Conference Proceedings of Science and Technology, 2(3), 2019, 189–193

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

Deferred Statistical Convergence in Metric Spaces

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

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Abstract: In this paper, the concept of deferred statistical convergence is generalized to general metric spaces, and some inclusion relations between deferred strong Cesàro summability and deferred statistical convergence are given in general metric spaces.

Keywords: Metric space, Statistical convergence, Deferred statistical convergence.

1 Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [2] and Fast [3] and then reintroduced independently by Schoenberg [4]. Over the years and under different names, statistical convergence has been discussed in the Theory of Fourier Analysis, Ergodic Theory, Number Theory, Measure Theory, Trigonometric Series, Turnpike Theory and Banach Spaces. Later on it was further investigated from the sequence spaces point of view and linked with summability theory by Gupta and Bhardwaj [5], Braha et al. [6], Çınar et al. [7], Connor [8], Et et al. ([9],[10],[11],[12],[13]), Fridy [14], Işık et al. ([15],[16],[17]), Mohiuddine et al. [18], Mursaleen et al. [19], Nuray [20], Nuray and Aydın [21], Salat [22], Şengül et al. ([23],[24],[25],[26]), Srivastava et al. ([27],[28]) and many others.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_{\mathbb{E}}(k),$$

provided that the limit exists, where $\chi_{\mathbb{E}}$ is the characteristic function of the set $\mathbb{E}$. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and that

$$\delta\left(\mathbb{E}^c\right) = 1 - \delta\left(\mathbb{E}\right).$$

A sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if, for every $\varepsilon > 0$, we have

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0.$$

In this case, we write

$$x_k \xrightarrow{\text{stat}} L \quad \text{as} \quad k\to\infty \qquad \text{or} \qquad S - \lim_{k\to\infty} x_k = L.$$

In 1932, Agnew [29] introduced the concept of deferred Cesàro mean of real (or complex) valued sequences $x = (x_k)$ defined by

$$(D_{p,q}x)_n = \frac{1}{(q(n) - p(n))}\sum_{k=p(n)+1}^{q(n)} x_k,\ n = 1, 2, 3, \ldots,$$

where $p = \{p(n)\}$ and $q = \{q(n)\}$ are the sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty.$$

Let K be a subset of $\mathbb{N}$ and denote the set $\{k : p(n) < k \leq q(n), k \in K\}$ by $K_{p,q}(n)$.

Deferred density of K is defined by

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |K_{p,q}(n)|, \text{ provided the limit exists,}$$

where, vertical bars indicate the cardinality of the enclosed set $K_{p,q}(n)$. If $q(n) = n$, $p(n) = 0$, then the deferred density coincides with natural density of K.

A real valued sequence $x = (x_k)$ is said to be deferred statistically convergent to L, if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_{p,q} - \lim x_k = L$. If $q(n) = n$, $p(n) = 0$, for all $n \in \mathbb{N}$, then deferred statistical convergence coincides with usual statistical convergence [30].

2 Main Results

In this section, we give some inclusion relations between statistical convergence, deferred strong Cesàro summability and deferred statistical convergence in general metric spaces.

Definition 1 Let (X, d) be a metric space and $\{p(n)\}$ and $\{q(n)\}$ be two sequences as above. A metric valued sequence $x = (x_k)$ is said to be $DS^d_{p,q}-$convergent (or deferred $d-$statistically convergent) to a if there is a real number $a \in X$ such that

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}| = 0.$$

In this case we write $DS^d_{p,q} - \lim x_k = a$ or $x_k \to a \left(DS^d_{p,q}\right)$. The set of all $DS^d_{p,q}-$statistically convergent sequences will be denoted by $DS^d_{p,q}$. If $q(n) = n$ and $p(n) = 0$, then deferred $d-$statistical convergence coincides $d-$statistical convergence.

Definition 2 Let (X, d) be a metric space and $\{p(n)\}$ and $\{q(n)\}$ be two sequences as above. A metric valued sequence $x = (x_k)$ is said to be strongly $Dw^d_{p,q}-$summable (or deferred strongly $d-$Cesàro summable) to a if there is a real number $a \in X$ such that

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} \sum_{p(n)+1}^{q(n)} d(x_k, a) = 0.$$

In this case we write $Dw^d_{p,q} - \lim x_k = a$ or $x_k \to a \left(Dw^d_{p,q}\right)$. The set of all strongly $Dw^d_{p,q}-$summable sequences will be denoted by $Dw^d_{p,q}$. If $q(n) = n$ and $p(n) = 0$, for all $n \in \mathbb{N}$, then deferred strong $d-$Cesàro summability coincides strong $d-$Cesàro summability.

