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Research Article

On the Dual Jacobsthal and Dual Jacobsthal-Lucas Sedenions

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Abctract

The sedenions form a 16-dimensional non-associative and non-commutative algebra over the set of . $\mathbb{R}$. The main object of this paper is to present a systematic investigation of new classes of sedenion numbers associated with the familiar Jacobsthal numbers. The various results obtained here for these classes of sedenion numbers include recurrence relations, Binet formula, generating function, exponentinal generating functions, poisson generating functions and also we presented the Cassini identity, Catalan's identities and d'Ocagne's identity by their Binet forms

Keywords: Sedenion numbers, Jacobsthal numbers, Dual numbers

Dual Jacobsthal ve Dual Jacobsthal-Lucas Sedeniyonlar Üzerine

Öz

Sedeniyonlar $\mathbb{R}$ üzerinde birleşmeli ve değişmeli olmayan 16 boyutlu bir cebirdir. Bu çalışmanın temel amacı bilinen Jacobsthal sayıları ile ilgili sedeniyon sayıların yeni bir sınıfını sunmaktır. Rekürans ilişkilerini içeren sedeniyon sayıların bu sınıfı için; Binet formülleri, üreteç fonksiyonlar, üstel üreteç fonksiyonlar, poisson üreteç fonksiyonlar gibi çeşitli sonuçlar elde edildi ve aynı zamanda bu sayıların Binet formülleri yardımıyla Cassini özdeşliği, Catalan özdeşlikleri ve d'Ocagne's özdeşliği sunuldu.

Anahtar Kelimeler: Sedeniyon sayılar, Jacobsthal sayılar, Dual sayılar.

1. Introduction and Preliminaries

Sedenions appear in many areas of science, such as electromagnetic theory and linear gravity. Sedenion algebra, which is usually denoted by $\mathbf{S}$, the structure of the sedenion algebra is a non-associative, non-commutative, and non-alternative but power-associative $16-$dimensional Cayley-Dickson algebra over the $\mathbb{R}$. Because of their zero divisors, sedenions do not form a composition algebra or a division algebra. They are hyper-complex numbers, similar to quaternions and octonions.

Throughout this paper, we take the basis elements of $\mathbf{S}$ as $\{e_0, e_1, ..., e_{15}\}$ where $e_1, e_2, ..., e_{15}$ are imaginaries. A sedenion $\mathbf{S}$ can be written as

$$\mathbf{S} = \sum_{i=0}^{15} a_i e_i \tag{1.1}$$

where $a_1, a_2, ..., a_{15}$ are reals.

By setting $i \equiv e_i$, where $i = 0,1,2,...,15$, (Cawagas, 2004) constructed multiplication table for the basis of $\mathbf{S}$.

In [Cariow and Cariowa, 2015], the authors derived an algorithm for the fast multiplication of two sedenions. In [Bilgici, et al., 2017], the authors defined as the following the Fibonacci and Lucas sedenions over the sedenion algebra $\mathbf{S}$.

The famous Fibonacci numbers are second order recursive relation and used in various disciplines. Some lesser known second order recursive relations are Lucas numbers, Pell and Pell-Lucas numbers, Jacobsthal and Jacobsthal-Lucas numbers, etc..

In [Horadam, 1996], Horadam defined the

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classic Jacobsthal numbers and Jacobsthal-Lucas numbers for all nonnegative integers, by

$$J_n = J_{n-1} + 2J_{n-2},\ J_0 = 0,\ J_1 = 1. \tag{1.2}$$

$$j_n = j_{n-1} + 2j_{n-2},\ j_0 = 2,\ j_1 = 1 \tag{1.3}$$

respectively.

For convenience initial Jacobsthal numbers and Jacobsthal-Lucas numbers are presented in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
J_n	0	1	1	3	5	11	21	43	85	171	341
j_n	2	1	5	7	17	31	65	127	257	511	1025

The following properties given for Jacobsthal numbers and Jacobsthal-Lucas numbers play important roles in this paper (see (Horadam, 1996)).

$$j_n J_n = J_{2n}, \tag{1.4}$$

$$J_n + j_n = 2J_{n+1}, \tag{1.5}$$

$$3J_n + j_n = 2^{n+1}, \tag{1.6}$$

$$j_{n+1} + 2j_{n-1} = 9j_n, \tag{1.7}$$

$$J_m j_n + J_n j_m = 2J_{n+m}, \tag{1.8}$$

$$J_n = \frac{1}{3}\left(2^n - (-1)^n\right), \tag{1.9}$$

$$j_n = 2^n + (-1)^n, \tag{1.10}$$

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}, \tag{1.11}$$

$$j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3.2^n, \tag{1.12}$$

$$\begin{aligned} j_{n+r} - j_{n-r} &= 3(J_{n+r} - J_{n-r}) \\ &= 2^{n+r} + 2^{n-r}, \end{aligned} \tag{1.13}$$

$$\begin{aligned} j_{n+1} - j_n &= 3(J_{n+1} - J_n) + 4(-1)^{n+1} \\ &= 2^n + 2(-1)^{n+1}, \end{aligned} \tag{1.14}$$

$$\begin{aligned} j_{n+r} + j_{n-r} &= 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} \\ &= 2^{n+r} + 2^{n-r} + 2.(-1)^{n-r} \end{aligned} \tag{1.15}$$

