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Ulam Stability for A Singular Fractional $2D$ Nonlinear System

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Abstract

In this paper, we study a singular fractional $2D$ nonlinear system. We investigate the existence and uniqueness of solutions in addition to the existence of at least one solution by means of Schauder fixed point theorem, and the contraction mapping principle. Moreover, we define and study the Ulam-Hyers stability and the generalized Ulam-Hyers stability of solutions for such systems. Some applications are presented to illustrate our main results.

Keywords: *Caputo derivative, fixed point, singular fractional differential equation, existence and uniqueness, generalized Ulam-Hyers stability.*
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1. Introduction and Preliminaries

The arbitrary order of the derivatives provides an additional degree of freedom to fit a specific behavior. It provides several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. For more details, we refer the reader to the monographs of K.S. Miller and B. Ross in [11], R. Hilfer in [7], A.A. Kilbas et al. in [9].
On the other hand, Ulam-Hyers stability is one of the important issues in the thory of differential equations and their applications. Considerable work have been done in this field of research, for instance, see the papers of J. Wang in [25], S. Abbas et al. in [1], S. Harikrishnan et al. in [8], E.C. de Oliveira et al. in [12] and J.V.C. Sousa et al in [13, 14, 15] . In [10], R. Li studied the existence of solutions for nonlinear singular fractional differential equations. Furthermore, A. Taïeb and Z. Dahmani have established the existence and uniqueness of solutions in addition to some types of Ulam stability for some fractional systems. The reader may refer to the following papers [2, 3, 4, 5, 6, 16, 17, 18, 19, 20, 21] and the recent contributions of A. Taïeb in [22, 23, 24].
In this paper, we are concerned with the following singular fractional $2D$ nonlinear system:

$$\begin{cases} D^{\alpha_n}u(t)=f\left(t,u(t),v(t),D^{\beta_1}v(t),...,D^{\beta_{n-1}}v(t)\right), \quad D^{\beta_n}v(t)=g\left(t,u(t),v(t),D^{\alpha_1}u(t),...,D^{\alpha_{n-1}}u(t)\right), \\ 0<t\leq 1, k-1<\alpha_k,\beta_k<k, \quad k=1,2,...,n, \quad u^{(j)}(0)=a_j, \quad v^{(j)}(0)=b_j, \quad j=0,1,...,n-2, \\ u^{(n-1)}(0)=D^{\eta}u(1), \quad v^{(n-1)}(0)=D^{\kappa}v(1), \quad n-2<\eta,\kappa<n-1, \end{cases} \tag{1.1}$$

where $n\in\mathbb{N}-\{0,1\}$, $f,g:(0,1]\times\mathbb{R}^{n+1}\to\mathbb{R}$ are continuous functions, singular at $t=0$, and $\lim_{t\to 0^+}f(t)=\infty$, $\lim_{t\to 0^+}g(t)=\infty$. The operators D^{α_k}, D^{β_k}, D^{η}, D^{κ} are the derivatives in the sense of Caputo, defined by:

$$D^{\gamma}u(t)=\frac{1}{\Gamma(m-\gamma)}\int_0^t(t-s)^{m-\gamma-1}u^{(m)}(s)\,ds=J^{m-\gamma}u^{(m)}(t), \quad m-1<\gamma<m, \quad m\in\mathbb{N}-\{0\}. \tag{1.2}$$

We recall that: The Riemann-Liouville fractional integral J^{α} of order $\alpha\geq 0$ for a continuous function f on $[0,+\infty)$ is defined by:

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds,\ \alpha>0,\ t\geq 0, \quad \Gamma(\alpha):=\int_0^{\infty}e^{-x}x^{\alpha-1}dx. \tag{1.3}$$

Also, we list some well known properties of the fractional calculus theory which can be found in [7, 9, 11].

(i) : For $\alpha,\beta>0$; $n-1<\alpha<n$, we have $D^{\alpha}t^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha-1}$, $\beta>n$, and $D^{\alpha}t^{j}=0$, $j=0,1,...,n-1$.

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$(ii): D^{p}J^{q}f(t)=J^{q-p}f(t)$, where $q>p>0$ and $f\in L^{1}([a,b])$.

(iii) : Let $n\in\mathbb{N}-\{0\}$, $n-1<\alpha<n$, and $D^{\alpha}u(t)=0$. Then, $u(t)=\sum_{j=0}^{n-1}c_{j}t^{j}$, and $J^{\alpha}D^{\alpha}u(t)=u(t)+\sum_{j=0}^{n-1}c_{j}t^{j}$, $(c_{j})_{j=0,1,\ldots,n-1}\in\mathbb{R}$.
The following Lemma is fundamental to prove our existence results

Lemma 1.1. *[7, 9, 11] (Shauder fixed point theorem) Let (E,d) be a complete metric space, let X be a closed convex subset of E, and let $A:E\to E$ be a mapping such that the set $Y:=\{Ax:x\in X\}$ is relatively compact in E. Then, A has at least one fixed point.*

Now, we will import the solution of system (1.1) by proving the following auxiliary result.

Lemma 1.2. *Let given $n\in\mathbb{N}-\{0,1\}$, $n-1<\alpha_n,\beta_n<n$, and $(U,V)\in C([0,1],\mathbb{R})$. Then, the unique solution of*

$$\begin{cases} D^{\alpha_n}u(t)=U(t), \quad D^{\beta_n}v(t)=V(t), \quad 0<t<1, u^{(j)}(0)=a_j, \quad v^{(j)}(0)=b_j, \quad j=0,1,\ldots,n-2, \\ u^{(n-1)}(0)=D^{\eta}u(1), \quad v^{(n-1)}(0)=D^{\kappa}v(1), \quad n-2<\eta,\kappa<n-1, \end{cases} \tag{1.4}$$

is given by $(u,v)(t)$;

$$u(t) = \int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}U(s)\,ds+\sum_{j=0}^{n-2}\frac{a_j}{j!}t^j+\frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)}\int_0^1(1-s)^{\alpha_n-\eta-1}U(s)\,ds, \tag{1.5}$$

and

$$v(t) = \int_0^t \frac{(t-s)^{\beta_n-1}}{\Gamma(\beta_n)}V(s)\,ds+\sum_{j=0}^{n-2}\frac{b_j}{j!}t^j+\frac{\Gamma(n-\kappa)t^{n-1}}{(n-1)!(\Gamma(n-\kappa)-1)\Gamma(\beta_n-\kappa)}\int_0^1(1-s)^{\beta_n-\kappa-1}V(s)\,ds. \tag{1.6}$$

Proof. By the property (iii), we can write system (1.4) to an equivalent integral equations:

$$u(t)=\int_0^t\frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}U(s)\,ds-\sum_{j=0}^{n-1}c_j^1t^j, \quad v(t)=\int_0^t\frac{(t-s)^{\beta_n-1}}{\Gamma(\beta_n)}V(s)\,ds-\sum_{j=0}^{n-1}c_j^2t^j, \tag{1.7}$$

where, $\begin{pmatrix} c_0^1 & c_1^1 & \ldots & c_{n-1}^1 \\ c_0^2 & c_1^2 & \ldots & c_{n-1}^2 \end{pmatrix}\in M_n(\mathbb{R})$. Then, we observe that

$$\begin{cases} u^{(j)}(0)=-j!c_j^1, \quad v^{(j)}(0)=-j!c_j^2, \quad j=0,1,\ldots,n-2, \quad u^{(n-1)}(0)=-(n-1)!c_{n-1}^1, \quad v^{(n-1)}(0)=-(n-1)!c_{n-1}^2, \\ D^{\eta}u(1)=\int_0^1\frac{(1-s)^{\alpha_n-\eta-1}}{\Gamma(\alpha_n-\eta)}U(s)\,ds-\frac{\Gamma(n)}{\Gamma(n-\eta)}c_{n-1}^1, \quad D^{\kappa}v(1)=\int_0^1\frac{(1-s)^{\beta_n-\kappa-1}}{\Gamma(\beta_n-\kappa)}V(s)\,ds-\frac{\Gamma(n)}{\Gamma(n-\kappa)}c_{n-1}^2. \end{cases} \tag{1.8}$$

From the conditions

$$u^{(j)}(0)=a_j, \quad v^{(j)}(0)=b_j, \quad j=0,1,\ldots,n-2, \quad u^{(n-1)}(0)=D^{\eta}u(1), \quad v^{(n-1)}(0)=D^{\kappa}v(1),$$

we get

$$c_j^1=\begin{cases} -\frac{a_j}{j!}, \quad j=0,1,\ldots,n-2, \\ \frac{\Gamma(n-\eta)}{(n-1)!(1-\Gamma(n-\eta))}\int_0^1\frac{(1-s)^{\alpha_n-\eta-1}}{\Gamma(\alpha_n-\eta)}U(s)\,ds, \quad j=n-1, \end{cases} \qquad c_j^2=\begin{cases} -\frac{b_j}{j!}, \ j=0,1,\ldots,n-2, \\ \frac{\Gamma(n-\kappa)}{(n-1)!(1-\Gamma(n-\kappa))}\int_0^1\frac{(1-s)^{\beta_n-\kappa-1}}{\Gamma(\beta_n-\kappa)}V(s)\,ds, \quad j=n-1. \end{cases} \tag{1.9}$$

Substituting Eq. (1.9) in Eq. (1.7), we get Eq. (1.5) and Eq. (1.6). This completes the proof. □

We introduce the Banach space: $B:=\left\{(u,v):u,v\in C([0,1],\mathbb{R}), \quad D^{\alpha_k}u,D^{\beta_k}v\in C([0,1],\mathbb{R}), \quad k=1,2,\ldots,n-1\right\}$, where $n\in\mathbb{N}-\{0,1\}$, endowed with the norm:

$\|(u,v)\|_B=\max_{1\le k\le n-1}\left(\|u\|_\infty,\|v\|_\infty,\|D^{\alpha_k}u\|_\infty,\left\|D^{\beta_k}v\right\|_\infty\right)$; $\|u\|_\infty=\max_{t\in[0,1]}|u(t)|$, $\|v\|_\infty=\max_{t\in[0,1]}|v(t)|$, $\|D^{\alpha_k}u\|_\infty=\max_{t\in[0,1]}|D^{\alpha_k}u(t)|$, $\left\|D^{\beta_k}v\right\|_\infty=\max_{t\in[0,1]}\left|D^{\beta_k}v\right|$.