Theorem 3 Let (X, d) be a linear metric space and $x = (x_k)$, $y = (y_k)$ be metric valued sequences, then

(i) If $DS^d_{p,q} - \lim x_k = x_0$ and $DS^d_{p,q} - \lim y_k = y_0$, then $DS^d_{p,q} - \lim (x_k + y_k) = x_0 + y_0$,

(ii)If $DS^d_{p,q} - \lim x_k = x_0$ and $c \in \mathbb{C}$, then $DS^d_{p,q} - \lim (cx_k) = cx_0$,

(iii) If $DS^d_{p,q} - \lim x_k = x_0$, $DS^d_{p,q} - \lim y_k = y_0$ and $x, y \in \ell_\infty$, then $DS^d_{p,q} - \lim (x_k y_k) = x_0 y_0$.

Theorem 4 $Dw^d_{p,q} \subseteq DS^d_{p,q}$ and the inclusion is strict.

Proof. First part of proof is easy, so omitted. To show the strictness of the inclusion, choose $q(n) = n, p(n) = 0$, for all $n \in \mathbb{N}$ and $a = 0$ and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \frac{\sqrt{n}}{2}, & k = n^2 \\ 0, & k \neq n^2 \end{cases}.$$

Then for every $\varepsilon > 0$, we have

$$\frac{1}{(q(n) - p(n))} |\{p(n) < k \leq q(n) : d(x_k, 0) \geq \varepsilon\}| \leq \frac{[\sqrt{n}]}{n} \to 0, \text{ as } n \to \infty,$$

where $d(x, y) = |x - y|$, that is $x_k \to 0 \left(DS^d_{p,q}\right)$. At the same time, we get

$$\frac{1}{(q(n) - p(n))} \sum_{p(n)+1}^{q(n)} d(x_k, 0) \leq \frac{[\sqrt{n}]\,[\sqrt{n}]}{n} \to 1,$$

i.e. $x_k \nrightarrow 0 \left(Dw^d_{p,q}\right)$. Therefore, $Dw^d_{p,q} \subseteq DS^d_{p,q}$ is strict.

Theorem 5 If $\liminf_n \frac{q(n)}{p(n)} > 1$, then $S^d \subset DS^d_{p,q}$.

Proof. Suppose that $\liminf_n \frac{q(n)}{p(n)} > 1$; then there exists a $\nu > 0$ such that $\frac{q(n)}{p(n)} \geq 1 + \nu$ for sufficiently large n, which implies that

$$\frac{q(n) - p(n)}{q(n)} \geq \frac{\nu}{1+\nu} \Longrightarrow \frac{1}{q(n)} \geq \frac{\nu}{(1+\nu)} \frac{1}{(q(n) - p(n))}.$$

If $x_k \to a \left(S^d\right)$, then for every $\varepsilon > 0$ and for sufficiently large n, we have

$$\begin{aligned}\frac{1}{q(n)} \left|\{k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right| &\geq \frac{1}{q(n)} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right| \\ &\geq \frac{\nu}{(1+\nu)} \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right|.\end{aligned}$$

This proves the proof.
"In the following theorem, by changing the conditions on the sequences (p_n) and (q_n) we give the same relation with Theorem 5."

Theorem 6 If $\lim_{n\to\infty} \inf \dfrac{(q(n) - p(n))}{n} > 0$ and $q(n) < n$, then $S^d \subseteq DS^d_{p,q}$.

Proof. Let $\lim_{n\to\infty} \inf \dfrac{(q(n) - p(n))}{n} > 0$ and $q(n) < n$, then for each $\varepsilon > 0$ the inclusion

$$\{k \leq n : d(x_k, a) \geq \varepsilon\} \supset \{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}$$

is satisfied and so we have the following inequality

$$\begin{aligned}\frac{1}{n} \left|\{k \leq n : d(x_k, a) \geq \varepsilon\}\right| &\geq \frac{1}{n} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right| \\ &= \frac{(q(n) - p(n))}{n} \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right|.\end{aligned}$$

Therefore $S^d \subseteq DS^d_{p,q}$.

