

APPLICATION OF MML METHODOLOGY TO AN α -SERIES PROCESS WITH WEIBULL DISTRIBUTION

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Abstract

In an α -series process, explicit estimators of the parameters α , μ and σ^2 are obtained by using the methodology of modified maximum likelihood (MML) when the distribution of the first occurrence time of an event is assumed to be Weibull. Monte Carlo simulations are performed to compare the efficiencies of the MML estimators with the corresponding nonparametric (NP) estimators. We also apply the MML methodology to two real life data sets to show the performance of the MML estimators compared to the NP estimators.

Keywords: α -series process, Modified likelihood, Nonparametric, Efficiency, Monte Carlo simulation.

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1. Introduction

In the statistical literature, a counting process (CP) $\{N(t), t \geq 0\}$ is a common method of modeling the total number of events that have occurred in the interval $(0, t]$. If the data consist of independent and identically distributed (iid) successive interarrival times (i.e., there is no trend), a renewal process (RP) can be used. However, this is not always the case because it is more reasonable to assume that the successive operating times will follow a monotone trend due to the ageing effect and accumulated wear, see Chan *et al.* [8]. Using a nonhomogeneous Poisson process with a monotone intensity function is one approach to modeling these trends, see Cox and Lewis [9] and Ascher and Feingold [2]. A more direct approach is to apply a monotone counting process model, see, for example, Lam [13, 14], Lam and Chan [15], and Chan *et al.* [8]. Braun *et al.* [6] defined such a monotone process as below.

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1.1. Definition. Let X_k be the time between the $(k-1)^{\text{th}}$ and k^{th} event of a counting process $\{N(t), t \geq 0\}$ for $k = 1, 2, \dots$. The counting process $\{N(t), t \geq 0\}$ is said to be an α -series process with parameter α if there exists a real number α such that $k^\alpha X_k, k = 1, 2, \dots$ are iid random variables with a distribution function F .

The α -series process was first introduced by Braun *et al.* [6] as a possible alternative to the geometric process (GP) in situations where the GP is inappropriate. They applied it to some reliability and scheduling problems. Some theoretical properties of the α -series process are given in Braun *et al.* [6, 7]. Clearly, the X_k 's form a stochastically increasing sequence when $\alpha < 0$; they are stochastically decreasing when $\alpha > 0$. When $\alpha = 0$, all of the X_k are identically distributed and an α -series process reduces to a RP.

If F is an exponential distribution function and $\alpha = 1$, then the α -series process $\{N(t), t \geq 0\}$ is a linear birth process [7].

Assume that the distribution function F for an α -series process has positive mean $\mu(F(0) < 1)$, and finite variance σ^2 . Then

$$(1.1) \quad E(X_k) = \frac{\mu}{k^\alpha} \text{ and } \text{Var}(X_k) = \frac{\sigma^2}{k^{2\alpha}}, k = 1, 2, \dots$$

Thus, α, μ and σ^2 are the most important parameters for an α -series process because these parameters completely determine the mean and variance of X_k .

In this paper, we study the statistical inference problem for α -series process with Weibull distribution and obtain the explicit estimators of the parameters α, μ and σ^2 by adopting the method of modified likelihood. We recall that the mean and variance of the Weibull distribution are given by

$$(1.2) \quad \mu = \Gamma\left(1 + \frac{1}{a}\right) b \text{ and } \sigma^2 = \left[\Gamma\left(1 + \frac{2}{a}\right) - \Gamma^2\left(1 + \frac{1}{a}\right)\right] b^2,$$

respectively. Here, a and b are the shape and scale parameters of the Weibull distribution, respectively.

Since obtaining explicit estimators of the parameters is similar to that given in [25], we just reproduce the estimators and omit the details, see also [10], [11] and [23]. The motivation for this paper comes from the fact that the Weibull distribution is not easy to incorporate into the α -series process model because of difficulties encountered in solving the likelihood equations numerically. To the best of our knowledge, this is the first study applying the MML methodology to an α -series process when the distribution of the first occurrence time is Weibull.

The remainder of this paper is organized as follows: Likelihood equations for estimating the unknown parameters and the corresponding MML estimators are given in Section 2. Section 3 presents the elements of the inverse of the Fisher information matrix. Section 4 presents the simulation results for comparing the efficiencies of the MML estimators with the NP estimators. Two real life examples are given in Section 5. Concluding remarks are given in Section 6.