2. Existence and Uniqueness

In this section, we will establish sufficient conditions for the existence and uniqueness of solutions to system (1.1). Moreover, we will give some illustrative applications.
We list the following hypotheses:
(H_1) : Let $k-1<\alpha_k,\beta_k<k$, $k=1,2,...,n$, $n\in\mathbb{N}-\{0,1\}$, and $f,g:(0,1]\times\mathbb{R}^{n+1}\to\mathbb{R}$ be continuous, $\lim_{t\to0^+} f(t,...)=\infty$ and $\lim_{t\to0^+} g(t,...)=\infty$. Assume that there exist constants $0<\delta,\mu<1$, such that $t^{\delta}f(t,...)$ and $t^{\mu}g(t,...)$ are continuous on $[0,1]\times\mathbb{R}^{n+1}$.
(H_2) : There exist nonnegative constants $\left(\omega_j^1\right)_{j=1,...,n+1}$ and $\left(\omega_j^2\right)_{j=1,...,n+1}$, $n\in\mathbb{N}-\{0,1\}$, satisfying

$$t^{\delta}\left|f(t,x_1,...,x_{n+1})-f(t,y_1,...,y_{n+1})\right|\le\sum_{j=1}^{n+1}\omega_j^1\left|x_j-y_j\right|,\ t^{\mu}\left|g(t,x_1,...,x_{n+1})-g(t,y_1,...,y_{n+1})\right|\le\sum_{j=1}^{n+1}\omega_j^2\left|x_j-y_j\right|,$$

$\forall\, t\in[0,1]$, $\forall (x_1,...,x_{n+1}), (y_1,...,y_{n+1})\in\mathbb{R}^{n+1}$.
(H_3) : $\Theta:=\max_{1\le k\le n-1}\left(\sum_{j=1}^{n+1}\omega_j^1(\Upsilon_0,\Upsilon_k),\sum_{j=1}^{n+1}\omega_j^2(\Upsilon_0^*,\Upsilon_k^*)\right)<1$, where

$$\begin{aligned}
\Upsilon_0 &:= \frac{\Gamma(1-\delta)}{\Gamma(\alpha_n+1-\delta)}+\frac{\Gamma(n-\eta)\Gamma(1-\delta)}{(n-1)!\left|\Gamma(n-\eta)-1\right|\Gamma(\alpha_n-\eta+1-\delta)},\\
\Upsilon_k &:= \frac{\Gamma(1-\delta)}{\Gamma(\alpha_n-\alpha_k+1-\delta)}+\frac{\Gamma(n-\eta)\Gamma(1-\delta)}{\Gamma(n-\alpha_k)\left|\Gamma(n-\eta)-1\right|\Gamma(\alpha_n-\eta+1-\delta)},\\
\Upsilon_0^* &:= \frac{\Gamma(1-\mu)}{\Gamma(\beta_n+1-\mu)}+\frac{\Gamma(n-\kappa)\Gamma(1-\mu)}{(n-1)!\left|\Gamma(n-\kappa)-1\right|\Gamma(\beta_n-\kappa+1-\mu)},\\
\Upsilon_k^* &:= \frac{\Gamma(1-\mu)}{\Gamma(\beta_n-\beta_k+1-\mu)}+\frac{\Gamma(n-\kappa)\Gamma(1-\mu)}{\Gamma(n-\beta_k)\left|\Gamma(n-\kappa)-1\right|\Gamma(\beta_n-\kappa+1-\mu)}.
\end{aligned}$$

First, we define the nonlinear operator $T:B\to B$ by

$T(u,v)(t):=(T_1(u,v)(t),T_2(u,v)(t))$,

such that

$$\begin{aligned}
T_1(u,v)(t) &:= \int_0^t\frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)ds+\sum_{j=0}^{n-2}\frac{a_j}{j!}t^j+\frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)}\\
&\times\int_0^1(1-s)^{\alpha_n-\eta-1}f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)ds, \qquad (2.1)
\end{aligned}$$

and

$$\begin{aligned}
T_2(u,v)(t) &:= \int_0^t\frac{(t-s)^{\beta_n-1}}{\Gamma(\beta_n)}g\left(s,u(s),v(s),D^{\alpha_1}u(s),...,D^{\alpha_{n-1}}u(s)\right)ds+\sum_{j=0}^{n-2}\frac{b_j}{j!}t^j+\frac{\Gamma(n-\kappa)t^{n-1}}{(n-1)!(\Gamma(n-\kappa)-1)\Gamma(\beta_n-\kappa)}\\
&\times\int_0^1(1-s)^{\beta_n-\kappa-1}g\left(s,u(s),v(s),D^{\alpha_1}u(s),...,D^{\alpha_{n-1}}u(s)\right)ds. \qquad (2.2)
\end{aligned}$$

for all $t\in[0,1]$, and $n\in\mathbb{N}-\{0,1\}$.

Lemma 2.1. *Let $n-1<\alpha_n,\beta_n<n$, $n\in\mathbb{N}-\{0,1\}$. Assume that $F,G:(0,1]\to\mathbb{R}$ are continuous, $\lim_{t\to0^+}F(t)=\infty$, $\lim_{t\to0^+}G(t)=\infty$, and there exist constants $0<\delta,\mu<1$, such that $t^{\delta}F(t)$ and $t^{\mu}G(t)$ are continuous for all $t\in[0,1]$. Then,*

$$u(t)=\int_0^t\frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}F(s)ds+\sum_{j=0}^{n-2}\frac{a_j}{j!}t^j+\frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)}\int_0^1(1-s)^{\alpha_n-\eta-1}F(s)ds,$$

and

$$v(t)=\int_0^t\frac{(t-s)^{\beta_n-1}}{\Gamma(\beta_n)}G(s)ds+\sum_{j=0}^{n-2}\frac{b_j}{j!}t^j+\frac{\Gamma(n-\kappa)t^{n-1}}{(n-1)!(\Gamma(n-\kappa)-1)\Gamma(\beta_n-\kappa)}\int_0^1(1-s)^{\beta_n-\kappa-1}G(s)ds,$$

are continuous on $[0,1]$.

Proof. By the continuity of $t^{\delta}F(t)$, $t^{\mu}G(t)$,

$$u(t)=\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}s^{\delta}F(s)ds+\sum_{j=0}^{n-2}\frac{a_j}{j!}t^j+\frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)}\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}s^{\delta}F(s)ds,$$

and

$$v(t)=\int_0^t\frac{(t-s)^{\beta_n-1}s^{-\mu}}{\Gamma(\beta_n)}s^{\mu}G(s)ds+\sum_{j=0}^{n-2}\frac{b_j}{j!}t^j+\frac{\Gamma(n-\kappa)t^{n-1}}{(n-1)!(\Gamma(n-\kappa)-1)\Gamma(\beta_n-\kappa)}\int_0^1(1-s)^{\beta_n-\kappa-1}s^{-\mu}s^{\mu}G(s)ds,$$

it is clear that $u(0)=a_0$ and $v(0)=b_0$. Now, let us divide the proof into three cases.
Case 1: For $t_0=0$ and $\forall t\in(0,1]$, since $t^{\delta}F(t)$ and $t^{\mu}G(t)$ are continuous, there exist $A_1,A_2>0:\left|t^{\delta}F(t)\right|\leq A_1$ and $|t^{\mu}G(t)|\leq A_2$, $\forall t\in[0,1]$. Then,

$$\begin{aligned}|u(t)-u(0)|= & \left|\int_0^t \frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}s^{\delta}F(s)\,ds+\sum_{j=1}^{n-2}\frac{a_j}{j!}t^j+\frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)}\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}s^{\delta}F(s)\,ds\right|\\
\leq & \frac{A_1}{\Gamma(\alpha_n)}\int_0^t(t-s)^{\alpha_n-1}s^{-\delta}ds+\sum_{j=1}^{n-2}\frac{|a_j|}{j!}t^j+\frac{A_1\Gamma(n-\eta)t^{n-1}}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)}\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}ds\\
\leq & \frac{A_1t^{\alpha_n-\delta}}{\Gamma(\alpha_n)}\int_0^1(1-w)^{\alpha_n-1}w^{-\delta}dw+\sum_{j=1}^{n-2}\frac{|a_j|}{j!}t^j+\frac{A_1\Gamma(n-\eta)t^{n-1}}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)}\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}ds\\
\leq & \frac{A_1Be(\alpha_n,1-\delta)t^{\alpha_n-\delta}}{\Gamma(\alpha_n)}+\sum_{j=1}^{n-2}\frac{|a_j|}{j!}t^j+\frac{A_1\Gamma(n-\eta)Be(\alpha_n-\eta,1-\delta)t^{n-1}}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)},\end{aligned}$$

such that Be denotes the Beta function. Therefore,

$$\begin{aligned}|u(t)-u(0)| & \leq \left(\frac{A_1\Gamma(1-\delta)t^{\alpha_n-\delta}}{\Gamma(\alpha_n+1-\delta)}+\sum_{j=1}^{n-2}\frac{|a_j|}{j!}t^j+\frac{A_1\Gamma(n-\eta)\Gamma(1-\delta)t^{n-1}}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta+1-\delta)}\right)\\
& \to 0,\ \text{as } t\to 0. \end{aligned}\tag{2.3}$$

