Fractional order mixed difference operator and its applications in angular approximation

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Abstract

Lebesgue spaces are considered with Muckenhoupt weights. Fractional order mixed difference operator is investigated to obtain mixed fractional modulus of smoothness in these spaces. Using this modulus of smoothness we give the proof of direct and inverse estimates of angular trigonometric approximation. Also we obtain an equivalence between fractional mixed modulus of smoothness and fractional mixed K-functional.

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1. Introduction

Let $\mathbb{T} := [0, 2\pi]$, $\mathbb{T}^2 := \mathbb{T} \times \mathbb{T}$, and $L^p_\omega := L^p_\omega(\mathbb{T}^2)$ be the weighted Lebesgue space of functions $f(x, y) : \mathbb{T}^2 \to \mathbb{R}$, 2π-periodic with respect to each variable x, y, and

$$\|f\|^p_{p,\omega} := \int_{\mathbb{T}^2} |f(x, y)|^p \, \omega(x, y) \, dx dy < \infty.$$

A function $\omega : \mathbb{T}^2 \to [0, \infty)$ is called a weight on $\mathbb{T}^2$ if $\omega(x, y)$ is measurable and positive almost everywhere on $\mathbb{T}^2$. We denote by $A_p := A_p(\mathbb{T}^2, \mathbb{J})$, $1 < p < \infty$, the collection of locally integrable weight functions ω such that $\omega(x, y)$ is 2π-periodic with respect to each variable x, y and

$$C := \sup_{G \in \mathbb{J}} \frac{1}{|G|} \int_G \omega(x, y) \, dx dy \left(\frac{1}{|G|} \int_G [\omega(x, y)]^{\frac{-1}{p-1}} \, dx dy \right)^{p-1} < \infty, \qquad (1.1)$$

where $\mathbb{J}$ is the set of rectangles in $\mathbb{T}^2$ with sides parallel to coordinate axes. Least constant C in (1.1) will be called the Muckenhoupt's constant of ω and denoted by $[\omega]_{A_p}$. In the present paper we considered approximation properties of the two dimensional Fourier series in Lebesgue spaces L^p_ω with weights ω belonging to the Muckenhoupt's class A_p. We consider a weighted mixed modulus of smoothness and obtain several angular trigonometric approximation inequalities involving angular trigonometric approximation errors and modulus of smoothness in L^p_ω with $\omega \in A_p$, $1 < p < \infty$.

In the nonweighted case, on $\mathbb{T}^2$, with the classical nonweighted mixed modulus of smoothness of functions given in the classical Lebesgue spaces $L^p := L^p(\mathbb{T}^2)$, approximation by "angle" were investigated by several mathematicians. One can see the papers ([16–20]) written by M. K. Potapov. Also in the works [22, 24], transformed Fourier series and mixed modulus of smoothness were investigated. Embedding problems of the Besov-Nikolskii and Weyl-Nikolskii classes are studied in [21, 23]. Ul'yanov type inequalities were considered in [25] and [26]. Mixed K-functionals were considered by C. Cottin in [8]. For the univariate case one can see the papers [10, 12–14, 29].

In the weighted spaces, generally, ordinary translation is not suitable to construct difference operator and modulus of smoothness. The modulus of smoothness defined here is applicable in some weighted spaces.

Let

$$Y_{m_1,m_2}(f)_{p,\omega} = \inf_{T_i}\left\{\left\|f - \sum_{i=1}^{2} T_i\right\|_{p,\omega} : T_i \in \mathfrak{T}_{m_i}\right\},$$

where $\mathfrak{T}_{m_i}$ is the set of all two dimensional trigonometric polynomial of degree at most m_i with respect to variable x_i $(i = 1, 2)$.

Define the following Steklov averages

$$\sigma_{h,\circ}f(x,y) := \frac{1}{h}\int\limits_{x-h/2}^{x+h/2} f(t,y)\,dt, \quad \sigma_{\circ,k}f(x,y) := \frac{1}{k}\int\limits_{y-k/2}^{y+k/2} f(x,\tau)\,d\tau,$$

$$\sigma_{h,k}f(x,y) := \frac{1}{hk}\int\limits_{x-h/2}^{x+h/2}\int\limits_{y-k/2}^{y+k/2} f(t,\tau)\,dt d\tau,$$

$$\sigma_{0,\circ}f(x,y) = \sigma_{\circ,0}f(x,y) = \sigma_{0,0}f(x,y) := f(x,y).$$

Let $x, y \in \mathbb{T}$, $r, h, k > 0$, $p \in (1, \infty)$, $\omega \in A_p$, and $f \in L^p_\omega$. Define the quantities

$$\nabla_{h,\circ}^{r,\circ}f(x,\cdot) \quad : \quad = (\mathbb{I} - \sigma_{h,\circ})^r f(x,\cdot) = \sum_{i=0}^{\infty}\binom{r}{i}(-1)^i(\sigma_{h,\circ}f)^i(x,\cdot) \tag{1.2}$$

$$\nabla_{\circ,k}^{\circ,r}f(\cdot,y) \quad : \quad = (\mathbb{I} - \sigma_{\circ,k})^r f(\cdot,y) = \sum_{j=0}^{\infty}\binom{r}{j}(-1)^j(\sigma_{\circ,k}f)^j(\cdot,y) \tag{1.3}$$

$$\nabla_{h,k}^{r,r}f(x,y) \quad : \quad = \nabla_{h,\circ}^{r,\circ}\left(\nabla_{\circ,k}^{\circ,r}f\right)(x,y) \tag{1.4}$$

where $\mathbb{I}$ is identity operator on $\mathbb{T}^2$, $\binom{r}{j} := \frac{r(r-1)\ldots(r-j+1)}{j!}$ for $j > 1$, $\binom{r}{1} := r$ and $\binom{r}{0} := 1$ are binomial coefficients.

Definition 1.1. The fractional weighted mixed modulus of smoothness of $f \in L^p_\omega$, $1 < p < \infty$, $\omega \in A_p$, $r \in \{0\} \cup \mathbb{R}^+$, defined as

$$\Omega_r(f,\delta_1,\delta_2)_{p,\omega} = \begin{cases} \sup\limits_{h,k}\{\left\|\nabla_{h,k}^{r,r}f\right\|_{p,\omega} : 0 \le h \le \delta_1, 0 \le k \le \delta_2\} & , r > 0, \\ \|f\|_{p,\omega} & , r = 0. \end{cases} \tag{1.5}$$

In the present work we obtain main properties of the weighted fractional order mixed modulus of smoothness (1.5). The first of them is given in following approximation error estimate (direct theorem of angular trigonometric approximation):

Theorem 1.2. *If $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$ and $r \in \mathbb{R}^+$, then there exists a constant $C_{[\omega]_{A_p},p,r}$ depending only on Muckenhoupt's constant $[\omega]_{A_p}$ of ω and p, r such that*

$$Y_{m_1,m_2}(f)_{p,\omega} \leq C_{[\omega]_{A_p},p,r} \Omega_r \left(f, \frac{1}{m_1}, \frac{1}{m_2} \right)_{p,\omega} \tag{1.6}$$

holds for $m_1, m_2 \in \mathbb{N}$.

We use the notation for fractional Weyl type derivatives $f^{(r,s)} := \frac{\partial^{r+s}f}{\partial x^r \partial y^s}$; $f^{(r,\circ)} := \frac{\partial^r f}{\partial x^r}$; $f^{(\circ,s)} := \frac{\partial^s f}{\partial y^s}$. Let $W_{p,\omega}^{r,s}$, $r, s \in \mathbb{N}$, (respectively $W_{p,\omega}^{r,\circ}$; $W_{p,\omega}^{\circ,s}$) be denote the collection of integrable functions f with $f^{(r,s)} \in L_\omega^p$ (respectively $f^{(r,\circ)} \in L_\omega^p$; $f^{(\circ,s)} \in L_\omega^p$).

Definition 1.3. The quantity

$$K(f, \delta, \xi, p, \omega, r, s) := \inf_{g_1,g_2,g} \left\{ \|f - g_1 - g_2 - g\|_{p,\omega} + \delta^r \left\| \frac{\partial^r g_1}{\partial x^r} \right\|_{p,\omega} + \right.$$

$$\left. + \xi^s \left\| \frac{\partial^s g_2}{\partial y^s} \right\|_{p,\omega} + \delta^r \xi^s \left\| \frac{\partial^{r+s} g}{\partial x^r \partial y^s} \right\|_{p,\omega} \right\}$$

is known as weighted mixed K-functional, where infimum is taken over g_1, g_2, g so that $g_1 \in W_{p,\omega}^{r,\circ}$, $g_2 \in W_{p,\omega}^{\circ,s}$, $g \in W_{p,\omega}^{r,s}$, where $r, s \in \mathbb{R}^+ := (0, \infty)$, $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$.

By Theorem 1.2 we get the following equivalence between $\Omega_r(f, \delta_1, \delta_2)_{p,\omega}$ and mixed K-functional.