2. Likelihood equations

As mentioned in Section 1, $Y_k = k^\alpha X_k, k = 1, 2, \dots, n$ are iid Weibull random variables. The likelihood equations, $\frac{\partial \ln L}{\partial \alpha} = 0, \frac{\partial \ln L}{\partial a} = 0$ and $\frac{\partial \ln L}{\partial b} = 0$ are obtained by taking the first derivatives of the log-likelihood function with respect to the parameters α, a and b . For example, we see that the likelihood equation for the parameter α is

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{k=1}^n \ln k - \sum_{k=1}^n \left(\frac{k^\alpha X_k}{b}\right)^\alpha \ln k = 0.$$

The solutions of the likelihood equations give us the ML estimators of the parameters α , a and b . However, the ML estimators cannot be obtained explicitly or numerically because the first derivatives of the likelihood function involve power functions of the parameter α , as well as the shape parameter a .

To overcome these difficulties, we take the logarithm of the Y_k 's as shown below:

$$(2.1) \quad \ln Y_k = \alpha \ln k + \ln X_k, \quad k = 1, 2, \dots, n.$$

It is known that the $\ln Y_k$'s are iid extreme value (EV) random variables with the probability density function given by

$$(2.2) \quad f(w) = \frac{1}{\eta} \exp\left(\frac{w - \delta}{\eta}\right) \exp\left(-\exp\left(\frac{w - \delta}{\eta}\right)\right), \quad w \in \mathbb{R}; \quad \eta > 0, \quad \delta \in \mathbb{R},$$

where $\delta = \ln b$ is the location parameter and $\eta = 1/a$ is the scale parameter.

We first obtain the estimators of the unknown parameters in the extreme value distribution, and then obtain the estimators of the Weibull parameters by using the following inverse transformations:

$$(2.3) \quad a = 1/\eta \text{ and } b = \exp(\delta).$$

The likelihood function for $\ln X_k$, $k = 1, 2, \dots, n$ is found by using (2.1) and (2.2):

$$(2.4) \quad L(\alpha, \delta, \eta) = \frac{1}{\eta^n} \exp\left(\sum_{k=1}^n \frac{\ln X_k + \alpha \ln k - \delta}{\eta} - \exp\left(\frac{\ln X_k + \alpha \ln k - \delta}{\eta}\right)\right).$$

For the sake of simplicity, we use the notations c_k instead of $\ln k$ in (2.4).

Then, it is obvious that maximizing $L(\alpha, \delta, \eta)$ with respect to the unknown parameters α , δ and η is equivalent to estimating the parameters in (2.5),

$$(2.5) \quad \ln x_k = \delta - \alpha c_k + \varepsilon_k \quad (1 \leq k \leq n),$$

when $\varepsilon_k \sim \text{EV}(0, \eta)$.

Then the likelihood equations are given by

$$(2.6) \quad \begin{aligned} \frac{\partial \ln L}{\partial \alpha} &= \frac{1}{\eta} \sum_{k=1}^n c_k - \frac{1}{\eta} \sum_{k=1}^n g(z_k) c_k = 0 \\ \frac{\partial \ln L}{\partial \delta} &= \frac{1}{\eta} \sum_{k=1}^n g(z_k) - \frac{n}{\eta} = 0 \end{aligned}$$

and

$$\frac{\partial \ln L}{\partial \eta} = \frac{1}{\eta} \sum_{k=1}^n g(z_k) z_k - \frac{1}{\eta} \sum_{k=1}^n z_k - \frac{n}{\eta} = 0,$$

where

$$z_k = \frac{\varepsilon_k}{\eta}, \quad (k = 1, 2, \dots, n) \text{ and } g(z) = \exp(z).$$

The ML estimators of the extreme value distribution parameters are the solutions of the equations in (2.6). Because of the awkward function $g(z)$, it is not easy to solve these equations. The ML estimates are, therefore, elusive. Since it is impossible to obtain estimators of the unknown parameters in a closed form, we resort to iterative methods. However that can be problematic for reasons of

- i) multiple roots,
- ii) non-convergence of iterations, or
- iii) convergence to wrong values;

see, Barnett [4] and Vaughan [24]. If the data contain outliers, the iterations for the likelihood equations might never converge, see Puthenpura and Sinha [17].