The set of dual numbers

$$D = \left\{a = a + \varepsilon a^* : \varepsilon \neq 0,\ \varepsilon^2 = 0,\ a, a^* \in \mathbb{R}\right\}$$

is a commutative ring with a unit. Dual numbers were introduced by Clifford (Clifford, 1873) in the 19th century.

They were applied to describe rigid body motions in three- dimensional space by Koltelnikov (Koltelnikov, 1895). With the help of dual numbers, Yaglom (Yaglom, 1879) described geometrical objects in three-dimensional space. The notion of dual angle is defined by Study (Study, 1903). Recently, dual numbers have found applications in many areas such as in kinematics, dynamics, robotics, computer aided geometrical design, mechanism design and modeling of rigid bodies, group theory, and field theory.

Since a dual quaternion is usually described as a quaternion with dual numbers as coefficient, dual Fibonacci quaternions and octonions can be defined in a similar way. That is a dual quaternion P can be written as

$$P = p + q\varepsilon, \quad \varepsilon^2 = 0,\ \varepsilon \neq 0,\ p, q \in \mathbf{H}. \tag{1.16}$$

Like dual quaternions, dual octonions are also useful tool for geometry (Kabadayi, 2016) and electromagnetism [Bhupesh, et al., 2016].

Any dual octonion can be defined as

$$K = (k, l) = k + l\varepsilon, \varepsilon^2 = 0, \varepsilon \neq 0,\ k, l \in \mathbf{O}. \tag{1.17}$$

In [Halici, 2015], Halici investigated some fundamental algebraic properties of the dual Fibonacci octonions and quaternions and also give the Binet formulas and the generating functions of them. In [Ünal, et al., 2017], the authors investigated dual Fibonacci and dual Lucas octonions and also obtain some identities for these sequences such as Catalan's, Cassini's and d'Ocagne's identities.

In [Cimen and Ipek, 2017a], Cimen and İpek defined the Jacobsthal and Jacobsthal-Lucas octonions over the octonion algebra. They present generating functions and Binet formulas for the Jacobsthal and Jacobsthal-Lucas octonions, and derive some identities of Jacobsthal and Jacobsthal-Lucas octonions. In [Cimen and İpek, 2017b], Cimen and Ipek defined the Jacobsthal and Jacobsthal-Lucas sedenions over the sedenion algebra $\mathbf{S}$. Also, they present generating functions and Binet formulas for the Jacobsthal and Jacobsthal-Lucas sedenions, and derive some identities of Jacobsthal and Jacobsthal-Lucas sedenions.

In this study, we are mainly interested dual Jacobsthal sequence as a generalization of linear recurrence equations of order two. The various results obtained here for these classes of sedenion numbers include recurrence relations, Binet formula, generating function, exponentinal generating functions, poisson generating functions and also we presented the Cassini identity, sum formula and norm formula by their Binet forms.

2. Algebraic Properties of the Dual Jacobsthal and Dual Jacobsthal-Lucas Sedenions

In this section, we define new kinds of sequences of Jacobsthal and Jacobsthal-Lucas numbers called as Jacobsthal and Jacobsthal-Lucas Sedenions. We give some properties of these sedenions. Also, we investigate Binet formula, generating function, exponentinal generating functions, poisson generating functions, Cassini identity, summation formula and norm value for dual Jacobsthal and Jacobsthal-Lucas Sedenions.

Now, in the following, we define the *nth* dual Jacobsthal sedenion and dual Jacobsthal-Lucas sedenion numbers, respectively, by the following recurrence relations:

$$\begin{aligned} SJ_n &= (SJ_n, SJ_{n+1}) = SJ_n + SJ_{n+1}\varepsilon \\ &= (J_n + J_{n+1}\varepsilon)e_0 + (J_{n+1} + J_{n+2}\varepsilon)e_1 \\ &\quad +\ldots+(J_{n+15} + J_{n+16}\varepsilon)e_{15}, \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} Sj_n &= (Sj_n, Sj_{n+1}) = Sj_n + Sj_{n+1}\varepsilon \\ &= (j_n + j_{n+1}\varepsilon)e_0 + (j_{n+1} + j_{n+2}\varepsilon)e_1 \\ &\quad +\ldots+(j_{n+15} + j_{n+16}\varepsilon)e_{15}, \end{aligned} \tag{1.19}$$

where J_n and j_n are the *nth* Jacobsthal number and Jacobsthal-Lucas number.

