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Smarandache Curves According to Sabban Frame of the anti-Salkowski Indicatrix Curve

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Abstract

The aim of this paper is to define Smarandache curves according to the Sabban frame belonging to the spherical indicatrix curve of the anti-Salkowski curve. We also illustrate these curves with the Maple program and calculate the geodesic curvatures of these curves.

1. Introduction

Erich Salkowski (1881-1943), a German mathematician. In 1909, he defined curve families with non-constant τ and constant curvature κ [1]. Later J. Monterde constructed a method for closed curves and the properties of anti-Salkowski curve used in [2]. For authors worked on the anti-Salkowski curve also can be seen in [3]-[7]. When the Frenet vectors of any curve are taken as the position vector, then the regular curves generated by these vectors are called Smarandache curves [8]. Smarandache curves in Euclidean 3-space are defined and some features of these curves are given in [9]. For some authors worked on the Smarandache curve also may be seen in [10, 11]. In 1990, the geodesic curve of a spherical curve is calculated by J. Koenderink with the Sabban frame of the spherical indicatrix curves in [12]. Then the Smarandache curves obtained from Sabban frame are defined and geodesic curvatures of these curves are given in [13].
In this study, Smarandache curves are defined according to the Sabban frames belonging to the spherical indicatrix curves of each of the T,N,B Frenet vectors of the anti-Salkowski curve. The geodesic curvatures of these curves are then calculated.

2. Preliminaries

In the Euclidean 3-space E^3, the Frenet frame of any curve α is given by $\{T,N,B\}$. For an arbitrary curve $\alpha \in E^3$, with the first and second curvatures, κ and τ respectively, the Frenet apparatus are given by [14]

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.$$

Accordingly, the spherical indicatrix curves of Frenet vectors are (T), (N) and (B) respectively.These equations of curves are given by [14]

$$\alpha_T(s) = T(s), \quad \alpha_N(s) = N(s), \quad \alpha_B(s) = B(s).$$

Let $\gamma : I \to S^2$ be a unit speed spherical curve. We denote s as the arc-length parameter of γ. Let us denote by [14]

$$\gamma(s) = \gamma(s), \quad t(s) = \gamma'(s), \quad d(s) = \gamma(s) \wedge t(s).$$

We call $t(s)$ a unit tangent vector of γ. $\{\gamma, t, d\}$ frame is called the Sabban frame of γ on S^2. Then we have the following spherical Frenet formulae of γ :

$$\gamma' = t, \quad t' = -\gamma + \kappa_g d, \quad d' = -\kappa_g t \tag{2.1}$$

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where is called the geodesic curvature of κ_g on S^2 and

$$\kappa_g = \langle t', d \rangle, \tag{2.2}$$

[12, 13].

Definition 2.1. *(anti-Salkowski curve) [2]. For any $m \in \mathbb{R}$ with $m \neq \mp\frac{1}{\sqrt{3}}$, 0, let us define the space curve*

$$\begin{aligned} \beta_m(s) &= \Bigg(\frac{n}{2(4n^2-1)m}\Big(n(1-4n^2+3\cos(2ns))\cos(s)+(2n^2+1)\sin(s)\sin(2ns)\Big), \\ &\quad \frac{n}{2(4n^2-1)m}\Big(n(1-4n^2+3\cos(2ns))\sin(s)-(2n^2+1)\cos(s)\sin(2ns), \frac{n^2-1}{4n}(2ns+\sin(2ns))\Big)\Bigg) \end{aligned}$$

where $n = \frac{m}{\sqrt{1+m^2}}$. *The Frenet apparatus are*

$$\begin{cases} \kappa = \tan(ns), \quad \tau = 1, \quad \|\gamma_m(s)\| = \frac{cos(ns)}{\sqrt{1+m^2}} \\ T(s) = -\Big(\cos(s)\sin(ns)-n\sin(s)\cos(ns), \sin(s)\sin(ns)+n\cos(s)\cos(ns), \frac{n}{m}\cos(ns)\Big), \\ N(s) = n\left(\frac{\sin(s)}{m}, -\frac{\cos(s)}{m}, 1\right), \\ B(s) = \Big(-\cos(s)\cos(ns)-n\sin(s)\sin(ns), -\sin(s)\cos(ns)+n\cos(s)\sin(ns), \frac{n}{m}\sin(ns)\Big). \end{cases}$$

The shape of this curve is given in Figure (2.1)

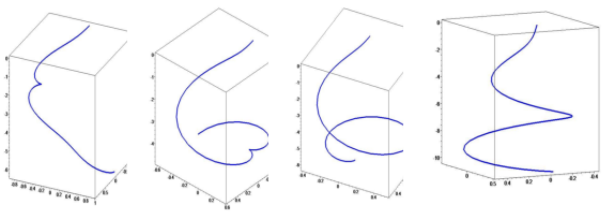

Figure 2.1: anti-Salkowski Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

Let $(\alpha), (\delta)$ and (ζ) be spherical indicatrix curves of tangent, principal normal and binormal vectors belonging to anti-Salkowski curve, respectively. Using the equations (2.1) and (2.2), Sabban apparatus belonging to these curves is given by

$$\begin{aligned} T &= T\,, \quad T_T = N\,, \quad T \wedge T_T = B, \\ T' &= T_T, \quad T_T' = -T + \frac{1}{\tan(ns)}(T \wedge T_T), \quad (T \wedge T_T)' = -\frac{1}{\tan(ns)} T_T, \\ K_g^T &= \frac{1}{\kappa} = \frac{1}{\tan(ns)}. \end{aligned} \tag{2.3}$$

$$T(s) = (\cos(s)\sin(ns) - n\sin(s)\cos(ns), \sin(s)\sin(ns) + n\cos(s)\cos(ns), \frac{n}{m}\cos(ns)), \tag{2.4}$$

$$T_T(s) = n\left(\frac{\sin(s)}{m}, -\frac{\cos(s)}{m}, 1\right),$$

$$(T \wedge T_T)(s) = -(\cos(s)\cos(ns) + n\sin(s)\sin(ns), \sin(s)\cos(ns) - n\cos(s)\sin(ns), \frac{n}{m}\sin(ns)).$$

$$\begin{aligned} N &= N\,, \quad T_N = \frac{-\tan(ns)T + B}{\sqrt{\tan^2(ns)+1}}\,, \quad N \wedge T_N = \frac{T + \tan(ns)B}{\sqrt{\tan^2(ns)+1}}, \\ N' &= T_N\,, \quad T_N' = \frac{\tan(ns)'}{\sqrt{\tan^2(ns)+1}} N + N \wedge T_N\,, \quad (T \wedge T_T)' = \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}} T_N, \\ K_g^N &= \frac{-\kappa'}{\sqrt{\kappa^2+1}} = \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}. \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
N(s) &= \Big(\frac{n\sin(s)}{m}, -\frac{n\cos(s)}{m}, n\Big),\\
T_N(s) &= \frac{1}{\sqrt{\tan^2(ns)+1}}\Big(-\cos(s)\cos(ns)-n\sin(s)\sin(ns)-\tan(ns)(-\cos(s)\sin(ns)+n\sin(s)\cos(ns)),\\
&\quad -\tan(ns)(-\sin(s)\sin(ns)-n\cos(s)\cos(ns))-\sin(s)\cos(ns)+n\cos(s)\sin(ns), \frac{2n}{m}\sin(ns)\Big), \qquad (2.6)\\
(N\wedge T_N)(s) &= \frac{1}{\sqrt{\tan^2(ns)+1}}\Big(\tan(ns)(-\cos(s)\cos(ns)-n\sin(s)\sin(ns))-\cos(s)\sin(ns)+n\sin(s)\cos(ns), -\sin(s)\sin(ns)\\
&\quad +\tan(ns)(-\sin(s)\cos(ns)+n\cos(s)\sin(ns))-n\cos(s)\cos(ns), \frac{n}{m}\tan(ns)\sin(ns)-\frac{n}{m}\cos(ns)\Big).
\end{aligned}
$$