To overcome these difficulties, we use the method of modified maximum likelihood introduced by Tiku [19, 20], see also Tiku *et al.* [22]. The method linearizes the intractable terms in (2.6) and calls the resulting equations the “modified” likelihood equations. The solutions of these equations are the following closed form MML estimators (see [10, 11, 23] and [25]):

$$(2.7) \quad \hat{\alpha} = \hat{\eta}D - E, \quad \hat{\delta} = \ln \bar{x}_{[.]} + \hat{\alpha}\bar{c}_{[.]} + \frac{\Delta}{m}\hat{\eta} \quad \text{and} \quad \hat{\eta} = \frac{B + \sqrt{B^2 + 4nC}}{2\sqrt{n(n-2)}} \quad (\text{bias corrected})$$

where

$$D = \frac{\sum_{k=1}^n (1 - a_k)(c_{[k]} - \bar{c}_{[.]})}{\sum_{k=1}^n b_k (c_{[k]} - \bar{c}_{[.]})^2}, \quad E = \frac{\sum_{k=1}^n b_k (\ln x_{[k]} - \ln \bar{x}_{[.]})(c_{[k]} - \bar{c}_{[.]})}{\sum_{k=1}^n b_k (c_{[k]} - \bar{c}_{[.]})^2},$$

$$\ln \bar{x}_{[.]} = \frac{\sum_{k=1}^n b_k \ln x_{[k]}}{m}, \quad \bar{c}_{[.]} = \frac{\sum_{k=1}^n b_k c_{[k]}}{m}, \quad \Delta = \sum_{k=1}^n (a_k - 1), \quad m = \sum_{k=1}^n b_k,$$

$$B = \sum_{k=1}^n (a_k - 1) [(\ln x_{[k]} - \ln \bar{x}_{[.]}) - E(c_{[k]} - \bar{c}_{[.]})] \quad \text{and}$$

$$C = \sum_{k=1}^n b_k [(\ln x_{[k]} - \ln \bar{x}_{[.]}) - E(c_{[k]} - \bar{c}_{[.]})]^2.$$

Then, by (1.2) and (2.3), the MML estimators of μ and σ^2 are obtained as

$$(2.8) \quad \hat{\mu} = \Gamma(1 + \hat{\eta}) \exp(\hat{\delta}) \quad \text{and} \quad \hat{\sigma}^2 = [\Gamma(1 + 2\hat{\eta}) - \Gamma^2(1 + \hat{\eta})] \exp(2\hat{\delta})$$

respectively.

It should be noted that the divisor n in the denominator of $\hat{\eta}$ was replaced by $\sqrt{n(n-2)}$ as a bias correction. See Vaughan and Tiku [25] and Tiku and Akkaya [23] for the asymptotic and small samples properties of the MML estimators.

3. The Fisher information matrix

The elements of the inverse of the Fisher information matrix (I^{-1}) are given by

$$V_{11} = \eta^2 \left/ \left(\sum_{k=1}^n c_k^2 - \left(\sum_{k=1}^n c_k \right)^2 / n \right) \right/ n,$$

$$V_{12} = \eta^2 \sum_{k=1}^n c_k \left/ \left(n \sum_{k=1}^n c_k^2 - \left(\sum_{k=1}^n c_k \right)^2 \right) \right/ n, \quad V_{13} = 0,$$

$$V_{22} = \eta^2 \sum_{k=1}^n c_k^2 \left/ \left(n \sum_{k=1}^n c_k^2 - \left(\sum_{k=1}^n c_k \right)^2 \right) \right/ n + (1 - \gamma)^2 / n\pi^2,$$

$$V_{23} = -6\eta^2(1 - \gamma) / n\pi^2 \quad \text{and} \quad V_{33} = 6\eta^2 / n\pi^2,$$

where γ is the Euler constant.

It should be noted that the Fisher information matrix I is symmetric, so $V_{21} = V_{12}, V_{31} = V_{13}$ and $V_{32} = V_{23}$. The diagonal elements in I^{-1} provide the asymptotic variances of the ML estimators, i.e., $V(\hat{\alpha}), V(\hat{\delta})$ and $V(\hat{\eta})$. These variances are also known as the minimum variance bounds (MVBs) for estimating α, δ and η . The MML estimators

are asymptotically equivalent to the ML estimators when the regularity conditions hold, see Bhattacharyya [5] and Vaughan and Tiku [25]. Therefore, we can conclude that the MML estimators are fully efficient, i.e., they are asymptotically unbiased and their covariance matrix is asymptotically the same as the inverse of I . See Table 1 for the simulated variances of the MML estimators in (2.7) and the corresponding MVB values.

Table 1. The simulated variances of the MML estimators and the corresponding MVB values

n	Simulated variances			MVB		
	$\hat{\alpha}$	$\hat{\delta}$	$\hat{\eta}$	$\hat{\alpha}$	$\hat{\delta}$	$\hat{\eta}$
30	0.0132	0.0908	0.0054	0.0119	0.0828	0.0051
50	0.0069	0.0650	0.0031	0.0065	0.0623	0.0030
100	0.0030	0.0424	0.0015	0.0029	0.0415	0.0015

It is clear from Table 1 that the simulated variances of the MML estimators and the corresponding MVB values become close as n increases. Therefore, the MML estimators are highly efficient estimators.

