication* module if the mapping λ is surjective, i.e., for each submodule N of M there exist an ideal I of R such that $N = IM$. In this case, we can take $I = (N : M)$ (see, for example, [4, 7]).

Theorem 2.8. *Let M be a faithful multiplication R-module. Then η is a lattice epimorphism.*
In particular, $\mathfrak{S}_p(M)$ is isomorphic to a quotient of $\mathfrak{S}_p(R)$ for all prime ideals p of R.

Proof. Since M is a faithful multiplication R-module, M is a λ-module by [22, Theorem 2.12]. Thus by [22, Lemma 2.1], $(I \cap J)M = IM \cap JM$ for all ideals I and J of R. It follows that, by Lemma 2.1 (1),

$$S_p((I \cap J)M) = S_p(IM \cap JM) = S_p(IM) \cap S_p(JM)$$

for all ideals I and J and prime ideals p of R. Hence by Theorem 2.2, η is a lattice homomorphism. Now, let p be a prime ideal of R and $N \in \mathfrak{S}_p(M)$. Since M is a multiplication module, we have

$$\eta((N : M)) = S_p((N : M)M) = S_p(N) = N$$

and therefore η is an epimorphism. Now, we define the relation $\sim$ on $\mathfrak{S}_p(R)$ by

$$I{\sim}J \Leftrightarrow S_p(IM) = S_p(JM).$$

It is evident that $\sim$ is an equivalence relation on $\mathfrak{S}_p(R)$. We show that $\sim$ is a congruence relation. Assume that $I_1{\sim}J_1$ and $I_2{\sim}J_2$. Thus we have $S_p(I_1M) = S_p(J_1M)$ and $S_p(I_2M) = S_p(J_2M)$. Since M is a faithful multiplication module,

$$\begin{aligned} S_p((I_1 \cap J_1)M) &= S_p(I_1M) \cap S_p(J_1M) \\ &= S_p(I_2M) \cap S_p(J_2M) \\ &= S_p((I_2 \cap J_2)M), \end{aligned}$$

and therefore $I_1 \wedge J_1{\sim}I_2 \wedge J_2$. Also, by Lemma 2.1 (2),

$$\begin{aligned} S_p(S_p(I_1 + J_1)M) &= S_p(S_p(I_1M) + S_p(J_1M)) \\ &= S_p(S_p(I_2M) + S_p(J_2M)) \\ &= S_p(S_p(I_2 + J_2)M) \end{aligned}$$

which follows that $I_1 \vee J_1{\sim}I_2 \vee J_2$. Thus $\mathfrak{S}_p(R)/{\sim}$, the set of equivalence classes with respect to $\sim$, is a lattice with the following operations:

$$I/{\sim}\ \tilde{\vee}\ J/{\sim} = I \vee J/{\sim} \ \text{ and } \ I/{\sim}\ \tilde{\wedge}\ J/{\sim} = I \wedge J/{\sim}.$$

Now, the mapping $\bar{\eta} : \mathfrak{S}_p(R)/{\sim} \to \mathfrak{S}_p(M)$ given by $\bar{\eta}(I/{\sim}) = \eta(I) = S_p(IM)$ is a lattice isomorphism. □

Recall that $\theta : \mathfrak{S}_p(M) \to \mathfrak{S}_p(R)$ defined by $\theta(N) = (N : M)$ is the restriction of the mapping $\mu : \mathcal{L}(_RM) \to \mathcal{L}(_RR)$ to $\mathfrak{S}_p(M)$ given in [22]. Thus every μ-module is a θ-module.

Theorem 2.9. *Let R be a ring and M an R-module. Consider the following statements:*

(1) *M is a θ-module over R;*
(2) *$(L + N : M) = (L : M) + (N : M)$ for all saturated submodules L and N of M;*
(3) *$(L_p + N_p : M_p) = (L_p : M_p) + (N_p : M_p)$ for all submodules L and N of M and for all prime ideals p of R;*
(4) *$(L + N : M) = (L : M) + (N : M)$ for all submodules L and N of M;*
(5) *M is a μ-module over R.*

Then (1) $\Leftrightarrow$ (2) and (4) $\Leftrightarrow$ (5).
In particular, if M is a finitely generated R-module, then all of the above statements are equivalent.