$$
\begin{aligned}
B &= B, \quad T_B=-N, \quad B\wedge T_B = T,\\
B' &= T_B\ , \quad B_T' = -B+\tan(ns)(B\wedge T_B)\ ,\\
(B\wedge T_B)' &= \tan(ns)T_B, \quad K_g^B=\kappa=\tan(ns). \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
B(s) &= -\Big(\cos(s)\cos(ns)+n\sin(s)\sin(ns), \sin(s)\cos(ns)-n\cos(s)\sin(ns), \frac{n}{m}\sin(ns)\Big),\\
T_B(s) &= -\Big(\frac{n\sin(s)}{m}, -\frac{n\cos(s)}{m}, n\Big), \qquad (2.8)\\
(B\wedge T_B)(s) &= -\Big(\cos(s)\sin(ns)-n\sin(s)\cos(ns), \sin(s)\sin(ns)+n\cos(s)\cos(ns), -\frac{n}{m}\cos(ns)\Big).
\end{aligned}
$$

3. Smarandache curves according to the Sabban frame belonging to spherical indicatrix curve of the anti-Salkowski curve

Definition 3.1. *Let $\alpha=\alpha(s)$ be a curve and $\{T, T_T, T\wedge T_T\}$ be Sabban frame of this curve. Then TT_T-Smarandache curve is given by*

$$\alpha_1(s) = \frac{1}{\sqrt{2}}(T+T_T). \qquad (3.1)$$

According to equation (2.4) we can parameterize the $\alpha_1(s)$-Smarandache curve as in the following form

$$\alpha_1(s) = \frac{1}{\sqrt{2}}\Big(-\cos(s)\sin(ns)+n\sin(s)\cos(ns)+\frac{n}{m}\sin(s), -\sin(s)\sin(ns)-n\cos(s)\cos(ns)-\frac{n}{m}\cos(s), -\frac{n}{m}\cos(ns)+n\Big).$$

The shape of this curve is given in Figure (3.1)

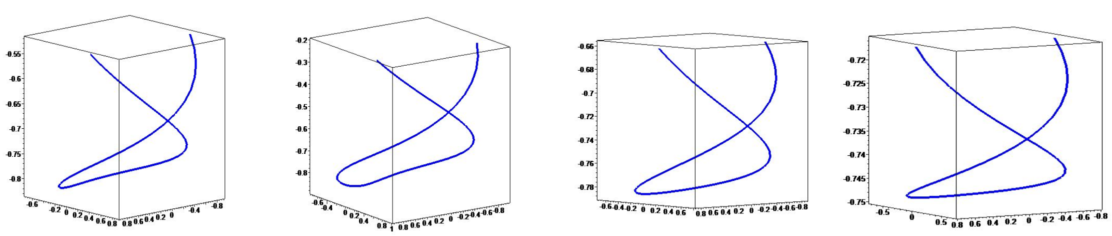

Figure 3.1: TT_T-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

Theorem 3.2. *The geodesic curvature $K_g^{\alpha_1}$ according to $\alpha_1(s)$-Smarandache curve is*

$$K_g^{\alpha_1} = \frac{\tan^4(ns)}{(2\tan(ns)+1)^{\frac{5}{2}}}(\chi_1-\chi_2+2\tan(ns)\chi_3),$$

where the coefficients χ_1, χ_2 and χ_3 are

$$
\begin{aligned}
\chi_1 &= -2-\frac{1}{\tan^2(ns)}+\frac{1}{\tan(ns)}(\frac{1}{\tan(ns)})',\\
\chi_2 &= -2-\frac{1}{\tan(ns)}(\frac{1}{\tan(ns)})'-\frac{3}{\tan^2(ns)}-\frac{1}{\tan^4(ns)},\\
\chi_3 &= \frac{2}{\tan(ns)}+(\frac{2}{\tan(ns)})'+\frac{1}{\tan^3(ns)}.
\end{aligned}
$$

Proof. If we take the derivative of (3.1) and from the equation (2.3) we get

$$(T_T)_{\alpha_1}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-T+T_T+\frac{1}{\tan(ns)}(T\wedge T_T)), \tag{3.2}$$

if we take the norm of (3.2) we have

$$\frac{ds^*}{ds}=\frac{\sqrt{2\tan^2(ns)+1}}{\tan(ns)\sqrt{2}}.$$

We obtain the tangent of $\alpha_1(s)$-Smarandahce curve as in

$$(T_T)_{\alpha_1} = \frac{1}{\sqrt{2\tan^2(ns)+1}}(-\tan(ns)T+\tan(ns)T_T+(T\wedge T_T)). \tag{3.3}$$

The derivative of (3.2) is

$$(T_T)'_{\alpha_1} = \frac{1}{\sqrt{2\tan^2(ns)+1}}(\chi_1 T+\chi_2 T_T+\chi_3(T\wedge T_T)).$$

From equations (3.1) and (3.3) we have

$$(T\wedge T_T)_{\alpha_1} = \frac{1}{\sqrt{2\tan^2(ns)+1}}(T-T_T+2\tan(ns)(T\wedge T_T)).$$

So the geodesic curvature from the equation (2.3) is

$$K_g^{\alpha_1} = \frac{\tan^4(ns)}{(2\tan(ns)+1)^{\frac{5}{2}}}\left(\chi_1-\chi_2+2\tan(ns)\chi_3\right).$$

□

Definition 3.3. *Let $\alpha=\alpha(s)$ be a curve and $\{T,T_T,T\wedge T_T\}$ be Sabban frame of this curve. Then $T(T\wedge T_T)$-Smarandache curve is given by*

$$\alpha_2(s) = \frac{1}{\sqrt{2}}(T+(T\wedge T_T)). \tag{3.4}$$

According to equation (2.4) we can parameterize the $\alpha_2(s)$-Smarandache curve as in the following form

$$\alpha_2(s) = \frac{1}{\sqrt{2}}\Big(-\cos(s)(\cos(ns)-\sin(ns))+n\sin(s)(\cos(ns)+\sin(ns)),$$
$$\sin(s)(\cos(ns)-\sin(ns))-n\cos(s)(\cos(ns)+\sin(ns)),-\frac{n}{m}(\cos(ns)+\sin(ns))\Big).$$

The shape of this curve is given in Figure (3.2)

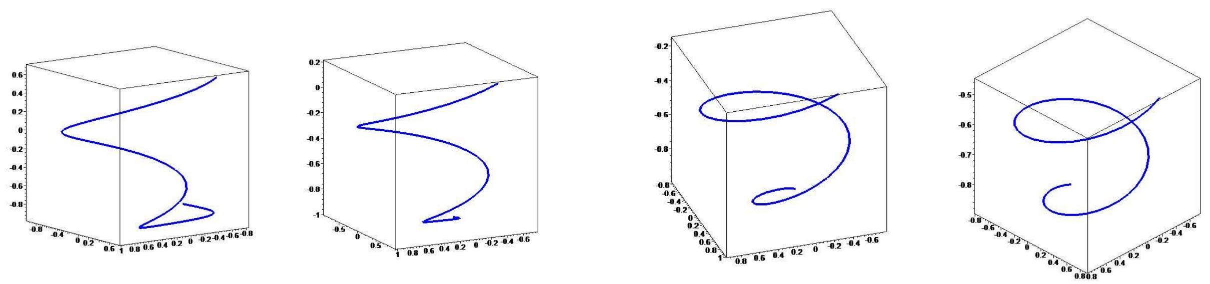

Figure 3.2: $T(T\wedge T_T)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

Theorem 3.4. *The geodesic curvature $K_g^{\alpha_2}$ according to $\alpha_2(s)$-Smarandache curve is given by*

$$K_g^{\alpha_2} = \frac{\tan(ns)+1}{\tan(ns)}.$$

Proof. If we take the derivative of (3.4) and from the equation (2.3) we get,

$$(T_T)_{\alpha_2}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(T_T-\frac{1}{\tan(ns)}T_T), \tag{3.5}$$

if we take the norm of (3.5), $\frac{ds^*}{ds}=\frac{\tan(ns)-1}{\tan(ns)\sqrt{2}}$ we have, We obtain the tangent of $\alpha_2(s)$-Smarandahce curve as in

$$(T_T)_{\alpha_2}=T_T. \tag{3.6}$$

The derivative in the (3.6) is

$$(T_T)'_{\alpha_2} \cdot \frac{ds^*}{ds} = \frac{\sqrt{2}}{\tan(ns)-1}(-\tan(ns)T + (T \wedge T_T)).$$