4. Simulation results

An extensive Monte Carlo simulation study was carried out in order to compare the efficiencies of the MML estimators and the NP estimators given below; see Aydogdu and Kara [3].

$$(4.1) \quad \tilde{\alpha} = \frac{\sum_{k=1}^n \ln k \sum_{k=1}^n \ln X_k - n \sum_{k=1}^n \ln X_k \ln k}{n \sum_{k=1}^n (\ln k)^2 - \left(\sum_{k=1}^n \ln k\right)^2},$$

$$\tilde{\sigma}^2 = \exp(2\tilde{\beta})\tilde{\sigma}_e^2$$

and

$$\tilde{\mu} = \begin{cases} \text{root of } 2\tilde{\mu}^2 \ln \tilde{\mu} - 2\tilde{\beta}\tilde{\mu}^2 - \tilde{\sigma}^2 = 0, & \tilde{\alpha} < 0 \\ \sum_{k=1}^n k^{-\tilde{\alpha}} X_k / \sum_{k=1}^n k^{-2\tilde{\alpha}}, & 0 < \tilde{\alpha} \leq 0.7 \\ \sum_{k=1}^n X_k / \sum_{k=1}^n k^{-\tilde{\alpha}}, & \tilde{\alpha} > 0.7 \end{cases}$$

where

$$\tilde{\beta} = \frac{\sum_{k=1}^n \ln k \sum_{k=1}^n \ln X_k \ln k - \sum_{k=1}^n (\ln k)^2 \sum_{k=1}^n \ln X_k}{\left(\sum_{k=1}^n \ln k\right)^2 - n \sum_{k=1}^n (\ln k)^2}$$

and

$$\tilde{\sigma}_e^2 = \frac{\left(\sum_{k=1}^n (\ln X_k)^2 - \frac{1}{n} \left(\sum_{k=1}^n \ln X_k\right)^2\right) - \tilde{\alpha}^2 \left(\sum_{k=1}^n (\ln k)^2 - \frac{1}{n} \left(\sum_{k=1}^n \ln k\right)^2\right)}{n - 2}.$$

All the computations were conducted in MATLAB 7. The means and the MSEs were calculated for different sample sizes, α parameters and shape parameters, based on $[100,000/n]$ Monte Carlo simulations. Here, $[\cdot]$ represents the greatest integer value. The scale parameter b was taken to be 1 throughout the study. We consider the sample

sizes $n = 30, 50, 100$, the α parameter values $\alpha = -0.5, 0.5, 0.9$ and the shape parameters $a = 1.5, 2$ and 3 . The simulated means, the MSEs and the relative efficiencies (REs)

$$RE(\hat{\mu}/\tilde{\mu}) = \frac{MSE(\hat{\mu})}{MSE(\tilde{\mu})} \times 100, \quad RE(\hat{\sigma}^2/\tilde{\sigma}^2) = \frac{MSE(\hat{\sigma}^2)}{MSE(\tilde{\sigma}^2)} \times 100,$$

$$RE(\hat{\alpha}/\tilde{\alpha}) = \frac{MSE(\hat{\alpha})}{MSE(\tilde{\alpha})} \times 100$$

of the NP and MML estimators are given in Table 2.

Table 2. The simulated means, nxMSEs and REs for the MML and NP estimators of the parameters μ, σ^2 and α