Analogously, we get

$$\begin{aligned}|v(t)-v(0)| & \leq \left(\frac{A_2\Gamma(1-\mu)t^{\beta_n-\mu}}{\Gamma(\beta_n+1-\mu)}+\sum_{j=1}^{n-2}\frac{|b_j|}{j!}t^j+\frac{A_2\Gamma(n-\kappa)\Gamma(1-\mu)t^{n-1}}{(n-1)!\,|\Gamma(n-\kappa)-1|\,\Gamma(\beta_n-\kappa+1-\mu)}\right)\\
& \to 0,\ \text{as } t\to 0. \end{aligned}\tag{2.4}$$

Case 2: For $t_0\in(0,1)$ and $\forall t\in(t_0,1]$,

$$\begin{aligned}&|u(t)-u(t_0)|\\
\leq & \left|\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}s^{\delta}F(s)\,ds-\int_0^{t_0}\frac{(t_0-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}s^{\delta}F(s)\,ds\right|+\sum_{j=1}^{n-2}\frac{|a_j|}{j!}\left(t^j-t_0^j\right)+\frac{\Gamma(n-\eta)\left(t^{n-1}-t_0^{n-1}\right)}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)}\\
&\times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}s^{\delta}F(s)\,ds\\
\leq & \frac{A_1}{\Gamma(\alpha_n)}\left(\int_0^t(t-s)^{\alpha_n-1}s^{-\delta}ds-\int_0^{t_0}(t_0-s)^{\alpha_n-1}s^{-\delta}ds\right)+\sum_{j=1}^{n-2}\frac{|a_j|}{j!}\left(t^j-t_0^j\right)\\
&+\frac{A_1\Gamma(n-\eta)\left(t^{n-1}-t_0^{n-1}\right)}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)}\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}ds.\end{aligned}$$

Thus,

$$\begin{aligned}|u(t)-u(t_0)| & \leq \frac{A_1\Gamma(1-\delta)\left(t^{\alpha_n-\delta}-t_0^{\alpha_n-\delta}\right)}{\Gamma(\alpha_n+1-\delta)}+\sum_{j=1}^{n-2}\frac{|a_j|\left(t^j-t_0^j\right)}{j!}+\frac{A_1\Gamma(n-\eta)\Gamma(1-\delta)\left(t^{n-1}-t_0^{n-1}\right)}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta+1-\delta)}\\
& \to 0,\ \text{as } t\to t_0.\end{aligned}\tag{2.5}$$

Analogously,

$$\begin{aligned}|v(t)-v(t_0)| & \leq \frac{A_2\Gamma(1-\mu)\left(t^{\beta_n-\mu}-t_0^{\beta_n-\mu}\right)}{\Gamma(\beta_n+1-\mu)}+\sum_{j=1}^{n-2}\frac{|b_j|\left(t^j-t_0^j\right)}{j!}+\frac{A_2\Gamma(n-\kappa)\Gamma(1-\mu)\left(t^{n-1}-t_0^{n-1}\right)}{(n-1)!\,|\Gamma(n-\kappa)-1|\,\Gamma(\beta_n-\kappa+1-\mu)}\\
& \to 0,\ \text{as } t\to t_0.\end{aligned}\tag{2.6}$$

Case 3: For t_0 $(0,1]$ and $\forall t\in[0,t_0)$, the proof is similar to that of case 2, we leave it. This ends the proof. □

Lemma 2.2. *If the the hypothesis (H_1) is satisfied, then, $D^{\alpha_k}T_1(u,v)$ and $D^{\beta_k}T_2(u,v)$ are continuous on $[0,1]\times\mathbb{R}^{n+1}$, such that:*

$$\begin{aligned}D^{\alpha_k}T_1(u,v)(t)= & \int_0^t\frac{(t-s)^{\alpha_n-\alpha_k-1}}{\Gamma(\alpha_n-\alpha_k)}f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)ds+\sum_{j=k}^{n-2}\frac{a_j}{\Gamma(j+1-\alpha_k)}t^{j-\alpha_k}\\
&+\frac{\Gamma(n-\eta)t^{n-1-\alpha_k}}{\Gamma(n-\alpha_k)(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)}\times\int_0^1(1-s)^{\alpha_n-\eta-1}f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)ds,\end{aligned}\tag{2.7}$$

for $k=1,2,...,n-2,$

$$D^{\alpha_{n-1}}T_1(u,v)(t) = \int_0^t \frac{(t-s)^{\alpha_n-\alpha_{n-1}-1}}{\Gamma(\alpha_n-\alpha_{n-1})} f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)ds + \frac{\Gamma(n-\eta)t^{n-1-\alpha_{n-1}}}{\Gamma(n-\alpha_{n-1})(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)} \times \int_0^1 (1-s)^{\alpha_n-\eta-1} f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)ds, \tag{2.8}$$

and

$$D^{\beta_k}T_2(u,v)(t) = \int_0^t \frac{(t-s)^{\beta_n-\beta_k-1}}{\Gamma(\beta_n-\beta_k)} g(s,u(s),v(s),D^{\alpha_1}u(s),...,D^{\alpha_{n-1}}u(s))ds + \sum_{j=k}^{n-2}\frac{b_j}{\Gamma(j+1-\beta_k)}t^{j-\beta_k} + \frac{\Gamma(n-\kappa)t^{n-1-\beta_k}}{\Gamma(n-\beta_k)(\Gamma(n-\kappa)-1)\Gamma(\beta_n-\kappa)}\int_0^1 (1-s)^{\beta_n-\kappa-1} g(s,u(s),v(s),D^{\alpha_1}u(s),...,D^{\alpha_{n-1}}u(s))ds, \tag{2.9}$$

for $k=1,2,...,n-2,$

$$D^{\beta_{n-1}}T_2(u,v)(t) = \int_0^t \frac{(t-s)^{\beta_n-\beta_{n-1}-1}}{\Gamma(\beta_n-\beta_{n-1})} g(s,u(s),v(s),D^{\alpha_1}u(s),...,D^{\alpha_{n-1}}u(s))ds + \frac{\Gamma(n-\kappa)t^{n-1-\beta_{n-1}}}{\Gamma(n-\beta_{n-1})(\Gamma(n-\kappa)-1)\Gamma(\beta_n-\kappa)} \times \int_0^1 (1-s)^{\beta_n-\kappa-1} g(s,u(s),v(s),D^{\alpha_1}u(s),...,D^{\alpha_{n-1}}u(s))ds. \tag{2.10}$$

Proof. Let $(u,v)\in B$, then $u(t),v(t)\in C([0,1])$, and $D^{\alpha_k}u(t),D^{\beta_k}v(t)\in C([0,1])$, $k=1,2,...,n-1$. So, there exist $l_k,l_k^{'}>0: |u(t)|\le l_0$, $|v(t)|\le l_0^{'}$, $|D^{\alpha_k}u(t)|\le l_k$, $\left|D^{\beta_k}v(t)\right|\le l_k^{'}$, $k=1,2,...,n-1$, $\forall t\in[0,1]$. Since $t^\delta f(t,...)$ and $t^\mu g(t,...)$ are continuous on $[0,1]\times\mathbb{R}^{n+1}$, there exist $M_1,M_2>0:$

$$M_1 = \left\|t^\delta f\left(t,u(t),v(t),D^{\beta_1}v(t),...,D^{\beta_{n-1}}v(t)\right)\right\|_\infty, M_2 = \|t^\mu g(t,u(t),v(t),D^{\alpha_1}u(t),...,D^{\alpha_{n-1}}u(t))\|_\infty,$$

for $-l_0\le u\le l_0$, $-l_0^{'}\le v\le l_0^{'}$, $-l_k\le D^{\alpha_k}u\le l_k$, $-l_k^{'}\le D^{\beta_k}v\le l_k^{'}$. Then, we have

$$\begin{aligned}
&|D^{\alpha_k}T_1(u,v)(t)| \\
&= \left| \int_0^t \frac{(t-s)^{\alpha_n-\alpha_k-1}s^{-\delta}}{\Gamma(\alpha_n-\alpha_k)} s^\delta f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)ds + \sum_{j=k}^{n-2}\frac{a_j}{\Gamma(j+1-\alpha_k)}t^{j-\alpha_k} + \frac{\Gamma(n-\eta)t^{n-1-\alpha_k}}{\Gamma(n-\alpha_k)(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)} \times \int_0^1 (1-s)^{\alpha_n-\eta-1}s^{-\delta}s^\delta f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)ds, \right| \\
&\le \frac{M_1}{\Gamma(\alpha_n-\alpha_k)}\int_0^t (t-s)^{\alpha_n-\alpha_k-1}s^{-\delta}ds + \sum_{j=k}^{n-2}\frac{|a_j|}{\Gamma(j+1-\alpha_k)}t^{j-\alpha_k} + \frac{\Gamma(n-\eta)M_1t^{n-1-\alpha_k}}{\Gamma(n-\alpha_k)|\Gamma(n-\eta)-1|\Gamma(\alpha_n-\eta)}\int_0^1 (1-s)^{\alpha_n-\eta-1}s^{-\delta}ds \\
&\le \frac{M_1\Gamma(1-\delta)t^{\alpha_n-\alpha_k-\delta}}{\Gamma(\alpha_n-\alpha_k+1-\delta)} + \sum_{j=k}^{n-2}\frac{|a_j|}{\Gamma(j+1-\alpha_k)}t^{j-\alpha_k} + \frac{M_1\Gamma(n-\eta)\Gamma(1-\delta)t^{n-1-\alpha_k}}{\Gamma(n-\alpha_k)|\Gamma(n-\eta)-1|\Gamma(\alpha_n-\eta+1-\delta)},
\end{aligned} \tag{2.11}$$

and

$$|D^{\alpha_{n-1}}T_1(u,v)(t)| \le \frac{M_1\Gamma(1-\delta)t^{\alpha_n-\alpha_{n-1}-\delta}}{\Gamma(\alpha_n-\alpha_{n-1}+1-\delta)} + \frac{M_1\Gamma(n-\eta)\Gamma(1-\delta)t^{n-1-\alpha_{n-1}}}{\Gamma(n-\alpha_{n-1})|\Gamma(n-\eta)-1|\Gamma(\alpha_n-\eta+1-\delta)}. \tag{2.12}$$