From equations (3.4) and (3.6) we have

$$(T \wedge T_T)_{\alpha_2} = \frac{1}{\sqrt{2}}(-T + (T \wedge T_T)).$$

So the geodesic curvature from the equation (2.3) is

$$K_g^{\alpha_2} = \frac{\tan(ns)+1}{\tan(ns)}.$$

□

Definition 3.5. *Let $\alpha = \alpha(s)$ be a curve and $\{T, T_T, T \wedge T_T\}$ be Sabban frame of this curve. Then $T_T(T \wedge T_T)$-Smarandache curve is given by*

$$\alpha_3(s) = \frac{1}{\sqrt{2}}(T_T + (T \wedge T_T)). \tag{3.7}$$

According to equation (2.4) we can parameterize the $\alpha_4(s)$-Smarandache curve as in the following form

$$\alpha_3(s) = \frac{1}{\sqrt{2}}\Big(\cos(s)\cos(ns) + n\sin(s)\sin(ns) + \frac{n}{m}\sin(s), \sin(s)\cos(ns) - n\cos(s)\sin(ns) - \frac{n}{m}\cos(s), -\frac{n}{m}\sin(ns) + n\Big).$$

The shape of this curve is given in Figure (3.3)

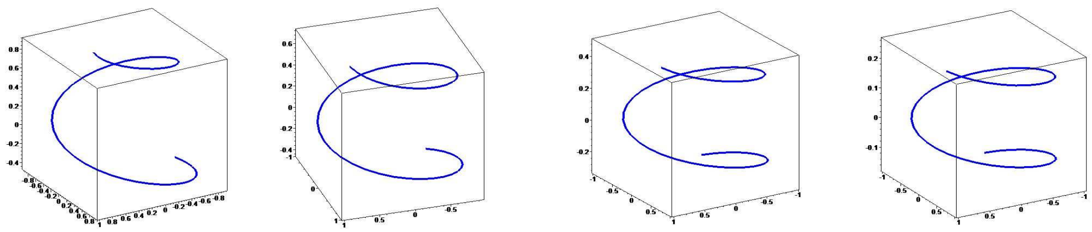

Figure 3.3: $T_T(T \wedge T_T)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5..5]$

Theorem 3.6. *The geodesic curvature $K_g^{\alpha_3}$ according to $\alpha_3(s)$-Smarandache curve is given by*

$$K_g^{\alpha_3} = \frac{\tan^4(ns)}{(1+2\tan^2(ns))^{\frac{5}{2}}}\left(2\chi_4 - \tan(ns)\chi_5 + \tan(ns)\chi_6\right),$$

where the coefficients χ_4, χ_5 and χ_6 are

$$\chi_4 = \frac{2}{\tan(ns)}(\frac{1}{\tan(ns)})' + \frac{1}{\tan(ns)} + \frac{2}{\tan^3(ns)},$$
$$\chi_5 = -1 - (\frac{1}{\tan(ns)})' - \frac{3}{\tan^2(ns)} - \frac{2}{\tan^4(ns)},$$
$$\chi_6 = -\frac{1}{\tan^2(ns)} + (\frac{1}{\tan(ns)})' - \frac{2}{\tan^4(ns)}.$$

Proof. If we take the derivative of (3.7) and from the equation (2.3) we get

$$(T_T)_{\alpha_3} \cdot \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-T - \frac{1}{\tan(ns)}T_T + \frac{1}{\tan(ns)}(T \wedge T_T)), \tag{3.8}$$

if take the norm of (3.8) we have, $\frac{ds^*}{ds} = \frac{\sqrt{\tan^2(ns)+2}}{\tan(ns)\sqrt{2}}$. We obtain the tangent of $\alpha_3(s)$-Smarandahce curve as in

$$(T_T)_{\alpha_3} = \frac{1}{\sqrt{\tan^2(ns)+2}}(-\tan(ns)T - T_T + (T \wedge T_T)). \tag{3.9}$$

The derivative of (3.9) is

$$(T_T)'_{\alpha_3} = \frac{\tan^4(ns)\sqrt{2}}{(\tan^2(ns)+2)^2}(\chi_4 T + \chi_5 T_T + \chi_6(T \wedge T_T)).$$

From equations (3.7) and (3.9) we have

$$(T \wedge T_T)_{\alpha_3} = \frac{1}{\sqrt{2(\tan^2(ns)+2)}}(2T - \tan(ns)T_T + \tan(ns)(T \wedge T_T)).$$

So the geodesic curvature from the equation (2.3) is

$$K_g^{\alpha_3} = \frac{\tan^4(ns)}{(1+2\tan^2(ns))^{\frac{5}{2}}}\left(2\chi_4 - \tan(ns)\chi_5 + \tan(ns)\chi_6\right).$$

□

Definition 3.7. *Let $\alpha = \alpha(s)$ be a curve and $\{T, T_T, T \wedge T_T\}$ be Sabban frame of this curve. Then $TT_T(T \wedge T_T)$-Smarandache curve is given by*

$$\alpha_4(s) = \frac{1}{\sqrt{3}}(T + T_T + (T \wedge T_T)). \tag{3.10}$$

According to equation (2.4) we can parameterize the $\alpha_1(s)$-Smarandache curve as in the following form

$$\begin{aligned}\alpha_4(s) = \frac{1}{\sqrt{3}}\Big(&\cos(s)(\cos(ns)-\sin(ns)) + n\sin(s)(\cos(ns)+\sin(ns)) + \frac{n}{m}\sin(s),\\ &\sin(s)(\cos(ns)-\sin(ns)) - n\cos(s)(\cos(ns)+\sin(ns)) - \frac{n}{m}\cos(s), -\frac{n}{m}(\cos(ns)+\sin(ns)) + n\Big).\end{aligned}$$

The shape of this curve is given in Figure (3.4)

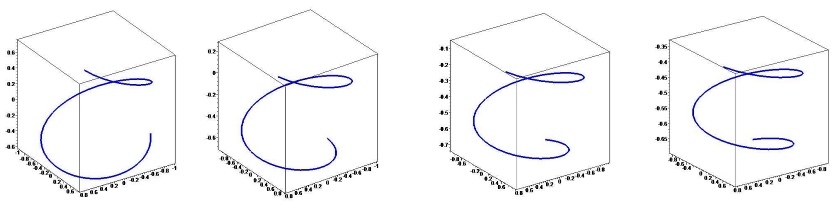

Figure 3.4: $TT_T(T \wedge T_T)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

