Gazi University Journal of Science
GU J Sci
25(4): 863-867 (2012)

ORIGINAL ARTICLE

Extensions of Baer and Principally Projective Modules

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Received: 25.02.2012 Accepted:10.08.2012

ABSTRACT

In this note, we investigate extensions of Baer and principally projective modules. Let R be an arbitrary ring with identity and M a right R-module. For an abelian module M, we show that M is Baer (resp. principally projective) if and only if the polynomial extension of M is Baer (resp. principally projective) if and only if the power series extension of M is Baer (resp. principally projective) if and only if the Laurent polynomial extension of M is Baer (resp. principally projective) if and only if the Laurent power series extension of M is Baer (resp. principally projective).

Key words: Abelian modules, Baer modules, principally projective modules.

2010 Mathematics Subject Classification: 13C11, 13C99.

1. INTRODUCTION

Throughout this paper R denotes an associative ring with identity, and modules are unitary right R-modules. In [3], *Baer rings* are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. A ring R is called *right (left) principally projective* if the right (left) annihilator of every element of R is generated by an idempotent [2]. For a module M, $S=\mathrm{End}_R(M)$ denotes the ring of endomorphisms of M. Then M is a left S-module, right R-module and (S, R)-bimodule. In this work, for any rings S and R and any (S, R)-bimodule M, $r_R(.)$ and $l_M(.)$ denote the right annihilator of a subset of M in R and the left annihilator of a subset of R in M, respectively. Similarly, $l_S(.)$ and $r_M(.)$ will be the left annihilator of a subset of M in S and the right annihilator of a subset of S in M, respectively. According to Rizvi and Roman [5], M is called a *Baer module* if the right annihilator in M of any left ideal of S is generated by an idempotent of S, i.e., for any left ideal I of S, $r_M(I)=eM$ for some $e^2 = e \in S$ (or equivalently, for all R-submodules N of M, $l_S(N)=Se$ with $e^2 = e \in S$). In [5], it is proved that any direct summand of a Baer module is again a Baer module, and the endomorphism ring of a Baer module is a Baer ring. Several results for a direct sum of Baer modules to be a Baer module are also given in [5].

We write $R[x]$, $R[[x]]$, $R[x,x^{-1}]$ and $R[[x,x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over R, respectively.

Lee and Zhou [4] introduced the following notations. For a module M, we consider;

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$$M[x]=\left\{\sum_{i=0}^{s}m_ix^i\ :\ s\geq 0, m_i\in M\right\}, M[[x]]=\left\{\sum_{i=0}^{\infty}m_ix^i\ : m_i\in M\right\},$$
$$M[x,x^{-1}]=\left\{\sum_{i=-s}^{t}m_ix^i\ :\ s\geq 0, t\geq 0, m_i\in M\right\}, M[[x,x^{-1}]]=\left\{\sum_{i=-s}^{\infty}m_ix^i\ :\ s\geq 0, m_i\in M\right\}.$$

Each of these is an abelian group under an obvious addition operation. Moreover $M[x]$ becomes a module over $R[x]$ where, for

$$m(x)=\sum_{i=0}^{s}m_ix^i\in M[x], f(x)=\sum_{i=0}^{t}a_ix^i\in R[x],\qquad m(x)f(x)=\sum_{k=0}^{s+t}\left(\sum_{i+j=k}m_ia_j\right)x^k.$$

The modules $M[x]$ and $M[[x]]$ are called the *polynomial extension* and the *power series extension of* M, respectively. With a similar scalar product, $M[x,x^{-1}]$ (resp. $M[[x,x^{-1}]]$) becomes a module over $R[x,x^{-1}]$ (resp. $R[[x,x^{-1}]]$). The modules $M[x,x^{-1}]$ and $M[[x,x^{-1}]]$ are called the *Laurent polynomial extension* and the *Laurent power series extension of* M, respectively. In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/\mathbb{Z}n$ we denote, respectively, integers, rational numbers, the ring of integers and the $\mathbb{Z}$-module of integers modulo n.

2. EXTENSIONS OF BAER AND PRINCIPALLY PROJECTIVE MODULES

In this section we investigate extensions of Baer and principally projective modules. Following Roos [6], a module M is called *abelian* if all idempotents of $S=\text{End}_R(M)$ are central. First, we mention some examples of abelian modules.