Let SG_n and SM_n be two dual Jacobsthal sedenions such that $SJ_n = SJ_n + SJ_{n+1}\varepsilon,$ and $QM_n = QM_n + QM_{n+1}\varepsilon$. Therefore, the addition, substraction of these sedenions directly are obtained by from (1.12), (1.18), respectively, as following

$$\begin{aligned} SJ_n \pm QM_n &= (SJ_n + QM_n) \\ &\quad + (SJ_{n+1} + QM_{n+1})\varepsilon. \end{aligned} \tag{1.20}$$

The conjugate of SJ_n and Sj_n are defined by

$$\overline{SJ_n} = \overline{SJ_n} + \overline{SJ_{n+1}}\varepsilon, \tag{1.21}$$

$$\overline{Sj_n} = \overline{Sj_n} + \overline{Sj_{n+1}}\varepsilon \tag{1.22}$$

respectively.

The norm of SJ_n is defined by

$$\begin{aligned} N_{SJ_n} &= SJ_n\overline{SJ_n} \\ &= (SJ_n + SJ_{n+1}\varepsilon)\left(\overline{SJ_n} + \overline{SJ_{n+1}}\varepsilon\right). \end{aligned} \tag{1.23}$$

Lemma 1 For $n \geq 1,$ we have the following identities:

$$SJ_n + \overline{SJ_n} = 2SJ_n, \tag{1.24}$$

$$SJ_n^2 + SJ_n.\overline{SJ_n} = 2SJ_nSJ_n. \tag{1.25}$$

Proof. From (1.18), (1.20) and (1.21), we get

$$\begin{aligned} SJ_n + \overline{SJ_n} &= SJ_n + SJ_{n+1}\varepsilon + SJ_n - SJ_{n+1}\varepsilon \\ &= 2SJ_n. \end{aligned}$$

On the other hand, from the equation (1.24), we have

$$\begin{aligned} SJ_n^{\ 2} &= SJ_n.SJ_n = SJ_n\left(2SJ_n - \overline{SJ_n}\right) \\ &= 2SJ_nSJ_n - SJ_n.\overline{SJ_n} \end{aligned}$$

and so

$$SJ_n^2 + SJ_n.\overline{SJ_n} = 2SJ_nSJ_n.$$

Lemma 2 For $n \geq 1,$ we have the following identities:

$$SJ_n + Sj_n = 2SJ_{n+1}, \tag{1.26}$$

$$3SJ_n + Sj_n = 2^{n+1}\left(e_0 + 2e_1 + \ldots + 2^{15}e_{15}\right)(1+2\varepsilon), \tag{1.27}$$

$$Sj_{n+1} + 2Sj_{n-1} = 9SJ_n. \tag{1.28}$$

Proof. By using of the equalities (1.18), (1.19) and (1.20), we obtain,

$$\begin{aligned} SJ_n + Sj_n &= SJ_n + SJ_{n+1}\varepsilon + Sj_n + Sj_{n+1}\varepsilon \\ &= (SJ_n + Sj_n) + (SJ_{n+1} + Sj_{n+1})\varepsilon \\ &= \big[(J_n + j_n)e_0 + (J_{n+1} + j_{n+1})e_1 \\ &\quad +\ldots+(J_{n+15} + j_{n+15})e_{15}\big] \\ &\quad + \big[(J_{n+1} + j_{n+1})e_0 + (J_{n+2} + j_{n+2})e_1 \\ &\quad +\ldots+(J_{n+16} + j_{n+16})e_{15}\big]\varepsilon \\ &= (2J_{n+1}e_0 + 2J_{n+2}e_1 + \ldots + 2J_{n+16}e_{15}) \\ &\quad + (2J_{n+2}e_0 + 2J_{n+3}e_1 + \ldots + 2J_{n+17}e_{15})\varepsilon \\ &= 2(SJ_{n+1} + SJ_{n+2}\varepsilon) \\ &= 2\widetilde{SJ_{n+1}}. \end{aligned}$$

In a similar way we can show the second equality. By using of the identities $3J_n + j_n = 2^{n+1}$ and $3SJ_n + Sj_n = 2^{n+1}\left(e_0 + 2e_1 + 2^2e_2 + \ldots + 2^{15}e_{15}\right)$ are obtained in [Horadam(1996), Cimen and İpek (2017b)], we have

$3SJ_n + Sj_n = 2^{n+1}\left(e_0 + 2e_1 + 2^2e_2 + \ldots + 2^{15}e_{15}\right)(1+2\varepsilon),$ which is the assertion. By using of the identities $j_{n+1} + 2j_{n-1} = 9J_n$ and