Similarly, we obtain:

$$\left|D^{\beta_k}T_2(u,v)(t)\right| \le \frac{M_2\Gamma(1-\mu)t^{\beta_n-\beta_k-\mu}}{\Gamma(\beta_n-\beta_k+1-\mu)} + \sum_{j=k}^{n-2}\frac{|b_j|}{\Gamma(j+1-\beta_k)}t^{j-\beta_k} + \frac{M_2\Gamma(n-\kappa)\Gamma(1-\mu)t^{n-1-\beta_k}}{\Gamma(n-\beta_k)|\Gamma(n-\kappa)-1|\Gamma(\beta_n-\kappa+1-\mu)}, \tag{2.13}$$

where $k=1,2,...,n-2$, and

$$\left|D^{\beta_{n-1}}T_2(u,v)(t)\right| \le \frac{M_2\Gamma(1-\mu)t^{\beta_n-\beta_{n-1}-\mu}}{\Gamma(\beta_n-\beta_{n-1}+1-\mu)} + \frac{M_2\Gamma(n-\kappa)\Gamma(1-\mu)t^{n-1-\beta_{n-1}}}{\Gamma(n-\beta_{n-1})|\Gamma(n-\kappa)-1|\Gamma(\beta_n-\kappa+1-\mu)}. \tag{2.14}$$

From the inequalities (2.11), (2.12), (2.13) and (2.14), we see that: $t^{\alpha_n-\alpha_k-\delta}$, $t^{j-\alpha_k}$, $t^{n-1-\alpha_k}$, $t^{\alpha_n-\alpha_{n-1}-\delta}$, $t^{n-1-\alpha_{n-1}}$, $t^{\beta_n-\beta_k-\mu}$, $t^{j-\beta_k}$, $t^{n-1-\beta_k}$, $t^{\beta_n-\beta_{n-1}-\mu}$ and $t^{n-1-\beta_{n-1}}$ are continuous on $[0,1]$. Hence, we can show that $D^{\alpha_k}T_1(u,v)$ and $D^{\beta_k}T_2(u,v)$ are continuous on $[0,1]$, for all $k=1,2,...,n-1$, by the same method as in Lemma 2.1. □

Lemma 2.3. *Let* (H_1) *holds. Then, the operator* $T:B\to B$ *is completely continuous.*

Proof. Let $(u,v)\in B$, then $T(u,v)(t)=(T_1(u,v)(t),T_2(u,v)(t))$, where $T_1(u,v)(t)$ and $T_2(u,v)$, are defining in Eq. (2.1) and Eq. (2.2). It follows Lemma 2.1 and Lemma 2.2, that $T:B\to B$.
Now, we devide the proof into three steps.
(1) : We show that $T:B\to B$ is continuous.
Let $(u_0,v_0)\in B:\|(u_0,v_0)\|_B=w_0$, and let $(u,v)\in B:\|(u,v)-(u_0,v_0)\|_B<1$, which implies that $\|(u,v)\|_B<1+w_0=w$. Then, by the continuity of $t^{\delta}f(t,...)$ and $t^{\mu}g(t,...)$, we see that $t^{\delta}f(t,...)$ and $t^{\mu}g(t,...)$ are uniformly continuous on $[0,1]\times[-w,w]^{n+1}$.
Therefore, $\forall\, t\in[0,1]$, $\forall\,\varepsilon>0$, there exist $\gamma>0$ $(\gamma<1)$:

$$\left|t^{\delta}f\left(t,u(t),v(t),D^{\beta_1}v(t),...,D^{\beta_{n-1}}v(t)\right)-t^{\delta}f\left(t,u_0(t),v_0(t),D^{\beta_1}v_0(t),...,D^{\beta_{n-1}}v_0(t)\right)\right| < \varepsilon, \tag{2.15}$$

$$\left|t^{\mu}g(t,u(t),v(t),D^{\alpha_1}u(t),...,D^{\alpha_{n-1}}u(t))-t^{\mu}g(t,u_0(t),v_0(t),D^{\alpha_1}u_0(t),...,D^{\alpha_{n-1}}u_0(t))\right| < \varepsilon, \tag{2.16}$$

where $(u,v)\in B$, with $\|(u,v)-(u_0,v_0)\|_B<\gamma$.
Using inequality (2.15), we get

$$\begin{aligned}
&\|T_1(u,v)-T_1(u_0,v_0)\|_\infty\\
\le\ & \max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}\left|s^{\delta}f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)-s^{\delta}f\left(s,u_0(s),v_0(s),D^{\beta_1}v_0(s),...,D^{\beta_{n-1}}v_0(s)\right)\right|ds\\
&+\frac{\Gamma(n-\eta)}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)}\max_{t\in[0,1]}t^{n-1}\\
&\times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}\left|s^{\delta}f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)-s^{\delta}f\left(s,u_0(s),v_0(s),D^{\beta_1}v_0(s),...,D^{\beta_{n-1}}v_0(s)\right)\right|ds\\
\le\ & \frac{\varepsilon\Gamma(1-\delta)}{\Gamma(\alpha_n+1-\delta)}\max_{t\in[0,1]}t^{\alpha_n-\delta}+\frac{\varepsilon\Gamma(n-\eta)\Gamma(1-\delta)}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta+1-\delta)}.
\end{aligned}$$

Thus,

$$\|T_1(u,v)-T_1(u_0,v_0)\|_\infty\le\varepsilon\Upsilon_0. \tag{2.17}$$

And for all $k=1,2,...,n-1$, we get

$$\begin{aligned}
&\|D^{\alpha_k}(T_1(u,v)-T_1(u_0,v_0))\|_\infty\\
\le\ & \max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_n-\alpha_k-1}s^{-\delta}}{\Gamma(\alpha_n-\alpha_k)}\left|s^{\delta}f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)-s^{\delta}f\left(s,u_0(s),v_0(s),D^{\beta_1}v_0(s),...,D^{\beta_{n-1}}v_0(s)\right)\right|ds\\
&+\frac{\Gamma(n-\eta)}{\Gamma(n-\alpha_k)\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)}\max_{t\in[0,1]}t^{n-\alpha_k-1}\\
&\times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}\left|s^{\delta}f\left(s,u(s),v(s),D^{\beta_1}v(s),...,D^{\beta_{n-1}}v(s)\right)-s^{\delta}f\left(s,u_0(s),v_0(s),D^{\beta_1}v_0(s),...,D^{\beta_{n-1}}v_0(s)\right)\right|ds,\\
\le\ & \frac{\varepsilon\Gamma(1-\delta)}{\Gamma(\alpha_n-\alpha_k+1-\delta)}\max_{t\in[0,1]}t^{\alpha_n-\alpha_k-\delta}+\frac{\varepsilon\Gamma(n-\eta)\Gamma(1-\delta)}{\Gamma(n-\alpha_k)\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta+1-\delta)}.
\end{aligned}$$

Then,

$$\|D^{\alpha_k}(T_1(u,v)-T_1(u_0,v_0))\|_\infty\le\varepsilon\Upsilon_k. \tag{2.18}$$

Similarly by inequality (2.16), we get

$$\|T_2(u,v)-T_2(u_0,v_0)\|_\infty\le\varepsilon\Upsilon_0^* \tag{2.19}$$

and

$$\left\|D^{\beta_k}(T_2(u,v)-T_2(u_0,v_0))\right\|_\infty\le\varepsilon\Upsilon_k^*. \tag{2.20}$$

Thanks to inequalities (2.17), (2.18), (2.19) and (2.20), we get $\|T(u,v)-T(u_0,v_0)\|_B\le\varepsilon\max_{1\le k\le n-1}\left(\Upsilon_0,\Upsilon_k,\Upsilon_0^*,\Upsilon_k^*\right)$.
Therefore, $\|T(u,v)-T(u_0,v_0)\|_B\to0$ as $\|(u,v)-(u_0,v_0)\|_B\to0$. Hence, $T:B\to B$ is continuous.
(2) : Let $F:=\{(u,v)\in B:\|(u,v)\|_B\le\xi\}$; $\xi>0$. We show that $T(F)$ is bounded.
Since $t^{\delta}f(t,...)$ and $t^{\mu}g(t,...)$ are continuous on $[0,1]\times[-\xi,\xi]^{n+1}$, there exist $L_1,L_2>0:\forall t\in[0,1]$, $\forall(u,v)\in F$,

$$\left|t^{\delta}f\left(t,u(t),v(t),D^{\beta_1}v(t),...,D^{\beta_{n-1}}v(t)\right)\right| \le L_1, \tag{2.21}$$

$$\left|t^{\mu}g(t,u(t),v(t),D^{\alpha_1}u(t),...,D^{\alpha_{n-1}}u(t))\right| \le L_2, \tag{2.22}$$