		$\hat{\mu}$			$\hat{\sigma}^2$			$\hat{\alpha}$		
$a = 1.5; \alpha = -0.5; \mu = 0.9027; \sigma^2 = 0.3757$										
n	Estimators	Mean	nxMSE	RE	Mean	nxMSE	RE	Mean	nxMSE	RE
30	MML	0.8965	3.43	55	0.4316	3.49	30	-0.5117	0.71	65
	NP	0.9468	6.22		0.5160	11.73		-0.4931	1.10	
50	MML	0.8993	4.18	56	0.4005	4.94	44	-0.5087	0.62	66
	NP	0.9179	7.43		0.4483	11.33		-0.5020	0.94	
100	MML	0.9040	5.66	58	0.3808	5.38	43	-0.5048	0.49	61
	NP	0.8945	9.72		0.3974	12.51		-0.4964	0.80	
$a = 2; \alpha = -0.5; \mu = 0.8862; \sigma^2 = 0.2146$										
30	MML	0.8737	1.85	55	0.2188	0.60	28	-0.5127	0.38	62
	NP	0.9217	3.34		0.2745	2.16		-0.4943	0.61	
50	MML	0.8797	2.74	57	0.2234	0.88	38	-0.5091	0.39	66
	NP	0.9105	4.77		0.2612	2.34		-0.4984	0.58	
100	MML	0.8813	3.40	60	0.2186	0.99	41	-0.5057	0.33	66
	NP	0.8957	5.68		0.2471	2.39		-0.5018	0.50	
$a = 3; \alpha = -0.5; \mu = 0.8930; \sigma^2 = 0.1053$										
30	MML	0.8759	0.91	66	0.1037	0.06	25	-0.5077	0.18	72
	NP	0.9154	1.37		0.1366	0.25		-0.4991	0.26	
50	MML	0.8864	1.17	65	0.1031	0.09	26	-0.5048	0.17	69
	NP	0.9084	1.78		0.1352	0.33		-0.5004	0.25	
100	MML	0.8879	1.42	62	0.1040	0.09	26	-0.5036	0.13	65
	NP	0.9074	2.29		0.1211	0.34		-0.4986	0.20	
$a = 1.5; \alpha = 0.5; \mu = 0.9027; \sigma^2 = 0.3757$										
30	MML	0.8901	3.43	76	0.4201	4.18	31	0.4878	0.74	70
	NP	0.9318	4.50		0.5266	13.32		0.5041	1.06	
50	MML	0.8971	4.66	72	0.4048	4.06	47	0.4897	0.61	67
	NP	0.9216	6.48		0.4492	8.56		0.5031	0.91	
100	MML	0.8993	5.81	64	0.3861	4.76	52	0.4934	0.52	65
	NP	0.9111	9.06		0.4082	9.11		0.5016	0.80	
$a = 2; \alpha = 0.5; \mu = 0.8862; \sigma^2 = 0.2146$										
30	MML	0.8671	1.93	76	0.2204	0.62	32	0.4838	0.40	70
	NP	0.9156	2.54		0.2757	1.93		0.5059	0.57	
50	MML	0.8801	2.44	72	0.2166	0.72	38	0.4916	0.36	68
	NP	0.8968	3.39		0.2415	1.90		0.4976	0.53	
100	MML	0.8826	3.45	67	0.2162	0.84	37	0.4949	0.32	67
	NP	0.8941	5.13		0.2363	2.25		0.4986	0.48	

Table 2 (continued)

		$\hat{\mu}$			$\hat{\sigma}^2$			$\hat{\alpha}$		
$a = 3; \alpha = 0.5; \mu = 0.8930; \sigma^2 = 0.1053$										
n	Estimators	Mean	nxMSE	RE	Mean	nxMSE	RE	Mean	nxMSE	RE
30	MML	0.8754	0.92	77	0.1012	0.07	28	0.4901	0.17	68
	NP	0.9058	1.20		0.1368	0.25		0.5016	0.25	
50	MML	0.8829	1.08	71	0.1040	0.08	31	0.4948	0.15	65
	NP	0.8957	1.52		0.1236	0.26		0.5026	0.23	
100	MML	0.8878	1.50	68	0.1049	0.10	31	0.4967	0.14	70
	NP	0.8890	2.22		0.1187	0.32		0.5018	0.20	
$a = 1.5; \alpha = 0.9; \mu = 0.9027; \sigma^2 = 0.3757$										
30	MML	0.8908	3.47	73	0.4103	3.67	43	0.8807	0.75	69
	NP	0.9497	4.77		0.4890	8.73		0.9020	1.08	
50	MML	0.8930	4.40	72	0.4031	4.34	49	0.8896	0.61	64
	NP	0.9313	6.10		0.4443	8.92		0.8981	0.95	
100	MML	0.8943	5.87	70	0.3924	4.89	49	0.8939	0.57	62
	NP	0.9221	8.40		0.4088	9.95		0.9018	0.92	
$a = 2; \alpha = 0.9; \mu = 0.8862; \sigma^2 = 0.2146$										
30	MML	0.8686	2.01	76	0.2239	0.67	36	0.8849	0.41	71
	NP	0.9258	2.66		0.2809	1.88		0.9064	0.58	
50	MML	0.8764	2.47	70	0.2211	0.73	38	0.8901	0.36	67
	NP	0.8987	3.52		0.2658	1.94		0.9017	0.54	
100	MML	0.8825	3.39	71	0.2155	0.87	40	0.8929	0.31	66
	NP	0.8943	4.80		0.2513	2.19		0.8976	0.47	
$a = 3; \alpha = 0.9; \mu = 0.8930; \sigma^2 = 0.1053$										
30	MML	0.8742	0.93	72	0.1028	0.07	26	0.8912	0.18	67
	NP	0.9166	1.30		0.1354	0.27		0.9020	0.27	
50	MML	0.8865	1.17	71	0.1027	0.08	24	0.8941	0.16	67
	NP	0.9030	1.64		0.1274	0.33		0.8990	0.24	
100	MML	0.8878	1.60	68	0.1030	0.10	32	0.8949	0.15	68
	NP	0.8986	2.36		0.1176	0.31		0.8989	0.22	

The results given in Table 2 show that the MML estimators are more efficient than the NP estimators in every case.