Theorem 1.4. *If $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$, and $r \in \mathbb{R}^+$, then there exist constants $c_{[\omega]_{A_p},p,r} > 0$, $C_{[\omega]_{A_p},p,r} > 0$, depending only on Muckenhoupt's constant $[\omega]_{A_p}$ of ω and p, r, so that the equivalence*

$$\Omega_r(f, \delta_1, \delta_2)_{p,\omega} \leq C_{[\omega]_{A_p},p,r} K(f, \delta_1, \delta_2, p, \omega, 2r) \leq c_{[\omega]_{A_p},p,r} \Omega_r(f, \delta_1, \delta_2)_{p,\omega}$$

holds for $\delta_1, \delta_2 \geq 0$.

Theorem 1.4 gives the following corollary.

Corollary 1.5. *If $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$, and $r \in \mathbb{R}^+$ then, there exist constants depending only on $[\omega]_{A_p}$ and p, r such that*

$$\Omega_r(f, \lambda\delta, \eta\xi)_{p,\omega} \leq c(1 + \lfloor \lambda \rfloor)^{2r}(1 + \lfloor \eta \rfloor)^{2r} \Omega_r(f, \delta, \xi)_{p,\omega}, \quad \delta, \xi > 0,$$

$$\frac{\Omega_r(f, \delta_1, \delta_2)_{p,\omega}}{\delta_1^{2r} \delta_2^{2r}} \leq C \frac{\Omega_r(f, t_1, t_2)_{p,\omega}}{t_1^{2r} t_2^{2r}} \quad \text{for } 0 < t_i \leq \delta_i; \quad i = 1, 2$$

where $\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leq x\}$.

Converse estimate to (1.6) is given in the next theorem.

Theorem 1.6. *If $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$ and $r \in \mathbb{R}^+$, then there exist constants depending only on $[\omega]_{A_p}$ and p, r so that*

$$\Omega_r \left(f, \frac{1}{m_1}, \frac{1}{m_2} \right)_{p,\omega} \leq \frac{C_{[\omega]_{A_p},p,r}}{\prod_{i=1}^2 m_i^{2r}} \sum_{l_{i_1}=0}^{m_1} \sum_{l_{i_2}=0}^{m_2} \prod_{j=1}^2 \left[\left(l_{i_j} + 1 \right) \right]^{2r-1} Y_{l_{i_1},l_{i_2}}(f)_{p,\omega}.$$

In this paper, we will denote positive constant $C_{u,v,\ldots}$, depending only on the parameters $u, v, \ldots$ so that it can be different in different places.

2. Preliminary definition and results

Suppose that L^1 is the collection of the Lebesgue integrable functions $f(x,y) : \mathbb{T}^2 \to (-\infty, \infty)$ such that $f(x,y)$ is 2π-periodic with respect to each variable x, y respectively. Let $\mathcal{T}_{m,\circ}$ (respectively $\mathcal{T}_{\circ,n}$) be the set of all trigonometric polynomial of degree at most m (at most n) with respect to variable x (variable y). We set $\mathcal{T}_{m,n}$ as the collection of all trigonometric polynomial of degree at most m with respect to variable x and of degree at most n with respect to variable y. The best angular trigonometric approximation error is defined as

$$Y_{m,n}(f)_{p,\omega} = \inf \left\{ \|f - T - U\|_{p,\omega} : T \in \mathcal{T}_{m,\circ}, U \in \mathcal{T}_{\circ,n} \right\}$$

where $1 < p < \infty$, $\omega \in A_p$, and $f \in L^p_\omega$.

Using [15, Theorem 6] we have

$$\left\| f - C^\alpha_{m,n} f \right\|_{p,\omega} \to 0, \quad \text{as } m, n \to \infty,$$

where $C^\alpha_{m,n} f$ is αth Cesàro mean of f. From this we can obtain that $C(\mathbb{T}^2)$, the class of continuous functions on $\mathbb{T}^2$, is a dense subset of L^p_ω for $1 < p < \infty$, $\omega \in A_p$. Then

$$Y_{m,n}(f)_{p,\omega} \le C_{[\omega]_{A_p},p,r} \left\| f - C^\alpha_{m,n} f \right\|_{p,\omega} \to 0$$

and hence

$$Y_{m,n}(f)_{p,\omega} \searrow 0, \quad \text{as } m, n \to \infty.$$

This shows that approximation problems make sense in L^p_ω for $1 < p < \infty$, $\omega \in A_p$.

Define Steklov type operators

$$\mathcal{S}_{\lambda,\tau;\circ} \ f(x,y) := \lambda \int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(u,y)du,$$

$$\mathcal{S}_{\circ;\theta,\rho} \ f(x,y) := \theta \int_{y+\rho-1/(2\theta)}^{y+\rho+1/(2\theta)} f(x,v)dv,$$

$$\mathcal{S}_{\lambda,\tau;\theta,\rho} \ f(x,y) = \mathcal{S}_{\lambda,\tau;\circ}(\mathcal{S}_{\circ;\theta,\rho}f(x,y)) = \lambda\theta \int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} \int_{y+\rho-1/(2\theta)}^{y+\rho+1/(2\theta)} f(u,v)dudv.$$

Theorem 2.1. *We suppose that $1 < p < \infty$ and $\omega \in A_p$.*

(i) *If $1 \le \lambda < \infty$ and $|\tau| \le \pi\lambda^{-1}$, then, the family of operators $\{\mathcal{S}_{\lambda,\tau;\circ}\}_{1\le\lambda<\infty}$ is uniformly bounded (in λ, τ) in L^p_ω :*

$$\|\mathcal{S}_{\lambda,\tau;\circ}f\|_{p,\omega} \le 108^{\frac{1}{p}} \pi^{\frac{2}{p}} [\omega]_{A_p}^{\frac{1}{p}} \|f\|_{p,\omega} .$$

(ii) *If $1 \le \theta < \infty$ and $|\rho| \le \pi\theta^{-1}$, then, the family of operators $\{\mathcal{S}_{\circ;\theta,\rho}\}_{1\le\theta<\infty}$ is uniformly bounded (in θ, ρ) in L^p_ω :*

$$\|\mathcal{S}_{\circ;\theta,\rho}f\|_{p,\omega} \le 108^{\frac{1}{p}} \pi^{\frac{2}{p}} [\omega]_{A_p}^{\frac{1}{p}} \|f\|_{p,\omega} .$$

(iii) *If $1 \le \lambda, \theta < \infty$, $|\tau| \le \pi\lambda^{-1}$, $|\rho| \le \pi\theta^{-1}$, then, the family of operators $\{\mathcal{S}_{\lambda,\tau;\theta,\rho}\}_{1\le\lambda,\theta<\infty}$ is uniformly bounded (in λ, τ and θ, ρ) in L^p_ω :*

$$\|\mathcal{S}_{\lambda,\tau;\theta,\rho}f\|_{p,\omega} \le 108^{\frac{2}{p}} \pi^{\frac{4}{p}} [\omega]_{A_p}^{\frac{2}{p}} \|f\|_{p,\omega} .$$

In this case Theorem 2.1 yields the following lemma.

Lemma 2.2 ([11, Theorem 3.3], [4]). *If $1 < p < \infty$, $\omega \in A_p$, and $f \in L^p_\omega$, then*

$$\left\{ \|\sigma_{h,k}f\|_{p,\omega}, \|\sigma_{h,\circ}f\|_{p,\omega}, \|\sigma_{\circ,k}f\|_{p,\omega} \right\} \le C_{[\omega]_{A_p},p} \|f\|_{p,\omega}, \tag{2.1}$$

with constants depend only on $[\omega]_{A_p}$ and p.

2.1. Transference result

At this stage we will need a transference result. If f is 2π periodic locally integrable function on $\mathbb{T}^2$, then (see Theorem 11.1 on page 211 of [9])

$$\lim_{(h,k)\to(0,0)} \frac{1}{hk} \int_{x-h/2}^{x+h/2} \int_{y-k/2}^{y+k/2} f(t,\tau)\,dt d\tau = f(x,y)$$

for almost every $(x,y) \in \mathbb{T}^2$. Then, for any $\zeta > 0$ one can find $h_0, k_0 \le 1$ such that

$$\mathcal{S}_{\frac{1}{h_0},0;\frac{1}{k_0},0} f(x,y) = \frac{1}{h_0 k_0} \int_{x-h_0/2}^{x+h_0/2} \int_{y-k_0/2}^{y+k_0/2} f(t,\tau)\,dt d\tau > f(x,y) - \zeta \tag{2.2}$$

almost every $(x,y) \in \mathbb{T}^2$. Throughout this work we will fix these h_0, k_0.

Let $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$,

$$q := \frac{p}{p-1}, \tag{2.3}$$

$$G \in L_\omega^q\left(\mathbb{T}^2\right), \qquad \|G\|_{q,\omega} = 1$$

and define, with h_0, k_0 of (2.2),

$$F_f(u,v) := \int_{\mathbb{T}^2} \mathcal{S}_{\frac{1}{h_0},0;\frac{1}{k_0},0} f(x+u, y+v)\,|G(x,y)|\,\omega(x,y)\,dxdy$$

for $u,v \in \mathbb{T}$ satisfying $|u| \le h_0, |v| \le k_0$.

