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# A note on constacyclic and skew constacyclic codes over the ring $\mathbb{Z}_p[u,v]/\langle u^2-u, v^2-v, uv-vu\rangle^*$

**Research Article**

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**Abstract:** For odd prime $p$, this paper studies $(1+(p-2)u)$-constacyclic codes over the ring $R=\mathbb{Z}_p[u,v]/\langle u^2-u, v^2-v, uv-vu\rangle$. We show that the Gray images of $(1+(p-2)u)$-constacyclic codes over $R$ are cyclic and permutation equivalent to a quasi cyclic code over $\mathbb{Z}_p$. We derive the generators for $(1+(p-2)u)$-constacyclic and principally generated $(1+(p-2)u)$-constacyclic codes over $R$. Among others, we extend our results for skew $(1+(p-2)u)$-constacyclic codes over $R$ and exhibit the relation between skew $(1+(p-2)u)$-constacyclic codes with the other linear codes. Finally, as an application of our study, we compute several non trivial linear codes by using the Gray images of $(1+(p-2)u)$-constacyclic codes over this ring $R$.

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## 1. Introduction

In algebraic coding theory, one of the main goals is to produce good error-correcting linear codes by means of larger minimum distance and code rate. Towards this, the cyclic code is one of important class of linear codes and researchers have been studying it for last six decades due to their successful applications in the theory of error-correcting codes. The constacyclic code is one of the prominent generalization of cyclic codes by which many good error-correcting codes can be developed over finite fields and rings. In 2006, Qian et al. [12] introduced constacyclic codes over $\mathbb{F}_2+u\mathbb{F}_2$. Later, Abualrub and Siap [1] also studied structural properties of constacyclic codes over $\mathbb{F}_2+u\mathbb{F}_2$. In 2011, Karadeniz and Yildiz [9] studied $(1+v)$-constacyclic codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$ and constructed some new optimal binary codes as the Gray images of $(1+v)$-constacyclic codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$. In 2015, Ashraf and

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Mohammed [2] studied $(1+2u)$-constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$. Recently, researchers [11, 13, 14] have studied constacyclic codes over the extension ring $\mathbb{Z}_4 + u\mathbb{Z}_4$ of $\mathbb{Z}_4$ and obtained several new linear codes over $\mathbb{Z}_4$ from Gray images of these codes. For more works on the topic, we refer [3, 4, 6–8].

Motivated by the above works, we study $\lambda = (1+(p-2)u)$-constacyclic codes over the finite non-chain ring $R = \mathbb{Z}_p[u,v]/\langle u^2-u, v^2-v, uv-vu\rangle$ and find some good codes over $\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the finite field with $p$ elements for odd prime $p$. This ring can also be seen as $\mathbb{Z}_p + u\mathbb{Z}_p + v\mathbb{Z}_p + uv\mathbb{Z}_p$, where $u^2 = u, v^2 = v, uv = vu$.

The paper is organized as follows: In Section 2, we define two Gray maps over the ring $R$. Section 3 contains some results on Gray images of $\lambda = (1+(p-2)u)$-constacyclic codes over the ring $R$. In Section 4, we derive the generators for $\lambda = (1+(p-2)u)$-constacyclic and one-generator $\lambda = (1+(p-2)u)$-constacyclic codes, respectively over $R$. In Section 5, we extend our results for skew $\lambda = (1+(p-2)u)$-constacyclic codes over $R$ and finally, in Section 6 some non trivial examples are included by using our Gray maps.

## 2. Preliminaries

For an odd prime $p$, let $R = \mathbb{Z}_p + u\mathbb{Z}_p + v\mathbb{Z}_p + uv\mathbb{Z}_p$, where $u^2 = u, v^2 = v, uv = vu$. Then $R$ is a commutative non-chain semi-local ring with maximal ideals $\langle u, v\rangle$ and $\langle u, 1-v\rangle$. Recall that a non empty subset $C$ of $R^n$ is called a linear code of length $n$ over $R$ if it forms an $R$-submodule of $R^n$, and elements of $C$ are referred as codewords. A linear code $C$ of length $n$ over $R$ is said to be a $\lambda$-constacyclic code if $C$ is closed under the constacyclic shift operator $\Upsilon : R^n \longrightarrow R^n$, defined by $\Upsilon(c_0, c_1, \dots, c_{n-1}) = (\lambda c_{n-1}, c_0, \dots, c_{n-2})$, where $\lambda$ is a unit in $R$. Note that a constacyclic code is a cyclic code for $\lambda = 1$ and a negacyclic code for $\lambda = -1$. By identifying a codeword $c = (c_0, c_1, \dots, c_{n-1}) \in R^n$ to a polynomial $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ in $\frac{R[x]}{\langle x^n-\lambda\rangle}$, a linear code $C$ is a $\lambda$-constacyclic code of length $n$ over $R$ if and only if it is an ideal of the ring $\frac{R[x]}{\langle x^n-\lambda\rangle}$. For the rest of this article, we denote $\lambda = (1+(p-2)u)$.

