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On Extensions of Extended Gauss Hypergeometric Function

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Abstract
The aim of this paper is to introduce a new extensions of extended Gauss hypergeometric function. Certain integral representations, transformation and summation formulas for extended Gauss hypergeometric function are presented and some special cases are also discussed.

Keywords: Extended hypergeometric function, Integral representations, Mittag-Leffler function, Summation formulas, Transformation formulas
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1. Introduction

The classical Beta function $B(x,y)$ is defined by:

$$B(x,y) = \begin{cases} \int_0^1 t^{x-1}(1-t)^{y-1}dt & , \quad (Re(x)>0\ ,\ Re(y)>0) \\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} & , \quad Re(x)>0\ ,\ Re(y)>0, \end{cases} \tag{1.1}$$

where $\Gamma(x)$ is the familiar Gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad (Re(x)>0).$$

The generalized hypergeometric function ${}_pF_q$ with p numerator parameters and q denominator parameters is defined by (see [1])

$${}_pF_q\left[\begin{array}{lll} a_1,\ldots,a_p & ; & \\ & & z \\ b_1,\ldots,b_q & ; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(b_p)_n}{(a_1)_n\ldots(b_q)_n}\frac{z^n}{n!}, \tag{1.2}$$

where $(\lambda)_n$ is the well-known Pochhammer symbol. The case $p=2$ and $q=1$ of (1.2), yields the Gauss's hypergeometric function ${}_2F_1(z)$.

The Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[x,y]$ is defined by (see[1])

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{l}(a_p)\ :\ (b_q)\ ;\ (c_k)\ ;\\ \\ (e_l)\ :\ (f_m)\ ;\ (g_n)\ ;\end{array}\ x\ ,\ y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\ \prod_{j=1}^{q}(b_j)_r\ \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(e_j)_{r+s}\ \prod_{j=1}^{m}(f_j)_r\ \prod_{j=1}^{n}(g_j)_s}\ \frac{x^r}{r!}\ \frac{y^s}{s!}.$$

In 1903, Gosta Mittag-Leffler [2] introduced the function $E_\alpha(z)$ defined as:

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha>0\ ,\ z\in\mathbb{C}.$$

In 1905, Wiman [3] defined the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ as follows:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad (\alpha,\beta\in\mathbb{C}\ ,\ Re(\alpha)>0\ ,\ Re(\beta)>0).$$

Afterward, Prabhakar [4] defined the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ as follows:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad (\alpha,\beta,\gamma\in\mathbb{C}\ ,\ Re(\alpha)>0\ ,\ Re(\beta)>0\ ,\ Re(\gamma)>0). \tag{1.3}$$

Clearly,

$$E^{1}_{\alpha,\beta}=E_{\alpha,\beta}(z),\quad E^{1}_{\alpha,1}=E_\alpha(z),\quad E^{1}_{1,1}=E_1(z)=e^z.$$

In recent years, some extensions of Beta function and Gauss hypergeometric function have been considered by several authors (see [5, 6, 7, 8, 9, 10, 11]).

The following extended Beta function and extended Gauss hypergeometric function are introduced by Chaudhry *et al*. [12] and Chaudhry *et al*. [13] respectively:

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}exp\left(\frac{-p}{t(1-t)}\right)dt,\quad (Re(p)>0,\ Re(x)>0,\ Re(y)>0)$$

and

$$F_p(a,b;c;z)=\sum_{n=0}^{\infty}\frac{B_p(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!},\quad (Re(c)>Re(b)>0,\ p\geq 0).$$

Choi *et al*. [14] introduced the extended Beta and extended Gauss hypergeometric functions as follows:

$$B(x,y;p;q)=\int_0^1 t^{x-1}(1-t)^{y-1}exp\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)dt,\quad (Re(p)>0,\ Re(q)>0) \tag{1.4}$$

and

$$F_{p,q}(a,b;c;z)=\sum_{n=0}^{\infty}\frac{B_{p,q}(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!},\quad (Re(c)>Re(b)>0,\ p,q\geq 0). \tag{1.5}$$