Examples 2.1. (1) If M is a duo module, then M is abelian. For if $f\in\text{End}_R(M)$ and $e^2=e\in\text{End}_R(M)$, then $f(1-e)M\leq(1-e)M$ implies $ef(1-e)=0$. From $fe(M)\leq eM$ we have $efe=fe$. Hence $ef=fe$ for all $f\in S$.

(2) Let M be a finitely generated torsion $\mathbb{Z}$-module. Then M is isomorphic to the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p_1^{n_1})\oplus(\mathbb{Z}/\mathbb{Z}p_2^{n_2})\oplus\ldots\oplus(\mathbb{Z}/\mathbb{Z}p_t^{n_t})$ where p_i $(i=1,\ldots,t)$ are distinct prime numbers and n_i $(i=1,\ldots,t)$ are positive integers. $\text{End}_{\mathbb{Z}}(M)$ is isomorphic to the commutative ring $(\mathbb{Z}_{p_1^{n_1}})\oplus(\mathbb{Z}_{p_2^{n_2}})\oplus\ldots\oplus(\mathbb{Z}_{p_t^{n_t}})$. So M is abelian.

We introduce a class of modules that is a generalization of principally projective rings and Baer modules. A module M is called *principally projective* if for any $m\in M$, $l_S(m)=Se$ (which is equal to $l_S(mR)$) for some $e^2=e\in S$. It is obvious that the R-module R is principally projective if and only if the ring R is left principally projective.

In [1], a module M is called *Armendariz* if for any $m(x)=\sum_{i=0}^{n}m_ix^i\in M[x]$ and $f(x)=\sum_{j=0}^{s}a_jx^j\in S[x]$, $f(x)m(x)=0$ implies $a_jm_i=0$ for all i and j.

Lemma 2.2. Let M be a module. If M is Armendariz, then it is abelian. The converse holds if M is a principally projective module.

Proof. Let $m\in M, e^2=e\in S$ and $g\in S$. Consider

$$m_1(x)=(1-e)m+eg(1-e)mx,\quad m_2(x)=em+(1-e)gemx\in M[x]$$

$$h_1(x)=e-eg(1-e)x,\ \ h_2(x)=(1-e)-(1-e)gex\in S[x].$$

Then $h_i(x)m_i(x) = 0$ for $i = 1, 2$. Since M is Armendariz, $eg(1-e)m = 0$ and $(1-e)gem = 0$. Therefore $egm = gem$ for all $m \in M$. Hence M is abelian.

Suppose that M is a principally projective and abelian module. Let $m(t) = \sum_{i=0}^{s} m_i t^i \in M[t]$ and $f(t) = \sum_{j=0}^{t} f_j t^j \in S[t]$. If $f(t)m(t) = 0$, then

$$f_0 m_0 = 0 \qquad (1)$$

$$f_0 m_1 + f_1 m_0 = 0 \qquad (2)$$

$$f_0 m_2 + f_1 m_1 + f_2 m_0 = 0 \qquad (3)$$

$$\ldots \qquad \ldots$$

By hypothesis, there exists an idempotent $e_0 \in S$ such that $l_S(m_0) = Se_0$. Then (1) implies $f_0 e_0 = f_0$. Multiplying (2) by e_0 from the left, we have $0 = e_0 f_0 m_1 + e_0 f_1 m_0 = e_0 f_0 m_1 = f_0 m_1$. By (2) $f_1 m_0 = 0$ and so $f_1 e_0 = f_1$. Let $l_S(m_1) = Se_1$. Then $f_0 e_1 = f_0$. We multiply (3) by $e_0 e_1$ from the left and use S being abelian and $e_1 e_0 f_0 m_2 = f_0 m_2$, we have $f_0 m_2 = 0$. Then (3) becomes $f_1 m_1 + f_2 m_0 = 0$. Multiplying this equation by e_0 from the left and using $e_0 f_2 m_0 = 0$ and $e_0 f_1 m_1 = f_1 m_1$ we have $f_1 m_1 = 0$. From (3) we have $f_2 m_0 = 0$. Continuing in this way, we may conclude that $f_j m_i = 0$ for all $0 \le i \le s$ and $0 \le j \le t$. Hence M is Armendariz. This completes the proof.

Corollary 2.3. If M is an Armendariz module, then it is abelian. The converse holds if M is a Baer module.