Here, we define two new Gray maps over $R$. The first Gray map is $\phi_1 : R \to \mathbb{Z}_p^2$ define by

$$\phi_1(a+ub+vc+uvd) = (a+b+c+d, (p-1)(a+b+c+d)), \tag{1}$$

where $a, b, c, d \in \mathbb{Z}_p$. It is easy to see that $\phi_1$ is a $\mathbb{Z}_p$-linear map and can be extended component-wise as follows.

$$\phi_1 : R^n \to \mathbb{Z}_p^{2n}$$

$$\begin{aligned}(r_0, r_1, \dots, r_{n-1}) \longmapsto (a_0+b_0+c_0+d_0, \dots, a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1}, (p-1)(a_0+b_0 \\ +c_0+d_0), \dots, (p-1)(a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1})),\end{aligned}$$

where $r_i = a_i + ub_i + vc_i + uvd_i \in R$ and $a_i, b_i, c_i, d_i \in \mathbb{Z}_p$ for $i = 0, 1, \dots, n-1$.

The next Gray map is $\phi_2 : R \to \mathbb{Z}_p^3$ define by

$$\phi_2(a+ub+vc+uvd) = (b+c+d, (p-1)(2a+b+c+d), a), \tag{2}$$

where $a, b, c, d \in \mathbb{Z}_p$. This $\phi_2$ is also a $\mathbb{Z}_p$-linear map and can be extended component-wise like the $\phi_1$-Gray map.

Note that these two Gray maps are $\mathbb{Z}_p$-linear but not bijection, similar to the Gray map in [13]. The Lee weight of any element $a+ub+vc+uvd \in R$ is defined as $w_L(r) = w_H(\phi_i(a+ub+vc+uvd)), i = 1, 2$, where $w_H$ denotes the Hamming weight over $\mathbb{Z}_p$. Lee weight for $r = (r_0, r_1, \dots, r_{n-1}) \in R^n$ is defined by $w_L(r) = \sum_{i=0}^{n-1} w_L(r_i)$. The Lee distance between any two elements $r', r'' \in R^n$ is $d_L(r', r'') = w_L(r'-r'')$ and the minimum Lee distance of $C$ is defined as $d_L(C) = min\{d_L(r', r'') \mid r' \neq r''; r', r'' \in C\}$. By this discussion, one can check that $\phi_1, \phi_2$ are distance preserving $\mathbb{Z}_p$-linear maps from $(R^n, d_L)$ to $(\mathbb{Z}_p^{2n}, d_H)$ and $(\mathbb{Z}_p^{3n}, d_H)$, respectively, where $d_H$ denotes the minimum Hamming distance of codes over $\mathbb{Z}_p$.

## 3. Gray images of $(1+(p-2)u)$-constacyclic codes

In this section, we explore the connection between cyclic, quasi cyclic and $(1+(p-2)u)$-constacyclic codes via the Gray maps $\phi_1$ and $\phi_2$, defined in the previous section.

**Proposition 3.1.** *Let $\Upsilon$ be the $(1+(p-2)u)$-constacyclic shift on $R^n$ and $\rho$ be the cyclic shift on $\mathbb{Z}_p^{2n}$. If $\phi_1$ is the Gray map from $R^n$ to $\mathbb{Z}_p^{2n}$ as defined in equation (1), then $\phi_1\Upsilon=\rho\phi_1$.*

***Proof.*** Let $r=(r_0,r_1,\dots,r_{n-1})\in R^n$, where $r_i=a_i+ub_i+vc_i+uvd_i$ and $a_i,b_i,c_i,d_i\in\mathbb{Z}_p$ for $i=0,1,\dots,n-1$. Then

$$\begin{aligned}\phi_1(r)=&(a_0+b_0+c_0+d_0,\dots,a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1},(p-1)(a_0+b_0+c_0+d_0),\\&\dots,(p-1)(a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1})).\end{aligned}$$

Applying $\rho$ on both sides, we get

$$\begin{aligned}\rho\phi_1(r)=&((p-1)(a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1}),a_0+b_0+c_0+d_0,\dots,a_{n-1}\\&+b_{n-1}+c_{n-1}+d_{n-1},(p-1)(a_0+b_0+c_0+d_0),\dots,\\&(p-1)(a_{n-2}+b_{n-2}+c_{n-2}+d_{n-2})).\end{aligned}$$

On the other hand,

$$\begin{aligned}\phi_1\Upsilon(r)=&((p-1)(a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1}),a_0+b_0+c_0+d_0,\dots,a_{n-2}+b_{n-2}\\&+c_{n-2}+d_{n-2},a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1},(p-1)(a_0+b_0+c_0+d_0),\\&\dots,(p-1)(a_{n-2}+b_{n-2}+c_{n-2}+d_{n-2})).\end{aligned}$$

Therefore, $\phi_1\Upsilon=\rho\phi_1$. $\square$

The derived relation in Proposition 3.1, will help us to find the Gray images of $(1+(p-2)u)$-constacyclic codes of $R$. In that way, we have the following result.