Using inequality (2.21), we get

$$\begin{aligned}
&\|T_1(u,v)\|_\infty \\
\leq\ & \max_{t\in[0,1]}\int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}s^{-\delta}\left|s^\delta f\left(s,u(s),v(s),D^{\beta_1}v(s),\dots,D^{\beta_{n-1}}v(s)\right)\right|ds+\sum_{j=0}^{n-2}\frac{|a_j|}{j!}\max_{t\in[0,1]}t^j \\
&+\frac{\Gamma(n-\eta)}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)}\max_{t\in[0,1]}t^{n-1}\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}\left|s^\delta f\left(s,u(s),v(s),D^{\beta_1}v(s),\dots,D^{\beta_{n-1}}v(s)\right)\right|ds \\
\leq\ & L_1\Upsilon_0+\sum_{j=0}^{n-2}\frac{|a_j|}{j!}, \qquad (2.23)
\end{aligned}$$

$$\begin{aligned}
&\|D^{\alpha_k}T_1(u,v)\|_\infty \\
\leq\ & \max_{t\in[0,1]}\int_0^t \frac{(t-s)^{\alpha_n-\alpha_k-1}}{\Gamma(\alpha_n-\alpha_k)}s^{-\delta}\left|s^\delta f\left(s,u(s),v(s),D^{\beta_1}v(s),\dots,D^{\beta_{n-1}}v(s)\right)\right|ds+\sum_{j=k}^{n-2}\frac{|a_j|}{\Gamma(j+1-\alpha_k)}\max_{t\in[0,1]}t^{j-\alpha_k} \\
&+\frac{\Gamma(n-\eta)\,t^{n-1-\alpha_k}}{\Gamma(n-\alpha_k)\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)}\max_{t\in[0,1]}t^{n-1-\alpha_k}\times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}\left|s^\delta f\left(s,u(s),v(s),D^{\beta_1}v(s),\dots,D^{\beta_{n-1}}v(s)\right)\right|ds \\
\leq\ & L_1\Upsilon_k+\sum_{j=k}^{n-2}\frac{|a_j|}{\Gamma(j+1-\alpha_k)},\ k=1,2,\dots,n-2, \qquad (2.24)
\end{aligned}$$

and

$$\|D^{\alpha_{n-1}}T_1(u,v)\|_\infty\leq L_1\Upsilon_{n-1}. \qquad (2.25)$$

Similarly using inequality (2.22), we get

$$\begin{aligned}
\|T_2(u,v)\|_\infty &\leq L_2\Upsilon_0^*+\sum_{j=0}^{n-2}\frac{|b_j|}{j!}, && (2.26)\\
\left\|D^{\beta_k}T_2(u,v)\right\|_\infty &\leq L_2\Upsilon_k^*+\sum_{j=k}^{n-2}\frac{|b_j|}{\Gamma(j+1-\beta_k)},\ k=1,2,\dots,n-2, && (2.27)\\
\left\|D^{\beta_{n-1}}T_2(u,v)\right\|_\infty &\leq L_2\Upsilon_{n-1}^*. && (2.28)
\end{aligned}$$

It follows from inequalities (2.23), (2.24), (2.25), (2.26), (2.27), and (2.28), that

$$\|T(u,v)\|_B\leq\max_{1\leq k\leq n-1}\left(L_1\Upsilon_0+\sum_{j=0}^{n-2}\frac{|a_j|}{j!},L_1\Upsilon_k+\sum_{j=k}^{n-2}\frac{|a_j|}{\Gamma(j+1-\alpha_k)},L_1\Upsilon_{n-1},L_2\Upsilon_0^*+\sum_{j=0}^{n-2}\frac{|b_j|}{j!},L_2\Upsilon_k^*+\sum_{j=k}^{n-2}\frac{|b_j|}{\Gamma(j+1-\beta_k)},L_2\Upsilon_{n-1}^*\right). \qquad (2.29)$$

So, $T(F)$ is bounded.
(3) : We shall show that $T(F)$ is equicontinuous.
Let $(u,v)\in F$, and $t_1,t_2\in[0,1]:t_1<t_2$. Then,

$$\begin{aligned}
&\|T_1(u,v)(t_2)-T_1(u,v)(t_1)\|_\infty \\
\leq\ & \frac{L_1\Gamma(1-\delta)\left(t_2^{\alpha_n-\delta}-t_1^{\alpha_n-\delta}\right)}{\Gamma(\alpha_n+1-\delta)}+\sum_{j=0}^{n-2}\frac{|a_j|\left(t_2^j-t_1^j\right)}{j!}+\frac{L_1\Gamma(n-\eta)\Gamma(1-\delta)\left(t_2^{n-1}-t_1^{n-1}\right)}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta+1-\delta)}. \qquad (2.30)
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
&\|D^{\alpha_k}(T_1(u,v)(t_2)-T_1(u,v)(t_1))\|_\infty \\
\leq\ & \frac{L_1\Gamma(1-\delta)\left(t_2^{\alpha_n-\alpha_k-\delta}-t_1^{\alpha_n-\alpha_k-\delta}\right)}{\Gamma(\alpha_n-\alpha_k+1-\delta)}+\sum_{j=k}^{n-2}\frac{|a_j|\left(t_2^{j-\alpha_k}-t_1^{j-\alpha_k}\right)}{\Gamma(j+1-\alpha_k)}+\frac{L_1\Gamma(n-\eta)\Gamma(1-\delta)\left(t_2^{n-1-\alpha_k}-t_1^{n-1-\alpha_k}\right)}{\Gamma(n-\alpha_k)\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta+1-\delta)}, \qquad (2.31)
\end{aligned}$$

$$\begin{aligned}
&\|D^{\alpha_{n-1}}(T_1(u,v)(t_2)-T_1(u,v)(t_1))\|_\infty \\
\leq\ & \frac{L_1\Gamma(1-\delta)\left(t_2^{\alpha_n-\alpha_{n-1}-\delta}-t_1^{\alpha_n-\alpha_{n-1}-\delta}\right)}{\Gamma(\alpha_n-\alpha_{n-1}+1-\delta)}+\frac{L_1\Gamma(n-\eta)\Gamma(1-\delta)\left(t_2^{n-1-\alpha_{n-1}}-t_1^{n-1-\alpha_{n-1}}\right)}{\Gamma(n-\alpha_{n-1})\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta+1-\delta)}, \qquad (2.32)
\end{aligned}$$

$$\begin{aligned}
&\|T_2(u,v)(t_2)-T_2(u,v)(t_1)\|_\infty \\
\leq\ & \frac{L_2\Gamma(1-\mu)\left(t_2^{\beta_n-\mu}-t_1^{\beta_n-\mu}\right)}{\Gamma(\beta_n+1-\mu)}+\sum_{j=0}^{n-2}\frac{|b_j|\left(t_2^j-t_1^j\right)}{j!}+\frac{L_2\Gamma(n-\kappa)\Gamma(1-\mu)\left(t_2^{n-1}-t_1^{n-1}\right)}{(n-1)!\,|\Gamma(n-\kappa)-1|\,\Gamma(\beta_n-\kappa+1-\mu)}, \qquad (2.33)
\end{aligned}$$

$$\left\|D^{\beta_k}\left(T_2(u,v)(t_2)-T_2(u,v)(t_1)\right)\right\|_\infty$$
$$\leq \frac{L_2\Gamma(1-\mu)\left(t_2^{\beta_n-\beta_k-\mu}-t_1^{\beta_n-\beta_k-\mu}\right)}{\Gamma(\beta_n-\beta_k+1-\mu)}+\sum_{j=k}^{n-2}\frac{|b_j|\left(t_2^{j-\beta_k}-t_1^{j-\beta_k}\right)}{\Gamma(j+1-\beta_k)}+\frac{L_2\Gamma(n-\kappa)\Gamma(1-\mu)\left(t_2^{n-1-\beta_k}-t_1^{n-1-\beta_k}\right)}{\Gamma(n-\beta_k)|\Gamma(n-\kappa)-1|\Gamma(\beta_n-\kappa+1-\mu)}, \tag{2.34}$$

and

$$\left\|D^{\beta_{n-1}}\left(T_2(u,v)(t_2)-T_2(u,v)(t_1)\right)\right\|_\infty$$
$$\leq \frac{L_2\Gamma(1-\mu)\left(t_2^{\beta_n-\beta_{n-1}-\mu}-t_1^{\beta_n-\beta_{n-1}-\mu}\right)}{\Gamma(\beta_n-\beta_{n-1}+1-\mu)}+\frac{L_2\Gamma(n-\kappa)\Gamma(1-\mu)\left(t_2^{n-1-\beta_{n-1}}-t_1^{n-1-\beta_{n-1}}\right)}{\Gamma(n-\beta_{n-1})|\Gamma(n-\kappa)-1|\Gamma(\beta_n-\kappa+1-\mu)}. \tag{2.35}$$

The right-hand sides of inequalities (2.30), (2.31), (2.32), (2.33), (2.34),and (2.35), are independent of (u,v) and tend to zero as $t_1\to t_2$, we state that $T(F)$ is equicontinuous. Then, by Arzela-Ascoli theorem, we deduce that T is completely continuous. □

Theorem 2.4. *Let (H_2) and (H_3) hold. Then, system (1.1) has a unique solution on $[0,1]$.*