5. Illustrative examples

In this section, the parameters α, μ and σ^2 in an α -series process are estimated on two different real data sets by using the MML estimators and the NP estimators. The first example uses data on coal-mining disasters and the second one is about the aircraft 6 data.

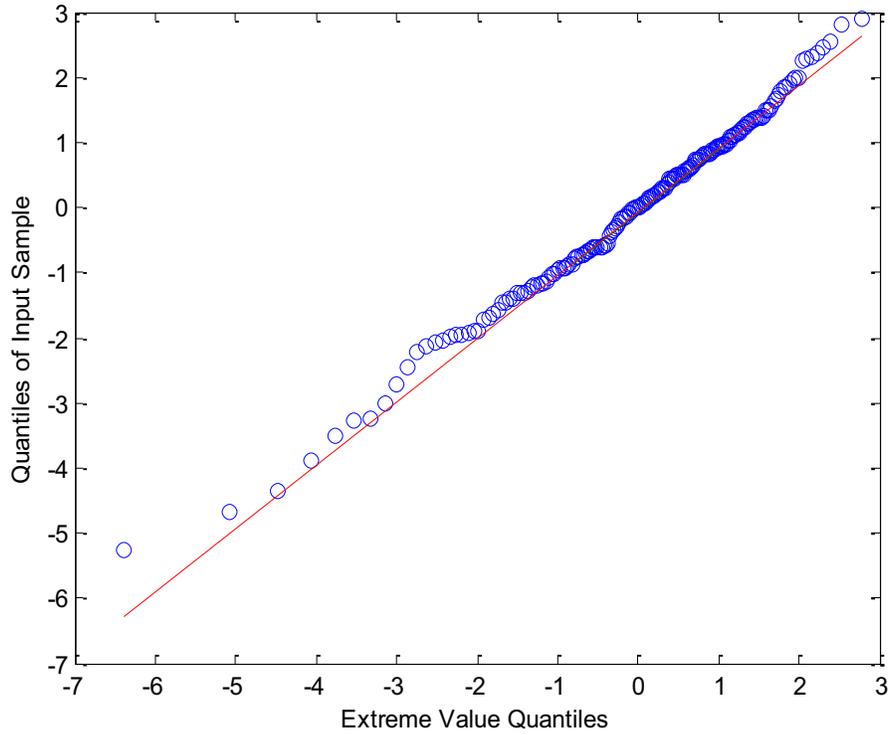
5.1. Coal-mining disaster data. This data set has 190 observations showing the interval in days between successive disasters in Great Britain, see Andrews and Herzberg [1]. The data contain one “zero” because there were two accidents on the same day. The zero is replaced by 0.5 since two accidents on the same day usually are not at the same time of the day, an approximate time interval between them should be 0.5 days, see Jarrett [12].

To obtain an idea about the underlying distribution of ε_k in the following model

$$\ln x_k = \delta - \alpha c_k + \varepsilon_k \quad (1 \leq k \leq 190),$$

we first constructed a Q-Q plot of the 190 residuals, see Figure 1. It should be noted that the EV Q-Q plot of the residuals are constructed by plotting the quantiles of the EV distribution against the ordered residuals $\tilde{\varepsilon}_k = \ln x_k - \delta + \tilde{\alpha}c_k$.

Figure 1. EV Q-Q plot of the coal-mining disaster data



It is clear from Figure 1 that the data points do not deviate much from a straight line, therefore we conclude that EV is the most appropriate distribution for ε_k . This is also supported by the well-known Z^* test statistic proposed by Tiku [21] (i.e., $Z^*_{\text{calculated}} = 0.9499$ and $p - \text{value} = 0.1916$), see also Surucu [18].

The estimates of the parameters α, μ and σ^2 obtained by using the MML estimators and the NP estimators are given in Table 3.

