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Research Article

Mus-Sasaki Metric and Complex Structures

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Abstract
In this paper we study the geometry of some paracomplex structures on tangent fiber bundle TM equipped with a Mus-Sasaki metrics.

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1. Introduction

The geometry of the tangent bundle TM equipped with Sasaki metric has been studied by many authors Sasaki [13], K. Yano and S. Ishihara [16], P. Dombrowski [5] , A. Salimov, A. Gezer and N. Cengiz [2] [6] [11] [9] [12] etc... The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM. J. Cheeger and D. Gromoll has introduced the notion of Cheeger-Gromoll metric [3], this metric has been studied also by many authors (see [1] [4] [7] [10] [14]).

In this paper, we introduce the Mus-Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM and we investigate the geometry of some paracomplex structures on tangent fiber bundle TM. First we construct some almost paracomplex Norden structures on TM (see 15, Theorem 19 and Theorem 20) and we characterize the para-Kähler-Norden structures on TM (see Theorem 17 and Theorem 22), also we construct an almost product connection and we give the necessary and sufficient conditions for the almost product connection to be symmetric (see Theorem 27).

2. Preliminaries

2.1 Horizontal and Vertical Lifts on TM.

Let (M,g) be an m-dimensional Riemannian manifold and (TM,π,M) be its tangent bundle. A local chart $(U,x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U),x^i,y^i)_{i=1...n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.
We have two complementary distributions on TM, the vertical distribution $\mathscr{V}$ and the horizontal distribution $\mathscr{H}$, defined by:

$$\begin{aligned}\mathscr{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)}; \quad a^i \in \mathbb{R}\},\\ \mathscr{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)}; \quad a^i \in \mathbb{R}\},\end{aligned}$$

where $(x,u) \in TM$, such that $T_{(x,u)}TM = \mathscr{H}_{(x,u)} \oplus \mathscr{V}_{(x,u)}$.
Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \}. \tag{2}$$

For consequences, we have the following remark:

Remark 1.

1. $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ *and* $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$.
2. $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1\ldots n}$ *is a local adapted frame in* TTM.
3. *if* $u = u^i \frac{\partial}{\partial x^i} \in T_xM$ *then its vertical and horizontal lifts are defined by:*

$$u^V = u^i \frac{\partial}{\partial y^i} \in \mathscr{V}_{(x,u)}.$$

$$u^H = u^i \{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \} \in \mathscr{H}_{(x,u)}.$$

4. *if* $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial y^j} \in T_{(x,u)}TM$, *then* $w = w^h + w^v$, *where the horizontal and vertical parts of* w *are defined by*

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} = (w^i \frac{\partial}{\partial x^i})^H \in \mathscr{H}_{(x,u)}.$$

$$w^v = \{ \overline{w}^k + w^i u^j \Gamma_{ij}^k \} \frac{\partial}{\partial y^k} = (\{ \overline{w}^k + w^i u^j \Gamma_{ij}^k \} \frac{\partial}{\partial x^k})^V \in \mathscr{V}_{(x,u)}.$$

5. $[X^H, Y^H]_p = [X,Y]_p^H - (R_x(X,Y)u)^V$,
6. $[X^H, Y^V]_p = (\nabla_X Y)_p^V$,
7. $[X^V, Y^V]_p = 0$.

For more detail see [16].

2.2 Complex Structure

Definition 2. *Let* M^{2m} *be a manifold of dimension* $2m$. *An almost product structure is a morphism* $J : TM \to TM$ *such that*

$$J^2 = Id_{TM}, \quad (J \neq \mp Id_{TM}).$$

(M^{2m}, J) *is called almost product manifold (see [8] and [16]).*

Definition 3. *An almost paracomplex manifold is an almost product manifold* (M^{2m}, J) *such that*

$$dim(Ker(J - Id_{TM})) = dim(Ker(J + Id_{TM})).$$

Definition 4. *An almost paracomplex Norden manifold* (M^{2m}, J, g) *is an almost paracomplex manifold* (M^{2m}, J) *equipped with a Riemannian metric* g, *such that*

$$g(JX, Y) = g(X, JY)$$

or

$$g(JX, JY) = g(X, Y)$$

for all $X, Y \in \Gamma(TM)$.