Proof. We will prove that T is contractive on B. Let $(u_1,v_1),(u_2,v_2)\in B$ and $t\in[0,1]$.
Thanks to (H_1), we get

$$\|T_1(u_1,v_1)-T_1(u_2,v_2)\|_\infty$$
$$\leq \max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}s^{\delta}\left|f\left(s,u_1(s),v_1(s),D^{\beta_1}v_1(s),...,D^{\beta_{n-1}}v_1(s)\right)-f\left(s,u_2(s),v_2(s),D^{\beta_1}v_2(s),...,D^{\beta_{n-1}}v_2(s)\right)\right|ds$$
$$+\frac{\Gamma(n-\eta)}{(n-1)!|\Gamma(n-\eta)-1|\Gamma(\alpha_n-\eta)}\max_{t\in[0,1]}t^{n-1}$$
$$\times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}s^{\delta}\left|f\left(s,u_1(s),v_1(s),D^{\beta_1}v_1(s),...,D^{\beta_{n-1}}v_1(s)\right)-f\left(s,u_1(s),v_2(s),D^{\beta_1}v_2(s),...,D^{\beta_{n-1}}v_2(s)\right)\right|ds$$
$$\leq\left(\omega_1^1\|u_1-u_2\|_\infty+\omega_2^1\|v_1-v_2\|_\infty+\omega_3^1\left\|D^{\beta_1}(v_1-v_2)\right\|_\infty+...+\omega_{n+1}^1\left\|D^{\beta_{n-1}}(v_1-v_2)\right\|_\infty\right)\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}ds$$
$$+\left(\omega_1^1\|u_1-u_2\|_\infty+\omega_2^1\|v_1-v_2\|_\infty+\omega_3^1\left\|D^{\beta_1}(v_1-v_2)\right\|_\infty+...+\omega_{n+1}^1\left\|D^{\beta_{n-1}}(v_1-v_2)\right\|_\infty\right)$$
$$\times\frac{\Gamma(n-\eta)}{(n-1)!|\Gamma(n-\eta)-1|\Gamma(\alpha_n-\eta)}\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}ds$$
$$\leq\sum_{j=1}^{n+1}\omega_j^1\max\left(\|u_1-u_2\|_\infty,\|v_1-v_2\|_\infty,...,\left\|D^{\beta_{n-1}}(v_1-v_2)\right\|_\infty\right)$$
$$\times\left(\frac{\Gamma(1-\delta)}{\Gamma(\alpha_n+1-\delta)}\max_{t\in[0,1]}t^{\alpha_n-\delta}+\frac{\Gamma(n-\eta)\Gamma(1-\delta)}{(n-1)!|\Gamma(n-\eta)-1|\Gamma(\alpha_n-\eta+1-\delta)}\right).$$

Thus,

$$\|T_1(u_1,v_1)-T_1(u_2,v_2)\|_\infty\leq\sum_{j=1}^{n+1}\omega_j^1\Upsilon_0\|(u_1-u_2,v_1-v_2)\|_B. \tag{2.36}$$

Also by (H_1), we get

$$\|D^{\alpha_k}(T_1(u_1,v_1)-T_1(u_2,v_2))\|_\infty$$
$$\leq\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_n-\alpha_k-1}s^{-\delta}}{\Gamma(\alpha_n-\alpha_k)}s^{\delta}\left|f\left(s,u_1(s),v_1(s),D^{\beta_1}v_1(s),...,D^{\beta_{n-1}}v_1(s)\right)-f\left(s,u_2(s),v_2(s),D^{\beta_1}v_2(s),...,D^{\beta_{n-1}}v_2(s)\right)\right|ds$$
$$+\frac{\Gamma(n-\eta)}{(n-\alpha_k)|\Gamma(n-\eta)-1|\Gamma(\alpha_n-\eta)}\max_{t\in[0,1]}t^{n-1-\alpha_k}$$
$$\times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}s^{\delta}\left|f\left(s,u_1(s),v_1(s),D^{\beta_1}v_1(s),...,D^{\beta_{n-1}}v_1(s)\right)-f\left(s,u_1(s),v_2(s),D^{\beta_1}v_2(s),...,D^{\beta_{n-1}}v_2(s)\right)\right|ds$$
$$\leq\sum_{j=1}^{n+1}\omega_j^1\max\left(\|u_1-u_2\|_\infty,\|v_1-v_2\|_\infty,...,\left\|D^{\beta_{n-1}}(v_1-v_2)\right\|_\infty\right)$$
$$\times\left(\frac{\Gamma(1-\delta)}{\Gamma(\alpha_n-\alpha_k+1-\delta)}\max_{t\in[0,1]}t^{\alpha_n-\alpha_k-\delta}+\frac{\Gamma(n-\eta)\Gamma(1-\delta)}{(n-\alpha_k)|\Gamma(n-\eta)-1|\Gamma(\alpha_n-\eta+1-\delta)}\right).$$

Therefore,

$$\|D^{\alpha_k}(T_1(u_1,v_1)-T_1(u_2,v_2))\|_\infty\leq\sum_{j=1}^{n+1}\omega_j^1\Upsilon_k\|(u_1-u_2,v_1-v_2)\|_B,\quad k=1,2,...,n-1. \tag{2.37}$$

Similarly for the other hand, we get

$$\|T_2(u_1,v_1)-T_2(u_2,v_2)\|_\infty \leq \sum_{j=1}^{n+1}\omega_j^2\Upsilon_0^*\|(u_1-u_2,v_1-v_2)\|_B, \quad (2.38)$$

$$\left\|D^{\beta_k}(T_2(u_1,v_1)-T_2(u_2,v_2))\right\|_\infty \leq \sum_{j=1}^{n+1}\omega_j^2\Upsilon_k^*\|(u_1-u_2,v_1-v_2)\|_B. \quad (2.39)$$

It follows inequalities (2.36), (2.37), (2.38), and (2.39), that $\|T(u_1,v_1)-T(u_2,v_2)\|_B \leq \Theta\|(u_1-u_2,v_1-v_2)\|_B$.
Using (H_3), we deduce that T is contractive. By Banach fixed point theorem, we state that T has a fixed point which is the unique solution of system (1.1). □

Example 2.5. *Consider the following system:*

$$\begin{cases} D^{\frac{15}{4}}u(t)=\dfrac{\left|u(t)+v(t)+D^{\frac{1}{2}}v(t)+D^{\frac{4}{3}}v(t)+D^{\frac{7}{3}}v(t)\right|}{60\pi^2t^{\frac{2}{9}}\left(1+\left|u(t)+v(t)+D^{\frac{1}{2}}v(t)+D^{\frac{4}{3}}v(t)+D^{\frac{7}{3}}v(t)\right|\right)}, \quad D^{\frac{11}{3}}v(t)=\dfrac{\sin u(t)-\cos v(t)+\sin D^{\frac{3}{4}}u(t)+\sin D^{\frac{3}{2}}u(t)+\sin D^{\frac{9}{4}}u(t)}{125\pi t^{\frac{1}{4}}}, \\ 0<t\leq 1, \\ u(0)=\sqrt{2}, \quad u'(0)=1, \quad u''(0)=2\sqrt{3}, \quad u'''(0)=D^{\frac{11}{5}}u(1), \quad v(0)=\sqrt{3}, \quad v'(0)=1, \quad v''(0)=5\sqrt{2}, \quad v'''(0)=D^{\frac{14}{5}}v(1). \end{cases} \quad (2.40)$$

Here, we have: $n=4$, $\alpha_4=\frac{15}{4}$, $\beta_1=\frac{1}{2}$, $\beta_2=\frac{4}{3}$, $\beta_3=\frac{7}{3}$, $a_0=\sqrt{2}$, $a_1=1$, $a_2=2\sqrt{3}$, $\eta=\frac{11}{5}$, $\beta_4=\frac{11}{3}$, $\alpha_1=\frac{3}{4}$, $\alpha_2=\frac{3}{2}$, $\alpha_3=\frac{9}{4}$, $b_0=\sqrt{3}$, $b_1=1$, $b_2=5\sqrt{2}$, $\kappa=\frac{14}{5}$.
For all $t\in[0,1]$ *and* $(x_1,\ldots,x_5),(y_1,\ldots,y_5)\in\mathbb{R}^5$, *we get:*

$$t^{\frac{4}{9}}|f(t,x_1,\ldots,x_5)-f(t,y_1,\ldots,y_5)|\leq\frac{t^{\frac{2}{9}}}{60\pi^2}\sum_{i=1}^5|x_i-y_i|, t^{\frac{3}{4}}|g(t,x_1,\ldots,x_5)-g(t,y_1,\ldots,y_5)|\leq\frac{t^{\frac{1}{2}}}{125\pi}\sum_{i=1}^5|x_i-y_i|, \quad \delta=\frac{4}{9}, \quad \mu=\frac{3}{4}.$$

So, we can take

$$\omega_j^1=\frac{1}{60\pi^2},\ \omega_j^2=\frac{1}{125\pi},\ j=1,\ldots,5,\ \sum_{j=1}^5\omega_j^1=\frac{1}{12\pi^2},\ \sum_{j=1}^5\omega_j^2=\frac{1}{25\pi}.$$

On the other hand, we obtain

$$\Upsilon_0=3.6314,\ \Upsilon_1=8.5776,\ \Upsilon_2=16.5292,\ \Upsilon_3=24.0831, \Upsilon_0^*=7.8493,\ \Upsilon_1^*=14.1526,\ \Upsilon_2^*=31.1912,\ \Upsilon_3^*=51.7577.$$

Indeed,

$$\sum_{j=1}^5\omega_j^1\Upsilon_0 = 0.0307,\ \sum_{j=1}^5\omega_j^1\Upsilon_1=0.0724,\ \sum_{j=1}^5\omega_j^1\Upsilon_2=0.1396,\ \sum_{j=1}^5\omega_j^1\Upsilon_3=0.1902,$$
$$\sum_{j=1}^5\omega_j^1\Upsilon_0^* = 0.0999,\ \sum_{j=1}^5\omega_j^1\Upsilon_1^*=0.1802,\ \sum_{j=1}^5\omega_j^1\Upsilon_2^*=0.3971,\ \sum_{j=1}^5\omega_j^1\Upsilon_3^*=0.6590.$$

So, we get $\Theta<1$. *Then, system* (2.40) *has a unique solution on* $[0,1]$.

