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A STUDY OF ORDERED BI-GAMMA-HYPERIDEALS IN ORDERED GAMMA-SEMIHYPERGROUPS

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Abstract

The main purpose of this paper is to investigate ordered Γ-semihypergroups in the general terms of ordered Γ-hyperideals. We introduce ordered (generalized) (m, n)-Γ-hyperideals in ordered Γ-semihypergroups. Then, we characterize ordered Γ-semihypergroup by ordered (generalized) $(0, 2)$-Γ-hyperideals, ordered (generalized) $(1, 2)$-Γ–hyperideals and ordered (generalized) 0-minimal $(0, 2)$-Γ-hyperideals. Furthermore, we investigate the notion of ordered (generalized) $(0, 2)$-bi-Γ-hyperideals, ordered 0-$(0, 2)$ bisimple ordered Γ-semihypergroups and ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-hyperideals in ordered Γ-semihyperoups. It is proved that an ordered Γ-semihypergroup S with a zero 0 is 0-$(0, 2)$-bisimple if and only if it is left 0-simple.

Keywords: Algebraic hyperstructure; Γ-subsemihypergroup; bisimple; ordered Γ-semihypergroup; ordered bi-Γ-hyperideal; ordered (m, n)-Γ–hyperideal; ordered $(0, 2)$-Γ-hyperideal.

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1 Introduction

The theory of (m, n)-ideal in semigroups was given by Lajos [44] as a generalization of left(resp. right) ideals in semigroups. Thereafter, the notion of generalized bi-ideal [(or generalized (1,1)-ideal] was introduced in semigroups also by Lajos [43] as a generalization of bi-ideals in semigroups. Then, various authors investigated these concepts [1], [2], [19], [27], [28], [29], [30], [31]. Akram, Yaqoob and Khan studied (m, n)-hyperideals in LA-semihypergroups [25]. Hila et al. [23], [47] investigated quasi-hyperideals and bi-hyperideals in semihypergroups.

The concept of hyperstructure was given by Marty [20], at the 8th Congress of Scandinavian Mathematics. He formulated hypergroups and began to derive its properties and results. Now, the notion of algebraic hyperstructures has become a highly fruitful branch in algebraic theory and it has wide applications in various branches of mathematics and applied science. For detailed review of the notion of hyperstructures, readers are referred to [8], [13], [18], [23], [33], [35], [37], [38], [39], [40], [42].

Recently, Basar et al. studied different aspects of ideal theoretic results in ordered semihypergroups [3], [4], [5], [6], [7] [41].

Later on, many algebraists have developed semihypergroups as the simplest algebraic hyperstructures with closure and associative properties. Semihypergroups (hypergroups) have been found useful for dealing with problems in different domains of algebraic hyperstructures. Many mathematicians studied various aspects of semihypergroups (hypergroups), for instance, Kondo and Lekkoksung [26], Bonansinga and Corsini [35], Leoreanu [49], Davvaz [8], Pibaljommee and Davvaz [9], Davvaz [10], [11], Freni [14], and Salvo [32]. The applications of semihypergroups (hypergroups) to areas such as graph theory, optimization theory, theory of discrete event dynamical systems, automata theory, generalized fuzzy computation, formal language theory, coding theory and analysis of computer programs have been extensively studied in the literature [12].

Then connection between hyperstructures and ordered sets has been investigated by many researchers. Heidari and Davvaz [15], [18] studied ordered hyperstructures. One main aspect of this theory, known as El-hyperstructures, was studied by Chvalina and Novak [21], [34]. Conard studied ordered semigroups [36]. The concept of ordered semihypergroups was studied in [9], [22], [47], [48]. Heideri et al. [16], [17], [45], [46] studied Γ-semihypergroups. We assume that the reader is familiar with some terminology in theory of semihypergroup and other related notions. What follows now are some definitions and preliminaries in the theory of Γ-semihypergroups that we need for formulation and proof of our main results.

Let H be a nonempty set, then the mapping $\circ : H \times H \to H$ is called hyperoperation or join operation on H, where $P^{\star}(H) = P(H) \setminus \{0\}$ is the set of all nonempty subsets of H. Let A and B be two nonempty sets. Then a hypergroupoid $(S, \circ)$ is called a Γ-semihypergroups if for every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$,

$$x \circ \alpha \circ (y \circ \beta \circ z) = (x \circ \alpha \circ y) \circ \beta \circ z,$$

i.e.,

$$\bigcup_{u \in y \circ \alpha \circ z} x \circ \alpha \circ u = \bigcup_{v \in x \circ \alpha \circ y} v \circ \beta \circ z.$$