Let $\tilde{F}_f(u,v)$ be a continuous function defined on $\mathbb{T}^2$ such that
i) $\tilde{F}_f(u,v)$ coincides with $F_f(u,v)$ on

$$I_{h_0}^{k_0} := \left\{ (u,v) \in \mathbb{T}^2 : |u| \le h_0, |v| \le k_0 \right\}.$$

ii) $\max_{(u,v)\in\mathbb{T}^2} \left| \tilde{F}_f(u,v) \right| \le \max_{(u,v)\in I_{h_0}^{k_0}} |F_f(u,v)|.$

Let $C\left(\mathbb{T}^2\right)$ denote the collection of continuous functions $f : \mathbb{T}^2 \to \mathbb{R}$ with

$$\|f\|_{C(\mathbb{T}^2)} := \max\left\{ |f(x,y)| : x, y \in \mathbb{T} \right\} < \infty.$$

Lemma 2.3. *If $1 < p < \infty$, $\omega \in A_p$ and $f \in L_\omega^p$, then the function $F_f(u,v)$, defined above, is uniformly continuous on $I_{h_0}^{k_0}$.*

Lemma 2.4. *Let $1 < p < \infty$, $q := p/(p-1)$ and γ be a weight on $\mathbb{T}^2$. Then*

$$\sup_{G\in L_\omega^q : \|G\|_{q,\omega}=1} \int_{\mathbb{T}^2} f(x,y)\,G(x,y)\,\omega(x,y)\,dxdy = \|f\|_{p,\omega} \tag{2.4}$$

for $f \in L_\omega^p$.

Theorem 2.5. *If $1 < p < \infty$, $\omega \in A_p$, $f,g \in L_\omega^p$ and*

$$\|F_g\|_{C\left(I_{h_0}^{k_0}\right)} \le c \|F_f\|_{C\left(I_{h_0}^{k_0}\right)},$$

then

$$\|g\|_{p,\omega} \le c108^{\frac{2}{p}} \pi^{\frac{4}{p}} [\omega]_{A_p}^{\frac{2}{p}} \|f\|_{p,\omega}.$$

2.2. Fractional order modulus of smoothness

Now, we consider the fractional smoothness $\Omega_r \left(\cdot, \delta, \xi \right)_{p,\omega}$, $r > 0$, suitable for some weighted spaces. Note that classical non-weighted fractional smoothness $\omega_r \left(f, \cdot \right)_p$, $r > 0$, was defined by P. L. Butzer, H. Dyckhoff, E. Görlich, R. L. Stens [7], and R. Taberski [28] and may be some others. See also [27].

Firstly we discuss boundedness of Steklov operators.

Remark 2.6. (i) If $\mathcal{F} \in C \left(\mathbb{T}^2 \right)$ then we know that

$$\left\| \sigma_{h,\circ} \mathcal{F} \right\|_{C(\mathbb{T}^2)} \leq \left\| \mathcal{F} \right\|_{C(\mathbb{T}^2)}, \quad \left\| \sigma_{\circ,k} \mathcal{F} \right\|_{C(\mathbb{T}^2)} \leq \left\| \mathcal{F} \right\|_{C(\mathbb{T}^2)}, \quad \left\| \sigma_{h,k} \mathcal{F} \right\|_{C(\mathbb{T}^2)} \leq \left\| \mathcal{F} \right\|_{C(\mathbb{T}^2)}.$$

Hence

$$\left\| \nabla_{h,\circ}^{r,\circ} \mathcal{F} \right\|_{C(\mathbb{T}^2)} \leq 2^r \left\| \mathcal{F} \right\|_{C(\mathbb{T}^2)}, \quad \left\| \nabla_{\circ,k}^{\circ,r} \mathcal{F} \right\|_{C(\mathbb{T}^2)} \leq 2^r \left\| \mathcal{F} \right\|_{C(\mathbb{T}^2)}$$

and these give that

$$\left\| \nabla_{h,k}^{r,r} \mathcal{F} \right\|_{C(\mathbb{T}^2)} \leq 2^{2r} \left\| \mathcal{F} \right\|_{C(\mathbb{T}^2)}$$

for $\mathcal{F} \in C \left(\mathbb{T}^2 \right)$.

(ii) Using (i)

$$\left\| F_{\nabla_{h,\circ}^{r,\circ} \mathcal{F}} \right\|_{C \left(I_{h_0}^{k_0} \right)} = \max_{u,v \in I_{h_0}^{k_0}} \left| \int_{\mathbb{T}^2} \nabla_{h,\circ}^{r,\circ} \mathcal{S}_{\frac{1}{h_0},0;\frac{1}{k_0},0} \mathcal{F} \left(x+u, y+v \right) \left| G \left(x,y \right) \right| \omega \left(x,y \right) dxdy \right|$$

$$= \max_{u,v \in I_{h_0}^{k_0}} \left| \nabla_{h,\circ}^{r,\circ} \int_{\mathbb{T}^2} \mathcal{S}_{\frac{1}{h_0},0;\frac{1}{k_0},0} \mathcal{F} \left(x+u, y+v \right) \left| G \left(x,y \right) \right| \omega \left(x,y \right) dxdy \right| = \max_{u,v \in I_{h_0}^{k_0}} \left| \nabla_{h,\circ}^{r,\circ} F_{\mathcal{F}} \right|$$

$$\leq 2^r \max_{u,v \in I_{h_0}^{k_0}} \left| F_{\mathcal{F}} \right| = 2^r \left\| F_{\mathcal{F}} \right\|_{C \left(I_{h_0}^{k_0} \right)}.$$

The same method gives that

$$\left\| F_{\nabla_{\circ,k}^{\circ,r} \mathcal{F}} \right\|_{C \left(I_{h_0}^{k_0} \right)} \leq 2^r \left\| F_{\mathcal{F}} \right\|_{C \left(I_{h_0}^{k_0} \right)}.$$

Using the last two inequalities we find

$$\left\| F_{\nabla_{h,k}^{r,r} \mathcal{F}} \right\|_{C \left(I_{h_0}^{k_0} \right)} \leq 2^{2r} \left\| F_{\mathcal{F}} \right\|_{C \left(I_{h_0}^{k_0} \right)}.$$

Using Theorem 2.5 we find

$$\left\| \nabla_{h,\circ}^{r,\circ} f \right\|_{p,\omega} \leq 108^{\frac{2}{p}} \pi^{\frac{4}{p}} \left[\omega \right]_{A_p}^{\frac{2}{p}} 2^r \left\| f \right\|_{p,\omega}, \quad \left\| \nabla_{\circ,k}^{\circ,r} f \right\|_{p,\omega} \leq 108^{\frac{2}{p}} \pi^{\frac{4}{p}} \left[\omega \right]_{A_p}^{\frac{2}{p}} 2^r \left\| f \right\|_{p,\omega} \tag{2.5}$$

and, therefore

$$\left\| \nabla_{h,k}^{r,r} f \right\|_{p,\omega} \leq 2^{2r} 108^{\frac{4}{p}} \pi^{\frac{8}{p}} \left[\omega \right]_{A_p}^{\frac{4}{p}} \left\| f \right\|_{p,\omega} \tag{2.6}$$

for $f \in L_\omega^p$.

The last remark implies the following.

Corollary 2.7. *Let $p \in (1, \infty)$, $\omega \in A_p$, $r \in \mathbb{R}^+$ and $f \in L_\omega^p$. Then*

(i) *There exists a constant $C_{p,\omega,r} > 0$, independent of h, k, such that*

$$\left\{ \left\| \nabla_{h,\circ}^{r,\circ} f \right\|_{p,\omega}, \left\| \nabla_{\circ,k}^{\circ,r} f \right\|_{p,\omega}, \left\| \nabla_{h,k}^{r,r} f \right\|_{p,\omega} \right\} \leq C_{p,\omega,r} \left\| f \right\|_{p,\omega} \tag{2.7}$$

holds for $r > 0$.

(ii) *There holds*

$$\Omega_r \left(f, \delta_1, \delta_2 \right)_{p,\omega} \leq C_{\left[\omega \right]_{A_p}, p, r} \left\| f \right\|_{p,\omega}$$

with constant depending only on $\left[\omega \right]_{A_p}$ and p, r.

(iii) $\Omega_r (f, 0, 0)_{p,\omega} = 0.$

(iv) $\Omega_r (f, \delta_1, \delta_2)_{p,\omega}$ *is subadditive with respect to* f.