Theorem 2.6. *Let* (H_1) *holds. Then, system* (1.1) *has at least one solution on* $[0,1]$.

Proof. Let $\Omega:=\{(u,v)\in B:\|(u,v)\|_B\leq r\}$, where

$$A_1 = \max_{t\in[0,1]}t^\delta\left|f\left(t,u(t),v(t),D^{\beta_1}v(t),\ldots,D^{\beta_{n-1}}v(t)\right)\right|, \quad (2.41)$$

$$A_2 = \max_{t\in[0,1]}t^\mu\left|g\left(t,u(t),v(t),D^{\alpha_1}u(t),\ldots,D^{\alpha_{n-1}}u(t)\right)\right|, \quad (2.42)$$

$$r=\max_{1\leq k\leq n-2}\left(A_1\Upsilon_0+\sum_{j=0}^{n-2}\frac{|a_j|}{j!},\ A_1\Upsilon_k+\sum_{j=k}^{n-2}\frac{|a_j|}{\Gamma(j+1-\alpha_k)},A_1\Upsilon_{n-1},A_2\Upsilon_0^*+\sum_{j=0}^{n-2}\frac{|b_j|}{j!},A_2\Upsilon_k^*+\sum_{j=k}^{n-2}\frac{|b_j|}{\Gamma(j+1-\beta_k)},A_2\Upsilon_{n-1}^*\right). \quad (2.43)$$

We show that $T:\Omega\to\Omega$. Let $(u,v)\in\Omega$ and $t\in[0,1]$.
Considering Eq. (2.41) and Eq. (2.42), we can state by inequality (2.29) that

$$\|T(u,v)\|_B\leq\max_{1\leq k\leq n-2}\left(A_1\Upsilon_0+\sum_{j=0}^{n-2}\frac{|a_j|}{j!},A_1\Upsilon_k+\sum_{j=k}^{n-2}\frac{|a_j|}{\Gamma(j+1-\alpha_k)},A_1\Upsilon_{n-1},A_2\Upsilon_0^*+\sum_{j=0}^{n-2}\frac{|b_j|}{j!},A_2\Upsilon_k^*+\sum_{j=k}^{n-2}\frac{|b_j|}{\Gamma(j+1-\beta_k)},A_2\Upsilon_{n-1}^*\right). \quad (2.44)$$

Thus, $\|T(u,v)\|_B\leq r$. It is clear that for $(u,v)\in\Omega$, we get $T(u,v)\in\Omega$. Moreover, it follows Lemma 2.1. and Lemma 2.2. that $T_1(u,v),T_2(u,v)\in C([0,1])$ and $D^{\alpha_k}T_1(u,v)\in C([0,1])$, $D^{\beta_k}T_2(u,v)\in C([0,1])$. Hence, $T:\Omega\to\Omega$.
By Lemma 2.3. we have T is completely continuous. As a consequence of Lemma 1.1. system (1.1) has at least one solution on $[0,1]$. □

Example 2.7. *Consider the following system:*

$$\begin{cases} D^{\frac{9}{2}}u(t) = \dfrac{t^{-\frac{3}{8}}\left(\cos u(t)\cos v(t) + \sin D^{\frac{1}{3}}v(t)\sin D^{\frac{3}{2}}v(t)\right)}{2\pi e^{t} + \left|\cos D^{\frac{9}{4}}v(t) - \sin D^{\frac{19}{6}}v(t)\right|}, \quad D^{\frac{14}{3}}v(t) = \dfrac{t^{-\frac{1}{4}}\sin u(t)\cos u(t)}{2\pi - \sin\left(D^{\frac{2}{3}}u(t) + D^{\frac{5}{4}}u(t)\right)\cos\left(D^{\frac{8}{3}}u(t)D^{\frac{15}{4}}u(t)\right)}, \\ 0 < t \le 1, u(0) = 1, \quad u'(0) = -\sqrt{3}, \quad u''(0) = \sqrt{5}, \quad u'''(0) = \frac{1}{\pi}, \ u^{(4)}(0) = D^{\frac{10}{3}}u(1), \\ v(0) = \sqrt{2}, \quad v'(0) = -1, \quad v''(0) = 3\sqrt{2}, \quad v'''(0) = \pi, \quad v^{(4)}(0) = D^{\frac{11}{3}}v(1). \end{cases} \tag{2.45}$$

We have: $n = 5$, $\alpha_5 = \frac{9}{2}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{3}{2}$, $\beta_3 = \frac{9}{4}$, $\beta_4 = \frac{19}{6}$, $a_0 = 1$, $a_1 = -\sqrt{3}$, $a_2 = \sqrt{5}$, $a_3 = \frac{1}{\pi}$, $\eta = \frac{10}{3}$, $\beta_5 = \frac{14}{3}$, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{5}{4}$, $\alpha_3 = \frac{8}{3}$, $\alpha_4 = \frac{15}{4}$, $b_0 = \sqrt{2}$, $b_1 = -1$, $b_2 = 3\sqrt{2}$, $b_3 = \pi$, $b_3 =$, $\kappa = \frac{11}{3}$.
For $\delta = \frac{1}{2}$ *and* $\mu = \frac{3}{4}$, *all the assumptios of Theorem 2.6 will be satisfied . Therefore,* (2.45) *has at least one solution on* $[0,1]$.

3. Generalized Ulam-Hyers Stability

In this section, we study the Ulam-Hyers stability and the generalized Ulam-Hyers stability for system (1.1).

Definition 3.1. *System* (1.1) *is Ulam-Hyers stable if there exists a constant* $\lambda_{f,g} > 0$, *such that for all* $(\varepsilon_1, \varepsilon_2) > 0$, *and for all solution* $(x,y) \in B$ *of*

$$\begin{cases} \left|D^{\alpha_n}x(t) - f\left(t, x(t), y(t), D^{\beta_1}y(t), ..., D^{\beta_{n-1}}y(t)\right)\right| \le \varepsilon_1, \quad \left|D^{\beta_n}y(t) - g\left(t, x(t), y(t), D^{\alpha_1}x(t), ..., D^{\alpha_{n-1}}x(t)\right)\right| \le \varepsilon_2, \\ 0 < t \le 1, \quad k-1 < \alpha_k, \beta_k < k, \quad k = 1,2,...,n, \end{cases} \tag{3.1}$$

where $x^{(j)}(0) = a_j$, $y^{(j)}(0) = b_j$, $j = 0,1,...,n-2$, $x^{(n-1)}(0) = D^{\eta}x(1)$, $y^{(n-1)}(0) = D^{\kappa}y(1)$, $n-2 < \eta, \kappa < n-1$, *there exists* $(u,v) \in B$ *of system* (1.1), *with* $\|(x-u, y-v)\|_B \le \lambda_{f,g}\varepsilon$, $\varepsilon > 0$.

Definition 3.2. *System* (1.1) *is generalized Ulam-Hyers stable if there exist* $\phi_{f,g} \in C\left(\mathbb{R}^+, \mathbb{R}^+\right)$, $\phi_{f,g}(0) = 0$, *such that for all* $\varepsilon > 0$, *and for each solution* $(x,y) \in B$ *of system* (3.1), *there exists* $(u,v) \in B$ *of system* (1.1) *with* $\|(x-u, y-v)\|_B \le \phi_{f,g}(\varepsilon)$, $\varepsilon > 0$.

Theorem 3.3. *Let* (H_2) *and* (H_3) *hold. Then, system* (1.1) *is generalized Ulam-Hyers stable in* B.

Proof. Let $(x,y) \in B$ be a solution of inequalities (3.1). Then, by integrating inequalities (3.1), we obtain

$$\begin{aligned} & \left| \begin{array}{l} x_k(t) - \int_0^t \frac{(t-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} f\left(s, x(s), y(s), D^{\beta_1}y(s), ..., D^{\beta_{n-1}}y(s)\right) ds - \sum\limits_{j=0}^{n-2} \frac{a_j}{j!} t^j \\ - \frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n - \eta)} \int_0^1 (1-s)^{\alpha_n - \eta - 1} f\left(s, x(s), y(s), D^{\beta_1}y(s), ..., D^{\beta_{n-1}}y(s)\right) ds \end{array} \right| \\ \le \ & J^{\alpha_n}\varepsilon_1 \\ \le \ & \frac{t^{\alpha_n}}{\Gamma(\alpha_n + 1)}\varepsilon_1, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} & \left| \begin{array}{l} y_k(t) - \int_0^t \frac{(t-s)^{\beta_n - 1}}{\Gamma(\beta_n)} g\left(s, x(s), y(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right) ds - \sum\limits_{j=0}^{n-2} \frac{b_j}{j!} t^j \\ - \frac{\Gamma(n-\kappa)t^{n-1}}{(n-1)!(\Gamma(n-\kappa)-1)\Gamma(\beta_n - \kappa)} \int_0^1 (1-s)^{\beta_n - \kappa - 1} g\left(s, x(s), y(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right) ds \end{array} \right| \\ \le \ & J^{\beta_n}\varepsilon_2 \\ \le \ & \frac{t^{\beta_n}}{\Gamma(\beta_n + 1)}\varepsilon_2. \end{aligned} \tag{3.3}$$