**Theorem 3.2.** *The $\phi_1$-Gray image of $(1+(p-2)u)$-constacyclic code of length $n$ over $R$ is a cyclic code of length $2n$ over $\mathbb{Z}_p$.*

***Proof.*** Let $C$ be a $(1+(p-2)u)$-constacyclic code of length $n$ over $R$. Then $\Upsilon(C)=C$, applying $\phi_1$ on both sides, we get $\phi_1\Upsilon(C)=\phi_1(C)$. By Proposition 3.1, $\rho\phi_1(C)=\phi_1\Upsilon(C)=\phi_1(C)$. Therefore, $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_p$. $\square$

**Definition 3.3.** *Let $a\in\mathbb{Z}_p^{mn}$, where $a=(a_1|a_2|\dots|a_{m-1}|a_m)$ and each $a_i\in\mathbb{Z}_p^n$ for $i=1,2,\dots,m$. Let $\eta_m$ be a map from $\mathbb{Z}_p^{mn}$ to $\mathbb{Z}_p^{mn}$ defined by $\eta_m(a)=(\rho(a_1)|\rho(a_2)|\dots|\rho(a_m))$, where $\rho$ is the cyclic shift from $\mathbb{Z}_p^n$ to $\mathbb{Z}_p^n$ and $'\,|'$ is the usual vector concatenation. A linear code $C$ of length $mn$ over $\mathbb{Z}_p$ is called a QC or quasi cyclic code of index $m$ if $\eta_m(C)=C$.*

Similar to Proposition 3.1, here we derive a relation based on the Gray map $\phi_2$, which will help us to find the $\phi_2$-Gray images of $(1+(p-2)u)$-constacyclic codes of $R$.

**Proposition 3.4.** *Let $\Upsilon$ be the $(1+(p-2)u)$-constacyclic shift on $R^n$, $\phi_2$, the Gray map from $R^n$ to $\mathbb{Z}_p^{3n}$ defined in equation (2), and $\eta_3$, the map defined in the preliminary section. Then $\phi_2\Upsilon=\delta\eta_3\phi_2$, where the permutation $\delta$ on $\mathbb{Z}_p^{3n}$ is defined as $\delta(x_1,x_2,\dots,x_{3n})=(x_{\beta(1)},x_{\beta(2)},\dots,x_{\beta(3n)})$ with the permutation $\beta=(1,n+1)$ on the set $\{1,2,\dots,3n\}$.*

***Proof.*** Let $r=(r_0,r_1,\dots,r_{n-1})\in R^n$, where $r_i=a_i+ub_i+vc_i+uvd_i$ and $a_i,b_i,c_i,d_i\in\mathbb{Z}_p$ for $i=0,1,\dots,n-1$. Then

$$\begin{aligned}\phi_2(r)=&(b_0+c_0+d_0,\dots,b_{n-1}+c_{n-1}+d_{n-1},(p-1)(2a_0+b_0+c_0+d_0),\\&\dots,(p-1)(2a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1}),a_0,\dots,a_{n-1}).\end{aligned}$$

Now, applying $\eta_3$ on both sides, we get

$$\begin{aligned}\eta_3\phi_2(r) =&(b_{n-1}+c_{n-1}+d_{n-1}, b_0+c_0+d_0, \dots, b_{n-2}+c_{n-2}+d_{n-2},\\ &(p-1)(2a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1}), (p-1)(2a_0+b_0+c_0+d_0),\\ &\dots, (p-1)(2a_{n-2}+b_{n-2}+c_{n-2}+d_{n-2}), a_{n-1}, a_0, \dots, a_{n-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\phi_2\Upsilon(r) =&((p-1)(2a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1}), b_0+c_0+d_0, \dots, b_{n-2}+c_{n-2}+d_{n-2},\\ &b_{n-1}+c_{n-1}+d_{n-1}, (p-1)(2a_0+b_0+c_0+d_0), \dots, (p-1)(2a_{n-2}+b_{n-2}\\ &+c_{n-2}+d_{n-2}), a_{n-1}, a_0, \dots, a_{n-2}).\end{aligned}$$

Now, applying $\delta$ on $\eta_3\phi_2(r)$, we get $\delta\eta_3\phi_2(r) = \phi_2\Upsilon(r)$. Therefore, $\phi_2\Upsilon = \delta\eta_3\phi_2$. □

In Theorem 3.2, we presented the $\phi_1$-Gray images of $(1+(p-2)u)$-constacyclic codes over $R$. Similar to that, using Proposition 3.4, we present the $\phi_2$-Gray image of $(1+(p-2)u)$-constacyclic code over $R$ as below.

**Theorem 3.5.** *The $\phi_2$-Gray image of $(1+(p-2)u)$-constacyclic code of length $n$ over $R$ is permutation equivalent to a QC code of index 3 over $\mathbb{Z}_p$.*

***Proof.*** Let $C$ be a $(1+(p-2)u)$-constacyclic code of length $n$ over $R$. Then $\Upsilon(C) = C$. By applying $\phi_2$, we have $\phi_2(\Upsilon(C)) = \phi_2(C)$. Now, by Proposition 3.4, $\phi_2(\Upsilon(C)) = \delta\eta_3(\phi_2(C)) = \phi_2(C)$, Therefore, $\phi_2(C)$ is permutation equivalent to a QC code of length $3n$ and index 3 over $\mathbb{Z}_p$. □

Now, we present the permutation version $\Phi_\pi$ of $\phi_1$ defined as

$$\begin{aligned}\Phi_\pi(r_0, r_1, \dots, r_{n-1}) =&(a_0+b_0+c_0+d_0, (p-1)(a_0+b_0+c_0+d_0), a_1+b_1+c_1+d_1,\\ &(p-1)(a_1+b_1+c_1+d_1), \dots, a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1},\\ &(p-1)(a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1})),\end{aligned}$$

where $r_i = a_i + ub_i + vc_i + uvd_i \in R$ and $a_i, b_i, c_i, d_i \in \mathbb{Z}_p$ for $i = 0, 1, \dots, n-1$.