(v) $\Omega_r (f, \delta_1, \delta_2)_{p,\omega} \leq \Omega_r (f, t_1, t_2)_{p,\omega}$ *for* $0 \leq \delta_i \leq t_i;$ $i = 1, 2.$

2.3. Some means of Fourier series

Let $1 < p < \infty$, $\omega \in A_p$, and $f \in L_\omega^p$, then one can find a $p^* \in (1, \infty)$ with $f \in L^{p^*} (\mathbb{T}^2)$. Hence,

$$L_\omega^p \subset L^{p^*}.$$

Let $1 < p < \infty$, $\omega \in A_p$ and

$$\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty A_{n_1,n_2} (x, y) \tag{2.8}$$

be the corresponding Fourier series for $f \in L_\omega^p \subset L^1$. For the Fourier series (2.8) of $f \in L_\omega^p$, $1 < p < \infty$, $\omega \in A_p$ we define

$$S_{m,\circ} (f) (x, y) = \sum_{n_1=0}^m \sum_{n_2=0}^\infty A_{n_1,n_2} (x, y, f), S_{\circ,n} (f) (x, y) = \sum_{n_1=0}^\infty \sum_{n_2=0}^n A_{n_1,n_2} (x, y, f),$$

$$S_{m,n} (f) (x, y) = S_{m,\circ} (S_{\circ,n} (f)) (x, y) = \sum_{n_1=0}^m \sum_{n_2=0}^n A_{n_1,n_2} (x, y, f),$$

and de la Valleè Poussin means of f

$$V_{m,\circ} (f) = \frac{1}{m + 1} \sum_{k=m}^{2m-1} S_{k,\circ} (f), V_{\circ,n} (f) = \frac{1}{n + 1} \sum_{l=n}^{2n-1} S_{\circ,l} (f), \tag{2.9}$$

$$V_{m,n} (f) = V_{m,\circ} (V_{\circ,n} (f)) = \frac{1}{(n + 1)(m + 1)} \sum_{k=m}^{2m-1} \sum_{l=n}^{2n-1} S_{k,l} (f). \tag{2.10}$$

In what follows, $A \lesssim B$ will mean that the inequality $A \leq CB$ holds. If $A \lesssim B$ and $B \lesssim A$ we will write $A \approx B$.

Lemma 2.8 ([4]). *If* $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$, *then*

(i)

$$\left\{ \|S_{m,\circ} (f)\|_{p,\omega}, \|S_{\circ,n} (f)\|_{p,\omega}, \|S_{m,n} (f)\|_{p,\omega} \right\} \lesssim \|f\|_{p,\omega},$$

(ii)

$$\left\{ \|V_{m,\circ} (f)\|_{p,\omega}, \|V_{\circ,n} (f)\|_{p,\omega}, \|V_{m,n} (f)\|_{p,\omega} \right\} \lesssim \|f\|_{p,\omega},$$

(iii)

$$\|f - W_{m,n} f\|_{p,\omega} \lesssim Y_{m,n} (f)_{p,\omega}$$

where

$$W_{m,n} f (x, y) := (V_{m,\circ} (f) + V_{\circ,n} (f) - V_{m,n} (f)) (x, y)$$

and all constants depending only on $[\omega]_{A_p}$ *and* p.

2.4. Bernstein inequalities

Lemma 2.9 (Bernstein's Inequality, [4]). *If* $1 < p < \infty$, $\omega \in A_p$, $T_1 \in \mathfrak{T}_{m,\circ}, T_2 \in \mathfrak{T}_{\circ,n}, T_3 \in \mathfrak{T}_{m,n}$, $j,l \in \mathbb{N}$, *then*

$$\left\| T_1^{(j,\circ)} \right\|_{p,\omega} \lesssim m^j \left\| T_1 \right\|_{p,\omega}, \quad \left\| T_2^{(\circ,l)} \right\|_{p,\omega} \lesssim n^l \left\| T_2 \right\|_{p,\omega},$$

and, as a result,

$$\left\| T_3^{(j,l)} \right\|_{p,\omega} \lesssim m^j n^l \left\| T_3 \right\|_{p,\omega}$$

with constants depending only on $[\omega]_{A_p}$ *and* p.

Suppose that $\| \cdot \|_{L_\omega^p}$ is the one dimensional norm in $L_\omega^p(\mathbb{T})$,

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt,$$

and $\mathfrak{U}_n$ is the collection of all one dimensional trigonometric polynomial of degree at most n.

Lemma 2.10 ([3]). *Let* $r \in \mathbb{R}^+$, $n \in \mathbb{N}$, $p \in (1,\infty)$, $\omega \in A_p$ *and* $U_n \in \mathfrak{U}_n$. *Then*

$$h^{2r} \left\| U_n^{(2r)} \right\|_{L_\omega^p} \lesssim \left\| (I - \sigma_h)^r U_n \right\|_{L_\omega^p}$$

holds for any $h \in (0, \pi/n]$ *with some constant depending only on* r, p *and* $[\omega]_{A_p}$.

From the last lemma we obtain that if $T_m \in \mathfrak{U}_m$ then

$$\left(\int_{\mathbb{T}} \left| T_m^{(r)}(x) \right|^p \omega(x)\, dx \right)^{1/p} \lesssim m^r \left(\int_{\mathbb{T}} \left| \left[T_m(x) - \sigma_{\frac{1}{m}} T_m(x) \right]^{r/2} \right|^p \omega(x)\, dx \right)^{1/p}$$

$$\lesssim m^r \left(\int_{\mathbb{T}} \left| T_m(x) \right|^p \omega(x)\, dx \right)^{1/p}$$

and, accordingly

$$\int_{\mathbb{T}} \left| T_m^{(r)}(x) \right|^p \omega(x)\, dx \lesssim m^{rp} \int_{\mathbb{T}} \left| T_m(x) \right|^p \omega(x)\, dx. \tag{2.11}$$

Lemma 2.11 (Fractional Bernstein Inequality). *Let* $1 < p < \infty$, $\omega \in A_p$, $T_1 \in \mathfrak{T}_{m,\circ}, T_2 \in \mathfrak{T}_{\circ,n}, T_3 \in \mathfrak{T}_{m,n}$, $j,l \in \mathbb{R}^+$. *Then*

$$\left\| T_1^{(j,\circ)} \right\|_{p,\omega} \lesssim m^j \left\| T_1 \right\|_{p,\omega}, \quad \left\| T_2^{(\circ,l)} \right\|_{p,\omega} \lesssim n^l \left\| T_2 \right\|_{p,\omega},$$

and, as a result,

$$\left\| T_3^{(j,l)} \right\|_{p,\omega} \lesssim m^j n^l \left\| T_3 \right\|_{p,\omega}$$

with constants depending only on $[\omega]_{A_p}$ *and* p.

As a corollary of Lemma 2.11 and [4] we have the following lemma.

Lemma 2.12. *For* $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$, $\varsigma, l \in \mathbb{R}^+$ *there exists a constant depending only on* $[\omega]_{A_p}$ *and* p *so that*

$$\left\| [\varphi_{i,j}(f)]^{(\varsigma,l)} \right\|_{p,\omega} \lesssim 2^{i\varsigma} 2^{jl} Y_{\lfloor 2^{i-1} \rfloor, \lfloor 2^{j-1} \rfloor}(f)_{p,\omega},$$

$$\left\| [\psi_{i,j}(f)]^{(\varsigma,\circ)} \right\|_{p,\omega} \lesssim 2^{i\varsigma} Y_{\lfloor 2^{i-1} \rfloor, 2^j}(f)_{p,\omega}, \quad \left\| [h_{i,j}(f)]^{(\circ,l)} \right\|_{p,\omega} \lesssim 2^{jl} Y_{2^i, \lfloor 2^{j-1} \rfloor}(f)_{p,\omega},$$

where

$$V_{2^i,2^j}(f) - V_{2^i,\lfloor 2^{j-1}\rfloor}(f) - V_{\lfloor 2^{i-1}\rfloor,2^j}(f) + V_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f) =: \varphi_{i,j}(f) \in \mathcal{T}_{2^{i+1}-1,2^{j+1}-1},$$

$$V_{2^i,\circ}\left(f - V_{\circ,2^j}(f)\right) - V_{\lfloor 2^{i-1}\rfloor,\circ}\left(f - V_{\circ,2^j}(f)\right) =: \psi_{i,j}(f) \in \mathcal{T}_{2^{i+1}-1,\circ},$$

$$V_{\circ,2^j}\left(f - V_{2^i,\circ}(f)\right) - V_{\circ,\lfloor 2^{j-1}\rfloor}\left(f - V_{2^i,\circ}(f)\right) =: h_{i,j}(f) \in \mathcal{T}_{\circ,2^{j+1}-1}.$$

3. Favard inequalities

Lemma 3.1. *Let $1 < p < \infty$, $\omega \in A_p$, and $r \in \mathbb{R}^+$. Then, there exist constants depending only on $[\omega]_{A_p}$ and p, r such that*

$$Y_{m,n}(g_1)_{p,\omega} \lesssim (m+1)^{-2r}\left\|g_1^{(2r,\circ)}\right\|_{p,\omega}, \quad g_1 \in W_{p,\omega}^{2r,\circ}, \tag{3.1}$$

$$Y_{m,n}(g_2)_{p,\omega} \lesssim (n+1)^{-2r}\left\|g_2^{(\circ,2r)}\right\|_{p,\omega}, \quad g_2 \in W_{p,\omega}^{\circ,2r},$$

$$Y_{m,n}(g)_{p,\omega} \lesssim (m+1)^{-2r}(n+1)^{-2r}\left\|g^{(2r,2r)}\right\|_{p,\omega}, \quad g \in W_{p,\omega}^{2r,2r}. \tag{3.2}$$