A Γ-semihypergroup $(S, \circ)$ together with a partial order "$\leq$" on S that is compatible with Γ-semihypergroup operation such that for all $x, y, z \in S$, we have

$$x \leq y \Rightarrow z \circ \alpha \circ x \leq z \circ \beta \circ y \text{ and } \mathrm{x} \circ \alpha \circ \mathrm{z} \leq \mathrm{y} \circ \beta \circ \mathrm{z},$$

ia called an ordered Γ-semihypergroup. For subsets A, B of an ordered Γ-semihypergroup S, the product set $A \circ \Gamma \circ B$ of the pair (A, B) relative to S is defined as below:

$$A \circ \Gamma \circ B = \{a \circ \gamma \circ b : a \in A, b \in B, \gamma \in \Gamma\},$$

and for $A \subseteq S$, the product set $A \circ \Gamma \circ A$ relative to S is defined as $A^2 = A \circ \Gamma \circ A$.

For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m$ for some $m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

Let $A \subseteq S$. Then, for a non-negative integer m, the power of A is defined by $A^m = A \circ \Gamma \circ A \circ \Gamma \circ A \circ \Gamma \circ A \cdots$, where A occurs m times. Note that the power vanishes if $m = 0$. So, $A^0 \circ \Gamma \circ S = S = S \circ \Gamma \circ A^0$.

In what follows we denote ordered Γ-semihypergroup $(S, \circ, \Gamma, \leq)$ by S unless otherwise specified.

Suppose S is an ordered Γ-semihypergroup and I is a nonempty subset of S. Then, I is called an ordered right (resp. left) Γ-hyperideal of S if

(i) $I \circ \Gamma \circ S \subseteq I (resp.\ S \circ \Gamma \circ I \subseteq I)$,

(ii) $a \in I, b \leq a$ for $b \in S \Rightarrow b \in I$.

Equivalent Definition:

(i) $I \circ \Gamma \circ S \subseteq I$ (resp. $S \circ \Gamma \circ I \subseteq I$).

(ii) $(I] = I$.

An ordered Γ-hyperideal I of S is both a right and a left ordered Γ-hyperideal of an ordered Γ-semihypergroup S. A right, left or (two-sided) ordered Γ-hyperideal I of S is called proper if $I \neq S$.

Definition 1.1: Let S be a Γ-semihypergroup and A be a nonempty subset of S, then A is called a generalized (m, n)-Γ-hyperideal of S if $A^m \Gamma S \Gamma A^n \subseteq A$, where m, n are arbitrary non-negative integers. Notice that if A is a sub-Γ-semihypergroup of S, then A is called an (m, n)-Γ-hyperideal of S.

Definition 1.2. Suppose A is a sub-Γ-semihypergroup (resp. nonempty subset) of an ordered Γ-semihypergroup S. Then, A is called an (resp. generalized) (m, n)-Γ-hyperideal of S if (i) $A^m \circ \Gamma \circ S \circ \Gamma \circ A^n \subseteq A$, and (ii) for $b \in A$, $s \in S$, $s \leq b \Rightarrow s \in A$.

Observe that in the above Definition 1.2., if we put $m = n = 1$, then A is called an ordered (generalized) bi-Γ-hyperideal of S. Furthermore, if $m = 0$ and $n = 2$, then we find an ordered (generalized) $(0, 2)$-Γ-hyperideal of S. In a similar manner, we can derive an ordered (generalized) $(1, 2)$-Γ-hyperideal and an ordered (generalized) $(2, 1)$-Γ-hyperideal of S.

Let $(S, \circ, \Gamma, \leq)$ be an ordered Γ-semihypergroup and A, B be nonempty subsets of S, then we easily have the following:

(i) $A \subseteq (A]$;

(ii) If $A \subseteq B$, then $(A] \subseteq (B]$;

(iii) $(A] \circ \Gamma \circ (B] \subseteq (A \circ \Gamma \circ B]$;

(iv) $(A] = ((A]]$;

(v) $((A] \circ \Gamma \circ (B]] = (A \circ \Gamma \circ B]$;

(vi) For every left (resp. right) ordered Γ-hyperideal T of S, $(T] = T$.