In the sequel, we investigate extensions of Baer modules and principally projective modules by using abelian modules. In case the module M is abelian, we show that there is a strong connection between Baer modules, principally projective modules and polynomial extension, power series extension, Laurent polynomial extension, Laurent power series extension of M.

For a module M, $M[x]$ is a left $S[x]$-module by the scalar product:

$$m(x) = \sum_{j=0}^{s} m_j x^j \in M[x],\ \alpha(x) = \sum_{i=0}^{t} f_i x^i \in S[x],\ \alpha(x)m(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} f_i m_j \right) x^k.$$

With a similar scalar product, $M[[x]]$, $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ become left modules over $S[[x]]$, $S[x, x^{-1}]$ and $S[[x, x^{-1}]]$, respectively.

To get rid of confusions we recall that $M[x]$ is an $S[x]$-*Baer module* if for any $R[x]$-submodule A of $M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(A) = S[x]e$, and while $M[x]$ is an $S[x]$-*principally projective module* if for any $m(x) \in M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(m(x)) = S[x]e$. Similarly, we may define $M[[x]]$ is an $S[[x]]$-Baer and $S[[x]]$-principally projective module, $M[x, x^{-1}]$ is an $S[x, x^{-1}]$-Baer and $S[x, x^{-1}]$-principally projective module and $M[[x, x^{-1}]]$ is an $S[[x, x^{-1}]]$-Baer module and $S[[x, x^{-1}]]$-principally projective module.

Theorem 2.4. Let M be a module. Then

(1) If $M[x]$ is an $S[x]$-Baer module, then M is a Baer module. The converse holds if M is abelian.

(2) If $M[[x]]$ is an $S[[x]]$-Baer module, then M is a Baer module. The converse holds if M is abelian.

(3) If $M[x,x^{-1}]$ is an $S[x,x^{-1}]$-Baer module, then M is a Baer module. The converse holds if M is abelian.

(4) If $M[[x,x^{-1}]]$ is an $S[[x,x^{-1}]]$-Baer module, then M is a Baer module. The converse holds if M is abelian.

Proof. (1) Assume that $M[x]$ is an $S[x]$-Baer module. Let N be an R-submodule of M. Then $l_S(N) \subseteq l_S(N)[x] = l_{S[x]}(N)$. Since $M[x]$ is $S[x]$-Baer, there exists $e(x)^2 = e(x) \in S[x]$ such that $l_{S[x]}(N) = S[x]e(x)$. Let $e(x) = \sum_{i=0}^{t} e_i x^i$ where all $e_i \in l_S(N)$. We show that $l_S(N) = Se_0$. Note that $e_0^2 = e_0$, because $e(x)$ is an idempotent in $S[x]$. Let $f \in l_S(N)$, then there exists $g(x) \in S[x]$ such that $f = g(x)e(x)$. So $fe(x) = f$. It follows that $fe_0 = f$. Hence $l_S(N) \subseteq Se_0$. Since $e_0 \in l_S(N)$, $l_S(N) = Se_0$. Therefore M is a Baer module. Conversely, assume that M is an abelian and Baer module. Let N be an $R[x]$-submodule of $M[x]$. We prove that there exists $e(x)^2 = e(x) \in S[x]$ such that $l_{S[x]}(N) = S[x]e(x)$. Let N^* be the right R-submodule of M generated by the coefficients of elements of N. Since M is Baer, there exists $e^2 = e \in S$ such that $l_S(N^*) = Se$. Then $eN^* = 0$ and so $eN = 0$. Hence $S[x]e \leq l_{S[x]}(N)$. To prove reverse inclusion, let $g(x) = g_0 + g_1 x + ... + g_n \in l_{S[x]}(N)$. Then $g(x)N = 0$. By Corollary 2.3, M is Armendariz. Then $g_i N^* = 0$, $g_i \in l_S(N^*) = Se$ and $g_i e = g_i$ for all $0 \leq i \leq n$. So $g(x)e = g(x) \in S[x]e$. Hence $l_{S[x]}(N) \leq S[x]e$. Therefore $l_{S[x]}(N) = S[x]e$ and so $M[x]$ is an $S[x]$-Baer module.

(2) is proved similarly as (1).