Theorem 3.8. *The geodesic curvature $K_g^{\alpha_4}$ according to $\alpha_4(s)$-Smarandache curve is given as*

$$K_g^{\alpha_4} = \frac{\tan^4(ns)\left((2-\tan(ns))\chi_7 - (1+\tan(ns))\chi_8 + (2\tan(ns)-1)\chi_9\right)}{(4\sqrt{2}(\tan^2(ns)-\tan(ns)+1)^2)^{\frac{5}{2}}},$$

where the coefficients χ_6, χ_7 and χ_8 are

$$\begin{aligned}\chi_7 &= -\left(\frac{1}{\tan(ns)}\right)' + \frac{2}{\tan(ns)}\left(\frac{1}{\tan(ns)}\right)' - 2 + \frac{4}{\tan(ns)} - \frac{4}{\tan^2(ns)} + \frac{2}{\tan^3(ns)},\\ \chi_8 &= -\left(\frac{1}{\tan(ns)}\right)' - \frac{1}{\tan(ns)}\left(\frac{1}{\tan(ns)}\right)' - 2 - \frac{4}{\tan^2(ns)} + \frac{2}{\tan(ns)} + \frac{2}{\tan^3(ns)} - \frac{2}{\tan^4(ns)},\\ \chi_9 &= \frac{1}{\tan(ns)}\left(\frac{1}{\tan(ns)}\right)' + \frac{2}{\tan(ns)} - \frac{4}{\tan^2(ns)} + \left(\frac{2}{\tan(ns)}\right)' + \frac{4}{\tan^3(ns)} - \frac{2}{\tan^4(ns)}.\end{aligned}$$

Proof. If we take the derivative of (3.10) and from the equation (2.3) we get,

$$(T_T)_{\alpha_4} \cdot \frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(-T + \left(1 - \frac{1}{\tan(ns)}\right)T_T + \frac{1}{\tan(ns)}(T \wedge T_T)\right), \tag{3.11}$$

if we take the norm of (3.11) we have,

$$\frac{ds^*}{ds} = \frac{\sqrt{2(\tan^2(ns)-\tan(ns)+1)}}{\tan(ns)\sqrt{3}}.$$

We obtain the tangent of $\alpha_4(s)$-Smarandahce curve as in

$$(T_T)_{\alpha_4} = \frac{(-\tan(ns)T + (\tan(ns)-1)T_T + (T \wedge T_T))}{\sqrt{2(\tan^2(ns)-\tan(ns)+1)}}. \tag{3.12}$$

The derivative of (3.12) is

$$(T_T)'_{\alpha_4} = \frac{\tan^2(ns)\sqrt{3}(\chi_7 T + \chi_8 T_T + \chi_9(T \wedge T_T))}{4(\tan^2(ns)-\tan(ns)+1)^2}.$$

From equations (3.10) and (3.12) we have

$$(T\wedge T_T)_{\alpha_4} = \frac{(-\tan(ns)+2)T-(\tan(ns)+1)T_T+(2\tan(ns)-1)(T\wedge T_T)}{\sqrt{6(\tan^2(ns)-\tan(ns)+1)}}.$$

So the geodesic curvature from the equation (2.3) is

$$K_g^{\alpha_4} = \frac{\tan^4(ns)\left((2-\tan(ns))\chi_7-(1+\tan(ns))\chi_8+(2\tan(ns)-1)\chi_9\right)}{(4\sqrt{2}(\tan^2(ns)-\tan(ns)+1)^2)^{\frac{5}{2}}}.$$

□

Definition 3.9. *Let $\delta=\delta(s)$ be a curve and $\{N, T_N, N\wedge T_N\}$ be Sabban frame of this curve. Then NT_N-Smarandache curve is given by*

$$\delta_1(s) = \frac{1}{\sqrt{2}}(N+T_N).$$

According to equation (2.6) we can parameterize the $\delta_1(s)$-Smarandache curve as in the following form

$$\begin{aligned}\delta_1(s) = \frac{1}{\sqrt{2}}\Big(&-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\sin(ns)+n\sin(s)\cos(ns))+\frac{n\sin(s)}{m}-\cos(s)\cos(ns)-n\sin(s)\sin(ns),\\ &-\sin(s)\cos(ns)+n\cos(s)\sin(ns)-\frac{n\cos(s)}{m}-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\sin(ns)-n\cos(s)\cos(ns)),\\ &\frac{n\tan(ns)}{m\sqrt{\tan^2(ns)+1}}\cos(ns)+n\Big).\end{aligned}$$

The shape of this curve is given in Figure (3.5)

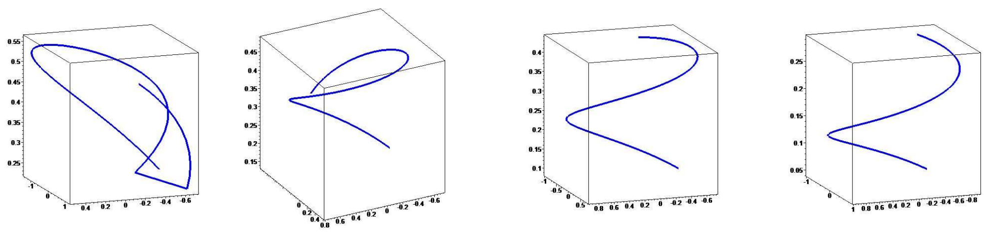

Figure 3.5: NT_N-Smarandache Curves , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

Theorem 3.10. *The geodesic curvature $K_g^{\delta_1}$ according to $\delta_1(s)$-Smarandache curve is given by*

$$K_g^{\delta_1} = \frac{(1+\tan^2(ns))\left(-\tan(ns)'\chi_{10}+\tan(ns)'\chi_{11}+2\sqrt{\tan^2(ns)+1}\chi_{12}\right)}{(2\sqrt{1+\tan^2(ns)}-(\tan(ns)')^2)^{\frac{5}{2}}},$$

where the coefficients χ_{10}, χ_{11} and χ_{12} are

$$\begin{aligned}\chi_{10} &= -2-\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^2+\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)',\\ \chi_{11} &= -2-\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)'-3\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^2-\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^4,\\ \chi_{12} &= \frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}}+2\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)'+\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^3.\end{aligned}$$

Proof. If we take the derivative of equation (3.13) and from the equation (2.5) we have

$$(T_N)_{\delta_1}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\Big(-N+T_N+\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(N\wedge T_N)\Big), \quad (3.13)$$

if we take the norm of equation (3.13) we get

$$\frac{ds^*}{ds} = \frac{\sqrt{2(1+\tan^2(ns))-(\tan(ns)')^2}}{\sqrt{2}\sqrt{1+\tan^2(ns)}}.$$

We obtain the tangent of $\delta_1(s)$-Smarandahce curve as in

$$(T_N)_{\delta_1} = \frac{-\sqrt{\tan^2(ns)+1}\,N+\sqrt{\tan^2(ns)+1}\,T_N-\tan(ns)'(N\wedge T_N)}{\sqrt{2(1+\tan^2(ns))-(\tan(ns)')^2}}. \tag{3.14}$$

The derivative of (3.13) is

$$(T_N)'_{\delta_1} = \frac{(\tan^2(ns)+1)\sqrt{2}(\chi_{10}N+\chi_{11}T_N+\chi_{12}(N\wedge T_N))}{(2(\tan^2(ns)+1)-(\tan(ns)')^2)^2}.$$

From equations (3.13) and (3.14) we have

$$(N\wedge T_N)_{\delta_1} = \frac{(1+\tan^2(ns))^4(-\tan(ns)'(N-T_N)+2\sqrt{1+\tan^2(ns)}(N\wedge T_N))}{\sqrt{2(2(1+\tan^2(ns))-(\tan(ns)')^2)}}.$$

So the geodesic curvature from the equation (2.5) is

$$K_g^{\delta_1} = \frac{(1+\tan^2(ns))\left(-\tan(ns)'\chi_{10}+\tan(ns)'\chi_{11}+2\sqrt{\tan^2(ns)+1}\chi_{12}\right)}{(2\sqrt{1+\tan^2(ns)}-(\tan(ns)')^2)^{\frac{5}{2}}}.$$