Theorem 3.2 ([5])**.** *Let $1 < p < \infty$, $\omega \in A_p$, $f \in L_\omega^p$, and $r \in \mathbb{N}$. Then there exist constants depending only on $[\omega]_{A_p}$ and p, r so that*

$$\Omega_r(g_1,\delta,\cdot)_{p,\omega} \lesssim \delta^2\Omega_{r-1}\left(g_1^{(2,\circ)},\delta,\cdot\right)_{p,\omega}, \quad g_1 \in W_{p,\omega}^{2,\circ}, \tag{3.3}$$

$$\Omega_r(g_2,\cdot,\xi)_{p,\omega} \lesssim \xi^2\Omega_{r-1}\left(g_2^{(\circ,2)},\cdot,\xi\right)_{p,\omega}, \quad g_2 \in W_{p,\omega}^{\circ,2}, \tag{3.4}$$

$$\Omega_r(g,\delta,\xi)_{p,\omega} \lesssim \delta^2\xi^2\Omega_{r-1}\left(g^{(2,2)},\delta,\xi\right)_{p,\omega}, \quad g \in W_{p,\omega}^{2,2}, \tag{3.5}$$

hold for $\delta, \xi > 0$.

Corollary 3.3. *Let $1 < p < \infty$, $m, n \in \mathbb{N}$, $\omega \in A_p$, and $r \in \mathbb{R}^+$. Then there exist constants depending only on $[\omega]_{A_p}$ and p, r such that*

$$\Omega_r(T_1,\pi/(m+1),\cdot)_{p,\omega} \lesssim (m+1)^{-2r}\left\|T_1^{(2r,\circ)}\right\|_{p,\omega}, \quad T_1 \in \mathcal{T}_{m,\circ},$$

$$\Omega_r(T_2,\cdot,\pi/(n+1))_{p,\omega} \lesssim (n+1)^{-2r}\left\|T_2^{(\circ,2r)}\right\|_{p,\omega}, \quad T_2 \in \mathcal{T}_{\circ,n}, \text{ and}$$

$$\Omega_r(T_3,\pi/(m+1),\pi/(n+1))_{p,\omega} \lesssim ((m+1)(n+1))^{-2r}\left\|T_3^{(2r,2r)}\right\|_{p,\omega}, \quad T_3 \in \mathcal{T}_{m,n}$$

hold.

For $r = 1$ Corollary 3.3 was proved in [4].

4. Proof of the results

Proof of Theorem 2.1. From [6, Th.10] we have, for almost every y,

$$\int_{\mathbb{T}} |\mathcal{S}_{\lambda,\tau;\circ}f(x,y)|^p\,\omega(x,y)\,dx \leq 108\pi^2[\omega]_{A_p}\int_{\mathbb{T}} |f(x,y)|^p\,\omega(x,y)\,dx.$$

The last inequality imply that

$$\int_{\mathbb{T}^2} |\mathcal{S}_{\lambda,\tau;\circ}f(x,y)|^p\,\omega(x,y)\,dxdy \leq 108\pi^2[\omega]_{A_p}\int_{\mathbb{T}^2} |f(x,y)|^p\,\omega(x,y)\,dxdy,$$

$$\|\mathcal{S}_{\lambda,\tau;\circ}f\|_{p,\omega} \leq 108^{\frac{1}{p}}\pi^{\frac{2}{p}}[\omega]_{A_p}^{\frac{1}{p}}\|f\|_{p,\omega}.$$

Completely similar arguments give

$$\|\mathcal{S}_{\circ;\theta,\rho}f\|_{p,\omega} \leq 108^{\frac{1}{p}}\pi^{\frac{2}{p}}[\omega]_{A_p}^{\frac{1}{p}}\|f\|_{p,\omega}.$$

Summing up obtained inequalities

$$\|\mathcal{S}_{\lambda,\tau;\theta,\rho}f\|_{p,\omega} = \|\mathcal{S}_{\lambda,\tau;\circ}(\mathcal{S}_{\circ;\theta,\rho}f)\|_{p,\omega}$$

$$\leq 108^{\frac{1}{p}}\pi^{\frac{2}{p}}[\omega]_{A_p}^{\frac{1}{p}}\|\mathcal{S}_{\circ;\theta,\rho}f\|_{p,\omega} \leq 108^{\frac{2}{p}}\pi^{\frac{4}{p}}[\omega]_{A_p}^{\frac{2}{p}}\|f\|_{p,\omega}$$

as desired. $\square$

Proof of Lemma 2.3. Since $C\left[\mathbb{T}^2\right]$ is a dense subset of $f \in L_\omega^p$, $1 < p < \infty$, $\omega \in A_p$, we consider only the case $f \in C\left[\mathbb{T}^2\right]$. Let $0 < h, k \leq 1$ be given. Suppose that $u, v \in I_{h_0}^{k_0}$ and $\varepsilon > 0$. When $\max\{|u_1|, |v_1|\} \to 0$ we have $x + u + u_1 \to x + u$ and $y + v + v_1 \to y + v$. Also

$$\sigma_{h,k}f(x + u + u_1, y + v + v_1) \to \sigma_{h,k}f(x + u, y + v).$$

Then one can find a $\delta(\varepsilon) > 0$ so that

$$|\sigma_{h,k}f(x + u + u_1, y + v + v_1) - \sigma_{h,k}f(x + u, y + v)| < \varepsilon$$

for $|u_1|, |v_1| < \delta$. Hence,

$$|F_f(u + u_1, v + v_1) - F_f(u, v)|$$

$$= \left|\int_{\mathbb{T}^2}[\sigma_{h,k}f(x + u + u_1, y + v + v_1) - \sigma_{h,k}f(x + u, y + v)]|G(x,y)|\,\omega(x,y)\,dxdy\right|$$

$$\leq \varepsilon\int_{\mathbb{T}^2}|G(x,y)|\,\omega(x,y)\,dxdy \leq \varepsilon\|\omega\|_{1,1}^{1/p}\|G\|_{q,\omega} = \varepsilon\|\omega\|_{1,1}^{1/p}$$

for $|u_1|, |v_1| < \delta$. A density result now give that $F_f(u, v)$ is uniformly continuous on $I_{h_0}^{k_0}$. From this we can write

$$\tilde{F}_f(u, v) \in C\left(\mathbb{T}^2\right).$$

$\square$

Proof of Lemma 2.4. (2.4) is known (see e.g., Proposition 3.1 of [9, p.250] for any measure $d\mu$:

$$\sup_{G \in L_{q,d\mu}:\|G\|_{q,d\mu}=1}\int_{\mathbb{T}^2}f(x,y)\,G(x,y)\,d\mu = \|f\|_{p,d\mu}$$

When $d\mu = \omega(x,y)\,dxdy$ (2.4) also holds. $\square$

Proof of Theorem 2.5. Let $0 < h, k \leq 1$, $\omega \in A_p$, $1 < p < \infty$ and $f, g \in L_\omega^p$. In this case

$$\|F_g\|_{C\left(I_{h_0}^{k_0}\right)} \leq c\|F_f\|_{C\left(I_{h_0}^{k_0}\right)}$$

$$= c\left\|\int_{\mathbb{T}^2}\mathcal{S}_{\frac{1}{h_0},0;\frac{1}{k_0},0}f(x + u, y + v)|G(x,y)|\,\omega(x,y)\,dxdy\right\|_{C\left(I_{h_0}^{k_0}\right)}$$

$$= c\left|\max_{u,v \in I_{h_0}^{k_0}}\int_{\mathbb{T}^2}\mathcal{S}_{\frac{1}{h_0},0;\frac{1}{k_0},0}f(x + u, y + v)|G(x,y)|\,\omega(x,y)\,dxdy\right|$$

$$\leq c\max_{u,v \in I_{h_0}^{k_0}}\left\|\mathcal{S}_{\frac{1}{h_0},0;\frac{1}{k_0},0}f(\cdot + u, \cdot + v)\right\|_{p,\omega}\|G\|_{q,\omega}$$

$$\leq c\max_{u,v \in I_{h_0}^{k_0}}\left\|\mathcal{S}_{\frac{1}{h_0},u;\frac{1}{k_0},v}\right\|_{L_\omega^p \to L_\omega^p}\|f\|_{p,\omega}, \qquad \text{(by Theorem 2.1)}$$

$$\leq c108^{\frac{2}{p}}\pi^{\frac{4}{p}}[\omega]_{A_p}^{\frac{2}{p}}\|f\|_{p,\omega}.$$