Proof. (1) ⇔ (2) Follows from definition.
(4) ⇔ (5) Follows from [22, Lemma 3.1].
(4) ⇒ (2) Clear.
(2) ⇒ (3) Suppose that M is finitely generated. Then $M = Rm_1 + \ldots + Rm_k$ for some $m_i \in M$ $(1 \leq i \leq k)$. Let L and N be two submodules of M. First we show that $(S_p(L)+S_p(N) : M)_p = ((L+N)_p : M_p)$ for all prime ideals p of R. Let p be a prime ideal of R and assume that $r/1 \in (S_p(L) + S_p(N) : M)_p$. It follows that $rM \subseteq S_p(L) + S_p(N)$. Thus $rm_i = x_i + y_i$ for some $x_i \in S_p(L)$, $y_i \in S_p(N)$ $(1 \leq i \leq k)$. Therefore $c_i x_i \in L$ and $d_i y_i \in N$ for some $c_i, d_i \in R \setminus p$ $(1 \leq i \leq k)$. Now, since $c_1 \ldots c_k d_1 \ldots d_k rM \subseteq L + N$, we have $r/1 \in ((L + N)_p : M_p)$, as requested. Hence, by using [15, Theorem 2.1], we have

$$\begin{aligned}(L_p : M_p) + (N_p : M_p) &= (S_p(L) : M)_p + (S_p(N) : M)_p \\ &= ((S_p(L) : M) + (S_p(N) : M))_p \\ &= (S_p(L) + S_p(N) : M)_p \\ &= ((L + N)_p : M_p) \\ &= (L_p + N_p : M_p).\end{aligned}$$

(3) ⇒ (4) Follows from [3, Proposition 3.8 and Corollaries 3.4 and 3.15].
(4) ⇒ (3) Follows from [3, Corollary 3.4 and Corollary 3.15]. □

Corollary 2.10. *For any ring R, every cyclic R-module is a θ-module.*

Proof. Follows from [22, Corollary 3.7] and Theorem 2.9. □

Corollary 2.11. *Let M be a finitely generated R-module. Then the following statements are equivalent:*

(1) *M is a θ-module over R;*
(2) *M_p is a θ-module over R_p for every prime ideal p of R;*
(3) *M_m is a θ-module over R_m for every maximal ideal m of R;*
(4) *M is a μ-module over R;*
(5) *M is a σ-module over R;*
(6) *M is a multiplication module over R.*

Proof. (1) ⇔ (4) By Theorem 2.9.
(4) ⇔ (5) ⇔ (6) By [20, Theorem 2.11 and Theorem 2.19].
(6) ⇔ (2) ⇔ (3) By [4, Lemma 2 (ii)], [20, Theorem 2.11] and Theorem 2.9. □

Corollary 2.12. *Let R be a ring. If M is a finitely generated θ-module over R and $((0) : M) = Re$ for some idempotent e of R, then M is an η-module over R. In particular, every finitely generated faithful θ-module is an η-module.*

Proof. By Corollary 2.11 M is a multiplication R-module, and then by [21, Theorem 11] M is a projective R-module. Thus by Corollary 2.6, M is an η-module over R. □

Now, we investigate conditions under which η and θ are injective or surjective.

Theorem 2.13. *Let η and θ be as before. Then*

(1) *$\eta\theta\eta = \eta$;*
(2) *$\theta\eta\theta = \theta$.*

Proof. (1) Let p be a prime ideal of R and $I \in \mathfrak{S}_p(R)$. Since $\eta\theta\eta(I) = S_p((S_p(IM) : M)M)$, we must show that $S_p((S_p(IM) : M)M) = S_p(IM)$. First note that, since $I \subseteq (S_p(IM) : M)$, we have $IM \subseteq (S_p(IM) : M)M$ and thus $S_p(IM) \subseteq S_p((S_p(IM) : M)M)$. The reverse inclusion follows from

$$S_p((S_p(IM) : M)M) \subseteq S_p(S_p(IM)) = S_p(IM).$$

(2) Let p be a prime ideal of R and $N \in \mathfrak{S}_p(M)$. Now, since $\theta\eta\theta(N) = (S_p((N : M)M) : M)$, we must show that $(S_p((N : M)M) : M) = (N : M)$. Since $(N : M)M \subseteq S_p((N : M)M)$, we have $(N : M) \subseteq (S_p((N : M)M) : M)$. The reverse inclusion follows from

$$(S_p((N : M)M) : M) \subseteq (S_p(N) : M) = (N : M).$$

□

Corollary 2.14. *Let η and θ be as before, and p be a prime ideal of R. Then the following statements are equivalent:*