□

Definition 3.11. *Let $\delta=\delta(s)$ be a curve and $\{N,T_N,N\wedge T_N\}$ be Sabban frame of this curve. Then $N(N\wedge T_N)$-Smarandache curve is given by*

$$\delta_2(s) = \frac{1}{\sqrt{2}}(N+(N\wedge T_N)). \tag{3.15}$$

According to equation (2.6) we can parameterize the $\delta_2(s)$-Smarandache curve as in the following form

$$\begin{aligned}\delta_2(s) = \frac{1}{\sqrt{2}}\Big(&\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\cos(ns)-n\sin(s)\sin(ns))-\cos(s)\sin(ns)+n\sin(s)\cos(ns)+\frac{n\sin(s)}{m},\\ &+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\cos(ns)+n\cos(s)\sin(ns))-\sin(s)\sin(ns)-n\cos(s)\cos(ns)-\frac{n\cos(s)}{m},\\ &\frac{n\tan(ns)}{m\sqrt{\tan^2(ns)+1}}\sin(ns)-\frac{n}{m}\cos(ns)+n\Big).\end{aligned}$$

The shape of this curve is given in Figure (3.6)

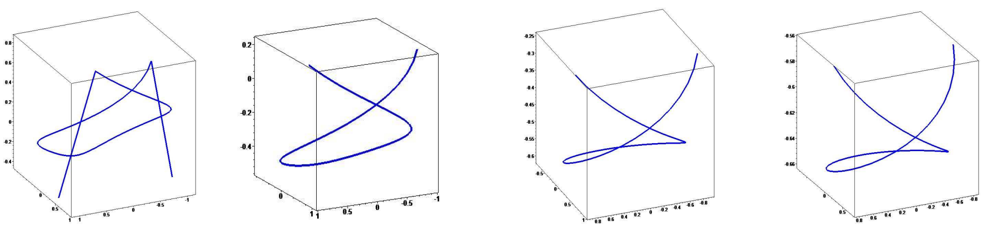

Figure 3.6: $N(N\wedge T_N)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

Theorem 3.12. *The geodesic curvature $K_g^{\delta_2}$ according to $\delta_2(s)$-Smarandache curve is given by*

$$K_g^{\delta_2} = \frac{\sqrt{\tan(ns)^2+1}-\tan(ns)'}{\sqrt{\tan(ns)^2+1}}.$$

Proof. If we take the derivative of equation (3.15) and from the equation (2.5) we get

$$(T_N)_{\delta_2}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(T_N-\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}T_N), \tag{3.16}$$

if we take the norm of equation (3.16) we have

$$\frac{ds^*}{ds}=\frac{\sqrt{\tan(ns)^2+1}+\tan(ns)'}{\sqrt{2}\sqrt{\tan(ns)^2+1}}.$$

We obtain the tangent of $\delta_2(s)$-Smarandahce curve as in

$$(T_N)_{\delta_2} = T_N. \tag{3.17}$$

The derivative of (3.17) is

$$(T_N)'_{\delta_2} = \frac{\sqrt{2}(-\sqrt{\tan^2(ns)+1}N - \tan(ns)'(N\wedge T_N))}{\sqrt{\tan(ns)^2+1}+\tan(ns)'}.$$

From equations (3.15) and (3.17) we have

$$(N\wedge T_N)_{\delta_2} = \frac{1}{\sqrt{2}}(-N+(N\wedge T_N)).$$

So the geodesic curvature from the equation (2.5) is

$$K_g^{\delta_2} = \frac{\sqrt{\tan(ns)^2+1}-\tan(ns)'}{\sqrt{\tan(ns)^2+1}}.$$

□

Definition 3.13. *Let $\delta=\delta(s)$ be a curve and $\{N, T_N, N\wedge T_N\}$ be Sabban frame of this curve. Then $T_N(N\wedge T_N)$-Smarandache curve ($\delta_3(s)$-Smarandache curve) is given by*

$$\delta_3(s) = \frac{1}{\sqrt{2}}(T_N+(N\wedge T_N)). \tag{3.18}$$

According to equation (2.6) we can parameterize the $\delta_3(s)$-Smarandache curve as in the following form

$$\begin{aligned}\delta_3(s) = \frac{1}{\sqrt{2}}\Big(&-\cos(s)\cos(ns)-n\sin(s)\sin(ns)-\cos(s)\sin(ns)+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\cos(ns)-n\sin(s)\sin(ns))\\ &-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\sin(ns)+n\sin(s)\cos(ns))+n\sin(s)\cos(ns), -\sin(s)\sin(ns)-n\cos(s)\cos(ns)\\ &-\sin(s)\cos(ns)+n\cos(s)\sin(ns)+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\cos(ns)+n\cos(s)\sin(ns))\\ &-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\sin(ns)-n\cos(s)\cos(ns)), \frac{n\tan(ns)}{m\sqrt{\tan^2(ns)+1}}(\cos(ns)+\sin(ns))-\frac{n}{m}\cos(ns)\Big)\end{aligned}$$

.

The shape of this curve is given in Figure (3.7)

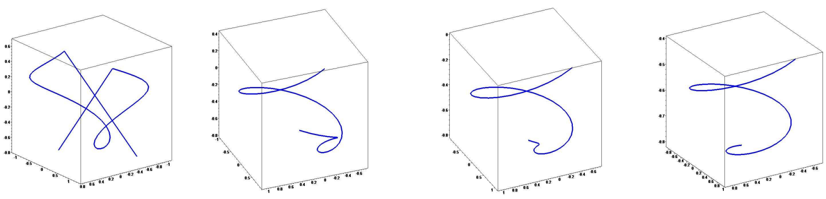

Figure 3.7: $T_N(N\wedge TN)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

Theorem 3.14. *The geodesic curvature $K_g^{\delta_3}$ according to $\delta_3(s)$-Smarandache curve is given by*

$$K_g^{\delta_3} = \frac{(\tan^2(ns)+1)^4((-2\tan(ns)')\chi_{13}-\sqrt{\tan(ns)^2+1}(\chi_{14}-\chi_{15}))}{(1+\tan^2(ns)+(-\tan(ns)')^2)^{\frac{5}{2}}},$$

where the coefficients χ_{13}, χ_{14} and χ_{15} are

$$\chi_{13} = 2\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)'+\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}+2\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^3,$$

$$\chi_{14} = -1-\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)'-\Big(\frac{-3\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^2-\Big(\frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^4,$$

$$\chi_{15} = -\Big(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^2+\Big(\frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)'-\Big(\frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}}\Big)^4.$$

Proof. If we take the derivative of equation (3.18) and from the equation (2.5) we get

$$(T_N)_{\delta_3}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\Big(-N-\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}T_N+\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(N\wedge T_N)\Big), \tag{3.19}$$

if we take the norm of equation (3.19) we have

$$\frac{ds^*}{ds}=\frac{\sqrt{1+\tan^2(ns)+2(-\tan(ns)')^2}}{\sqrt{2}\sqrt{\tan^2(ns)+1}}.$$

We obtain the tangent of $\delta_3(s)$-Smarandahce curve as in

$$(T_N)_{\delta_3} = \frac{-\sqrt{\tan^2(ns)+1}N+\tan(ns)'T_N-\tan(ns)'(N\wedge T_N)}{\sqrt{1+\tan^2(ns)+2(-\tan(ns)')^2}}. \tag{3.20}$$

The derivative of (3.20) is

$$(T_N)'_{\delta_3} = \frac{\sqrt{2}(\tan^2(ns)+1)^2}{(1+\tan^2(ns)+2(-\tan(ns)')^2)^2}(\chi_{13}N+\chi_{14}T_N+\chi_{15}(N\wedge T_N)).$$