Using similar arguments used in the proof of [11, Theorem 4.3], one can easily show the following result.

**Proposition 3.6.** *For any $r \in R^n$, we have $\Phi_\pi\rho(r) = \rho^2\Phi_\pi(r)$.*

The following corollary is a direct consequence of Proposition 3.6.

**Corollary 3.7.** *Let $C$ be a cyclic code of length $n$ over $R$. Then $\Phi_\pi(C)$ is equivalent to a quasi cyclic code of length $2n$ and index 2 over $\mathbb{Z}_p$.*

## 4. Generators of $(1+(p-2)u)$-constacyclic codes

In this section, we derive the generators of $\lambda = (1+(p-2)u)$-constacyclic codes of length $n$ over $R$, when $\gcd(n,p) = 1$. We start with the approach shown in [2, 11].
Let $n$ be an odd integer. Then

$$\Psi : R_n = \frac{R[x]}{\langle x^n - 1\rangle} \longrightarrow R_{n,\lambda} = \frac{R[x]}{\langle x^n - \lambda\rangle}$$

defined by $\Psi(c(x)) = c(\lambda x)$, is a ring isomorphism. By this isomorphism it is evident that $I$ is an ideal of the ring $R_n$ if and only if $\Psi(I)$ is an ideal of ring $R_{n,\lambda}$. Note that $\lambda^n = 1$ when $n$ is an even integer and $\lambda^n = \lambda$ when $n$ is an odd integer. Then the map $\mu : R^n \to R^n$ defined by

$$\mu(c_0, c_1, \ldots, c_{n-1}) = (c_0, \lambda c_1, \lambda^2 c_2, \ldots, \lambda^{n-1} c_{n-1}). \tag{3}$$

corresponds to the map $\Psi$ in the polynomial form. Then it is easy to see that $C$ is a cyclic code of odd length $n$ over $R$ if and only if $\mu(C)$ is a $\lambda$-constacyclic code of length $n$ over $R$.

**Theorem 4.1.** *[10, Theorem 3.4] Let $C$ be a cyclic code of length $n$ over $R$. If $\gcd(n, p) = 1$, then*

$$C = \langle g(x) + ua_1(x) + uvr_1(x), va_2(x) + uva_3(x)\rangle,$$

*where $a_1(x) \mid g(x) \mid (x^n - 1)$ and $a_3(x) \mid a_2(x) \mid g(x) \mid (x^n - 1)$.*

Theorem 4.1 gives the generators of cyclic codes of length $n$ over $R$, see [10]. Now, extending this result for $\lambda$-constacyclic codes of length $n$ over $R$, we present the generators for $\lambda$-constacyclic codes of length $n$ over $R$, where $\gcd(n, p) = 1$.

**Theorem 4.2.** *Let $C$ be a $\lambda$-constacyclic code of length $n$ over $R$. If $\gcd(n, p) = 1$, then $C$ is an ideal of $R_{n,\lambda}$ which is generated by*

$$C = \langle g(\bar{x}) + ua_1(\bar{x}) + uvr_1(\bar{x}), va_2(\bar{x}) + uva_3(\bar{x})\rangle,$$

*where $\bar{x} = \lambda x$ and $g(x), a_i(x), r_1(x)$ are polynomials in $\mathbb{Z}_p[x]/\langle x^n - 1\rangle$, satisfying the conditions $a_1(x) \mid g(x) \mid (x^n - 1)$ and $a_3(x) \mid a_2(x) \mid g(x) \mid (x^n - 1)$.*

Using the $\phi_1$-Gray map, we give the one-generated $\lambda$-constacyclic code of length $n$ over $R$ as follows:

**Theorem 4.3.** *Let $C = \langle a(x) + ub(x) + vc(x) + uvd(x)\rangle$ be a $\lambda$-constacyclic code of length $n$ over $R$, where $a(x), b(x), c(x), d(x) \in \mathbb{Z}_p[x]$. Then $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_p$ generated by the polynomial*

$$(a(x) + b(x) + c(x) + d(x)) + x^n[(p - 1)(a(x) + b(x) + c(x) + d(x))].$$

***Proof.*** The polynomial version of the Gray map $\phi_1$ is

$$\phi_1 : R[x]/\langle x^n - \lambda\rangle \longrightarrow \mathbb{Z}_p[x]/\langle x^n - 1\rangle \times \mathbb{Z}_p[x]/\langle x^n - 1\rangle$$

defined by

$$\begin{aligned}\phi_1(a(x) + ub(x) + vc(x) + uvd(x)) =&(a(x) + b(x) + c(x) + d(x),\\ &(p - 1)a(x) + b(x) + c(x) + d(x)).\end{aligned}$$