(3) Assume now that $M[x,x^{-1}]$ is an $S[x,x^{-1}]$-Baer module. Then the proof of being M a Baer module follows from the necessity of (1). Conversely, assume that M is a Baer and abelian module. Let N be an $R[x,x^{-1}]$-submodule of $M[x,x^{-1}]$. We prove that there exists $e(x)^2 = e(x) \in S[x,x^{-1}]$ such that $l_{S[x,x^{-1}]}(N) = S[x,x^{-1}]e(x)$. Let N^* be the right R-submodule of M generated by the coefficients of elements of N. By assumption $l_S(N^*) = Se$ for some $e^2 = e \in S$. Then $S[x,x^{-1}]e \leq l_{S[x,x^{-1}]}(N)$. For the reverse inclusion, let $g(x) = \sum_{i=-k}^{t} g_i x^i \in l_{S[x,x^{-1}]}(N)$ and so $g(x)N = 0$. If $f(x) = \sum_{j=-l}^{m} f_j x^j \in N$, then $g(x)f(x) = 0$. There exist positive integers u and v such that $x^u g(x) \in S[x]$ and $x^v f(x) \in N[x]$. By Corollary 2.3, M is Armendariz. Since $(x^u g(x))(x^v f(x)) = 0$, $g_i f_j = 0$ where $-k \leq i \leq t$ and $-l \leq j \leq m$. Then $g_i \in l_S(N^*)$ and so $g_i e = g_i$ for all $-k \leq i \leq t$. Thus $g(x)e = g(x) \in S[x,x^{-1}]e$.

(4) is proved similarly.

Theorem 2.5. Let M be a module. Then

(1) If $M[x]$ is an $S[x]$-principally projective module, then M is a principally projective module. The converse holds if M is abelian.

(2) If $M[[x]]$ is an $S[[x]]$-principally projective module, then M is a principally projective module. The converse holds if M is abelian.

(3) If $M[x,x^{-1}]$ is an $S[x,x^{-1}]$-principally projective module, then M is a principally projective module. The converse holds if M is abelian.

(4) If $M[[x,x^{-1}]]$ is an $S[[x,x^{-1}]]$-principally projective module, then M is a principally projective module. The converse holds if M is abelian.

Proof. (1) Assume that $M[x]$ is an $S[x]$-principally projective module and $m \in M$. There exists $e(x)^2 = e(x) \in S[x]$ such that $l_S(m) = l_S(mR) \leq l_S(mR)[x]$ and $l_S(mR)[x] = l_{S[x]}(mR) = S[x]e(x)$.

Write $e(x) = \sum_{i=0}^{t} e_i x^i$. Then $e(x)m = 0$ implies $e_i m = 0$ and so $e_i \in l_S(mR)$ for all $0 \leq i \leq t$. Let $a \in l_S(mR)$, then there exists $g(x) \in S[x]$ such that $a = g(x)e(x)$. So $ae(x) = a$. It follows that $ae_0 = a$. Hence $l_S(mR) \leq Se_0$. We have $Se_0 \leq l_S(mR)$ from $e_0 m = 0$ and $e_0^2 = e_0$ because $e(x)$ is an idempotent in $S[x]$. Therefore M is a principally projective module. Conversely, assume that M is a principally projective module and $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$. By hypothesis, there exist $e_i^2 = e_i \in S$ $(i = 0, 1, 2, ..., k)$ such that $l_S(m_i) = Se_i$. Let $e = e_0 e_1 e_2 ... e_k$. Since M is abelian, each $e_i (i = 0, 1, 2, ..., k)$ is central, and so e is a central idempotent in S. We prove $l_{S[x]}(m(x)) = S[x]e$. For if $f(x) = \sum_{j=0}^{t} f_j x^j \in l_{S[x]}(m(x))$, then $f(x)m(x) = 0$. By Lemma 2.2, $f_j m_i = 0$ for each $j = 0, 1, 2, ..., t$ and for each $i = 0, 1, 2, ..., k$. It follows that $f_j e_i = f_j$, $f_j e = f_j$ and $f(x)e = f(x)$. Hence $f(x) \in S[x]e$ and so $l_{S[x]}(m(x)) \leq S[x]e$. Let $g(x) \in S[x]e$. Since S is abelian, $em(x) = 0$ and $g(x)em(x) = 0$. Hence $S[x]e \leq l_{S[x]}(m(x))$. Thus $S[x]e = l_{S[x]}(m(x))$. Therefore $M[x]$ is an $S[x]$-principally projective module.

(2), (3) and (4) are proved similarly.

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