Using (H_2) and (H_3), there exists a solution $(u,v) \in B$ of system (1.1) :

$$\begin{aligned} u(t) \ = \ & \int_0^t \frac{(t-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} f\left(s, u(s), v(s), D^{\beta_1}v(s), ..., D^{\beta_{n-1}}v(s)\right) ds + \sum_{j=0}^{n-2} \frac{a_j}{j!} t^j + \frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n - \eta)} \\ & \times \int_0^1 (1-s)^{\alpha_n - \eta - 1} f\left(s, u(s), v(s), D^{\beta_1}v(s), ..., D^{\beta_{n-1}}v(s)\right) ds, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} v(t) \ = \ & \int_0^t \frac{(t-s)^{\beta_n - 1}}{\Gamma(\beta_n)} g\left(s, u(s), v(s), D^{\alpha_1}u(s), ..., D^{\alpha_{n-1}}u(s)\right) ds + \sum_{j=0}^{n-2} \frac{b_j}{j!} t^j + \frac{\Gamma(n-\kappa)t^{n-1}}{(n-1)!(\Gamma(n-\kappa)-1)\Gamma(\beta_n - \kappa)} \\ & \times \int_0^1 (1-s)^{\beta_n - \kappa - 1} g\left(s, u(s), v(s), D^{\alpha_1}u(s), ..., D^{\alpha_{n-1}}u(s)\right) ds. \end{aligned} \tag{3.5}$$

Then, we get

$$
\begin{aligned}
& |x(t)-u(t)| \\
= & \left| \begin{array}{l}
x(t)-\sum_{j=0}^{n-2}\frac{a_j}{j!}t^j-\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}s^{\delta}f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)ds-\frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)} \\
\times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}s^{\delta}f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)ds \\
+\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}s^{\delta}\left(f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)-f\left(s,u(s),v(s),D^{\beta_1}v(s),\dots,D^{\beta_{n-1}}v(s)\right)\right)ds \\
+\frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)} \\
\times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}s^{\delta}\left(f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)ds-f\left(s,u(s),v(s),D^{\beta_1}v(s),\dots,D^{\beta_{n-1}}v(s)\right)\right)ds
\end{array}\right|.
\end{aligned}
$$

Using inequality (3.2), we get

$$
\begin{aligned}
& \max_{t\in[0,1]}|x(t)-u(t)| \\
\leq & \frac{\varepsilon_1}{\Gamma(\alpha_n+1)}+\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\delta}}{\Gamma(\alpha_n)}s^{\delta}\left|f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)-f\left(s,u(s),v(s),D^{\beta_1}v(s),\dots,D^{\beta_{n-1}}v(s)\right)\right|ds \\
& +\frac{\Gamma(n-\eta)t^{n-1}}{(n-1)!\,|\Gamma(n-\eta)-1|\,\Gamma(\alpha_n-\eta)} \\
& \times\int_0^1(1-s)^{\alpha_n-\eta-1}s^{-\delta}s^{\delta}\left|f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)ds-f\left(s,u(s),v(s),D^{\beta_1}v(s),\dots,D^{\beta_{n-1}}v(s)\right)\right|ds,
\end{aligned}
\tag{3.6}
$$

which implies that,

$$
\|x-u\|_\infty\leq\frac{\varepsilon_1}{\Gamma(\alpha_n+1)}+\sum_{j=1}^{n+1}\omega_j^1\Upsilon_0\|(x-u,y-v)\|_B. \tag{3.7}
$$

Similarly, we get

$$
\|y-v\|_\infty\leq\frac{\varepsilon_2}{\Gamma(\beta_n+1)}+\sum_{j=1}^{n+1}\omega_j^2\Upsilon_0^*\|(x-u,y-v)\|_B. \tag{3.8}
$$

By differentiating inequality (3.2), we get

$$
\begin{aligned}
& \left| \begin{array}{l}
D^{\alpha_k}x_k(t)-\int_0^t\frac{(t-s)^{\alpha_n-\alpha_k-1}}{\Gamma(\alpha_n-\alpha_k)}f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)ds-\sum_{j=k}^{n-2}\frac{a_j}{\Gamma(j+1-\alpha_k)}t^{j-\alpha_k}-\frac{\Gamma(n-\eta)t^{n-1-\alpha_k}}{\Gamma(n-\alpha_k)(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)} \\
\times\int_0^1(1-s)^{\alpha_n-\eta-1}f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)ds
\end{array}\right| \\
\leq & J^{\alpha_n-\alpha_k}\varepsilon_1 \\
\leq & \frac{t^{\alpha_n-\alpha_k}}{\Gamma(\alpha_n-\alpha_k+1)}\varepsilon_1,\quad k=1,2,\dots,n-2
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
& \left| \begin{array}{l}
D^{\alpha_{n-1}}x_k(t)-\int_0^t\frac{(t-s)^{\alpha_n-\alpha_{n-1}-1}}{\Gamma(\alpha_n-\alpha_{n-1})}f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)ds-\frac{\Gamma(n-\eta)t^{n-1-\alpha_{n-1}}}{\Gamma(n-\alpha_{n-1})(\Gamma(n-\eta)-1)\Gamma(\alpha_n-\eta)} \\
\times\int_0^1(1-s)^{\alpha_n-\eta-1}f\left(s,x(s),y(s),D^{\beta_1}y(s),\dots,D^{\beta_{n-1}}y(s)\right)ds
\end{array}\right| \\
\leq & J^{\alpha_n-\alpha_{n-1}}\varepsilon_1 \\
\leq & \frac{t^{\alpha_n-\alpha_{n-1}}}{\Gamma(\alpha_n-\alpha_{n-1}+1)}\varepsilon_1,
\end{aligned}
\tag{3.10}
$$

Also, by differentiating inequality (3.3), we have

$$
\begin{aligned}
& \left| \begin{array}{l}
D^{\beta_k}y_k(t)-\int_0^t\frac{(t-s)^{\beta_n-\beta_k-1}}{\Gamma(\beta_n-\beta_k)}g(s,x(s),y(s),D^{\alpha_1}x(s),\dots,D^{\alpha_{n-1}}x(s))ds-\sum_{j=k}^{n-2}\frac{b_j}{\Gamma(j+1-\beta_k)}t^{j-\beta_k}-\frac{\Gamma(n-\kappa)t^{n-1-\beta_k}}{\Gamma(n-\beta_k)(\Gamma(n-\kappa)-1)\Gamma(\beta_n-\kappa)} \\
\times\int_0^1(1-s)^{\beta_n-\kappa-1}g(s,x(s),y(s),D^{\alpha_1}x(s),\dots,D^{\alpha_{n-1}}x(s))ds
\end{array}\right| \\
\leq & J^{\beta_n-\beta_k}\varepsilon_2 \\
\leq & \frac{t^{\beta_n-\beta_k}}{\Gamma(\beta_n-\beta_k+1)}\varepsilon_2,\quad k=1,2,\dots,n-2,
\end{aligned}
\tag{3.11}
$$

and

$$\begin{vmatrix} D^{\beta_{n-1}}y_k(t)-\int_0^t \frac{(t-s)^{\beta_n-\beta_{n-1}-1}}{\Gamma(\beta_n-\beta_{n-1})}g(s,x(s),y(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s))\,ds-\frac{\Gamma(n-\kappa)t^{n-1-\beta_{n-1}}}{\Gamma(n-\beta_{n-1})(\Gamma(n-\kappa)-1)\Gamma(\beta_n-\kappa)} \\ \times\int_0^1(1-s)^{\beta_n-\kappa-1}g(s,x(s),y(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s))\,ds \end{vmatrix}$$

$$\begin{aligned} &\leq J^{\beta_n-\beta_{n-1}}\varepsilon_2 \\ &\leq \frac{t^{\beta_n-\beta_{n-1}}}{\Gamma(\beta_n-\beta_{n-1}+1)}\varepsilon_2. \end{aligned} \tag{3.12}$$

Similarly as before, we can show that

$$\|D^{\alpha_k}(x-u)\|_\infty \leq \frac{\varepsilon_1}{\Gamma(\alpha_n-\alpha_k+1)}+\sum_{j=1}^{n+1}\omega_j^1\Upsilon_k\|(x-u,y-v)\|_B, \tag{3.13}$$

$$\left\|D^{\beta_k}(y-v)\right\|_\infty \leq \frac{\varepsilon_2}{\Gamma(\beta_n-\beta_k+1)}+\sum_{j=1}^{n+1}\omega_j^2\Upsilon_k^*\|(x-u,y-v)\|_B. \tag{3.14}$$

Using inequalities (3.7), (3.8), (3.13) and (3.14), we get

$$\begin{aligned} \|(x-u,y-v)\|_B &\leq \max_{1\leq k\leq n}\left(\frac{\varepsilon_1}{\Gamma(\alpha_n+1)},\frac{\varepsilon_1}{\Gamma(\alpha_n-\alpha_k+1)},\frac{\varepsilon_2}{\Gamma(\beta_n+1)},\frac{\varepsilon_2}{\Gamma(\beta_n-\beta_k+1)}\right)+\Theta\|(x-u,y-v)\|_B \\ &\leq \varepsilon\psi+\Theta\|(x-u,y-v)\|_B, \end{aligned} \tag{3.15}$$

where $\varepsilon=\max_{1\leq k\leq 2}\varepsilon_k$, $\quad\psi=\max_{1\leq k\leq n}\left(\frac{1}{\Gamma(\alpha_n+1)},\frac{1}{\Gamma(\alpha_n-\alpha_k+1)},\frac{1}{\Gamma(\beta_n+1)},\frac{1}{\Gamma(\beta_n-\beta_k+1)}\right)$.

Hence,

$$\|(x-u,y-v)\|_B \leq \frac{\varepsilon\psi}{(1-\Theta)} := \lambda_{f,g}\varepsilon, \quad \lambda_{f,g}=\frac{\psi}{(1-\Theta)}.$$

Thanks to (H_3), we get $\lambda_{f,g}>0$. That is system (1.1) is Ulam-Hyers stable. Taking $\phi_{f,g}(\varepsilon)=\lambda_{f,g}\varepsilon$, we receive the generalized Ulam-Hyers stability for system (1.1). □

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