(1) $\eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M)$ *is a surjection;*
(2) $\eta\theta = 1$*;*
(3) $S_p((N : M)M) = N$ *for all* $N \in \mathfrak{S}_p(M)$*;*
(4) $\theta : \mathfrak{S}_p(M) \to \mathfrak{S}_p(R)$ *is an injection.*

Proof. (1) ⇒ (2) and (4) ⇒ (2) follows from Theorem 2.13.
(2) ⇔ (3), (2) ⇒ (1) and (2) ⇒ (4) are clear. □

Corollary 2.15. *Let η and θ be as before, and p be a prime ideal of R. Then the following statements are equivalent:*

(1) $\eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M)$ *is an injection;*
(2) $\theta\eta = 1$*;*
(3) $(S_p(IM) : M) = I$ *for all* $I \in \mathfrak{S}_p(R)$*;*
(4) $\theta : \mathfrak{S}_p(M) \to \mathfrak{S}_p(R)$ *is a surjection.*

Proof. (1) ⇒ (2) and (4) ⇒ (2) follows from Theorem 2.13.
(2) ⇔ (3), (2) ⇒ (1) and (2) ⇒ (4) are clear. □

Corollary 2.16. *Let η and θ be as before. Then η is a bijection if and only if θ is a bijection. In this case η and θ are inverse of each other.*

Proof. By Corollaries 2.14 and 2.15. □

Corollary 2.17. *Let R be a ring and M be a finitely generated faithful multiplication R-module. Then the mappings η and θ are lattice isomorphisms. In particular, η and θ are inverse of each other, and therefore $\mathfrak{S}_p(R)$ and $\mathfrak{S}_p(M)$ are isomorphic lattices for all prime ideals p of R.*

Proof. Since M is a faithful multiplication R-module, η is an epimorphism by Theorem 2.8, and hence θ is a monomorphism by Corollary 2.14 and [22, Theorem 3.8]. On the other hand, by [15, Proposition 3.2], we have

$$(S_p(IM) : M) = S_p(IM : M) = S_p(I) = I,$$

for all prime ideals p of R and $I \in \mathfrak{S}_p(R)$. Hence, by Corollary 2.15, η is an injection and θ is a surjection. Hence η is an isomorphism and its inverse is θ. □

3. $\mathfrak{S}_p(M)$ as a Boolean algebra

We start this section by recalling the following basic definition.

Definition 3.1. Let R be a ring and p be a prime ideal of R. An R-module M is called a $\mathfrak{S}$-*distributive* module with respect to p, if $\mathfrak{S}_p(M)$ is a distributive lattice.

First note the following simple fact.

Lemma 3.2. *Let R be a ring, p a prime ideal of R and M be an R-module. Then the following statements are equivalent:*

(1) *M is $\mathfrak{S}$-distributive with respect to p;*
(2) $K \cap S_p(L + N) = S_p((K \cap L) + (K \cap N))$ *for all* $K, L, N \in \mathfrak{S}_p(M)$*;*

(3) $S_p(K+(L\cap N)) = S_p(K+L)\cap S_p(K+N)$ *for all* $K, L, N \in \mathfrak{S}_p(M)$.

Proof. By [5, Theorem I.3.2]. □

The following example shows that a ring R may be $\mathfrak{S}$-distributive with respect to a prime ideal and not with respect to another one.

Example 3.3. Let $R = K[X, Y]$ be the ring of polynomials with independent indeterminates X and Y over a field K. It is evident that R is $\mathfrak{S}$-distributive with respect to (0), since $\mathfrak{S}_{(0)}(R) = \{(0), R\}$. However, R is not $\mathfrak{S}$-distributive with respect to $m = RX+RY$. Let $p_1 = RX$, $p_2 = RY$, $p_3 = R(X+Y)$. Since p_1, p_2 and p_3 are prime ideals of R, these ideals are saturated with respect to m and hence $p_3\cap p_1$ and $p_3\cap p_2$ are saturated with respect to m by Lemma 2.1 (1). Now, since $p_3\cap(p_1+p_2) \nsubseteq (p_3\cap p_1)+(p_3\cap p_2)$, R is not $\mathfrak{S}$-distributive with respect to m by Lemma 3.2.