Theorem 7 Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be four sequences of non-negative integers such that

$$p'(n) < p(n) < q(n) < q'(n) \text{ for all } n \in \mathbb{N}, \tag{1}$$

then

(i) If

$$\lim_{n\to\infty} \frac{q(n) - p(n)}{q'(n) - p'(n)} = m > 0 \tag{2}$$

then $DS^d_{p',q'} \subseteq DS^d_{p,q}$,

(ii) If

$$\lim_{n\to\infty} \frac{q'(n) - p'(n)}{q(n) - p(n)} = 1 \tag{3}$$

then $DS^d_{p,q} \subseteq DS^d_{p',q'}$.

Proof. (i) Let (2) be satisfied. For given $\varepsilon > 0$ we have

$$\{p'(n) < k \leq q'(n) : d(x_k, a) \geq \varepsilon\} \supseteq \{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\},$$

and so

$$\begin{aligned}&\frac{1}{(q'(n) - p'(n))} \left|\{p'(n) < k \leq q'(n) : d(x_k, a) \geq \varepsilon\}\right| \\ &\geq \frac{(q(n) - p(n))}{(q'(n) - p'(n))} \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon\}\right|.\end{aligned}$$

Therefore $DS^d_{p',q'} \subseteq DS^d_{p,q}$.

(ii) Omitted.

Theorem 8 Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$be four sequences of non-negative integers defined as in (1).

(i) If (2) holds then $Dw^{d}_{p',q'} \subset Dw^{d}_{p,q}$,

(ii) If (3) holds and $x = (x_k)$ be a bounded sequence, then $Dw^{d}_{p,q} \subset Dw^{d}_{p',q'}$.

Proof. Omitted.

Theorem 9 Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be four sequences of non-negative integers defined as in (1). Then

(i) Let (2) holds, if a sequence is strongly $Dw^{d}_{p',q'}-$summable to a, then it is $DS^{d}_{p,q}-$convergent to a,

(ii) Let (3) holds and $x = (x_k)$ be a bounded sequence, if a sequence is $DS^{d}_{p,q}-$convergent to a then it is strongly $Dw^{d}_{p',q'}-$summable to a.

Proof. (i) Omitted.

(ii) Suppose that $DS^{d}_{p,q} - \lim x_k = a$ and $(x_k) \in \ell_\infty$. Then there exists some $M > 0$ such that $d(x_k, a) < M$ for all k, then for every $\varepsilon > 0$ we may write

$$
\begin{aligned}
&\frac{1}{(q'(n) - p'(n))} \sum_{p'(n)+1}^{q'(n)} d(x_k, a) \\
&= \frac{1}{(q'(n) - p'(n))} \sum_{q(n)-p(n)+1}^{q'(n)-p'(n)} d(x_k, a) + \frac{1}{(q'(n) - p'(n))} \sum_{p(n)+1}^{q(n)} d(x_k, a) \\
&\leq \frac{\big(q'(n) - p'(n)\big) - (q(n) - p(n))}{(q'(n) - p'(n))} M + \frac{1}{(q'(n) - p'(n))} \sum_{p(n)+1}^{q(n)} d(x_k, a) \\
&\leq \left(\frac{q'(n) - p'(n)}{q(n) - p(n)} - 1 \right) M + \frac{1}{(q(n) - p(n))} \sum_{\substack{p(n)+1 \\ d(x_k,a)\geq\varepsilon}}^{q(n)} d(x_k, a) \\
&+ \frac{1}{(q(n) - p(n))} \sum_{\substack{p(n)+1 \\ d(x_k,a)<\varepsilon}}^{q(n)} d(x_k, a) \\
&\leq \left(\frac{q'(n) - p'(n)}{q(n) - p(n)} - 1 \right) M + \frac{M}{(q(n) - p(n))} \left| \{ p(n) < k \leq q(n) : d(x_k, a) \geq \varepsilon \} \right| \\
&+ \frac{q'(n) - p'(n)}{q(n) - p(n)} \varepsilon .
\end{aligned}
$$

This completes the proof.

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