Rahman *et al*. [15] introduced the following extensions of (1.4) and (1.5) as follows:

$$B^{\alpha}_{p,q}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}E_\alpha\left(-\frac{p}{t}\right)E_\alpha\left(-\frac{q}{(1-t)}\right)dt,\quad (Re(\alpha)>0,\ p,q\geq 0) \tag{1.6}$$

and

$$F^{\alpha}_{p,q}(a,b;c;z)=\sum_{n=0}^{\infty}\frac{B^{\alpha}_{p,q}(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!},\quad (Re(c)>Re(b)>0,\ Re(\alpha)>0,\ p,q\geq 0).$$

Further generalizations of (1.6) are introduced by Atash *et al*. [16] and Barahmah [17] as follows:

$$B_{p,q}^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}\left(-\frac{p}{t}\right)E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right)dt,\quad (Re(\alpha)>0,\ Re(\beta)>0,\ p,q\geq 0) \tag{1.7}$$

and

$$B_{p,q}^{(\alpha,\beta,\gamma)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{p}{t}\right)E_{\alpha,\beta}^{\gamma}\left(-\frac{q}{(1-t)}\right)dt,\quad (Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0). \tag{1.8}$$

In the present paper, we aim to introduce new extensions for extended Gauss hypergeometric function by using (1.7) and (1.8) as follows:

$$F_{p,q}^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},\quad (Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ p,q\geq 0) \tag{1.9}$$

and

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{p,q}^{(\alpha,\beta,\gamma)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},$$
$$(Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0). \tag{1.10}$$

Clearly,

$$F_{p,q}^{(\alpha,\beta,1)}=F_{p,q}^{(\alpha,\beta)},\ F_{p,q}^{(\alpha,1,1)}=F_{p,q}^{\alpha},\ F_{p,q}^{(1,1,1)}=F_{p,q},\ F_{p,p}^{(1,1,1)}=F_p,\ F_{0,0}^{(1,1,1)}={}_2F_1.$$

Further, if we use (1.7) in (1.9) and (1.8) in (1.10), we have respectively the following integral representations:

$$F_{p,q}^{(\alpha,\beta)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-\alpha}E_{\alpha,\beta}\left(-\frac{p}{t}\right)E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right)dt,$$
$$(Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ p,q\geq 0)$$

and

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-\alpha}E_{\alpha,\beta}^{\gamma}\left(-\frac{p}{t}\right)E_{\alpha,\beta}^{\gamma}\left(-\frac{q}{(1-t)}\right)dt,$$
$$(Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0). \tag{1.11}$$

2. Transformation and summation formulas

In this section, we present some transformation and summation formulas for extended Gauss hypergeometric function (1.10) as follows:

Theorem 2.1. *For* $(Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0)$*, the following transformation formula holds true:*

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;z)=(1-z)^{-a}F_{q,p}^{(\alpha,\beta,\gamma)}(a,c-b;c;\frac{-z}{1-z}). \tag{2.1}$$

Proof. Replacing t by $(1-t)$ in (1.11) and using the following result:

$$(1-z(1-t))^{-a}=(1-z)^{-a}(1-\frac{z}{z-1}t)^{-a},$$

we obtain

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}(1-\frac{z}{z-1}t)^{-a}E_{\alpha,\beta}\left(-\frac{q}{t}\right)E_{\alpha,\beta}\left(-\frac{p}{1-t}\right)dt,$$

which, by applying (1.11) yields the desired result. □

Remark 2.2. *Replacing* z *by* $1-\frac{1}{z}$ *and* $\frac{z}{1+z}$ *in (2.1), we have respectively*

Corollary 2.3.

$$F_{p,q}^{(\alpha,\beta,\gamma)}\left(a,b;c;1-\frac{1}{z}\right)=z^{a}F_{q,p}^{(\alpha,\beta,\gamma)}(a,c-b;c;1-z). \tag{2.2}$$

Corollary 2.4.

$$F_{p,q}^{(\alpha,\beta,\gamma)}\left(a,b;c;\frac{z}{1+z}\right)=(1+z)^{a}F_{q,p}^{(\alpha,\beta,\gamma)}(a,c-b;c;-z). \tag{2.3}$$