If A is a nonempty subset of S, $(A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$ is an ordered (generalized) bi-Γ-hyperideal of S, we depict the proof of it as follows:

$$
\begin{aligned}
((A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]] &= (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\
&\text{and } (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \circ \Gamma \circ S \circ \Gamma \circ \\
&(A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\
&= (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \circ \Gamma \circ \\
&(S] \circ \Gamma \circ (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\
&\subseteq (A^2 \circ \Gamma \circ S \circ \Gamma \circ A^2 \cup A^2 \circ \\
&\Gamma \circ S \circ \Gamma \circ A \circ \Gamma \circ S \circ \Gamma \circ A^2 \cup A \circ \\
&\Gamma \circ S \circ \Gamma \circ A^2 \circ \Gamma \circ S \circ \Gamma \\
&\circ A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2 \circ \Gamma \\
&\circ S \circ \Gamma \circ A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\
&\subseteq (A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\
&\subseteq (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2].
\end{aligned}
$$

2 Main Results

In the current section, we now study ideal theory in ordered Γ-semihypergroups. We obtain many equivalent conditions based on ordered Γ-hyperideal, ordered (0, 2)-Γ-hyperideal, ordered bi-Γ-hyperideal. We begin with the following:

Lemma 2.1: The following assertions are equivalent for a subset A of an ordered Γ-semihypergroup S:

(i) A is an ordered (generalized) $(0,2)$-Γ-hyperideal of S;

(ii) A is an ordered left Γ-hyperideal of some ordered left Γ-hyperideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose A is an ordered (generalized) $(0,2)$-Γ-hyperideal of an ordered Γ-semihypergroup S. Then, we obtain the following:

$$\begin{aligned}(A \cup S \circ \Gamma \circ A] \circ \Gamma \circ A &= (A^2 \cup S \circ \Gamma \circ A^2] \\ &\subseteq (A] \\ &= A,\end{aligned}$$

and

$$((A] = (A],$$

therefore, A is an ordered left Γ-hyperideal of ordered left Γ-hyperideal $(A \cup S \circ \Gamma \circ A]$ of S.
$(ii) \Rightarrow (i)$. Suppose L is an ordered left Γ-hyperideal of S and B is an ordered left Γ-hyperideal of L. Then, we have

$$\begin{aligned}S \circ \Gamma \circ A^2 &\subseteq S \circ \Gamma \circ L \circ \Gamma \circ \Gamma \circ A \\ &\subseteq L \circ \Gamma \circ A \\ &\subseteq A.\end{aligned}$$

Suppose $b \in A$ and $s \in S$ are such that $s \leq b$. As $b \in L$, we get $s \in L$ and so $s \in A$. Hence, A is an ordered (generalized) $(0,2)$-Γ-hyperideal of S.

Theorem 2.2. Let A be a subset of an ordered Γ-semihypergroup S. Then the following results are equivalent:

(i) A is an ordered (generalized) $(1,2)$-Γ-hyperideal of S;

(ii) A is an ordered left Γ-hyperideal of some ordered (generalized) bi-Γ-hyperideal of S;

(iii) A is an ordered (generalized) bi-Γ-hyperideal of some left ordered Γ-hyperideal of S;

(iv) A is an ordered (generalized) $(0,2)$-Γ-hyperideal of some ordered right Γ-hyperideal of S;

(v) A is an ordered right-Γ-hyperideal of some ordered (generalized) $(0,2)$-Γ-hyperideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose A is an ordered (generalized) $(1,2)$-Γ-hyperideal of S. This means A is a sub-Γ-semihypergroup (nonempty subset) of S and $A \circ \Gamma \circ S \circ \Gamma \circ A^2 \subseteq A$. Therefore,

$$\begin{aligned}(A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \circ \Gamma \circ A &= (B^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \circ \Gamma \circ (A] \\ &\subseteq (A^3 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^3] \\ &\subseteq (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A] = A.\end{aligned}$$

Clearly, if $b \in A$, $s \in (S^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$ so that $s \leq b$ then, $s \in A$. Hence, A is an ordered left Γ-hyperideal of ordered (generalized) bi-Γ-hyperideal $(A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2]$ of S.

$(ii) \Rightarrow (iii)$. Suppose A is an ordered left Γ-hyperideal of some ordered (generalized) bi-Γ-hyperideal B of S. Recall that $(A \cup S \circ \Gamma \circ A]$ is an ordered left Γ-hyperideal of S. According to our hypothesis,

$$\begin{aligned} A \circ (A \cup S \circ \Gamma \circ A] \circ B &\subseteq (A] \circ \Gamma \circ (A \cup S \circ \Gamma \circ A] \circ \Gamma \circ (A] \\ &\subseteq (A^3 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A \cup A \circ \Gamma \circ S \circ \Gamma \circ A \circ \Gamma \circ A] \\ &\subseteq (A \cup A \circ \Gamma \circ A] \\ &\subseteq (A] \\ &= A. \end{aligned}$$

Suppose $b \in A$, $s \in (A \cup S \circ \Gamma \circ A]$ such that $s \leq b$. As, $b \in A$, $b \in B$. So, $s \in B$ and therefore, $s \in A$. Hence, A is an ordered (generalized) bi-Γ-hyperideal of left ordered hyperideal $(A \cup S \circ \Gamma \circ A]$ of S.