From equations (3.18) and (3.20) we have

$$(N\wedge T_N)_{\delta_3} = \frac{(-2\tan(ns)'N-\sqrt{1+\tan^2(ns)}T_N+\sqrt{1+\tan^2(ns)}(N\wedge T_N))}{\sqrt{2(1+\tan^2(ns)+2(-\tan(ns)')^2)}}.$$

So the geodesic curvature from the equation (2.5) is

$$K_g^{\delta_3} = \frac{(\tan^2(ns)+1)^4((-2\tan(ns)')\chi_{13}-\sqrt{\tan(ns)^2+1}(\chi_{14}-\chi_{15}))}{(1+\tan^2(ns)+(-\tan(ns)')^2)^{\frac{5}{2}}}.$$

□

Definition 3.15. *Let $\delta=\delta(s)$ be a curve and $\{N,T_N,N\wedge T_N\}$ be Sabban frame of this curve. Then $NT_N(N\wedge T_N)$-Smarandache curve ($\delta_4(s)$-Smarandache curve) is given by*

$$\delta_4(s) = \frac{1}{\sqrt{3}}(N+T_N+(N\wedge T_N)). \tag{3.21}$$

According to equation (2.6) we can parameterize the $\delta_4(s)$-Smarandache curve as in the following form

$$\begin{aligned}\delta_4(s) = \frac{1}{\sqrt{3}}\Big(&-\cos(s)\cos(ns)-n\sin(s)\sin(ns)-\cos(s)\sin(ns)+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\cos(ns)-n\sin(s)\sin(ns))+\frac{n\sin(s)}{m}\\&-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\cos(s)\sin(ns)+n\sin(s)\cos(ns))+n\sin(s)\cos(ns),-\sin(s)\sin(ns)\\&+\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\cos(ns)+n\cos(s)\sin(ns))-n\cos(s)\cos(ns)-\sin(s)\cos(ns)+n\cos(s)\sin(ns)\\&-\frac{\tan(ns)}{\sqrt{\tan^2(ns)+1}}(-\sin(s)\sin(ns)-n\cos(s)\cos(ns))-\frac{n\cos(s)}{m},\frac{n\tan(ns)}{m\sqrt{\tan^2(ns)+1}}(\cos(ns)+\sin(ns))-\frac{n}{m}\cos(ns)+n\Big).\end{aligned}$$

The shape of this curve is given in Figure (3.8)

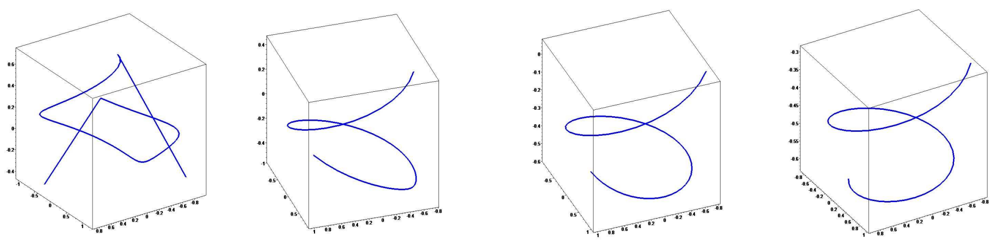

Figure 3.8: $TT_N(T\wedge T_N)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

Theorem 3.16. *The geodesic curvature $K_g^{\delta_4}$ according to $\delta_4(s)$-Smarandache curve is given by*

$$K_g^{\delta_4} = \frac{\left((-2\tan(ns)' - \sqrt{\tan(ns)^2+1})\chi_{16} - \chi_{17}(\sqrt{\tan(ns)^2+1} - \tan(ns)')\right)}{(4\sqrt{2}(1+\tan^2(ns)+\sqrt{1+\tan^2(ns)}\tan(ns)' + (-\tan(ns)')^2)^2)^{\frac{5}{2}}}$$

$$+\frac{(2\sqrt{\tan(ns)^2+1}+\tan(ns)')\chi_{18}}{(4\sqrt{2}(1+\tan^2(ns)+\sqrt{1+\tan^2(ns)}\tan(ns)' + (-\tan(ns)')^2)^2)^{\frac{5}{2}}},$$

where the coefficients χ_{16}, χ_{17} and χ_{18} are

$$\begin{aligned}
\chi_{16} &= (\frac{\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' + \frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' - 2 + \frac{-4\tan(ns)'}{\sqrt{\tan^2(ns)+1}} + (\frac{4\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^2 + (\frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^3,\\
\chi_{17} &= -(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' - \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' - 2 + (\frac{4\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^2 - \frac{2\tan(ns)'}{\sqrt{\tan^2(ns)+1}} - (\frac{2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^3\\
&\quad + (\frac{2\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^4,\\
\chi_{18} &= \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' + \frac{-2\tan(ns)'}{\sqrt{\tan^2(ns)+1}} - 4(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^2 + 2(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})' + 4(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^3\\
&\quad - 2(\frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})^4.
\end{aligned}$$

Proof. If we take the derivative of equation (3.21) and from the equation (2.5) we have

$$(T_N)_{\delta_4}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-N + (1 - \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}})T_N + \frac{-\tan(ns)'}{\sqrt{\tan^2(ns)+1}}(N\wedge T_N)), \tag{3.22}$$

if we take the norm of equation (3.22) we get

$$\frac{ds^*}{ds} = \frac{\sqrt{2(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)}}{\sqrt{3}\sqrt{\tan^2(ns)+1}}.$$

We obtain the tangent of $\delta_4(s)$-Smarandahce curve as in

$$(T_N)_{\delta_4} = \frac{-\sqrt{\tan^2(ns)+1}N + (\sqrt{\tan^2(ns)+1}+\tan(ns)')T_N - \tan(ns)'(N\wedge T_N)}{\sqrt{2(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)}}. \tag{3.23}$$

The derivative of (3.23) is

$$(T_N)'_{\delta_4} = \frac{\sqrt{3}(\chi_{16}N + \chi_{17}T_N + \chi_{18}(N\wedge T_N))}{4(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)^2}.$$

From equations (3.21) and (3.23) we have

$$(N\wedge T_N)_{\delta_4} = \frac{(-(\sqrt{\tan^2(ns)+1}+2\tan(ns)')N - (\sqrt{\tan^2(ns)+1}-\tan(ns)')T_N)}{\sqrt{6(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)}}$$

$$+\frac{(2\sqrt{\tan^2(ns)+1}+\tan(ns)')(N\wedge T_N)}{\sqrt{6(1+\tan^2(ns)+\tan(ns)'\sqrt{\tan^2(ns)+1}+(-\tan(ns)')^2)}}.$$

So the geodesic curvature from the equation (2.5) is

$$K_g^{\delta_4} = \frac{\left((-2\tan(ns)' - \sqrt{\tan(ns)^2+1})\chi_{16} - \chi_{17}(\sqrt{\tan(ns)^2+1} - \tan(ns)')\right)}{(4\sqrt{2}(1+\tan^2(ns)+\sqrt{1+\tan^2(ns)}\tan(ns)' + (-\tan(ns)')^2)^2)^{\frac{5}{2}}}$$

$$+\frac{(2\sqrt{\tan(ns)^2+1}+\tan(ns)')\chi_{18}}{(4\sqrt{2}(1+\tan^2(ns)+\sqrt{1+\tan^2(ns)}\tan(ns)' + (-\tan(ns)')^2)^2)^{\frac{5}{2}}}.$$