On the other hand, for any $\varepsilon > 0$ and appropriately chosen $G \in L_\omega^q(\mathbb{T}^2)$ with $\langle g, G \rangle = \int_{\mathbb{T}^2} g(x, y) G(x, y) \omega(x, y) \, dx dy \geq \|g\|_{p,\omega} - \varepsilon$, $\|G\|_{q,\omega} = 1$, (see Lemma 2.4 above), one can find

$$
\begin{aligned}
\|F_g\|_{C\left(I_{h_0}^{k_0}\right)} &\geq |F_g(0,0)| \geq \int_{\mathbb{T}^2} \mathcal{S}_{\frac{1}{h_0},0;\frac{1}{k_0},0} g(x,y) |G(x,y)| \omega(x,y) \, dx dy \\
&\geq \int_{\mathbb{T}^2} g(x,y) |G(x,y)| \omega(x,y) \, dx dy - \zeta \int_{\mathbb{T}^2} |G(x,y)| \omega(x,y) \, dx dy \\
&\geq \|g\|_{p,\omega} - \varepsilon - \zeta \|\omega\|_{1,1}^{1/p}.
\end{aligned}
$$

Since $\varepsilon, \zeta > 0$ is arbitrary, from the last inequality, we have

$$
\|F_g\|_{C\left(I_{h_0}^{k_0}\right)} \geq \|g\|_{p,\omega}
$$

and hence

$$
\|g\|_{p,\omega} \leq \|F_g\|_{C\left(I_{h_0}^{k_0}\right)} \leq c \|F_f\|_{C\left(I_{h_0}^{k_0}\right)} \leq c108^{\frac{2}{p}} \pi^{\frac{4}{p}} [\omega]_{A_p}^{\frac{2}{p}} \|f\|_{p,\omega}.
$$

This gives required result. $\qquad\square$

Proof of Theorem 2.7. Inequalities (2.7) hold with (2.5)-(2.6). $\qquad\square$

Proof of Lemma 2.11. From (2.11), we have for almost every y

$$
\int_{\mathbb{T}} \left|T_1^{(j,\circ)}(x,y)\right|^p \omega(x,y) \, dx \lesssim m^{jp} \int_{\mathbb{T}} |T_1(x,y)|^p \omega(x,y) \, dx.
$$

Hence

$$
\int_{\mathbb{T}^2} \left|T_1^{(j,\circ)}(x,y)\right|^p \omega(x,y) \, dx dy \lesssim m^{jp} \int_{\mathbb{T}^2} |T_1(x,y)|^p \omega(x,y) \, dx dy,
$$

$$
\left\|T_1^{(j,\circ)}\right\|_{p,\omega} \lesssim m^j \|T_1\|_{p,\omega}.
$$

From (2.11), we have for almost every x

$$
\int_{\mathbb{T}} \left|T_2^{(\circ,l)}(x,y)\right|^p \omega(x,y) \, dy \lesssim n^{lp} \int_{\mathbb{T}} |T_2(x,y)|^p \omega(x,y) \, dy.
$$

Then

$$
\int_{\mathbb{T}^2} \left|T_2^{(\circ,l)}(x,y)\right|^p \omega(x,y) \, dx dy \lesssim n^{lp} \int_{\mathbb{T}^2} |T_2(x,y)|^p \omega(x,y) \, dx dy,
$$

$$
\left\|T_2^{(\circ,l)}\right\|_{p,\omega} \lesssim n^l \|T_2\|_{p,\omega}
$$

holds. $\qquad\square$

Proof of Lemma 3.1. We consider (3.1). We have

$$
\|g_1 - S_{m,\circ}(g_1) - S_{\circ,n}(g_1) + S_{m,n}(g_1)\|_{p,\omega} = \left\|\sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} A_{i,j}(x,y,g_1)\right\|_{p,\omega}
$$

$$
= \left\|\sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{1}{i^{2r}} i^{2r} A_{i,j}\left(x + \frac{\pi}{2}, y, g_1\right) \cos \pi\right\|_{p,\omega}
$$

$$
= \left\|-\sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{1}{i^{2r}} A_{i,j}\left(x, y, g_1^{(2r,\circ)}\right)\right\|_{p,\omega}
$$

$$= \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{\left[S_{i,j}\left[g_1^{(2r,\circ)} \right] - S_{i,j-1}\left[g_1^{(2r,\circ)} \right] - S_{i-1,j}\left[g_1^{(2r,\circ)} \right] + S_{i-1,j-1}\left[g_1^{(2r,\circ)} \right] \right]}{i^{2r}} \right\|_{p,\omega}$$

$$= \left\| \sum_{i=m+1}^{\infty} \left[\frac{1}{(i+1)^{2r}} - \frac{1}{i^{2r}} \right] S_{i,m}\left(g_1^{(2r,\circ)} \right) + \frac{1}{(m+1)^2} S_{m,n}\left(g_1^{(2r,\circ)} \right) \right\|_{p,\omega}$$

$$\leq \sum_{i=m+1}^{\infty} \left| \frac{1}{(i+1)^{2r}} - \frac{1}{i^{2r}} \right| \left\| S_{i,m}\left(g_1^{(2r,\circ)} \right) \right\|_{p,\omega} + \frac{1}{(m+1)^{2r}} \left\| S_{m,n}\left(g_1^{(2r,\circ)} \right) \right\|_{p,\omega}$$

$$\leq \left\| g_1^{(2r,\circ)} \right\|_{p,\omega} \left(\sum_{i=m+1}^{\infty} \left(\frac{1}{i^{2r}} - \frac{1}{(i+1)^{2r}} \right) + \frac{1}{(m+1)^{2r}} \right) \leq \frac{C}{(m+1)^{2r}} \left\| g_1^{(2r,\circ)} \right\|_{p,\omega}.$$

Using

$$Y_{m,n}\left(g_1 \right)_{p,\omega} = Y_{m,n}\left(g_1 - S_{m,\circ}\left(g_1 \right) - S_{\circ,n}\left(g_1 \right) + S_{m,n}\left(g_1 \right) \right)_{p,\omega}$$

$$\leq \left\| g_1 - S_{m,\circ}\left(g_1 \right) - S_{\circ,n}\left(g_1 \right) + S_{m,n}\left(g_1 \right) \right\|_{p,\omega}$$

one can find (3.1). The same method give

$$\left\| g_2 - S_{\circ,n}\left(g_2 \right) - S_{m,\circ}\left(g_2 \right) + S_{m,n}\left(g_2 \right) \right\|_{p,\omega} \lesssim \frac{1}{(n+1)^{2r}} \left\| g_2^{(\circ,2r)} \right\|_{p,\omega}$$

and

$$Y_{m,n}\left(g_2 \right)_{p,\omega} = Y_{m,n}\left(g_2 - S_{\circ,n}\left(g_2 \right) - S_{m,\circ}\left(g_2 \right) + S_{m,n}\left(g_2 \right) \right)_{p,\omega}$$

$$\leq \left\| g_1 - S_{\circ,n}\left(g_1 \right) - S_{m,\circ}\left(g_1 \right) + S_{m,n}\left(g_1 \right) \right\|_{p,\omega} \lesssim \frac{1}{(n+1)^{2r}} \left\| g_2^{(\circ,2r)} \right\|_{p,\omega}.$$

Considering (3.2), we find

$$\left\| g - S_{m,\circ}\left(g \right) - S_{\circ,n}\left(g \right) + S_{m,n}\left(g \right) \right\|_{p,\omega}$$

$$= \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} A_{i,j}\left(x,y,g \right) \right\|_{p,\omega} = \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{i^{2r} j^{2r}}{i^{2r} j^{2r}} A_{i,j}\left(x + \frac{\pi}{2}, y + \frac{\pi}{2}, g \right) \cos^2 \pi \right\|_{p,\omega}$$

$$= \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{1}{i^{2r} j^{2r}} A_{i,j}\left(x,y,g^{(2r,2r)} \right) \right\|_{p,\omega} = \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{1}{i^{2r} j^{2r}} A_{i,j}\left(x,y,\Upsilon \right) \right\|_{p,\omega}$$

$$= \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{1}{i^{2r} j^{2r}} \left(S_{i,j}\left(\Upsilon \right) - S_{i,j-1}\left(\Upsilon \right) - S_{i-1,j}\left(\Upsilon \right) + S_{i-1,j-1}\left(\Upsilon \right) \right) \right\|_{p,\omega}$$

$$= \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \left[\frac{1}{(i+1)^{2r}} - \frac{1}{i^{2r}} \right] \left[\frac{1}{(j+1)^{2r}} - \frac{1}{j^{2r}} \right] S_{i,j}\left(\Upsilon \right) + \right.$$

$$+ \frac{1}{(m+1)^{2r}} \sum_{j=n+1}^{\infty} \left[\frac{1}{(j+1)^{2r}} - \frac{1}{j^{2r}} \right] S_{m,j}\left(\Upsilon \right) + \frac{1}{(n+1)^{2r}} \sum_{i=n+1}^{\infty} \left[\frac{1}{(i+1)^{2r}} - \frac{1}{i^{2r}} \right] S_{i,n}\left(\Upsilon \right)$$