Rest of this proof is straightforward. Note that for $r_i(x) \in \mathbb{Z}_p[x]$, we have

$$\begin{gathered}\phi_1((r_1 + ur_2 + vr_3 + uvr_4)(a_1 + ub_2 + vc_3 + uvd_4)) =\\ r_1[(a + b + c + d), (p - 1)(a + b + c + d)] + r_2[(a + b + c + d), (p - 1)(a + b + c + d)]\\ + r_3[(a + b + c + d), (p - 1)(a + b + c + d)] + r_4[(a + b + c + d), (p - 1)(a + b + c + d)],\end{gathered}$$

where $[(a(x)+b(x)+c(x)+d(x)), (p-1)(a(x)+b(x)+c(x)+d(x))]$ represents the element $(a(x)+b(x)+c(x) + d(x)) + x^n[(p - 1)(a(x) + b(x) + c(x) + d(x))]$ in $\mathbb{Z}_p[x]/\langle x^{2n} - 1\rangle$. □

**Example 4.4.** *Let $n = 4$, $p = 5$ and the one generated $(1 + 3u)$-constacyclic code be $C = \langle(1 + u + v + uv) + (1 + uv)x + (u + uv)x^2 + (v + uv)x^3\rangle$. By Theorem 4.3, $\phi_1(C)$ is a cyclic code of length $8$ over $\mathbb{Z}_5$ generated by the polynomial $3x^7 + 3x^6 + 3x^5 + x^4 + 2x^3 + 2x^2 + 2x + 4$ with minimum Lee distance $8$.*

Now, we study about a permutation over $\mathbb{Z}_p$, referred as Nechaev's permutation, which is defined as

$$\pi(c_0, c_1, \dots, c_{2n-1}) = (c_{\tau(0)}, c_{\tau(1)}, \dots, c_{\tau(2n-1)}),$$

where $n$ is odd and $\tau = (1, n+1)(3, n+3) \cdots (2i+1, n+2i+1) \cdots (n-2, 2n-2)$ is a permutation on the set $\{0, 1, \dots, 2n-1\}$.

**Proposition 4.5.** *Let $\mu$ be the map defined in the equation (3). If $\pi$ is the Nechaev permutation and $n$ is odd, then $\phi_1\mu = \pi\phi_1$.*

***Proof.*** Let $r = (r_0, r_1, \dots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + vc_i + uvd_i \in R$ and $a_i, b_i, c_i, d_i \in \mathbb{Z}_p$ for $i = 0, 1, \dots, n-1$. Since $n$ is odd, we have

$$\mu(r) = (r_0, (1 + (p-2)u)r_1, r_2, (1 + (p-2)u)r_3, r_4, \dots, (1 + (p-2)u)r_{n-2}, r_{n-1}).$$

Also, $\phi_1(r) = (a_0 + b_0 + c_0 + d_0, \dots, a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, (p-1)(a_0 + b_0 + c_0 + d_0), \dots, (p-1)(a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}))$. Therefore,

$$\begin{aligned} \phi_1\mu(r) =& (a_0 + b_0 + c_0 + d_0, (p-1)(a_1 + b_1 + c_1 + d_1), a_2 + b_2 + c_2 + d_2, \dots, a_{n-1} \\ &+ b_{n-1} + c_{n-1} + d_{n-1}, (p-1)(a_0 + b_0 + c_0 + d_0), a_1 + b_1 + c_1 + d_1, \\ &\dots, (p-1)(a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1})). \end{aligned}$$

On the other hand,

$$\begin{aligned} \pi\phi_1(r) =& (a_0 + b_0 + c_0 + d_0, (p-1)(a_1 + b_1 + c_1 + d_1), a_2 + b_2 + c_2 + d_2, \dots, a_{n-1} \\ &+ b_{n-1} + c_{n-1} + d_{n-1}, (p-1)(a_0 + b_0 + c_0 + d_0), a_1 + b_1 + c_1 + d_1, \\ &\dots, (p-1)(a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1})). \end{aligned}$$

Hence, $\phi_1\mu = \pi\phi_1$. □

**Corollary 4.6.** *Let $\pi$ be the Nechaev permutation and $n$ be an odd. If $\Lambda$ is the Gray image of a cyclic code of length $n$ over $R$, then $\pi(\Lambda)$ is a cyclic code.*

***Proof.*** Let $C$ be a cyclic code of length $n$ over $R$ and $\Lambda = \phi_1(C)$. Then by Proposition 4.5, $\phi_1\mu(C) = \pi\phi_1(C) = \pi(\Lambda)$. Also, from our discussion in the beginning of this section, $\mu(C)$ is a $(1 + (p-2)u)$-constacyclic code. Hence, by Theorem 3.2, $\phi_1(\mu(C)) = \pi(\Lambda)$ is a cyclic code. □

Analogous to Nechaev permutation, we present another permutation over $\mathbb{Z}_p$, defined as

$$\chi(c_1, \dots, c_{3n}) = (c_{\varrho(1)}, \dots, c_{\varrho(3n)}),$$

where $\varrho = (2, n+2)(4, n+4) \cdots (n-1, 2n-1)$ is a permutation on the set $\{1, 2, \dots, 3n\}$.