□

Definition 3.17. *Let $\zeta = \zeta(s)$ be a curve and $\{B, T_B, B \wedge T_B\}$ be Sabban frame of this curve. Then BT_B-Smarandache curve ($\zeta_1(s)$-Smarandache curve) is given by*

$$\zeta_1(s) = \frac{1}{\sqrt{2}}(B + T_B). \tag{3.24}$$

According to equation (2.8) we can parameterize the $\zeta_1(s)$-Smarandache curve as in the following form

$$\zeta_1(s) = \frac{1}{\sqrt{2}}\Big(-\cos(s)\cos(ns) - n\sin(s)\sin(ns) + \frac{n}{m}\sin(s), -\sin(s)\cos(ns) - n\cos(s)\sin(ns) - \frac{n}{m}\cos(s), \frac{n}{m}\sin(ns) + n\Big).$$

The shape of this curve is given in Figure (3.9)

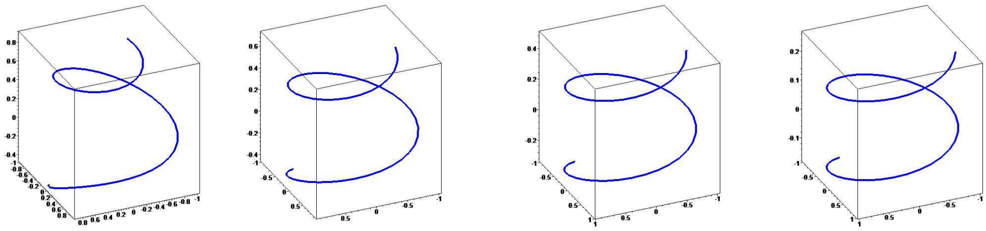

Figure 3.9: $BT_B(B \wedge T_B)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

Theorem 3.18. *The geodesic curvature $K_g^{\zeta_1}$ according to $\zeta_1(s)$-Smarandache curve is*

$$K_g^{\zeta_1} = \frac{1}{(2 + (\tan(ns))^2)^{\frac{5}{2}}}\left(\chi_{19}\tan(ns) - \chi_{20}\tan(ns) + 2\chi_{21}\right),$$

where the coefficients χ_{19}, χ_{20} and χ_{21} are

$$\begin{aligned}
\chi_{19} &= -2 - \tan^2(ns) + \tan(ns)\tan(ns)', \\
\chi_{20} &= -2 - \tan(ns)\tan(ns)' - 3\tan^2(ns) - \tan^4(ns), \\
\chi_{21} &= 2\tan(ns) + 2\tan(ns)' + \tan^3(ns).
\end{aligned}$$

Proof. If we take the derivative of equation (3.24) and from the equation (2.7) we get

$$(T_B)_{\zeta_1} \cdot \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-B + T_B + \tan(ns)(B \wedge T_B)), \tag{3.25}$$

if we take the norm of equation (3.25) we have

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{2 + \tan^2(ns)}.$$

We obtain the tangent of $\zeta_1(s)$-Smarandahce curve as in

$$(T_B)_{\zeta_1} = \frac{1}{\sqrt{2 + \tan^2(ns)}}(-B + T_N + \tan(ns)(B \wedge T_B)). \tag{3.26}$$

The derivative of (3.26) is

$$(T_B)'_{\zeta_1} \cdot \frac{ds^*}{ds} = \frac{\sqrt{2}}{(2 + \tan^2(ns))^2}(\chi_{19}B + \chi_{20}T_B + \chi_{21}(B \wedge T_B)).$$

From equations (3.24) and (3.26) we have

$$(B \wedge T_B)_{\zeta_1} = \frac{1}{\sqrt{4 + 2\tan^2(ns)}}(\tan(ns)N - \tan(ns)T_B + 2(B \wedge T_B)).$$

So the geodesic curvature from the equation (2.7) is

$$K_g^{\zeta_1} = \frac{1}{(2 + (\tan(ns))^2)^{\frac{5}{2}}}\left(\chi_{19}\tan(ns) - \chi_{20}\tan(ns) + 2\chi_{21}\right).$$

□

Definition 3.19. *Let $\zeta = \zeta(s)$ be a curve and $\{B, T_B, B \wedge T_B\}$ be Sabban frame of this curve. Then $B(B \wedge T_B)$-Smarandache curve ($\zeta_2(s)$-Smarandache curve) is given by*

$$\zeta_2(s) = \frac{1}{\sqrt{2}}(B + (B \wedge T_B)). \tag{3.27}$$

According to equation (2.8) we can parameterize the $\zeta_2(s)$-Smarandache curve as in the following form

$$\zeta_2(s) = \frac{1}{\sqrt{2}}\Big(-\cos(s)(\cos(ns) - \sin(ns)) - n\sin(s)(\cos(ns) + \sin(ns)), -\sin(s)(\cos(ns) - \sin(ns)) + n\cos(s)(\cos(ns) + \sin(ns)), \frac{n}{m}(\cos(ns) + \sin(ns))\Big).$$

The shape of this curve is given in Figure (3.10)

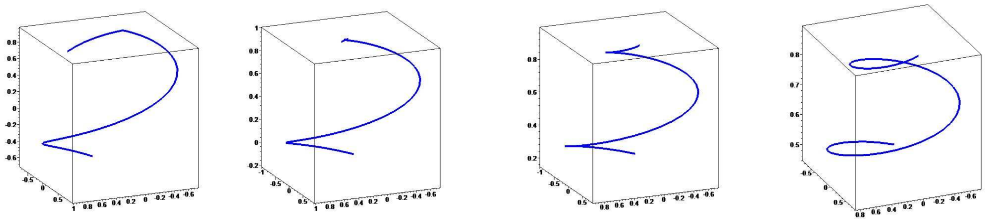

Figure 3.10: $B(B \wedge T_B)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

Theorem 3.20. *The geodesic curvature $K_g^{\zeta_2}$ according to $\zeta_2(s)$-Smarandache curve is*

$$K_g^{\zeta_2} = 1 + \tan(ns).$$

Proof. If we take the derivative of equation (3.27) and from the equation (2.7) we get

$$(T_B)_{\zeta_2} \cdot \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(T_B - \tan(ns)T_B), \tag{3.28}$$

if we take the norm in equation (3.28), $\frac{ds^*}{ds} = \frac{1-\tan(ns)}{\sqrt{2}}$. We obtain the tangent of $\zeta_2(s)$-Smarandahce curve as in

$$(T_B)_{\zeta_2} = T_B. \tag{3.29}$$

The derivative of (3.29) is

$$(T_B)'_{\zeta_2} \cdot \frac{ds^*}{ds} = -B + \tan(ns)(B \wedge T_B).$$

From eqnarrays (3.27) and (3.29) we have

$$(B \wedge T_B)_{\zeta_2} = \frac{1}{\sqrt{2}}(-B + (B \wedge T_B)).$$

So the geodesic curvature from the equation (2.7) is

$$K_g^{\zeta_2} = 1 + \tan(ns).$$

□

Definition 3.21. *Let $\zeta=\zeta(s)$ be a curve and $\{B,T_B,B\wedge T_B\}$ be Sabban frame of this curve. Then $T_B(B\wedge T_B)$-Smarandache curve ($\zeta_3(s)$-Smarandache curve) is given by*

$$\zeta_3(s) = \frac{1}{\sqrt{2}}(T_B+(B\wedge T_B)). \tag{3.30}$$

According to equation (2.8) we can parameterize the $\zeta_3(s)$-Smarandache curve as in the following form

$$\zeta_3(s) = \frac{1}{\sqrt{2}}\Big(\cos(s)\sin(ns)-n\sin(s)\cos(ns)+\frac{n}{m}\sin(s),\sin(s)\sin(ns)+n\cos(s)\cos(ns)-\frac{n}{m}\cos(s),\frac{n}{m}\cos(ns)+n\Big).$$

The shape of this curve is given in Figure (3.11)