Table 3. Parameter estimations for the coal-mining disaster data

Estimator	$\hat{\alpha}$	$\hat{\mu}$	$\hat{\sigma}^2$
MML	-0.393 (± 0.091)	36.311 (± 10.344)	1873.900
NP	-0.433 (± 0.108)	22.440 (± 12.210)	433.250

*Values given in parentheses are the standard errors (SE) of the estimators

Based on the simulation results given in Table 2, we conclude that the MML estimates of the parameters are preferable to the NP estimates according to the MSE criterion, since the MSE values of the MML estimators are smaller than the MSE values of the NP estimators when $\alpha < 0$.

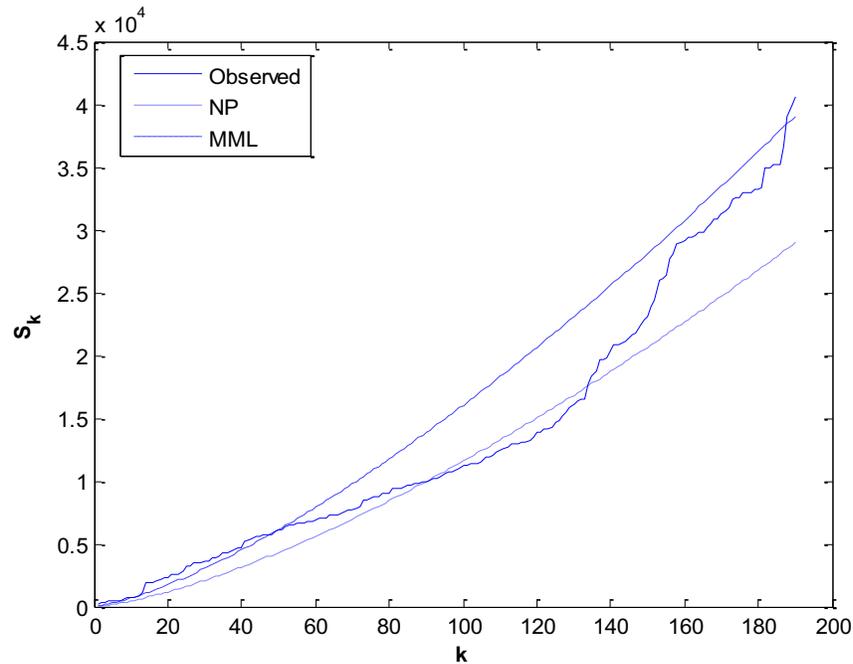
It should be noted that the variance of the Weibull distribution is very sensitive to changes in the value of a , especially for $a < 1$. Since the value of a is less than 1 for the coal-mining disaster data, the MML and NP estimators of the variance of Weibull distribution are quite different from each other. This happens frequently in practice. Similar statements can also be made for the Aircraft 6 data.

Let $S_k = X_1 + X_2 + \dots + X_k$, $k = 1, 2, \dots, n$. Then a fitted value of S_k may be defined by

$$\hat{S}_k = \hat{\mu} \sum_{j=1}^k 1/j^{\hat{\alpha}}.$$

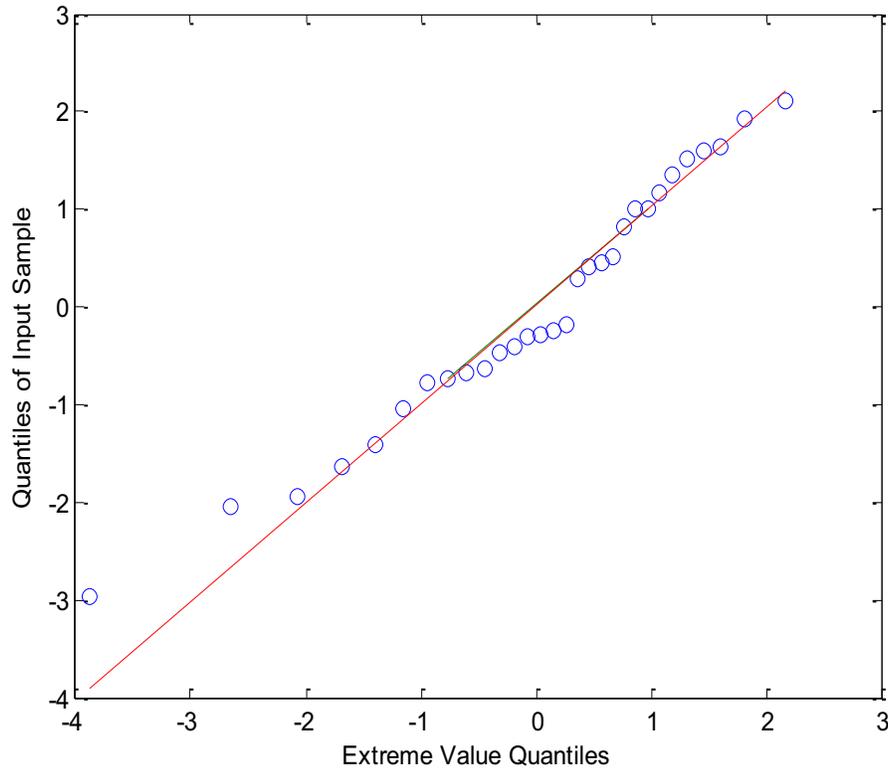
In Figure 2, we plot the coal-mining disaster times and their fitted times against the number of disasters by using NP and MML estimators, respectively. It can be seen that the MML estimators provide a better fit than the NP estimators.