**Proposition 4.7.** *Let $\mu$ and $\chi$ be the maps defined above. Then $\phi_2\mu = \chi\phi_2$. Moreover, if $n$ is odd and $\alpha$ is the $\phi_2$-Gray image of a cyclic code of length $n$ over $R$, then $\alpha$ is permutation equivalent to a QC code of length $3n$ and index 3 over $\mathbb{Z}_p$.*

***Proof.*** It can be easily shown using similar procedure adopted in the proof of [3, Proposition 4]. □

## 5. Skew $(1 + (p-2)u)$-constacyclic codes over $R$

In Section 3, we have derived some relations to study the $(1 + (p-2)u)$-constacyclic codes over $R$ in terms of cyclic and quasi cyclic codes over $\mathbb{Z}_p$. Extending these discussion, here we obtain some relations to study the skew $(1 + (p-2)u)$-constacyclic codes over $R$ as an extension in noncommutative set up.

**Definition 5.1.** *Let $Aut(R)$ be the set of all automorphisms defined over the ring $R$ and $\theta \in Aut(R)$. The set $R[x;\theta] = \{r_0+r_1x+\cdots+r_{n-1}x^{n-1} \mid r_i \in R^n\}$ forms a ring under the usual addition of polynomial and the multiplication defined as $(rx^i)(sx^j) = r\theta^i(s)x^{i+j}$. This ring is called skew polynomial ring over $R$. Also, it is a non-commutative ring unless $\theta$ is the identity.*

We define a non-trivial automorphism $\theta : R \longrightarrow R$

$$\theta(0) = 0,\ \theta(1) = 1,\ \theta(u) = v,\ \theta(v) = u;$$

i.e., $\theta(a + ub + vc + uvd) = a + vb + uc + uvd$ for $a, b, c, d \in \mathbb{Z}_p$. Note that the order of the automorphism is 2. For the rest of this section, we consider this automorphism $\theta$ over $R$.

**Definition 5.2.** *A subset $C$ of $R^n$ is called a skew $\lambda$-constacyclic code of length $n$ over $R$ if $C$ satisfies the following conditions:*
*i) $C$ is a $R$-submodule of $R^n$;*
*ii) If $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, then $\sigma_{\theta,\lambda}(c) = (\theta(\lambda c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C$.*

In polynomial representation of a skew $\lambda$-constacyclic code of length $n$ over $R$, identifying a codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in R^n$ by a polynomial $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ in $R[x;\theta]/\langle x^n - \lambda\rangle$, we have the following result.

**Theorem 5.3.** *Let $C$ be a linear code of length $n$ over $R$. Then $C$ is a skew $\lambda$-constacyclic code of length $n$ over $R$ if and only if $C$ is a left $R[x;\theta]$-submodule of $R[x;\theta]/\langle x^n - \lambda\rangle$.*

In Proposition 3.1 and Proposition 3.4, we derived relations using $\lambda$-constacyclic shift over $R$ to study the $\phi_1$ and $\phi_2$-Gray images of $\lambda$-constacyclic codes over $R$. Similarly, here also we derive some relations using skew $\lambda$-constacyclic shift over $R$ which will help us to study $\phi_1$ and $\phi_2$-Gray images of skew $\lambda$-constacyclic codes over $R$. These results can be seen as extension of Proposition 3.1 and Proposition 3.4.

**Proposition 5.4.** *If $\sigma_{\theta,\lambda}$ is the skew constacyclic shift on $R^n$ and $\phi_1, \phi_2$ are the Gray maps defined in equation (1) and equation (2), respectively, then*

1. $\rho\phi_1 = \phi_1\sigma_{\theta,\lambda}$.
2. $\delta\eta_3\phi_2 = \phi_2\sigma_{\theta,\lambda}$,

*where $\rho, \delta$ and $\eta_3$ are as in Proposition 3.1 and Proposition 3.4.*

***Proof.*** 1. From Proposition 3.1, we have

$$\begin{aligned}\rho\phi_1(r) =&((p-1)(a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}), a_0 + b_0 + c_0 + d_0, \ldots, a_{n-1}\\ &+ b_{n-1} + c_{n-1} + d_{n-1}, (p-1)(a_0 + b_0 + c_0 + d_0), \ldots,\\ &(p-1)(a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2})).\end{aligned}$$

Also, $\sigma_{\theta,\lambda}(r) = (\theta(\lambda r_{n-1}), \theta(r_0), \theta(r_1), \ldots, \theta(r_{n-2}))$. Applying $\phi_1$ on both sides, we get

$$\begin{aligned}\phi_1\sigma_{\theta,\lambda}(r) =&((p-1)(a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}), a_0 + b_0 + c_0 + d_0, \ldots, a_{n-1}\\ &+ b_{n-1} + c_{n-1} + d_{n-1}, (p-1)(a_0 + b_0 + c_0 + d_0), \ldots,\\ &(p-1)(a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2})).\end{aligned}$$

Hence, $\rho\phi_1 = \phi_1\sigma_{\theta,\lambda}$.