$(iii) \Rightarrow (iv)$. Suppose A is an ordered (generalized) bi-Γ-hyperideal of some left ordered Γ-hyperideal L of S. This implies that $B \subseteq L$, $A \circ \Gamma \circ L^1 \circ \Gamma \circ B \subseteq A$ and $S \circ \Gamma \circ L \subseteq L$. Therefore,

$$\begin{aligned} (A \cup A \circ \Gamma \circ S] \circ \Gamma \circ A^2 &\subseteq (A \cup A \circ \Gamma \circ S] \circ (A^2] \\ &\subseteq (A^3 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A \cup A \circ \Gamma \circ S \circ L \circ \Gamma \circ A] \\ &\subseteq (A \cup A \circ \Gamma \circ L \circ \Gamma \circ A] \\ &\subseteq (A] = A. \end{aligned}$$

Furthermore, suppose that $b \in A$, $s \in (A \cup A \circ \Gamma \circ S]$ such that $s \leq b$, so $b \in L$. Then, $s \in L$, therefore, $s \in A$. Hence, A is an ordered (generalized) $(0, 2)$-Γ-hyperideal of the ordered right Γ-hyperideal $(A \cup A \circ \Gamma \circ S]$ of S.

$(iv) \Rightarrow (v)$. Suppose A is an ordered (generalized) $(0, 2)$-Γ-hyperideal of some ordered right Γ-hyperideal R of S. This implies that $A \subseteq R$, $R \circ \Gamma \circ A^2 \subseteq A$ and $R \circ \Gamma \circ S \subseteq R$. Then,

$$\begin{aligned} A \circ \Gamma \circ (A \cup S \circ \Gamma \circ A^2] &\subseteq (A] \circ \Gamma \circ (A \cup S \circ \Gamma \circ A^2] \\ &\subseteq (A^2 \cup A \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A \cup R \circ \Gamma \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A \cup R \circ \Gamma \circ A^2] \\ &\subseteq (A] = A. \end{aligned}$$

Let $b \in A$, $s \in (A \cup S \circ \Gamma \circ A^2]$ such that $s \leq b$. Then, $b \in R$, so $s \in R$, thus $s \in B$. Hence, B is an ordered right Γ-hyperideal of the (generalized) $(0, 2)$-Γ-hyperideal $(B \cup S \circ \Gamma \circ B^2]$ of S.

$(v) \Rightarrow (i)$. Suppose A is an ordered right Γ-hyperideal of an ordered (generalized) $(0, 2)$-Γ-hyperideal R of S. This further shows that $A \subseteq R$, $A \circ \Gamma \circ R \subseteq A$ and $S \circ \Gamma \circ R^2 \subseteq R$. Then, we have the following:

$$\begin{aligned} A \circ S \circ \Gamma \circ A^2 &\subseteq A \circ \Gamma \circ S \circ \Gamma \circ R^2 \\ &\subseteq A \circ R \\ &\subseteq A. \end{aligned}$$

Suppose $b \in A$, $s \in S$ such that $s \leq b$. Since $b \in R$, so $s \in B$. Therefore, A is an ordered (generalized) $(1, 2)$-Γ-hyperideal of S. Hence, A is an ordered (generalized) bi-Γ-hyperideal of S.

Lemma 2.3. A sub-Γ-semihypergroup (nonempty subset) A of an ordered Γ-semihypergroup S such that $A = (A]$ is an ordered (generalized) $(1, 2)$-Γ-hyperideal of S if and only if there exists an ordered (generalized) $(0, 2)$-Γ-hyperideal L of S and an ordered right Γ-hyperideal R of S so that $R \circ \Gamma \circ L^2 \subseteq A \subseteq R \cap L$.

Proof. Suppose A is an ordered (generalized)$(1,2)$-Γ-hyperideal of S. We know that $(A\cup S\circ\Gamma\circ A^2]$ and $(A\cup A\circ\Gamma\circ S]$ are an ordered (generalized) $(0,2)$-Γ-hyperideal and an ordered right Γ-hyperideal of S, respectively. Furthermore, assume $L=(A\cup S\circ\Gamma\circ A^2]$ and $R=(A\cup A\circ\Gamma\circ S]$. Then, we have the following:

$$\begin{aligned}
R\circ\Gamma\circ L^2 &\subseteq (A^3\cup A^2\circ\Gamma\circ S\circ\Gamma\circ A^2\cup A\circ\Gamma\circ S\circ\Gamma\circ A^2\cup A\circ\Gamma\circ S\circ\Gamma\circ A\circ\Gamma\circ S\circ\Gamma\circ A^2]\\
&\subseteq (A^3\cup A\circ\Gamma\circ S\circ\Gamma\circ A^2]\\
&\subseteq (A]=A.
\end{aligned}$$

Hence, $R\subseteq R\cap L$.