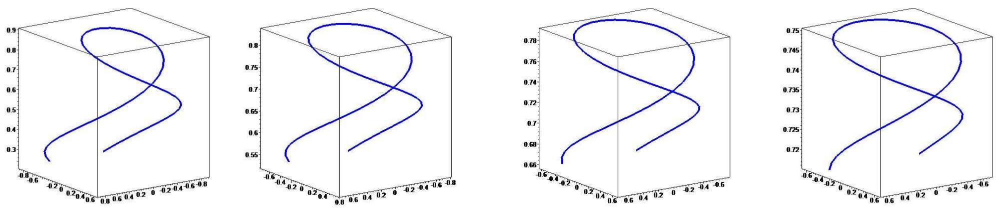

Figure 3.11: $T_B(B\wedge T_B)$-Smarandache Curve , $m=\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{16}$ and $s=[-5,5]$

Theorem 3.22. *The geodesic curvature $K_g^{\zeta_3}$ according to $\zeta_3(s)$-Smarandache curve is*

$$K_g^{\zeta_3} = \frac{1}{(1+2(\tan(ns))^2)^{\frac{5}{2}}}\left(2\tan(ns)\chi_{22}-\chi_{23}+\chi_{24}\right),$$

where the coefficients χ_{22}, χ_{23}, χ_{24} are

$$\begin{aligned}
\chi_{22} &= 2\tan(ns)\tan(ns)'+\tan(ns)+2\tan^3(ns),\\
\chi_{23} &= -1-\tan(ns)'-3\tan^2(ns)-2\tan^4(ns),\\
\chi_{24} &= -\tan^2(ns)+2\tan(ns)'-2\tan^4(ns).
\end{aligned}$$

Proof. If we take the derivative of equation (3.30) and from the equation (2.7) we get

$$(T_B)_{\zeta_3}\cdot\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-B-\tan(ns)T_B+\tan(ns)(B\wedge T_B)), \tag{3.31}$$

if we take the norm of eqnarray (3.31) we have

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{1+2\tan^2(ns)}.$$

We obtain the tangent of $\zeta_3(s)$-Smarandahce curve as in

$$(T_B)_{\zeta_3} = \frac{1}{\sqrt{1+2\tan^2(ns)}}(-B-\tan(ns)T_N+\tan(ns)(B\wedge T_B)). \tag{3.32}$$

The derivative of (3.32) is

$$(T_B)'_{\zeta_3}\cdot\frac{ds^*}{ds} = \frac{\sqrt{2}}{(1+2\tan^2(ns))^2}(\chi_{22}B+\chi_{23}T_B+\chi_{24}(B\wedge T_B)).$$

From equations (3.30) and (3.32) we have

$$(B\wedge T_B)_{\zeta_3} = \frac{1}{\sqrt{2+4\tan^2(ns)}}(2\tan(ns)B-T_B+(B\wedge T_B)).$$

So the geodesic curvature from the equation (2.7) is

$$K_g^{\zeta_3} = \frac{1}{(1+2(\tan(ns))^2)^{\frac{5}{2}}}\left(2\tan(ns)\chi_{22}-\chi_{23}+\chi_{24}\right).$$

□

Definition 3.23. *Let $\zeta = \zeta(s)$ be a curve and $\{B, T_B, B \wedge T_B\}$ be Sabban frame of this curve. Then $BT_B(B \wedge T_B)$-Smarandache curve ($\zeta_4(s)$-Smarandache curve) is given by*

$$\zeta_4(s) \quad = \quad \frac{1}{\sqrt{3}}(B + T_B + (B \wedge T_B)). \tag{3.33}$$

According to equation (2.8) we can parameterize the $\zeta_4(s)$-Smarandache curve as in the following form

$$\zeta_4(s) = \frac{1}{\sqrt{3}}\Big(-\cos(s)(\cos(ns) - \sin(ns)) - n\sin(s)(\cos(ns) + \sin(ns)) + \frac{n}{m}\sin(s), -\sin(s)(\cos(ns) - \sin(ns)) + n\cos(s)(\cos(ns) + \sin(ns)) - \frac{n}{m}\cos(s), \frac{n}{m}(\cos(ns) + \sin(ns)) + n\Big).$$

The shape of this curve is given in Figure (3.12)

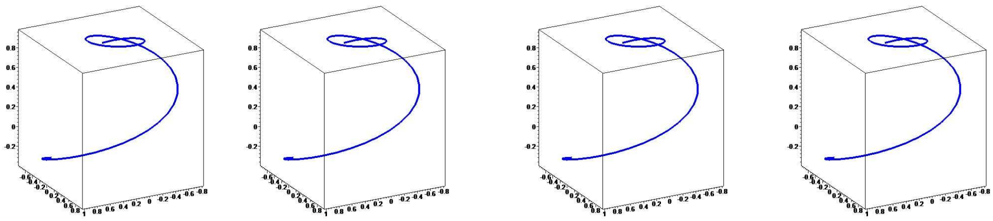

Figure 3.12: $BT_B(B \wedge T_B)$-Smarandache Curve , $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and $s = [-5, 5]$

Theorem 3.24. *The geodesic curvature $K_g^{\zeta_4}$ according to $\zeta_4(s)$-Smarandache curve is*

$$K_g^{\zeta_4} = \frac{(\chi_{25}(2\tan(ns) - 1) + \chi_{26}(-1 - \tan(ns)) + \chi_{27}(2 - \tan(ns)))}{(4\sqrt{2}(1 - \tan(ns) + \tan^2(ns))^2)^{\frac{5}{2}}},$$

where the coefficients χ_{25}, χ_{26}, χ_{27} are

$$\begin{aligned}
\chi_{25} &= -\tan(ns)' + 2\tan(ns)\tan(ns)' - 2 + 4\tan(ns) - 4\tan^2(ns) \\
&\quad + 2\tan^3(ns), \\
\chi_{26} &= -\tan(ns)' - \tan(ns)\tan(ns)' - 2 - 4\tan^2(ns) + 2\tan(ns) \\
&\quad + 2\tan^3(ns) - 2\tan^4(ns), \\
\chi_{27} &= \tan(ns)\tan(ns)' + 2\tan(ns) - 4\tan^2(ns) + 2\tan(ns)' + 4\tan^3(ns) \\
&\quad - 2\tan^4(ns).
\end{aligned}$$

Proof. If we take the derivative of equation (3.33) and from the equation (2.7) we have

$$(T_B)_{\zeta_4} \cdot \frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-B + (1 - \tan(ns))T_B + \tan(ns)(B \wedge T_B)), \tag{3.34}$$

if we take the norm of equation (3.34)

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\sqrt{2(1 - \tan(ns) + \tan^2(ns))}.$$

We obtain the tangent of $\zeta_4(s)$-Smarandahce curve as in

$$(T_B)_{\zeta_4} = \frac{1}{\sqrt{2(1 - \tan(ns) + \tan^2(ns))}}(-B + (1 - \tan(ns))T_B + \tan(ns)(B \wedge T_B)). \tag{3.35}$$

The derivative of (3.35) is

$$(T_B)'_{\zeta_3} \cdot \frac{ds^*}{ds} = \frac{\sqrt{3}}{4(1 - \tan(ns) + \tan^2(ns))^2}(\chi_{25}B + \chi_{26}T_B + \chi_{27}(B \wedge T_B)).$$

From equations (3.33) and (3.35) we have

$$(B \wedge T_B)_{\zeta_4} = \frac{((-1 + 2\tan(ns))B + (-1 - \tan(ns))T_B + (2 - \tan(ns))(B \wedge T_B))}{\sqrt{6(1 - \tan(ns) + \tan^2(ns))}}.$$

So the geodesic curvature from the equation (2.7) is

$$K_g^{\zeta_4} = \frac{(\chi_{25}(2\tan(ns) - 1) + \chi_{26}(-1 - \tan(ns)) + \chi_{27}(2 - \tan(ns)))}{(4\sqrt{2}(1 - \tan(ns) + \tan(ns)^2)^2)^{\frac{5}{2}}}.$$

□

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