2. From Proposition 3.4, we have

$$\begin{aligned}\delta\eta\phi_2(r) =&((p-1)(2a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1}), b_0+c_0+d_0, \dots, b_{n-2}+\\ &c_{n-2}+d_{n-2}, b_{n-1}+c_{n-1}+d_{n-1}, (p-1)(2a_0+b_0+c_0+d_0), \dots,\\ &(p-1)(2a_{n-2}+b_{n-2}+c_{n-2}+d_{n-2}), a_{n-1}, a_0, \dots, a_{n-2}).\end{aligned}$$

Also, $\sigma_{\theta,\lambda}(r) = (\theta(\lambda r_{n-1}), \theta(r_0), \theta(r_1), \dots, \theta(r_{n-2}))$. Applying $\phi_2$ on both sides, we get

$$\begin{aligned}\phi_2\sigma_{\theta,\lambda}(r) =&((p-1)(2a_{n-1}+b_{n-1}+c_{n-1}+d_{n-1}), b_0+c_0+d_0, \dots, b_{n-2}+\\ &c_{n-2}+d_{n-2}, b_{n-1}+c_{n-1}+d_{n-1}, (p-1)(2a_0+b_0+c_0+d_0),\\ &\dots, (p-1)(2a_{n-2}+b_{n-2}+c_{n-2}+d_{n-2}),\\ &a_{n-1}, a_0, \dots, a_{n-2}).\end{aligned}$$

Hence, $\delta\eta\phi_2 = \phi_2\sigma_{\theta,\lambda}$.

□

In Theorem 3.2 and Theorem 3.5, we discussed the $\phi_1$ and $\phi_2$-Gray images of $\lambda$-constacyclic codes over $R$. Extending these results for skew $\lambda$-constacyclic codes over $R$, we present the $\phi_1$ and $\phi_2$-Gray image of a skew $\lambda$-constacyclic code over $R$ as below:

**Theorem 5.5.** *Let $\phi_1, \phi_2$ be the Gray maps defined in equation (1) and equation (2), respectively. Then*

1. *$\phi_1$-Gray image of a skew $\lambda$-constacyclic code of length $n$ over $R$ is a cyclic code of length $2n$ over $\mathbb{Z}_p$.*
2. *$\phi_2$-Gray image of a skew $\lambda$-constacyclic code of length $n$ over $R$ is permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_p$.*

***Proof.*** 1. Let $C$ be a skew $(1+(p-2)u)$-constacyclic code of length $n$ over $R$. Then $\sigma_{\theta,\lambda}(C) = C$. Now, applying $\phi_1$ on both sides, we get $\phi_1\sigma_{\theta,\lambda}(C) = \phi_1(C)$. By Proposition 5.4, $\rho\phi_1(C) = \phi_1\sigma_{\theta,\lambda}(C) = \phi_1(C)$. Therefore, $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_p$.

2. Let $C$ be a skew $(1+(p-2)u)$-constacyclic code of length $n$ over $R$. Then $\sigma_{\theta,\lambda}(C) = C$. Now, applying $\phi_2$ on both sides, we get $\phi_2\sigma_{\theta,\lambda}(C) = \phi_2(C)$. Also, by Proposition 5.4, $\phi_2(\sigma_{\theta,\lambda}(C)) = \delta\eta(\phi_2(C)) = \phi_2(C)$. Thus, $\phi_2(C)$ is permutation equivalent to a $QC$ code of length $3n$ and index 3 over $\mathbb{Z}_p$.

□

## 6. Examples

For better understanding of our study, we present some non trivial examples which are computed by using the $\phi_1$ and $\phi_2$-Gray images of the $\lambda$-constacyclic codes over $R$. All computations of this section are carried out by Magma software [5].

**Example 6.1.** *Let $p = 3, \lambda = 1+u$ and $n = 8$. Following Theorem 4.2, assume that $C = \langle(1+u)x^5 + (2+2u)x^3 + (1+u)x^2 + 2 + u + uv, (v+uv)x^3 + 2vx^2 + vx + v + 2uv\rangle$ be the $(1+u)$-constacyclic code of length 8 over $R = \mathbb{Z}_3[u,v]/\langle u^2-u, v^2-v, uv-vu\rangle$. Therefore, the $\phi_2$-Gray image of $C$ is a $[24, 8, 4]$ linear code over $\mathbb{Z}_3$.*

**Example 6.2.** *Let $p = 17, \lambda = 1+15u$ and $n = 9$. Further, assume $C = \langle(1+15u)x^5 + 3x^4 + (1+15u)x^3 + 16x^2 + (1+15u)x + 16 + 16u + uv, (v+5uv)x^3 + 16uvx + 16v + 16uv\rangle$ is a $(1+15u)$-constacyclic code of length 9 over $R = \mathbb{Z}_{17}[u,v]/\langle u^2-u, v^2-v, uv-vu\rangle$. Then $\phi_1(C)$ and $\phi_2(C)$ have parameters $[18, 10, 5]$ and $[27, 10, 6]$ over $\mathbb{Z}_{17}$, respectively.*