$Sj_n + 2Sj_{n-1} = 9\left(J_n e_0 + J_{n+1} e_1 + ... + J_{n+15} e_{15}\right)$ in [Horadam (1996), Cimen and İpek (2017b)], we obtain

$$\begin{aligned} Sj_{n+1} + 2Sj_{n-1} &= \left[\left(j_{n+1} + 2j_{n-1}\right)e_0 + \left(j_{n+2} + 2j_n\right)e_1 \right. \\ &\quad \left. + ... + \left(j_{n+16} + 2j_{n+14}\right)e_{15}\right] \\ &\quad + \left[\left(j_{n+2} + 2j_n\right)e_0 + \left(j_{n+3} + 2j_{n+1}\right)e_1 \right. \\ &\quad \left. + ... + \left(j_{n+17} + 2j_{n+15}\right)e_{15}\right]\varepsilon \\ &= 9\left(J_n e_0 + J_{n+1} e_1 + ... + J_{n+15} e_{15}\right) \\ &\quad + 9\left(J_{n+1} e_0 + J_{n+3} e_1 + ... + J_{n+16} e_{15}\right)\varepsilon \\ &= 9\left(SJ_n + S\hat{J}_{n+1}\varepsilon\right) \\ &= 9SJ_n \end{aligned}$$

which is the assertion.

The characteristic equation of the classic dual Jacobsthal and dual Jacobsthal-Lucas numbers is

$$x^2 - x - 2 = 0. \qquad (1.29)$$

It is known that this equation has two real roots:

$$\alpha = 2 \text{ and } \beta = -1. \qquad (1.30)$$

Thus, Binet's formulas given in (1.31) and (1.32) are obtained for the dual Jacobsthal and dual Jacobsthal-Lucas sedenions such that

$$SJ_n = \frac{2^n}{3}A - \frac{(-1)^n}{3}B \qquad (1.31)$$

and

$$Sj_n = 2^n A + (-1)^n B, \qquad (1.32)$$

where $A = \sum_{s=0}^{15} 2^s e_s$ and $B = \sum_{s=0}^{15} (-1)^s e_s$ respectively.(Cimen and İpek (2017b)) Now, we will state the Binet's formulas for the dual Jacobsthal sedenions and dual Jacobsthal-Lucas sedenions,

$$\begin{aligned} SJ_n &= SJ_n + SJ_{n+1}\varepsilon \\ &= \sum_{s=0}^{15} J_{n+s} e_s + \sum_{s=0}^{15} J_{n+1+s} e_s \varepsilon \\ &= \left(\sum_{s=0}^{15} \frac{1}{3}\left(2^{n+s} - (-1)^{n+s}\right)e_s\right) \\ &\quad + \left(\sum_{s=0}^{15} \frac{1}{3}\left(2^{n+1+s} - (-1)^{n+1+s}\right)e_s\right)\varepsilon \\ &= \left(\frac{2^n}{3}A - \frac{(-1)^n}{3}B\right) + \left(\frac{2^{n+1}}{3}A - \frac{(-1)^{n+1}}{3}B\right)\varepsilon \end{aligned} \qquad (1.33)$$

and

$$\begin{aligned} Sj_n &= Sj_n + Sj_{n+1}\varepsilon \\ &= \sum_{i=0}^{15} j_{n+s} e_s + \sum_{i=0}^{15} j_{n+s+1} e_s \varepsilon \\ &= \left(\sum_{s=0}^{15} \left(2^{n+s} + (-1)^{n+s}\right)e_s\right) \\ &\quad + \left(\sum_{s=0}^{15} \left(2^{n+1+s} + (-1)^{n+1+s}\right)e_s\right)\varepsilon \\ &= \left(2^n A + (-1)^n B\right) + \left(2^{n+1} A + (-1)^{n+1} B\right)\varepsilon. \end{aligned} \qquad (1.34)$$

where $A = \sum_{s=0}^{15} 2^s e_s$ and $B = \sum_{s=0}^{15} (-1)^s e_s$, respectively.

The recurrence relations for the Jacobsthal sedenions of the *nth* dual Jacobsthal sedenion and *nth* dual Jacobsthal-Lucas sedenion are expressed in the following corollary.

Corollary 1 For $n \geq 1, r \geq 1,$ we have the following identities:

$$SJ_{n+1} + SJ_n = 2^n\left(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15}\right)(1 + 2\varepsilon), \qquad (1.35)$$

$$\begin{aligned} SJ_{n+1} - SJ_n &= \frac{2^n\left(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15}\right)}{3}(1 + 2\varepsilon) \\ &\quad + \frac{2(-1)^n\left(e_0 - e_1 + e_2 - e_3 + ... - e_{15}\right)}{3}(1 - \varepsilon) \end{aligned} \qquad (1.36)$$

$$\begin{aligned} SJ_{n+r} + SJ_{n-r} &= \frac{2^{n-r}\left(2^{2r} + 1\right)}{3} \\ &\quad \left(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15}\right)(1 + 2\varepsilon) \end{aligned} \qquad (1.37)$$

$$\begin{aligned} SJ_{n+r} - SJ_{n-r} &= \left(\frac{2^{n+r} - 2^{n-r}}{3}\right) \\ &\quad \left(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15}\right)(1 + 2\varepsilon). \end{aligned} \qquad (1.38)$$