Definition 5. *A para-Kähler-Norden (or paraholomorphic Norden) is an almost paracomplex Norden manifold* (M^{2m}, J, g)*, such that* J *is parallel, i.e:*

$$\nabla J = 0,$$

where ∇ *is the Levi-Civita connection of* g*.*

Definition 6. *An almost para-Hermitian manifold* (M^{2m}, J, h) *is an almost paracomplex manifold* (M^{2m}, J) *equipped with a pseudo-metric* h*, such that*

$$h(JX, Y) = -h(X, JY)$$

or

$$h(JX, JY) = -h(X, Y)$$

for all $X, Y \in \Gamma(TM)$*.*

Definition 7. *Let* (M^{2m}, J, h) *be an almost para-Hermitian manifold. The* 2*-form* Ω *defined by*

$$\Omega(X, Y) = h(X, JY) \tag{3}$$

is called a 2*-para-Kähler form. An almost para-Kähler manifold* (M^{2m}, J, h) *is an almost paracomplex manifold* (M^{2m}, J) *equipped with a pseudo-metric* h*, such that*

$$d\Omega = 0.$$

Lemma 8. *Let* (M^{2m}, J, g) *be an almost paracomplex Norden manifold. If* ϕ_J *denote the Tachibana operator, defined by*

$$\begin{aligned} (\phi^J g)(X, Y, Z) &= (JX)(g(Y, Z)) - X(g(JY, Z)) \\ &\quad + g((L_Y J)X, Z) + g((Y, L_Z J)X) \end{aligned} \tag{4}$$

then

$$\nabla J = 0 \Leftrightarrow \phi^J g = 0.$$

(see [15]).

3. Mus-Sasaki Metric and Complex Structures

Definition 9. *Let* (M, g) *be a Riemannian manifold and* $f : M \times \mathbb{R} \to]0, +\infty[$*. On the tangent bundle* TM*, we define a Mus-Sasaki metric noted* g_f^S *by*

1. $g_f^S(X^H, Y^H)_{(x,u)} = g_x(X, Y)$
2. $g_f^S(X^H, Y^V)_{(x,u)} = 0$
3. $g_f^S(X^V, Y^V)_{(x,u)} = f(x, r) g_x(X, Y)$

where $X, Y \in \Gamma(TM)$*,* $(x, u) \in TM$ *and* $r = g(u, u)$*.* f *is called twisting function (see [17], [18]).*

Note that, if $f = 1$ then g_f^S is the Sasaki metric [16]. If $f(x, r) = \frac{1}{\alpha(x)}$ then $\overline{g} = \alpha(x) g_f^S$ is the rescaled metric.

Lemma 10. *Let (M,g) be a Riemannian manifold, then for all* $x \in M$ *and* $u = u^i \frac{\partial}{\partial x^i} \in T_x M$*, we have the followings:*

1. $X^H(g(u, u))_{(x,u)} = 0$
2. $X^H(g(Y, u))_{(x,u)} = g(\nabla_X Y, u)_x$
3. $X^V(g(u, u)_{(x,u)} = 2g(X, u)_x$
4. $X^V(g(Y, u)_{(x,u)} = g(X, Y)_x$

(see [17], [18]).

Proof. Locally, if $U : x \in M \to U_x = u^i \frac{\partial}{\partial x^i} \in TM$ be a local vector field constant on each fiber T_xM, then we obtain:

$$\begin{aligned}
1.\quad X^H(g(u,u))_{(x,u)} &= [X^i \frac{\partial}{\partial x^i} g_{st} y^s y^t - \Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k} g_{st} y^s y^t]_{(x,u)} \\
&= X(g(U,U)_x - 2(\Gamma^k_{ij} X^i y^j g_{sk} y^s)_{(x,u)} \\
&= (X(g(U,U)_x - 2g(U,\nabla_X U))_x \\
&= 0. \\
2.\quad X^H(g(Y,u))_{(x,u)} &= [X^i \frac{\partial}{\partial x^i} g_{st} Y^s y^t - \Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k} g_{st} Y^s y^t]_{(x,u)} \\
&= X(g(Y,U)_x - (\Gamma^k_{ij} X^i y^j g_{sk} Y^s)_{(x,u)} \\
&= (X(g(Y,U)_x - g(Y,\nabla_X U))_x \\
&= g(\nabla_X Y, U))_x. \\
3.\quad X^V(g(u,u))_{(x,u)} &= [X^i \frac{\partial}{\partial y^i} g_{st} y^s y^t]_{(x,u)} = 2X^i g_{it} u^t = 2g(X,u)_x. \\
4.\quad X^V(g(Y,u))_{(x,u)} &= [X^i \frac{\partial}{\partial y^i} g_{st} Y^s y^t]_{(x,u)} = X^i g_{si} Y^s = g(X,Y)_x.
\end{aligned}$$