**Table 1. Linear codes as Gray images of $(1+(p-2)u)$-constacyclic codes over $\mathbb{Z}_p$**

| $\lambda$ | $n$ | $h(\bar{x})$ | $k(\bar{x})$ | $\phi_1(C)$ | $\phi_2(C)$ |
|---|---|---|---|---|---|
| $1+3u$ | 6 | $[1, 1+3u, u, 4+2u, 1+4u+uv]$ | $[v+3uv, 2v, 2v+4uv, v+uv]$ | $[12,5,2]_5$ | $[18,5,6]_5$ |
| $1+3u$ | 8 | $[1+3u, 3, 2, 4+uv]$ | $[4v, 3+uv]$ | $[16,8,2]_5$ | $[24,12,5]_5$ |
| $1+3u$ | 9 | $[1+3u, 4, 0, 1, 4+2u, 0, 1+2u, 3+uv]$ | $[v, 0, 0, v+3uv, 0, 0, v+uv]$ | $[18,5,2]_5$ | $[27,5,6]_5$ |
| $1+5u$ | 4 | $[1+5u, 1, 1+4u, 1+u+uv]$ | $[v, 0, v+uv]$ | $[8,4,2]_7$ | $[12,4,4]_7$ |
| $1+5u$ | 10 | $[1+5u, 0, 0, 0, 6u, 1+u+uv]$ | $[v, 6v+2uv, v, 6v+2uv, v+uv]$ | $[20,10,2]_7$ | $[30,11,5]_7$ |
| $1+9u$ | 5 | $[1+9u, 8, 10u, 8+10u+uv]$ | $[v, v+8uv, 3v+2uv]$ | $[10,5,2]_{11}$ | $[15,5,6]_{11}$ |
| $1+9u$ | 8 | $[1+9u, 2, 8+6u, 3+u, 7+8u, 1+u+uv]$ | $[v+9uv, 10v, v+8uv, 10v+10uv]$ | $[16,8,2]_{11}$ | $[24,8,6]_{11}$ |
| $1+11u$ | 4 | $[1, 9+7u, 8+u]$ | $[v+11uv, 8v+uv]$ | $[8,3,4]_{13}$ | $[12,5,4]_{13}$ |
| $1+11u$ | 7 | $[1+11u, 6, 1+11u, 0, 11+3u, 11+12u]$ | $[v, 8v+10uv, 4v+uv, 8v+5uv, v+uv]$ | $[14,5,4]_{13}$ | $[21,5,8]_{13}$ |
| $1+15u$ | 6 | $[1, 1+15u, 0, 16+u, 16+u]$ | $[v+15uv, 16v+uv]$ | $[12,5,2]_{17}$ | $[18,7,2]_{17}$ |
| $1+15u$ | 4 | $[1+15u, 1, 1+14u, 1+u]$ | $[v, 5v+6uv, 4v+4uv]$ | $[8,3,4]_{17}$ | $[12,3,6]_{17}$ |

We recall from Theorem 4.2 that generator of a $\lambda$-constacyclic code of length $n$ with $gcd(n,p)=1$ is given by $C = \langle h(\bar{x}), k(\bar{x})\rangle$, where $h(\bar{x}) = g(\bar{x}) + ua_1(\bar{x}) + uvr_1(\bar{x}), k(\bar{x}) = va_2(\bar{x}) + uva_3(\bar{x})$ with $a_1(x) \mid g(x) \mid (x^n-1)$ and $a_3(x) \mid a_2(x) \mid g(x) \mid (x^n-1)$, $\bar{x} = \lambda x$. In Table 1, we obtain some linear codes from the $\lambda$-constacyclic codes over $\mathbb{Z}_p$. First column includes the value of $\lambda$, second column is the length of the constacyclic code while third and fourth column gives the coefficients of generator polynomials $h(\bar{x}), k(\bar{x})$. Lastly, fifth and sixth column shows the parameters of $\phi_1$ and $\phi_2$-Gray images, respectively. We write coefficients of generator polynomials in decreasing order, for example, we write $[u, 0, 1+uv, v, 1+u+uv]$ to represent the polynomial $ux^4 + (1+uv)x^2 + vx + 1 + u + uv$.

# 7. Conclusion and future works

In this paper, we studied $\lambda = (1+(p-2)u)$-constacyclic codes over $R = \mathbb{Z}_p + u\mathbb{Z}_p + v\mathbb{Z}_p + uv\mathbb{Z}_p$ for odd prime $p$. We have constructed two new Gray maps over $R$ and have shown some results based on their definitions. We have derived generators for $\lambda = (1+(p-2)u)$-constacyclic and one-generated $\lambda = (1+(p-2)u)$-constacyclic codes over $R$. Using some permutation maps over $\mathbb{Z}_p$, we have shown some results to understand cyclic and quasi cyclic codes by $\lambda = (1+(p-2)u)$-constacyclic codes over this ring in simpler way. At last, we have discussed skew $(1+(p-2)u)$-constacyclic codes over this ring and extended our results from Section 3. As a future work, finding the generators of these skew $\lambda = (1+(p-2)u)$-constacyclic code over $R$ would be interesting. We hope our results would be useful to find some good codes over $\mathbb{Z}_p$ via these Gray maps over $R$.

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