Conversely, suppose R is an ordered right Γ-hyperideal of S and L is an ordered (generalized) $(0,2)$-Γ-hyperideal of S so that $R\circ\Gamma\circ L^2\subseteq A\subseteq R\cap L$. Then, we have the following:

$$\begin{aligned}
A\circ\Gamma\circ S\circ\Gamma\circ A^2 &\subseteq (R\cap L)\circ\Gamma\circ\Gamma\circ S\circ\Gamma\circ(R\cap L)\circ\Gamma\circ\Gamma\circ(R\cap L)\\
&\subseteq R\circ\Gamma\circ S\circ\Gamma\circ L^2\\
&\subseteq R\circ\Gamma\circ L^2\\
&\subseteq A.
\end{aligned}$$

Hence, A is an ordered (generalized) $(1,2)$-Γ-hyperideal of S.

Definition 2.4. An ordered (generalized) $(0,2)$-bi-Γ-hyperideal B of S is called 0-minimal if $B\neq\{0\}$, $\{0\}$ is the only ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S properly contained in B.

Lemma 2.5. Suppose L is an ordered 0-minimal left Γ-hyperideal of an ordered Γ-semihypergroup S with 0 and I is a sub-Γ-semihypergroup (nonempty subset) of L such that $I=(I]$. Then, I is an ordered (generalized) $(0,2)$-Γ-hyperideal of S contained in L if and only if $(I\circ\Gamma\circ I]=\{0\}$ or $I=L$.

Proof. Suppose I is an ordered (generalized) $(0,2)$-Γ-hyperideal of S contained in L. As $(S\circ\Gamma\circ I^2]$ is an ordered left Γ-hyperideal of S and $(S\circ\Gamma\circ I^2]\subseteq I\subseteq L$, we obtain the following:
$(S\circ\Gamma\circ I^2]=\{0\}$ or $(S\circ\Gamma\circ I^2]=\{L\}$.
If $(S\circ\Gamma\circ I^2]=L$, then $L=(S\circ\Gamma\circ I^2]\subseteq(I]$. So, $I=L$. Suppose $(S\circ I^2]=\{0\}$. As $S\circ(I^2]\subseteq(S\circ\Gamma\circ I^2]=\{0\}\subseteq(I^2]$, then $(I^2]$ is an ordered left Γ-hyperideal of S contained in L. By the minimality of L, we obtain $(I^2]=\{0\}$ or $(I^2]=L$. If $(I^2]=L$, then $I=L$. Therefore, $I^2=\{0\}$ or $I=L$.
The converse part is straightforward.

Lemma 2.6. Suppose M is an ordered 0-minimal (generalized) $(0,2)$-Γ-hyperideal of an ordered Γ-semihypergroup S with a zero 0. Then $(M^2]=\{0\}$ or M is an ordered 0-minimal left Γ-hyperideal of S.

Proof. Since $M^2\subseteq M$ and

$$\begin{aligned}
S\circ\Gamma\circ(M^2]^2 &= S\circ\Gamma\circ(M^2]\circ\Gamma\circ(M^2]\\
&\subseteq (S\circ\Gamma\circ M^2]\circ\Gamma\circ(M^2]\\
&\subseteq (M]\circ\Gamma\circ(M^2]\\
&\subseteq (M^2].
\end{aligned}$$

Then, we obtain $(M^2]$ is an ordered (generalized) $(0,2)$-Γ-hyperideal of S contained in M. Therefore, $(M^2]=\{0\}$ or $(M^2]=M$. Suppose $(M^2]=M$. Since

$$\begin{aligned}
S\circ M &= S\circ\Gamma\circ(M^2]\\
&\subseteq (S\circ\Gamma\circ M^2]\\
&\subseteq (M]=M.
\end{aligned}$$

It follows that M is an ordered left Γ-hyperideal of S. Suppose B is an ordered left Γ-hyperideal of S contained in M. Therefore,

$$\begin{aligned} S \circ \Gamma \circ B^2 &\subseteq B^2 \\ &\subseteq B \\ &\subseteq M. \end{aligned}$$

Hence, B is an ordered (generalized) $(0,2)$-Γ-hyperideal of S contained in M and so, $B = \{0\}$ or $B = M$.

Corollary 2.7 Suppose S is an ordered Γ-semihypergroup without a zero 0. Then, M is an ordered minimal (generalized) $(0,2)$-Γ-hyperideal of S if and only if M is an ordered minimal left Γ-hyperideal of S.

Proof. It follows by Lemma 2.5 and Lemma 2.6.