Proof. Write considering the equalities (1.18) and (1.20) and using of the equality $j_{n+1} + j_n = 3\left(J_{n+1} + J_n\right) = 3.2^n$, the following sum can be calculated

$$\begin{aligned} SJ_{n+1} + SJ_n &= \left(SJ_{n+1} + SJ_{n+2}\varepsilon\right) + \left(SJ_n + SJ_{n+1}\varepsilon\right) \\ &= \left(SJ_{n+1} + SJ_n\right) + \left(SJ_{n+2} + SJ_{n+1}\right)\varepsilon \\ &= 2^n\left(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15}\right) \\ &\quad + 2^{n+1}\left(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15}\right)\varepsilon \\ &= 2^n\left(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15}\right)(1 + 2\varepsilon). \end{aligned}$$

In a similar way we can show the equation (1.36). By using of the equalities (1.18) and (1.20) and

$$SJ_{n+1}-SJ_n=\tfrac{1}{3}\Big[2^n\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right)+2(-1)^n\left(e_0-e_1+e_2-e_3+...-e_{15}\right)\Big]$$

(Cimen and İpek (2017b)), we have

$$\begin{aligned}SJ_{n+1}-SJ_n&=\left(SJ_{n+1}+SJ_{n+2}\varepsilon\right)-\left(SJ_n+SJ_{n+1}\varepsilon\right)\\&=\left(SJ_{n+1}-SJ_n\right)+\left(SJ_{n+2}-SJ_{n+1}\right)\varepsilon\\&=\frac{1}{3}\Big[2^n\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right)(1+2\varepsilon)\\&\quad+2(-1)^n\left(e_0-e_1+e_2-e_3+...-e_{15}\right)(1-\varepsilon)\Big]\end{aligned}$$

and

$$\begin{aligned}SJ_{n+r}+SJ_{n-r}&=\left(SJ_{n+r}+SJ_{n+r+1}\varepsilon\right)+\left(SJ_{n-r+1}+SJ_{n-r+2}\varepsilon\right)\\&=\left(SJ_{n+r}+SJ_{n-r+1}\right)+\left(SJ_{n+r+1}+SJ_{n-r+2}\right)\varepsilon\\&=\Big[\tfrac{2^{n-r}\left(2^{2r}+1\right)}{3}\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right)\\&\quad+\tfrac{2(-1)^{n-r+1}}{3}\left(e_0-e_1+e_2-e_3+e_4-...-e_{15}\right)\Big]\\&\quad+\Big[\tfrac{2^{n-r+1}\left(2^{2r}+1\right)}{3}\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right)\\&\quad+\tfrac{2(-1)^{n-r+2}}{3}\left(e_0-e_1+e_2-e_3+e_4-...-e_{15}\right)\Big]\varepsilon\\&=\frac{2^{n-r}\left(2^{2r}+1\right)}{3}\left(e_0+2e_1+...+2^{15}e_{15}\right)(1+2\varepsilon)\\&\quad+\frac{2(-1)^{n-r+1}}{3}\left(e_0-e_1+e_2-...-e_{15}\right)(1-\varepsilon)\end{aligned}$$

and thus

$$\begin{aligned}SJ_{n+r}-SJ_{n-r}&=\left(SJ_{n+r}+SJ_{n+r+1}\varepsilon\right)-\left(SJ_{n-r}+SJ_{n-r+1}\varepsilon\right)\\&=\left(SJ_{n+r}-SJ_{n-r}\right)+\left(SJ_{n+r+1}-SJ_{n-r+1}\right)\varepsilon\\&=\left(\frac{2^{n+r}-2^{n-r}}{3}\right)\left(e_0+2e_1+...+2^{15}e_{15}\right)\\&\quad+\left(\frac{2^{n+r+1}-2^{n-r+1}}{3}\right)\left(e_0+2e_1+...+2^{15}e_{15}\right)\varepsilon\\&=\left(\frac{2^{n+r}-2^{n-r}}{3}\right)\\&\quad\left(e_0+2e_1+...+2^{15}e_{15}\right)(1+2\varepsilon).\end{aligned}$$