Figure 2. Plot of S_k against their fitted values using the NP and MML estimators for the coal-mining disaster data



5.2. The aircraft 6 data. This data set has 30 observations showing the intervals between successive failures of the air-conditioning equipment in Boeing 720 aircraft, see Cox and Lewis [9]. The data is also studied by Proschan [16] and Cox and Lewis [9].

Following similar steps to those for the coal-mining disaster data in Subsection 5.1, EV is again found to be the most appropriate distribution for the residuals, see Figure 3.

Figure 3. EV Q-Q plot of the aircraft 6 data

This is confirmed by the value of the Z^* test statistic (i.e., $Z^*_{\text{calculated}} = 0.9203$ and p -value = 0.4300), see also Tiku [21] and Surucu [18]. The estimates of the unknown parameters are given in Table 4.

Table 4. Parameter estimates for the aircraft 6 data set

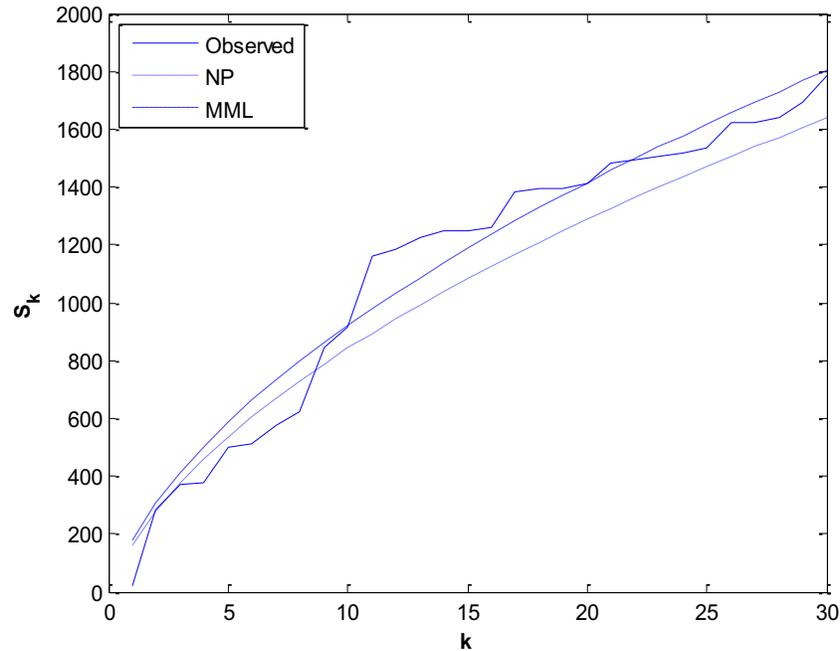
Estimator	$\hat{\alpha}$	$\hat{\mu}$	$\hat{\sigma}^2$
MML	0.4708 (± 0.249)	177.0362 (± 51.173)	40980
NP	0.4775 (± 0.272)	163.1353 (± 56.234)	15061

*Values given in parentheses are the standard errors (SE) of the estimators

Again, based on the simulation results given in Table 2, the MML estimators are chosen for estimating the parameters α , μ and σ^2 because the MML estimators outperform the NP estimators in terms of the MSE criterion as in the coal-mining disasters data.

In Figure 4, we plot the failure times of the air-conditioning equipment and their fitted times against the number of failures by using the NP and MML estimators, respectively. It can be seen that MML estimators provide a better fit than the NP estimators.

Figure 4. Plot of S_k against their fitted values using the NP and MML estimators for the aircraft 6 data



6. Conclusions

We have compared the efficiencies of the MML estimators with the NP estimators via a Monte Carlo simulation study. It is shown that the MML estimators have higher efficiencies than the NP estimators.

We also applied the MML methodology two real life data sets. The parameters α , μ and σ^2 for an α -series process with Weibull distribution have been estimated and compared with the NP estimators. We see that the MML estimators provide a better fit than NP estimators.

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