Lemma 2.8. Suppose S is an ordered Γ-semihypergroup without a zero 0. Further, suppose that M is a nonempty subset of S. Then, the following results are equivalent:

(i) M is an ordered (generalized) minimal $(2,1)$-Γ-hyperideal of S;

(ii) M is an ordered (generalized) minimal bi-Γ-hyperideal of S.

Proof. Suppose S is an ordered Γ-semihypergroup without zero and M is an ordered minimal (generalized) $(2,1)$-Γ-hyperideal of S. Then, $(M^2 \circ \Gamma \circ S \circ \Gamma \circ M] \subseteq M$ and so $(M^2 \circ \Gamma \circ S \circ \Gamma \circ M]$ is an ordered (generalized) $(2,1)$-Γ-hyperideal of S. Therefore, we obtain $(M^2 \circ \Gamma \circ S \circ \Gamma \circ M] = M$. As

$$\begin{aligned} M \circ \Gamma \circ S \circ \Gamma \circ M &= (M^2 \circ \Gamma \circ S \circ \Gamma \circ M] \circ \Gamma \circ S \circ \Gamma \circ M \\ &\subseteq (M^2 \circ \Gamma \circ S \circ \Gamma \circ M \circ \Gamma \circ S \circ \Gamma \circ M] \\ &\subseteq (M^2 \circ \Gamma \circ S \circ \Gamma \circ M] = M, \end{aligned}$$

we have that M is an ordered (generalized) bi-Γ-hyperideal of S. Let there exist an ordered (generalized) bi-Γ-hyperideal A of S contained in M. Then, $A^2 \circ S \circ A \subseteq A^2 \subseteq A \subseteq M$, therefore, A is an ordered (generalized) $(2,1)$-Γ-hyperideal of S contained in M. Using the minimality of M, we obtain $A = M$.

Conversely, suppose M is an ordered minimal (generalized) bi-Γ-hyperideal of S. Then, M is an ordered (generalized) $(2,1)$-Γ-hyperideal of S. Suppose T is an ordered (generalized) $(2,1)$-hyperideal of S contained in M. As

$$\begin{aligned} (T^2 \circ \Gamma \circ S \circ \Gamma \circ T] \circ \Gamma \circ S \circ \Gamma \circ (T^2 \circ S \circ T] &\subseteq (T^2 \circ (S \circ T \circ \Gamma \circ S \circ \Gamma \circ T^2 \circ \Gamma \circ S) \circ \Gamma \circ T] \\ &\subseteq (T^2 \circ \Gamma \circ S \circ \Gamma \circ T], \end{aligned}$$

we obtain $(T^2 \circ \Gamma \circ S \circ \Gamma \circ T]$ is an ordered (generalized) bi-Γ-hyperideal of S. This shows that $(T^2 \circ \Gamma \circ S \circ \Gamma \circ T] = M$. As $M = (T^2 \circ \Gamma \circ S \circ \Gamma \circ T] \subseteq (T] = T$, $M = T$. Hence, M is an ordered minimal (generalized) $(2,1)$-Γ-hyperideal of S.

Definition 2.9. A sub-Γ-semihypergroup (nonempty subset) A of an ordered Γ-semihypergroup S is called an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S if A is an ordered (generalized) bi-Γ-hyperideal of S and also an ordered (generalized) $(0,2)$-Γ-hyperideal of S.

Lemma 2.10. Suppose A is a subset of an ordered Γ-semihypergroup S. Then, the following conditions are equivalent:

(i) B is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S;

(ii) B is an ordered Γ-hyperideal of some ordered left Γ-hyperideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose A is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S. This shows that $A \circ \Gamma \circ S \circ \Gamma \circ B \subseteq B$ and $S \circ \Gamma \circ A^2 \subseteq A$. Then, we have

$$\begin{aligned} S \circ (A^2 \cup S \circ \Gamma \circ A^2] &\subseteq (S \circ \Gamma \circ A^2 \cup S^2 \circ \Gamma \circ A^2] \\ &\subseteq (S \circ A^2] \\ &\subseteq (A^2 \cup S \circ \Gamma \circ A^2] \end{aligned}$$

Therefore, $(A^2 \cup S \circ \Gamma \circ A^2]$ is an ordered left Γ-hyperideal of S. As

$$\begin{aligned} A \circ (A^2 \cup S \circ \Gamma \circ A^2] &\subseteq (A^3 \cup A \circ S \circ \Gamma \circ A^2] \\ &\subseteq (A] \\ &= A, \end{aligned}$$

$(A^2 \cup S \circ \Gamma \circ A^2] \circ \Gamma \circ B \subseteq (A^3 \cup S \circ \Gamma \circ A^3] \subseteq (A] = A$. Hence, A is an ordered Γ-hyperideal of left ordered hyperideal $(A^2 \cup S \circ \Gamma \circ A^2]$ of S.
$(ii) \Rightarrow (i)$. Suppose A is an ordered Γ-hyperideal of some ordered left Γ-hyperideal L of S. By Lemma 2.1, A is an ordered (generalized) $(0,2)$-Γ-hyperideal of S, and hence, A is an ordered (generalized) bi-Γ-hyperideal of S.