Corollary 2 For $n\geq 1,\ r\geq 1,$ we have the following identities:

$$Sj_{n+1}+Sj_n=3.2^n\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right)(1+2\varepsilon), \tag{1.39}$$

$$\begin{aligned}Sj_{n+1}-Sj_n&=2^n\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right)\\&\quad+2(-1)^{n+1}\left(e_0-e_1+e_2-e_3+...-e_{15}\right)\\&\quad+\Big[2^{n+1}\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right)\\&\quad+2(-1)^n\left(e_0-e_1+e_2-e_3+...-e_{15}\right)\Big]\varepsilon,\end{aligned} \tag{1.40}$$

$$\begin{aligned}Sj_{n+r}+Sj_{n-r}&=2^{n-r}\left(2^{2r}+1\right)\Big(e_0+2e_1\\&\quad+...+2^{15}e_{15}\Big)(1+2\varepsilon)\\&\quad+2(-1)^{n-r+1}(e_0-e_1+e_2\\&\quad-...-e_{15})(1-\varepsilon),\end{aligned} \tag{1.41}$$

$$Sj_{n+r}-Sj_{n-r}=\left(2^{n+r}-2^{n-r}\right)\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right)(1+2\varepsilon). \tag{1.42}$$

Proof. The proof of the identities (1.39)-(1.42) of this corollary are similar to the proofs of the identities of Corollary 1, respectively, and are omitted here.

In the following theorem, we state to different Cassini identities which occur from non-commutativity of sedenion multiplication.

Theorem 1 For dual Jacobsthal sedenions and dual Jacobsthal-Lucas sedenions the following identities are hold*:*

$$\begin{aligned}SJ_{n+1}.SJ_{n-1}-SJ_n{}^2&=\frac{2^n(-1)^n}{3}\left[AB+\frac{BA}{2}\right]\\&\quad+\frac{1}{3}\Big[2^{n+1}A^2+(-1)^nB^2\\&\quad-\left(4AB-\frac{BA}{2}\right)\Big]\varepsilon,\end{aligned} \tag{1.43}$$

$$\begin{aligned}SJ_{n-1}.SJ_{n+1}-SJ_n{}^2&=\frac{2^n(-1)^n}{3}\left[\frac{AB}{2}+BA\right]\\&\quad+\frac{1}{9}\Big(2^{2n+1}A^2-B^2\\&\quad-2^n(-1)^n\left(\frac{AB}{2}+4BA\right)\Big)\varepsilon,\end{aligned} \tag{1.44}$$

$$Sj_{n+1}.Sj_{n-1}-Sj_n{}^2=3.2^{n-1}(-1)^{n+1}\left[2AB+BA\right](1-\varepsilon), \tag{1.45}$$

and

$$\begin{aligned}Sj_{n-1}.Sj_{n+1}-Sj_n{}^2&=3.2^{n-1}(-1)^{n+1}\\&\quad\left[AB+2BA\right](1+\varepsilon)\end{aligned} \tag{1.46}$$

where $A=\sum_{s=0}^{15}2^se_s$ *and* $B=\sum_{s=0}^{15}(-1)^se_s$.

Proof. Using of the Binet's formula for Jacobsthal sedenions in equation (1.43) and $SJ_{n+1}.SJ_{n-1}-SJ_n^2=\frac{2^n(-1)^n}{3}\left[AB+\frac{BA}{2}\right]$ (Cimen and İpek (2017b)), we get

$$\begin{aligned}SJ_{n+1}.SJ_{n-1}-SJ_n^{\;2} &= (SJ_{n+1}+SJ_{n+2}\varepsilon).(SJ_{n-1}+SJ_n\varepsilon)\\ &\quad -(SJ_n+SJ_{n+1}\varepsilon)\\ &= SJ_{n+1}.SJ_{n-1}-SJ_n^2\\ &\quad +(SJ_{n+2}.SJ_{n-1}-SJ_nSJ_{n+1})\varepsilon\\ &= \frac{2^n(-1)^n}{3}\left[AB+\frac{BA}{2}\right]\\ &\quad +\frac{2^n(-1)^n}{9}\left[2^{n+1}A^2-(-1)^{n+1}B^2\right.\\ &\quad \left.-\left(4AB-\frac{BA}{2}\right)\right]\varepsilon\\ &= \frac{2^n(-1)^n}{3}\left[\left(AB+\frac{BA}{2}\right)+\frac{1}{3}\left[2^{n+1}A^2+(-1)^nB^2\right.\right.\\ &\quad \left.\left.-\left(4AB-\frac{BA}{2}\right)\right]\varepsilon\right]\end{aligned}$$

In a similar way, using the Binet's formula in equation (1.44) and

$$SJ_{n-1}.SJ_{n+1}-SJ_n^2=\tfrac{2^n(-1)^n}{3}\left[\tfrac{AB}{2}+BA\right],$$

we obtain

$$\begin{aligned}SJ_{n-1}.SJ_{n+1}-SJ_n^{\;2} &= (SJ_{n-1}+SJ_n\varepsilon).(SJ_{n+1}+SJ_{n+2}\varepsilon)\\ &\quad -(SJ_n+SJ_{n+1}\varepsilon)\\ &= SJ_{n-1}.SJ_{n+1}-SJ_n^2+(SJ_{n-1}SJ_{n+2})\varepsilon\\ &= \frac{2^n(-1)^n}{3}\left[\frac{AB}{2}+BA\right]+\frac{1}{9}\left(2^{2n+1}A^2-B^2\right.\\ &\quad \left.-2^n(-1)^n\left(\frac{AB}{2}+4BA\right)\right)\varepsilon\end{aligned}$$

which is desired.