Theorem 2.5. *For* $(Re(c-a-b)>0,\ Re(k)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0)$*, the following summation formula holds true:*

$$F_{p,q}^{(k,\beta,\gamma)}(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c-a)\Gamma(c-b)} \tag{2.4}$$
$$\times F\begin{matrix}1 & : & 1 & ; & 1\\ 0 & : & 1+k & ; & 1+k\end{matrix}\left[\begin{matrix}1+a-c & : & \gamma & ; & \gamma & ; \\ - & : & 1-b,\Delta(k;\beta) & ; & 1+a+b-c,\Delta(k;\beta) & ;\end{matrix}\ \frac{-p}{k^k},\frac{-q}{k^k}\right],$$

where $\Delta(k;\beta)$ *is k-tuple*

$$\frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k}.$$

Proof. From (1.11), we have

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;1)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}E_{k,\beta}^{\gamma}\left(-\frac{p}{t}\right)E_{k,\beta}^{\gamma}\left(-\frac{q}{(1-t)}\right)dt.$$

Applying (1.3) and interchanging the order of summation and integration and then using (1.1), we obtain

$$F_{p,q}^{(\alpha,\beta,\gamma)}(a,b;c;1)=\frac{\Gamma(c)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c-b)}\times\sum_{r,s=0}^{\infty}\frac{(\gamma)_r(\gamma)_s(-p)^r(-q)^s\Gamma(b-r)\Gamma(c-a-b-s)}{(\beta)_{kr}(\beta)_{ks}r!s!}.$$

Now, using the following identities (see [1]):

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(-1)^n}{(1-\alpha)_n}$$

and

$$(\alpha)_{kn}=k^{kn}\prod_{j=1}^{k}\left(\frac{\alpha+j-1}{k}\right)_n,\quad n=1,2,3,\ldots,$$

we have

$$F_{p,q}^{(k,\beta,\gamma)}(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c-a)\Gamma(c-b)}\times\sum_{r,s=0}^{\infty}\frac{(1-c+a)_{r+s}(\gamma)_r(\gamma)_s(-p)^r(-q)^s}{k^{kr}\prod_{j=1}^{k}\left(\frac{\beta+j-1}{k}\right)_r k^{ks}\prod_{j=1}^{k}\left(\frac{\beta+j-1}{k}\right)_s(1-b)_r(1+a+b-c)_s r!s!}$$
$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c-a)\Gamma(c-b)}$$
$$\times F\begin{matrix}1 & : & 1 & ; & 1\\ 0 & : & 1+k & ; & 1+k\end{matrix}\left[\begin{matrix}1+a-c & : & \gamma & ; & \gamma & ; \\ - & : & 1-b,\frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k} & ; & 1+a+b-c,\frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k} & ;\end{matrix}\ \frac{-p}{k^k},\frac{-q}{k^k}\right].$$

This completes the proof of (2.4). □

Remark 2.6. *Putting* $a=-n$ *in (2.4), we obtain*

Corollary 2.7.

$$F_{p,q}^{(k,\beta,\gamma)}(-n,b;c;1)=\frac{\Gamma(c)\Gamma(c+n-b)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c+n)\Gamma(c-b)}$$

$$\times F\begin{array}{c}1\\0\end{array}\begin{array}{c}:\\:\end{array}\begin{array}{c}1\\1+k\end{array}\begin{array}{c};\\;\end{array}\begin{array}{c}1\\1+k\end{array}\left[\begin{array}{ccccccc}1-n-c & : & \gamma & ; & \gamma & ; & \\ & & & & & & \frac{-p}{k^k},\frac{-q}{k^k}\\ - & : & 1-b,\Delta(k;\beta) & ; & 1-n+b-c,\Delta(k;\beta) & ; & \end{array}\right]. \tag{2.5}$$

Remark 2.8. *Putting $a=-n$ and $b=a+n$ in (2.4), we obtain*

Corollary 2.9.