Theorem 2.11. Suppose A is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-hyperideal of an ordered Γ-semihypergroup S with a zero 0. Then, exactly one of the followings cases arises:

(i) $A = \{0, b\}$, $(b \circ \Gamma \circ S \circ \Gamma \circ b] = \{0\}$;

(ii) $A = (\{0, b\}]$, $b^2 = 0$, $(b \circ S \circ \Gamma \circ b] = A$;

(iii) $(S \circ \Gamma \circ b^2] = A$ for all $b \in A \setminus \{0\}$.

Proof. Suppose A is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-hyperideal of an ordered Γ-semihypergroup S. Furthermore, suppose $b \in A \setminus \{0\}$. Then, $(S \circ \Gamma \circ b^2] \subseteq A$ and $(S \circ \Gamma \circ b \circ \Gamma \circ b]$ is an ordered left Γ-hyperideal of S, therefore, $(S \circ b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S. Hence, $(S \circ \Gamma \circ b^2] = \{0\}$ or $(S \circ b^2] = A$.
Let $(S \circ \Gamma \circ b^2] = \{0\}$. As $b^2 \in A$, we obtain either $b^2 = b$ or $b^2 = 0$ or $b^2 \in A \setminus \{0, b\}$. If $b^2 = b$, then $b = 0$. This is a contradiction. Let $b^2 \in A \setminus \{0, b\}$. Then,

$$\begin{aligned} S \circ \Gamma \circ (\{0, b^2\}]^2 &\subseteq (\{0, S \circ \Gamma \circ b^2\}] = (\{0\}] \cup (S \circ \Gamma \circ b^2] \\ &= \{0\} \\ &\subseteq (\{0\} \cup b^2], \end{aligned}$$

$$\begin{aligned} (\{0\} \cup b^2] \circ \Gamma \circ S \circ \Gamma \circ (\{0\} \cup b^2] &\subseteq (b^2 \circ \Gamma \circ S \circ \Gamma \circ b^2] \\ &\subseteq (S \circ \Gamma \circ b^2] = \{0\} \\ &\subseteq \{0, b^2\}. \end{aligned}$$

So, $(\{0\} \cup b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S contained in A, and we obtain that $(\{0\} \cup b^2] \neq \{0\}$, $(\{0\} \cup b^2] \neq A$. This is also not possible as A is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-hyperideal of S. Therefore, $b^2 = \{0\}$ and hence by Lemma 2.10, $A = (\{0, b\}]$. Now, since we have $(b \circ \Gamma \circ S \circ \Gamma \circ b]$ is an ordered (generalized) $(0,2)$-bi-Γ-hyperideal of S contained in A, we get $(b \circ \Gamma \circ S \circ \Gamma \circ b] = \{0\}$ or $(b \circ \Gamma \circ S \circ \Gamma \circ b] = A$. So, $(S \circ \Gamma \circ b^2] = \{0\}$ and it implies that either $A = \{0, b\}$

and $(b \circ \Gamma \circ S \circ \Gamma \circ b] = \{0\}$ or $A = \{0, b\}$, $b^2 = \{0\}$ and $(b \circ \Gamma \circ S \circ \Gamma \circ b] = A$. If $(S \circ \Gamma \circ b^2] \neq \{0\}$, then $(S \circ \Gamma \circ b^2] = A$.

Corollary 2.12. Suppose B is an ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-hyperideal of an ordered Γ-semihypergroup S with a zero 0 so that $(B^2] \neq \{0\}$. Then, $B = (S \circ \Gamma \circ b^2]$ for every $b \in B \setminus \{0\}$.

Definition 2.13. An ordered Γ-semihypergroup S with a zero 0 is called 0-$(0, 2)$- bisimple if (i) $(S^2] \neq \{0\}$, and $\{0\}$ is the only ordered proper (generalized) $(0, 2)$-bi-Γ-hyperideal of S.

Corollary 2.14. An ordered Γ-semihypergroup S with a zero 0 is 0-$(0, 2)$-bisimple if and only if $(S \circ \Gamma \circ s^2] = S$ for every $s \in S \setminus \{0\}$.

Proof. If S is 0-$(0, 2)$-bisimple, then $(S \circ S] \neq \{0\}$ and S is an ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-hyperideal. By Corollary 2.12., we have $S = (S \circ \Gamma \circ s^2]$ for every $s \in S \setminus \{0\}$.