It is remarked that some classes of R-modules are characterized by using the localization with respect to all prime ideal of R (see for example [1]). In the next result, it is seen that the class of distributive modules has this property.

Corollary 3.4. *Let R be a ring and M be an R-module. Then the following conditions are equivalent:*

(1) *M is a distributive R-module;*
(2) *M is $\mathfrak{S}$-distributive with respect to any prime ideal p of R;*
(3) *M_p is a distributive R_p-module for all prime ideals p of R.*

Proof. (1) ⇒ (2) Let p be a prime ideal of R and $K, L, N \in \mathfrak{S}_p(M)$. By Lemma 2.1 (1) and the assumption, we have

$$S_p(K+L)\cap S_p(K+N) = S_p((K+L)\cap(K+N)) = S_p(K+(L\cap N)).$$

Thus, the result follows from Lemma 3.2 (3).
(2) ⇒ (3) Let p be a prime ideal of R and K, L and N be submodules of M. It suffices to show that $(K_p+L_p)\cap(K_p+N_p) \subseteq (K_p+(L_p\cap N_p))$ or equivalently, by [3, Corollary 3.4], $((K+L)\cap(K+N))_p \subseteq (K+(L\cap N))_p$. For this, let $x/s \in ((K+L)\cap(K+N))_p$. Thus there are elements $k_1, k_2 \in K$, $l \in L$, $n \in N$ and $s_1, s_2 \in R\setminus p$ such that $x/s = (k_1+l)/s_1 = (k_2+n)/s_2$. It follows that $uss_1s_2x = (k_1+l) = (k_2+n)$ for some $u \in R\setminus p$ so that $x \in S_p(K+L)\cap S_p(K+N)$. Hence by (2), $x \in S_p(K+(L\cap N))$. Therefore $cx \in K+(L\cap N)$ for some $c \in R\setminus p$ which implies that $x/s = cx/cs \in (K+(L\cap N))_p$, as required.
(3) ⇒ (1) Follows from [3, Corollary 3.4 and Proposition 3.8].

□

Corollary 3.5. *Let R be an arithmetical ring, and M be a multiplication R-module. Then M is a $\mathfrak{S}$-distributive R-module with respect to any prime ideal of R.*

Proof. By [8, Proposition 1.2] and Corollary 3.4. □

Our next example shows that M being a multiplication module is needed in Corollary 3.5.

Example 3.6. Let K be a field and $V = K\oplus K$ be the usual two-dimensional vector space over K. It is easy to see that every subspace of V is saturated with respect to (0). Now if $W_1 = K(1,0)$, $W_2 = K(0,1)$ and $W_3 = K(1,1)$. Then $W_3\cap(W_1+W_2) = W_3$ while $(W_3\cap W_1)+(W_3\cap W_2) = K(0,0)$. Thus V is not $\mathfrak{S}$-distributive

We recall that a distributive lattice $(L, \vee, \wedge)$ is a Boolean algebra if there is a unary operation $'$ on L and two constants 0 and 1 such that $x\wedge x' = 0$ and $x\vee x' = 1$.

Let M be a semisimple R-module and N a submodule of M. Then, by definition, there is a submodule L of M such that $M = N\oplus L$. We define the unary operation $'$ on $\mathfrak{S}_p(M)$ by $N' = S_p(L)$.

Theorem 3.7. *Let R be a semisimple ring, p a prime ideal of R and M a distributive R-module. Then the lattice $\mathfrak{S}_p(M)$ is a Boolean algebra with the unary operation $'$ defined above, $\mathbf{0} = S_p(0)$ and $\mathbf{1} = M$.*

Proof. By Corollary 3.4, M is a $\mathfrak{S}$-distributive R-module. By using Lemma 2.1 (1),

$$N \wedge N' = N \cap N' = S_p(N) \cap S_p(L) = S_p(N \cap L) = S_p(0) = \mathbf{0}.$$

Moreover, $M = N + L \subseteq S_p(N) + S_p(L) \subseteq S_p(S_p(N) + S_p(L))$, which implies

$$N \vee N' = S_p(N + N') = S_p(S_p(N) + S_p(L)) = M.$$

Hence $\mathfrak{S}_p(M)$ is a Boolean algebra. □

From now on, $\mathfrak{S}_p(M)$ is assumed to be a Boolean algebra with the above assumptions.