$$\left. + \frac{1}{(m+1)^{2r}} \frac{1}{(n+1)^{2r}} S_{m,n}\left(\Upsilon \right) \right\|_{p,\omega}$$

$$\leq \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \left(\frac{1}{i^{2r}} - \frac{1}{(i+1)^{2r}} \right) \left(\frac{1}{j^{2r}} - \frac{1}{(j+1)^{2r}} \right) \left\| S_{i,j}\left(\Upsilon \right) \right\|_{p,\omega} +$$

$$+ \frac{1}{(m+1)^{2r}} \sum_{j=n+1}^{\infty} \left[\frac{1}{j^{2r}} - \frac{1}{(j+1)^{2r}} \right] \left\| S_{m,j}\left(\Upsilon \right) \right\|_{p,\omega} + \frac{1}{(n+1)^{2r}} \times$$

$$\times \sum_{i=m+1}^{\infty} \left[\frac{1}{i^{2r}} - \frac{1}{(i+1)^{2r}} \right] \|S_{i,n}(\Upsilon)\|_{p,\omega} + \frac{1}{(m+1)^{2r}} \frac{1}{(n+1)^{2r}} \|S_{m,n}(\Upsilon)\|_{p,\omega}$$

$$\lesssim \frac{1}{(m+1)^{2r}(n+1)^{2r}} \|\Upsilon\|_{p,\omega} = \frac{1}{(m+1)^{2r}(n+1)^{2r}} \left\|g^{(2r,2r)}\right\|_{p,\omega},$$

and, hence,

$$Y_{m,n}(g)_{p,\omega} = Y_{m,n}(g - S_{m,\circ}(g) - S_{\circ,n}(g) + S_{m,n}(g))_{p,\omega}$$

$$\leq \|g - S_{m,\circ}(g) - S_{\circ,n}(g) + S_{m,n}(g)\|_{p,\omega} \lesssim \frac{1}{(m+1)^{2r}(n+1)^{2r}} \left\|g^{(2r,2r)}\right\|_{p,\omega}$$

which completes the proof.   □

Here we give the proof of Potapov type Theorem 1.2.

Proof of Theorem 1.2. For $r \in \mathbb{N}$, this was obtained in [5]:

$$Y_{m,n}(f)_{p,\omega} \leq C_{p,r,[\omega]_{A_p}} \Omega_r \left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}. \tag{4.1}$$

We suppose that $r \in \mathbb{R}^+ \backslash \mathbb{N}$. For $0 \leq \alpha \leq \beta \leq 1$

$$\int_{\mathbb{T}} \left| [I - \sigma_{h,\circ}]^{\beta} f(x,y) \right|^p \omega(x,y)\, dx \lesssim \int_{\mathbb{T}} |[I - \sigma_{h,\circ}]^{\alpha} f(x,y)|^p \omega(x,y)\, dx, \quad \text{for almost every } y,$$

$$\int_{\mathbb{T}} \left| [I - \sigma_{\circ,k}]^{\beta} f(x,y) \right|^p \omega(x,y)\, dx \lesssim \int_{\mathbb{T}} |[I - \sigma_{\circ,k}]^{\alpha} f(x,y)|^p \omega(x,y)\, dx, \quad \text{for almost every } x,$$

were proved in [1, (2.5)]. The same proof also holds for $0 \leq \alpha \leq \beta < \infty$. Hence

$$\Omega_{\beta}(f, \cdot, \cdot)_{p,\omega} \lesssim \Omega_{\alpha}(f, \cdot, \cdot)_{p,\omega}.$$

From (4.1) and the last inequality we have

$$Y_{m,n}(f)_{p,\omega} \leq c\Omega_{[r]+1}\left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} \leq C\Omega_r \left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}, \quad m,n \in \mathbb{N}.$$

                                                                                                          □

Proof of Corollary 3.3. Let $r \in \mathbb{R}^+$ and $p \in (1,\infty)$. Suppose that $\omega \in A_p(\mathbb{T})$, $T \in \mathfrak{U}_n$ and $0 < h < \pi/(n+1)$. From one dimensional inequality [1,2]

$$\|[I - \sigma_h]^r T\|_{L_\omega^p} \lesssim (n+1)^{-2r} \left\| T^{(2r)} \right\|_{L_\omega^p},$$

we obtain

$$\int_{\mathbb{T}} |[I - \sigma_h]^r T(x)|^p \omega(x)\, dx \lesssim (n+1)^{-2rp} \int_{\mathbb{T}} \left| T^{(2r)}(x) \right|^p \omega(x)\, dx. \tag{4.2}$$

From (4.2), we have for almost every y

$$\int_{\mathbb{T}} |[I - \sigma_{h,\circ}]^r T_1(x,y)|^p \omega(x,y)\, dx \lesssim (m+1)^{-2rp} \int_{\mathbb{T}} \left| T_1^{(2r,\circ)}(x,y) \right|^p \omega(x,y)\, dx.$$

Then

$$\int_{\mathbb{T}^2} |[I - \sigma_{h,\circ}]^r T_1(x,y)|^p \omega(x,y)\, dxdy \lesssim (m+1)^{-2rp} \int_{\mathbb{T}^2} \left| T_1^{(2r,\circ)}(x,y) \right|^p \omega(x,y)\, dxdy,$$

$$\Omega_r(T_1, \pi/(m+1), \cdot)_{p,\omega} \lesssim (m+1)^{-2r} \left\| T_1^{(2r,\circ)} \right\|_{p,\omega}$$

holds. Other inequalities can be proved by the same method.   □

Proof of Theorem 1.4 . For $\delta, \xi > 0$, there exist natural numbers m, n so that $\frac{m}{\pi} \leq 1/\delta < 2\frac{m}{\pi}$, $\quad \frac{n}{\pi} \leq 1/\xi < 2\frac{n}{\pi}$. Setting

$$\Theta_{m,n}(f) := S_{m,\circ}(f) - S_{\circ,n}(f) + S_{m,n}(f),$$

from Theorem 1.2 one can get

$$A_1 = \|f - S_{m,\circ}(f) - S_{\circ,n}(f) + S_{m,n}(f)\|_{p,\omega}$$

$$\lesssim Y_{m,n}(f)_{p,\omega} \lesssim \Omega_r\left(f, m^{-1}, n^{-1}\right)_{p,\omega} \lesssim \Omega_r\left(f, \pi m^{-1}, \pi n^{-1}\right)_{p,\omega}.$$

Secondly, we set $A_2 = \left\|S_{m,\circ}^{(2r,\circ)}(f - S_{\circ,n}(f))\right\|_{p,\omega}$ and $\gamma = f - S_{\circ,n}(f)$. By one dimensional inequality (see Lemma 2.10)

$$\delta^{2r}\left(\int_{\mathbb{T}} \left|T_m^{(2r)}(x)\right|^p \omega(x)\,dx\right)^{1/p} \lesssim \left(\int_{\mathbb{T}} \left|\left[T_m(x) - \sigma_{\frac{1}{m}} T_m(x)\right]^r\right|^p \omega(x)\,dx\right)^{1/p}$$

we get, for almost every y,

$$\delta^{2r}\left(\int_{\mathbb{T}} \left|S_{m,\circ}^{(2r,\circ)}(\gamma)\right|^p \omega(x,y)\,dx\right)^{1/p} \lesssim \left(\int_{\mathbb{T}} \left|\nabla_{\frac{1}{m},\circ}^{r,\circ} S_{m,\circ}(\gamma)\right|^p \omega(x,y)\,dx\right)^{1/p}, \quad \text{and}$$

$$\delta^{2rp}\int_{\mathbb{T}} \left|S_{m,\circ}^{(2r,\circ)}(\gamma)\right|^p \omega(x,y)\,dx \lesssim \int_{\mathbb{T}} \left|\nabla_{\frac{1}{m},\circ}^{r,\circ} S_{m,\circ}(\gamma)\right|^p \omega(x,y)\,dx.$$