The proofs of the identities (1.45) and (1.46) are similar to that of (1.43) and (1.44).

Theorem 2 For every nonnegative integer numbers n and r such that $r\le n,$ we get

$$\begin{aligned}SJ_{n+r}.SJ_{n-r}-SJ_n^{\;2} &= \frac{2^n(-1)^n}{9}\left((-1)^r-2^r\right)\\ &\quad \left[\left[AB(-1)^r-BA(2)^{-r}\right]\right.\\ &\quad \left.+2^{-r}(-1)^r\left[2^rAB-BA(-1)^r\right]\varepsilon\right]\end{aligned}\tag{1.47}$$

$$\begin{aligned}SJ_{n-r}.SJ_{n+r}-SJ_n^{\;2} &= \frac{2^n(-1)^n}{9}\left(2^r-(-1)^r\right)\\ &\quad \left[\left[AB(2)^{-r}-BA(-1)^{-r}\right]\right.\\ &\quad \left.+2^{-r}(-1)^r\left[AB(-1)^r-2^rBA\right]\varepsilon\right],\end{aligned}\tag{1.48}$$

$$\begin{aligned}Sj_{n+r}.Sj_{n-r}-Sj_n^{\;2} &= 2^n(-1)^n\left[AB\left(2^r(-1)^r-1\right)\right.\\ &\quad \left.+BA\left(2^{-r}(-1)^r-1\right)\right](1+\varepsilon),\end{aligned}\tag{1.49}$$

and

$$\begin{aligned}Sj_{n-r}.Sj_{n+r}-Sj_n &= 2^n(-1)^n\left[AB\left(2^{-r}(-1)^r-1\right)\right.\\ &\quad \left.+BA\left(2^r(-1)^r-1\right)\right](1+\varepsilon)\end{aligned}\tag{1.50}$$

where $A=\sum_{s=0}^{15}2^se_s$ *and* $B=\sum_{s=0}^{15}(-1)^se_s$.

Proof. The proofs of the identities (1.47)-(1.50) of this theorem are similar to the proofs of Theorem 1.

Theorem 3 For every nonnegative integer numbers n and m such that $m>n$ we get ,

$$\begin{aligned}SJ_m.SJ_{n+1}-SJ_{m+1}SJ_n &= \frac{1}{3}\left[2^m(-1)^nAB\right.\\ &\quad \left.-2^n(-1)^mBA\right](1+\varepsilon)\end{aligned}\tag{1.51}$$

and

$$\begin{aligned}Sj_m.Sj_{n+1}-Sj_{m+1}Sj_n &= 3\left[-2^m(-1)^nAB\right.\\ &\quad \left.+2^n(-1)^mBA\right](1+\varepsilon)\end{aligned}\tag{1.52}$$

where $A=\sum_{s=0}^{15}2^se_s$ and $B=\sum_{s=0}^{15}(-1)^se_s$

Proof. The proofs of (1.51) and (1.52) are similar to the previous theorems.

The ordinary generatig functions (OGF) of a sequence $\{b_n\}_{n=0}^{\infty}$ is given by

$$OGF(b_n,x)=\sum_{n=0}^{\infty}b_nx^n.$$

The exponential generating function of a sequence $\{b_n\}_{n=0}^{\infty}$ is given by

$$EG(b_n,l)=\sum_{n=0}^{\infty}\frac{b_nl^n}{n!}.$$

The poisson generating function of a sequence $\{b_n\}_{n=0}^{\infty}$ is given by

$$PG(b_n,x)=e^{-l}EG(b_n,x).$$

In the following theorem, we now derive the ordinary generating functions

$$\Im(x)=\sum_{n=0}^{\infty}SJ_nx^n \text{ and } \wp(x)=\sum_{n=0}^{\infty}Sj_nx^n,$$

exponential generating functions $E_{SJ}(l)=\sum_{n=0}^{\infty}\frac{SJ_n\,l^n}{n!}$ and $E_{Sj}(l)=\sum_{n=0}^{\infty}\frac{Sj_n\,l^n}{n!}$, poisson generating functions

$$P_{SJ_n}(l)=\sum_{n=0}^{\infty}\frac{SJ_n\,l^n}{n!}e^{-l}$$

and

$$P_{Sj_n}(l)=\sum_{n=0}^{\infty}\frac{Sj_n\,l^n}{n!}e^{-l}$$

for SJ_n and Sj_n defined by (1.18) and (1.19).