Corollary 3.8. *For any semisimple ring R, $\mathfrak{S}_p(R)$ is a Boolean algebra with respect to any prime ideal p of R.*

Proof. Let R be a semisimple ring and p a prime ideal of R. By [12, Exercise 1.2.5] R is an arithmetical ring. Thus by Theorem 3.7, $\mathfrak{S}_p(R)$ is a Boolean algebra. □

Corollary 3.9. *Let R be a semisimple ring and M be a distributive R-module. Then $\mathfrak{S}_p(M)$ is a Boolean ring with the following operations:*

$$L + N = S_p(L \cap S_p(\tilde{N}) + S_p(\tilde{L}) \cap N) \text{ and } L \cdot N = L \cap N,$$

where $M = L \oplus \tilde{L} = N \oplus \tilde{N}$.

Proof. Follows from Theorem 3.7 and [5, Theorem IV.2.3]. □

Corollary 3.10. *Let R be a semisimple ring, p a prime ideal of R and M a multiplication R-module. Then M is cyclic and the lattice $\mathfrak{S}_p(M)$ is a Boolean algebra.*

Proof. Since R is a semisimple ring, by [12, Corollary 2.6], R is an Artinian ring. Hence M is cyclic by [7, Corollary 2.9]. Also, by [12, Exercise 1.2.5], R is an arithmetical ring. Thus by [8, Proposition 1.2], M is a distributive R-module. Hence by Theorem 3.7, $\mathfrak{S}_p(M)$ is a Boolean algebra with respect to any prime ideal p of R. □

Theorem 3.11. *Let R be a ring, p a prime ideal of R, M an R-module and N a submodule of M. Then the followings hold:*

(1) *For any submodule L containing N, $S_p(L/N) = S_p(L)/N$. In particular, the assignment $L \mapsto L/N$ is a one to one corresponding between the set $\{L \mid L \in \mathfrak{S}_p(M), L \supseteq N\}$ and $\mathfrak{S}_p(M/N)$;*
(2) *If M is a $\mathfrak{S}$-distributive lattice over R with respect to p, then M/N is $\mathfrak{S}$-distributive over R with respect to p;*
(3) *If R is a semisimple ring and M a distributive R-module, then $\mathfrak{S}_p(M/N)$ is a Boolean algebra.*

Proof. (1) Clear.
(2) Let $\mathfrak{S}_p(M)$ be a distributive lattice with the operations $\vee$ and $\wedge$ and $\mathfrak{S}_p(M/N)$ be a lattice with the operations $\tilde{\vee}$ and $\tilde{\wedge}$. It is seen that $\tilde{\vee}$ and $\tilde{\wedge}$ are expressed by $\vee$ and $\wedge$ respectively as follows:

$$\begin{aligned} L/N \;\tilde{\vee}\; K/N &= S_p(L/N + K/N) \\ &= S_p((L+K)/N) \\ &= S_p(L+K)/N \\ &= (L \vee K)/N, \end{aligned}$$

and

$$L/N \;\tilde{\wedge}\; K/N = L/N \cap K/N = (L \cap K)/N = (L \wedge K)/N.$$

By these statements, the distributivity of $\mathfrak{S}_p(M/N)$ follows immediately from the distributivity of $\mathfrak{S}_p(M)$.
(3) Follows from Theorem 3.7 and (2). □

Theorem 3.12. *Let R be a ring, T a multiplicatively closed subset of R, M an R-module and N a submodule of M. Then the followings hold:*

1. *$S_{T^{-1}p}(T^{-1}N) = T^{-1}(S_p(N))$ for all prime ideals p disjoint from T. In particular, $N \in \mathfrak{S}_p(M)$ if and only if $T^{-1}N \in \mathfrak{S}_{T^{-1}p}(T^{-1}M)$ for all prime ideals p disjoint from T;*
2. *If M is a $\mathfrak{S}$-distributive lattice over R with respect to a prime ideal p of R such that $p \cap T = \emptyset$, then $T^{-1}M$ is $\mathfrak{S}$-distributive over $T^{-1}R$ with respect to $T^{-1}p$;*
3. *If R is a semisimple ring, p a prime ideal of R with $p \cap T = \emptyset$ and M a distributive R-module, then $\mathfrak{S}_{T^{-1}p}(T^{-1}M)$ is a Boolean algebra.*