$$F_{p,q}^{(k,\beta,\gamma)}(-n,a+n;c;1)=\frac{\Gamma(c)\Gamma(c-a)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c+n)\Gamma(c-a-n)}$$

$$\times F\begin{array}{c}1\\0\end{array}\begin{array}{c}:\\:\end{array}\begin{array}{c}1\\1+k\end{array}\begin{array}{c};\\;\end{array}\begin{array}{c}1\\1+k\end{array}\left[\begin{array}{ccccccc}1-n-c & : & \gamma & ; & \gamma & ; & \\ & & & & & & \frac{-p}{k^k},\frac{-q}{k^k}\\ - & : & 1-a-n,\Delta(k;\beta) & ; & 1+a-c,\Delta(k;\beta) & ; & \end{array}\right]. \tag{2.6}$$

Remark 2.10. *Putting $a=-n$ and $b=1-b-n$ in (2.4), we obtain*

Corollary 2.11.

$$F_{p,q}^{(k,\beta,\gamma)}(-n,1-b-n;c;1)=\frac{\Gamma(c)\Gamma(c+b-1+2n)}{\Gamma(\beta)\Gamma(\beta)\Gamma(c+n)\Gamma(c+b-1-n)}$$

$$\times F\begin{array}{c}1\\0\end{array}\begin{array}{c}:\\:\end{array}\begin{array}{c}1\\1+k\end{array}\begin{array}{c};\\;\end{array}\begin{array}{c}1\\1+k\end{array}\left[\begin{array}{ccccccc}1-n-c & : & \gamma & ; & \gamma & ; & \\ & & & & & & \frac{-p}{k^k},\frac{-q}{k^k}\\ - & : & b+n,\Delta(k;\beta) & ; & 2-b-c-2n,\Delta(k;\beta) & ; & \end{array}\right]. \tag{2.7}$$

Theorem 2.12. *For $(Re(b)>0,\ Re(k)>0,\ Re(\beta)>0,\ Re(\gamma)>0,\ p,q\geq 0)$, the following summation formula holds true:*

$$F_{p,q}^{(k,\beta,\gamma)}(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)=\frac{\Gamma(b+\frac{1}{2})\Gamma(b+n)}{\Gamma(\beta)\Gamma(\beta)\Gamma(b+\frac{n}{2})\Gamma(b+\frac{n}{2}+\frac{1}{2})}$$

$$\times F\begin{array}{c}1\\0\end{array}\begin{array}{c}:\\:\end{array}\begin{array}{c}1\\1+k\end{array}\begin{array}{c};\\;\end{array}\begin{array}{c}1\\1+k\end{array}\left[\begin{array}{ccccccc}\frac{1}{2}-\frac{n}{2}-b & : & \gamma & ; & \gamma & ; & \\ & & & & & & \frac{-p}{k^k},\frac{-q}{k^k}\\ - & : & (n/2)+(1/2),\Delta(k;\beta) & ; & 1-b-n,\Delta(k;\beta) & ; & \end{array}\right], \tag{2.8}$$

where $\Delta(k;\beta)$ is k-tuple

$$\frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k}.$$

The proof of the Theorem 2.12 is similar to that of the Theorem 2.5. Therefore, we omit the details.

3. Special cases

(i) Setting $\beta=\gamma=1$ in (2.1), we get the following corrected formula given by Rahman *et al*. [15]

$$F_{p,q}^{k}(a,b;c;z)=(1-z)^{-a}F_{q,p}^{k}(a,c-b;c;\frac{-z}{1-z}).$$

(ii) Setting $k=\beta=\gamma=1$ in (2.1), (2.2) and (2.3), we get a known transformation formulas of Choi *et al*. [14] for $F_{p,q}(a,b;c;z)$.
(iii) Setting $k=\beta=\gamma=1,\ p=q$ in (2.1), we get a known transformation formula of Chaudhry *et al*. [13] for $F_p(a,b;c;z)$.