Conversely, suppose $S = (S \circ \Gamma \circ s^2]$ for every element $s \in S \setminus \{0\}$ and further suppose that A is an ordered (generalized) $(0, 2)$-bi-Γ-hyperideal of S such that $A \neq \{0\}$. Suppose $b \in A \setminus \{0\}$. Then, $S = (S \circ \Gamma \circ b^2] \subseteq (S \circ \Gamma \circ A^2] \subseteq (A] = A$, therefore $S = A$. Since, $S = (S \circ \Gamma \circ b^2] \subseteq (S \circ \Gamma \circ S] = (S^2]$, we obtain $\{0\} \neq S = (S \circ \Gamma \circ S] = (S^2]$. Hence, S is 0-$(0, 2)$-bi-simple. The proof is complete.

Theorem 2.15. An ordered Γ-semihypergroup S with a zero 0 is 0-$(0, 2)$-bisimple if and only if S is left 0-simple.

Proof. We recall that every ordered left Γ-hyperideal A of an ordered Γ-semihypergroup S is an ordered 0-$(0, 2)$-bi-Γ-hyperideal of S. So, $A = \{0\}$ or $A = S$. Therefore, if S is 0-$(0, 2)$-bisimple then S is left 0-simple.

Conversely, if S is left 0-simple then, $(S \circ s] = S$ for every $s \in S \setminus \{0\}$ from which it follows that

$$\begin{aligned} S &= (S \circ \Gamma \circ s] \\ &= ((S \circ \Gamma \circ s] \circ \Gamma \circ s] \\ &\subseteq ((S \circ \Gamma \circ s^2]] \\ &= (S \circ \Gamma \circ s^2]. \end{aligned}$$

Therefore, using Corollary 2.14, S is 0-$(0, 2)$-bisimple. The proof is complete.

Theorem 2.16. Suppose A is an ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-hyperideal of an ordered Γ-semihypergroup S. Then, either $(A \circ \Gamma \circ A] = \{0\}$ or A is left 0-simple.

Proof. Suppose $(A \circ A] \neq \{0\}$. Then, by Corollary 2.12, we obtain $(S \circ \Gamma \circ b^2] = A$ for every $b \in A \setminus \{0\}$. As $b^2 \in A \setminus \{0\}$ for every $b \in A \setminus \{0\}$, we obtain $b^4 = (b^2)^2 \in A \setminus \{0\}$. Suppose $b \in A \setminus \{0\}$. As, $(A \circ \Gamma \circ b^2] \circ \Gamma \circ S \circ \Gamma \circ (A \circ \Gamma \circ b^2] \subseteq (A \circ \Gamma \circ A \circ \Gamma \circ b^2] \subseteq (A \circ \Gamma \circ b^2]$ and

$$\begin{aligned} S \circ (A \circ \Gamma \circ b^2]^2 &\subseteq (S \circ \Gamma \circ A \circ \Gamma \circ b^2 \circ \Gamma \circ A \circ \Gamma \circ b^2] \\ &\subseteq (S \circ \Gamma \circ A^2 \circ \Gamma \circ b^2] \\ &\subseteq (A \circ \Gamma \circ b^2], \end{aligned}$$

we get that $(A \circ b^2]$ is an ordered (generalized) $(0, 2)$-bi-Γ-hyperideal of S contained in A. Therefore, $(A \circ \Gamma \circ b^2] = \{0\}$ or $(A \circ \Gamma \circ b^2] = A$. As, $b^4 \in A \circ b^2 \subseteq (A \circ b^2]$, and $b^4 \in A \setminus \{0\}$, we obtain $(A \circ \Gamma \circ b^2] = A$. By Corollary 2.14 and Theorem 2.15, it follows that A is left 0-simple. The proof is complete.

3 Conclusion

In the current paper, we enriched ideal theory in ordered Γ-semihypergroups. We derived various equivalent conditions related to ordered Γ-hyperideals, ordered (0, 2)-Γ-hyperideals, ordered bi-Γ-hyperideals. We introduced ordered (generalized) (m, n)-Γ-hyperideals in ordered Γ-semihypergroups. Then, we characterized ordered Γ-semihypergroup in terms of ordered (generalized) $(0, 2)$-Γ-hyperideals, ordered (generalized) $(1, 2)$-Γ-hyperideals and ordered (generalized) 0-minimal $(0, 2)$-Γ-hyperideals. Furthermore, we studied the notion of ordered (generalized) $(0, 2)$-bi-Γ-hyperideals, ordered 0-$(0, 2)$ bisimple ordered Γ-semihypergroups and ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-hyperideals in ordered Γ-semihyperoups. It is shown that an ordered Γ-semihypergroup S with a zero 0 is 0-$(0, 2)$-bisimple if and only if it is left 0-simple.

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