Then

$$\delta^{2rp}\int_{\mathbb{T}^2} \left|S_{m,\circ}^{(2r,\circ)}(\gamma)\right|^p \omega(x,y)\,dxdy \lesssim \int_{\mathbb{T}^2} \left|\nabla_{\frac{1}{m},\circ}^{r,\circ} S_{m,\circ}(\gamma)\right|^p \omega(x,y)\,dxdy, \quad \text{and}$$

$$\delta^{2r}A_2 \lesssim \left\|\nabla_{\frac{1}{m},\circ}^{r,\circ} S_{m,\circ}(\gamma)\right\|_{p,\omega} = \left\|S_{m,\circ}\left(\nabla_{\frac{1}{m},\circ}^{r,\circ}\gamma\right)\right\|_{p,\omega}$$

$$\lesssim \left\|\nabla_{\frac{1}{m},\circ}^{r,\circ}\gamma\right\|_{p,\omega} = \left\|\nabla_{\frac{1}{m},\circ}^{r,\circ}(f - S_{\circ,n}(f))\right\|_{p,\omega} = \left\|\nabla_{\frac{1}{m},\circ}^{r,\circ}f - \nabla_{\frac{1}{m},\circ}^{r,\circ}S_{\circ,n}(f)\right\|_{p,\omega}$$

$$= \left\|\nabla_{\frac{1}{m},\circ}^{r,\circ}f - S_{0,\circ}\left(\nabla_{\frac{1}{m},\circ}^{r,\circ}f\right) - S_{\circ,n}\left(\nabla_{\frac{1}{m},\circ}^{r,\circ}f\right) + S_{0,n}\left(\nabla_{\frac{1}{m},\circ}^{r,\circ}f\right)\right\|_{p,\omega}$$

$$\lesssim Y_{0,n}\left(\nabla_{\frac{1}{m},\circ}^{r,\circ}f\right)_{p,\omega} \lesssim \Omega_r\left(\nabla_{\frac{1}{m},\circ}^{r,\circ}f, 1, \frac{1}{n}\right)_{p,\omega}$$

$$= \sup_{\substack{0 \leq h \leq 1 \\ 0 \leq k \leq 1/n}} \left\|\nabla_{h,\circ}^{r,\circ}\nabla_{\circ,k}^{\circ,r}\left(\nabla_{\frac{1}{m},\circ}^{r,\circ}f\right)\right\|_{p,\omega} \lesssim \sup_{0 \leq k \leq 1/n} \left\|\nabla_{\circ,k}^{\circ,r}\left(\nabla_{\frac{1}{m},\circ}^{r,\circ}f\right)\right\|_{p,\omega}$$

$$\lesssim \sup_{\substack{0 \leq h \leq 1/m \\ 0 \leq k \leq 1/n}} \left\|\nabla_{h,\circ}^{r,\circ}\left(\nabla_{\circ,k}^{\circ,r}f\right)\right\|_{p,\omega} = \Omega_r\left(f, \frac{1}{m}, \frac{1}{n}\right)_{p,\omega} \lesssim \Omega_r\left(f, \frac{\pi}{m}, \frac{\pi}{n}\right)_{p,\omega}.$$

Taking

$$A_3 := \left\|S_{\circ,n}^{(\circ,2r)}(f - S_{m,\circ}(f))\right\|_{p,\omega} \quad \text{and} \quad A_4 := \left\|S_{m,n}^{(2r,2r)}(f)\right\|_{p,\omega},$$

similar arguments give us

$$\xi^{2r}A_3 \quad \lesssim \quad \left\|S_{\circ,n}^{(\circ,2r)}(f - S_{m,\circ}(f))\right\|_{p,\omega} \lesssim \Omega_r\left(f, \pi m^{-1}, \pi n^{-1}\right)_{p,\omega},$$

$$\delta^{2r}\xi^{2r}A_4 \quad \lesssim \quad \left\|S_{m,n}^{(2r,2r)}(f)\right\|_{p,\omega} \lesssim \Omega_r\left(f, \pi m^{-1}, \pi n^{-1}\right)_{p,\omega}, \text{ and}$$

$$A_1 + \delta^{2r}A_2 + \xi^{2r}A_3 + \delta^{2r}\xi^{2r}A_4 \lesssim \Omega_r(f, \delta, \xi)_{p,\omega}.$$

Definition of K-functional gives

$$K(f, \delta, \xi, p, \omega, 2r) \leq A_1 + \delta^{2r} A_2 + \xi^{2r} A_3 + \delta^{2r} \xi^{2r} A_4 \lesssim \Omega_r (f, \delta, \xi)_{p,\omega}.$$

Consider reverse of the last inequality. For any $g_1 \in W_{p,\omega}^{r,\circ}$, $g_2 \in W_{p,\omega}^{\circ,s}$, $g \in W_{p,\omega}^{r,s}$ we have

$$\Omega_r (f, \delta, \xi)_{p,\omega} \leq \Omega_r (f - g_1 - g_2 - g, \delta, \xi)_{p,\omega} + \Omega_r (g_1, \delta, \xi)_{p,\omega} +$$

$$+ \Omega_r (g_2, \delta, \xi)_{p,\omega} + \Omega_r (g, \delta, \xi)_{p,\omega} \lesssim \|f - g_1 - g_2 - g\|_{p,\omega} +$$

$$+ \delta^{2r} \left\| g_1^{(2r,\circ)} \right\|_{p,\omega} + \xi^{2r} \left\| g_2^{(\circ,2r)} \right\|_{p,\omega} + \delta^{2r} \xi^{2r} \left\| g^{(2r,2r)} \right\|_{p,\omega}.$$

From the last inequality, taking infimum on $g_1 \in W_{p,\omega}^{r,\circ}$, $g_2 \in W_{p,\omega}^{\circ,s}$, $g \in W_{p,\omega}^{r,s}$, one gets

$$\Omega_r (f, \delta, \xi)_{p,\omega} \lesssim K(f, \delta, \xi, p, \omega, 2r).$$

$\square$

Proof of Theorem 1.6. Using the properties of $\Omega_r (f, \cdot, \cdot)_{p,\omega}$ we have

$$\Omega_r \left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} \leq \Omega_r \left(f - W_{2^\mu, 2^\nu} f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} + \Omega_r \left(W_{2^\mu, 2^\nu} f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}, \quad \text{and}$$

$$\Omega_r \left(f - W_{2^\mu, 2^\nu} f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} \lesssim \|f - W_{2^\mu, 2^\nu}\|_{p,\omega} \lesssim Y_{2^\mu, 2^\nu} (f)_{p,\omega}.$$

By the property

$$W_{2^\mu, 2^\nu} f - W_{0,0} f \leq \sum_{i=0}^{\mu} \left(W_{2^i, 2^\nu} f - W_{\lfloor 2^{i-1} \rfloor, 2^\nu} f \right) + \sum_{j=0}^{\nu} \left(W_{2^\mu, 2^j} f - W_{2^\mu, \lfloor 2^{j-1} \rfloor} f \right) -$$

$$- \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} W_{2^i, 2^j} f - W_{2^i, \lfloor 2^{j-1} \rfloor} f - W_{\lfloor 2^{i-1} \rfloor, 2^j} f + W_{\lfloor 2^{i-1} \rfloor, \lfloor 2^{j-1} \rfloor} f,$$

we find (see Lemma 2.12 for quantities $\psi_{i,\nu} (f)$, $h_{\mu,j} (f)$, $\varphi_{i,j} (f)$)

$$\Omega_r \left(W_{2^\mu, 2^\nu} f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} = \Omega_r \left(W_{2^\mu, 2^\nu} f - W_{0,0} f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}$$

$$\leq \sum_{i=0}^{\mu} \Omega_r \left(\psi_{i,\nu} (f), \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} + \sum_{j=0}^{\nu} \Omega_r \left(h_{\mu,j} (f), \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} +$$

$$+ \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} \Omega_r \left(\varphi_{i,j} (f), \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}$$

$$\lesssim \frac{1}{m^{2r}} \sum_{i=0}^{\mu} \left\| (\psi_{i,\nu} (f))^{(2r,\circ)} \right\|_{p,\omega} + \frac{1}{n^{2r}} \sum_{j=0}^{\nu} \left\| (h_{\mu,j} (f))^{(\circ,2r)} \right\|_{p,\omega} +$$

$$+ \frac{1}{m^{2r}} \frac{1}{n^{2r}} \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} \left\| (\varphi_{i,j} (f))^{(2r,2r)} \right\|_{p,\omega} \lesssim$$

$$\lesssim \frac{1}{m^{2r}} \sum_{i=0}^{\mu} 2^{2ri} Y_{\lfloor 2^{i-1} \rfloor, 2^j} (f)_{p,\omega} + \frac{1}{n^{2r}} \sum_{j=0}^{\nu} 2^{2rj} Y_{2^i, \lfloor 2^{j-1} \rfloor} (f)_{p,\omega} +$$

$$+ \frac{1}{m^{2r}} \frac{1}{n^{2r}} \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} 2^{2ir+2rj} Y_{\lfloor 2^{i-1} \rfloor, \lfloor 2^{j-1} \rfloor} (f)_{p,\omega}.$$

Suppose that m, n satisfy $2^\mu \leq m < 2^{\mu+1}$, $2^\nu \leq n < 2^{\nu+1}$. Then, one can get

$$\Omega_r \left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} \lesssim \frac{1}{m^{2r}} \frac{1}{n^{2r}} \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} 2^{2ri+2rj} Y_{\lfloor 2^{i-1} \rfloor, \lfloor 2^{j-1} \rfloor} (f)_{p,\omega}$$

$$\lesssim \frac{1}{m^{2r}n^{2r}} \sum_{i=0}^{\mu+1} \sum_{j=0}^{\nu+1} 2^{2ri+2rj} Y_{\lfloor 2^{i-1} \rfloor, \lfloor 2^{j-1} \rfloor} (f)_{p,\omega}$$

$$\lesssim \frac{1}{m^{2r}n^{2r}} \sum_{i=0}^{m} \sum_{j=0}^{n} [(i+1)(j+1)]^{2r-1} Y_{i,j} (f)_{p,\omega}.$$

$\square$

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