Theorem 4 For SJ_n and Sj_n defined by (1.18) and (1.19), the following is its ordinary generating functions:

$$\Im(x)=\frac{SJ_0+\left(-SJ_0+SJ_1\right)x}{1-x-2x^2}, \qquad (1.53)$$

$$\wp(x)=\frac{Sj_0-\left(Sj_0-Sj_1\right)x}{1-x-2x^2} \qquad (1.54)$$

and its the exponential generating fuctions:

$$E_{SJ}(l)=\sum_{n=0}^{\infty}\frac{SJ_n\ l^n}{n!}=\frac{1}{3}\left(\left[Ae^{2l}-Be^{-l}\right]-\left[2Ae^{2l}+Be^{-l}\right]\varepsilon\right), \qquad (1.55)$$

$$E_{Sj}(l)=\sum_{n=0}^{\infty}\frac{Sj_n\ l^n}{n!}=\frac{1}{3}\left(\left[Ae^{2l}-Be^{-l}\right]-\left[2Ae^{2l}+Be^{-l}\right]\varepsilon\right) \qquad (1.56)$$

and its the poisson generating functions:

$$P_{SJ_n}(l)=\sum_{n=0}^{\infty}\frac{SJ_n\ l^n}{n!}e^{-l}=\frac{e^{-l}}{3}\left(\left[Ae^{2l}-Be^{-l}\right]-\left[2Ae^{2l}+Be^{-l}\right]\varepsilon\right), \qquad (1.57)$$

$$P_{Sj_n}(l)=\sum_{n=0}^{\infty}\frac{Sj_n\ l^n}{n!}e^{-l}=e^{-l}\left(\left[Ae^{2l}+Be^{-l}\right]+\left[2Ae^{2l}+Be^{-l}\right]\varepsilon\right). \qquad (1.58)$$

Proof. Firstly, we need to write generating function for SJ_n;

$$\Im(x)=SJ_0x^0+SJ_1x+SJ_2x^2+...+SJ_nx^n+...$$

Secondly, we need to calculate $x\Im(x)$ and $2x^2\Im(x)$ as the following equations;

$$x\Im(x)=\sum_{n=0}^{\infty}SJ_nx^{n+1} \text{ and } 2x^2\Im(x)=2\sum_{n=0}^{\infty}SJ_nx^{n+2}.$$

Finally, if we made necessary calculations, then we have

$$\Im(x)=\frac{SJ_0+\left(-SJ_0+SJ_1\right)x}{1-x-2x^2}$$

which is the generating function for SJ_n
In a similar way, we can show generating function $\wp(x)$ for Sj_n.

Using Binet Formulas for SJ_n in the proof of $E_{SJ}(l)=\sum_{n=0}^{\infty}\frac{SJ_n\ l^n}{n!}$, we obtain

$$\begin{aligned}E_{SJ}(l)&=\sum_{n=0}^{\infty}\frac{SJ_n\ l^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{\left(\frac{2^n}{3}A-\frac{(-1)^n}{3}B\right)+\left(\frac{2^{n+1}}{3}A-\frac{(-1)^{n+1}}{3}B\right)\varepsilon}{n!}l^n\\&=\frac{1}{3}\left[\left(A\sum_{n=0}^{\infty}\frac{2^nl^n}{n!}-B\sum_{n=0}^{\infty}\frac{(-1)^nl^n}{n!}\right)\right.\\&\qquad\left.-\left(2A\sum_{n=0}^{\infty}\frac{2^nl^n}{n!}+B\sum_{n=0}^{\infty}\frac{(-1)^nl^n}{n!}\right)\varepsilon\right]\\&=\frac{1}{3}\left(\left[Ae^{2l}-Be^{-l}\right]-\left[2Ae^{2l}+Be^{-l}\right]\varepsilon\right).\end{aligned}$$

where $\sum_{n=0}^{\infty}\frac{2^nl^n}{n!}=e^{2l}$, $\sum_{n=0}^{\infty}\frac{(-1)^nl^n}{n!}=e^{-l}$.

In a similar way, we can show exponential generating function $E_{Sj}(l)=\sum_{n=0}^{\infty}\frac{Sj_n\ l^n}{n!}$ for Sj_n.

Using of the equality

$$PG\left(b_n,x\right)=e^{-l}EG\left(b_n,x\right),$$

the proofs completed of the identities (1.57)-(1.58) of this theorem.

3. Conclusions

In this study, we presented new classes of sedenion numbers (dual Jacobsthal and dual Jacobsthal -Lucas sedenions) associated with the familiar Jacobsthal and Jacobsthal-Lucas numbers. Also, we obtained various results obtained here for these classes of sedenion numbers include recurrence relations, Binet formula, generating function, exponentinal generating functions, poisson generating functions and also we presented the Cassini identity, Catalan's identities and d'Ocagne's identity by their Binet forms.

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