Proof. (1) Clear.
(2) Let p be a prime ideal of R such that $p \cap T = \emptyset$. Let $\mathfrak{S}_p(M)$ be a distributive lattice with the operations $\vee$ and $\wedge$ and $\mathfrak{S}_{T^{-1}p}(T^{-1}M)$ be a lattice with the operations $\tilde{\vee}$ and $\tilde{\wedge}$. It is seen that $\tilde{\vee}$ and $\tilde{\wedge}$ are expressed by $\vee$ and $\wedge$ respectively as follows:

$$\begin{aligned} T^{-1}L \,\tilde{\vee}\, T^{-1}N &= S_{T^{-1}p}(T^{-1}L + T^{-1}N) \\ &= S_{T^{-1}p}(T^{-1}(L+N)) \\ &= T^{-1}(S_p(L+N)) \\ &= T^{-1}(L \vee N), \end{aligned}$$

and

$$\begin{aligned} T^{-1}L \,\tilde{\wedge}\, T^{-1}N &= T^{-1}L \cap T^{-1}N \\ &= T^{-1}(L \cap N) \\ &= T^{-1}(L \wedge N). \end{aligned}$$

By these statements, the distributivity of $\mathfrak{S}_{T^{-1}p}(T^{-1}M)$ follows immediately from the distributivity of $\mathfrak{S}_p(M)$.
(3) Since R is a semisimple ring, then so is $T^{-1}R$. Thus the result follows from Theorem 3.7 and (2). □

Let A and B be Boolean algebras. A function $f : A \to B$ is called a *Boolean algebra homomorphism*, if f is a lattice homomorphism, $f(\mathbf{0}) = \mathbf{0}$, $f(\mathbf{1}) = \mathbf{1}$ and $f(a') = f(a)'$ for all $a \in A$. It is easily proved that a lattice homomorphism f preserves $\mathbf{0}$ and $\mathbf{1}$ if and only if it preserves $'$. Thus, in order to show that a function f between two Boolean algebras is a Boolean algebra homomorphism, it suffices to check that f preserves lattice operations $\vee$ and $\wedge$ and constants $\mathbf{0}, \mathbf{1}$.

Theorem 3.13. *Let R be a semisimple ring, p a prime ideal of R and M a distributive R-module. Then $\eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M)$ is a Boolean algebra homomorphism.*

Proof. First note that $\mathfrak{S}_p(M)$ and $\mathfrak{S}_p(R)$ are Boolean algebras, by Theorem 3.7 and Corollary 3.8 respectively. By Corollary 2.7, η is a lattice homomorphism. Also,

$$\eta(\mathbf{0}) = \eta(S_p(0)) = S_p(S_p(0)M) = S_p(0) = \mathbf{0},$$

and

$$\eta(\mathbf{1}) = \eta(R) = S_p(RM) = S_p(M) = M = \mathbf{1}.$$

Hence, as noted above, η is a Boolean algebra homomorphism. □

Corollary 3.14. *Let R be a semisimple ring, p a prime ideal of R and M a multiplication R-module. Then $\eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M)$ is a Boolean algebra epimorphism.*

Proof. By Corollaries 3.8 and 3.10, $\mathfrak{S}_p(R)$ and $\mathfrak{S}_p(M)$ are Boolean algebras respectively. Also, by the proof of Corollary 3.10, M is distributive. Thus by Theorem 3.13, η is a Boolean algebra homomorphism. Moreover, if $N \in \mathfrak{S}_p(M)$, then $(N : M) \in \mathfrak{S}_p(R)$ and

$$\eta(N : M) = S_p((N : M)M) = S_p(N) = N.$$

Thus, η is an epimorphism. □

Finally, we remark that if M is a faithful multiplication module over a semisimple ring R, then since M is cyclic by Corollary 3.10, we conclude that M is isomorphic to R. So it clearly follows that η and θ are Boolean algebra isomorphisms.

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