(iv) Setting $k=\beta=\gamma=1,\ p=q=0$ in (2.1), we get Euler transformation [18, 1]).
(v) Setting $k=\beta=\gamma=1$ in (2.4), we get

$$F_{p,q}^{(1,1,1)}(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\psi_2[1+a-c;1-b,1+a+b-c;-p,-q], \tag{3.1}$$

where ψ_2 is the Humbert's confluent hypergeometric function [1].
By setting $p=q$ in (3.1) and using the result [1]

$$\psi_2[a;b,c;x,x]={}_3F_3\left[a,\frac{b+c}{2},\frac{b+c-1}{2};b,c,b+c-1;4x\right], \tag{3.2}$$

equation (3.1) reduces to

$$F_{p,p}^{(1,1,1)}(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_2\left[\frac{a-c+1}{2},\frac{a-c+2}{2};1-b,1+a+b-c;-4p\right]. \tag{3.3}$$

Further, setting $p=0$ in (3.3), we get the well-known Gauss summation formula (see [18])

$$F_{0,0}^{(1,1,1)}(a,b;c;1)={}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

(vi) Setting $k=\beta=\gamma=1$ in (2.5), we get

$$F_{p,q}^{(1,1,1)}(-n,b;c;1)=\frac{\Gamma(c)\Gamma(c+n-b)}{\Gamma(c+n)\Gamma(c-b)}\psi_2[1-n-c;1-b,1-n+b-c;-p,-q]. \tag{3.4}$$

Further, setting $p=q=0$ in (3.4), we get a known result (see [18])

$$F_{0,0}^{(1,1,1)}(-n,b;c;1)={}_2F_1(-n,b;c;1)=\frac{(c-b)_n}{(c)_n}.$$

(vii) Setting $k=\beta=\gamma=1$ in (2.6), we get

$$F_{p,q}^{(1,1,1)}(-n,a+n;c;1)=\frac{\Gamma(c)\Gamma(c-a)}{\Gamma(c+n)\Gamma(c-a-n)}\psi_2[1-n-c;1-a-n,1+a-c;-p,-q]. \tag{3.5}$$

Further, setting $p=q=0$ in (3.5), we get a known result (see [18])

$$F_{0,0}^{(1,1,1)}(-n,a+n;c;1)={}_2F_1(-n,a+n;c;1)=\frac{(-1)^n(1+a-c)_n}{(c)_n}.$$

(viii) Setting $k=\beta=\gamma=1$ in (2.7), we get

$$F_{p,q}^{(1,1,1)}(-n,1-b-n;c;1)=\frac{\Gamma(c)\Gamma(b+c-1+2n)}{\Gamma(c+n)\Gamma(b+c-1-n)}\psi_2[1-n-c;b+n,2-b-c-2n;-p,-q]. \tag{3.6}$$

Further, setting $p=q=0$ in (3.6), we get a known result (see [18])

$$F_{0,0}^{(1,1,1)}(-n,1-b-n;c;1)={}_2F_1(-n,1-b-n;c;1)=\frac{(-1)^n(b+c-1)_{2n}}{(c)_n(b+c-1)_n}.$$

(viiii) Setting $k=\beta=\gamma=1$ in (2.8), we get

$$F_{p,q}^{(1,1,1)}(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)=\frac{\Gamma(b+\frac{1}{2})\Gamma(b+n)}{\Gamma(b+\frac{n}{2})\Gamma(b+\frac{n}{2}+\frac{1}{2})}\psi_2\left[\frac{1}{2}-\frac{n}{2}-b;\frac{1}{2}+\frac{n}{2},1-b-n;-p,-q\right],$$

which for $p=q$ and using the result (3.2) reduces to

$$F_{p,p}^{(1,1,1)}(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)=\frac{\Gamma(b+\frac{1}{2})\Gamma(b+n)}{\Gamma(b+\frac{n}{2})\Gamma(b+\frac{n}{2}+\frac{1}{2})}{}_2F_2\left[\frac{3}{4}-\frac{n}{4}-\frac{b}{2},\frac{1}{4}-\frac{n}{4}-\frac{b}{2};\frac{1}{2}+\frac{n}{2},1-b-n;-4p\right]. \tag{3.7}$$

Further, setting $p=0$ in (3.7) and using Legender's duplication formula (see [18])

$$\Gamma(b)\Gamma(b+\frac{1}{2})=2^{1-2b}\sqrt{\pi}\Gamma(2b),$$

we get a known result (see [18])

$$F_{0,0}^{(1,1,1)}(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)={}_2F_1(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1)=\frac{2^n(b)_n}{(2b)_n}.$$

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