■

Lemma 11. *Let (M,g) be a Riemannian manifold, $F : (s,t) \in \mathbb{R}^2 \to F(s,t) \in]0,+\infty[$, $\alpha : M \to]0,+\infty[$ and $\beta : \mathbb{R} \to]0,+\infty[$ be smooth functions. If $f(x,r) = F(\alpha(x), \beta(r))$, then we have the following*

1. $X^V(f)_{(x,u)} = 2\beta'(r) g_x(X,u) \frac{\partial F}{\partial t}(\alpha(x), \beta(r))$,
2. $X^H(f)_{(x,u)} = g_x(grad_M \alpha, X) \frac{\partial F}{\partial s}(\alpha(x), \beta(r))$,

where $(x,u) \in TM$ and $r = g_x(u,u)$.

In the following, we consider $f(x,r) = F(\alpha(x), \beta(r))$, where $F : (s,t) \in \mathbb{R}^2 \to F(s,t) \in]0,+\infty[$, $\alpha : M \to]0,+\infty[$ and $\beta : \mathbb{R} \to]0,+\infty[$ are smooth functions.

Theorem 12. *Let (M,g) be a Riemannian manifold. If $f(x,r) = F(\alpha(x), \beta(r))$ and ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M,g) (resp (TM, g^S_f)), then we have:*

$$\begin{aligned}
1.\quad (\widetilde{\nabla}_{X^H} Y^H)_p &= (\nabla_X Y)^H_p - \frac{1}{2}(R_x(X,Y)u)^V, \\
2.\quad (\widetilde{\nabla}_{X^H} Y^V)_p &= (\nabla_X Y)^V_p + \frac{f(x,r)}{2}(R_x(u,Y)X)^H \\
&\quad + \frac{1}{2f(x,r)} g_x(grad_M \alpha, X) \frac{\partial F}{\partial s}(\alpha(x), \beta(r)) Y^V_p, \\
3.\quad (\widetilde{\nabla}_{X^V} Y^H)_p &= \frac{f(x,r)}{2}(R_x(u,X)Y))^H + \frac{1}{2f(x,r)} g_x(grad_M \alpha, Y) \frac{\partial F}{\partial s}(\alpha(x), \beta(r)) X^V_p, \\
4.\quad (\widetilde{\nabla}_{X^V} Y^V)_p &= \frac{\beta'(r)}{f(x,r)} \frac{\partial F}{\partial t}(\alpha(x), \beta(r)) \Big[g_x(Y,U) X^V_p) + g_x(X,U) Y^V_p) - g_x(X,Y) U^V_p \Big] \\
&\quad - \frac{1}{2} g_x(X,Y) \frac{\partial F}{\partial t}(\alpha(x), \beta(r))(grad_M \alpha)^H_p,
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $p = (x,u) \in TM$, where R denote the curvature tensor of (M,g).

The proof of Theorem 12 follows directly from Kozul formula, Lemma 10 and Lemma 11 (see [17]).

Lemma 13. *The morphism $J_1 : TTM \to TTM$ such that:*

$$\begin{cases} J_1(X^H) &= -X^H \\ J_1(X^V) &= X^V \end{cases} \tag{5}$$

for any vector field $X \in \Gamma(TM)$ is an almost paracomplex structure.

Proof. Let $Z \in T_{(x,u)}TM$, form 1 we have $Z = X^H + Y^V$ and

$$\begin{aligned} J_1^2(Z) &= J_1^2(X^H) + J_1^2(Y^V) = X^H + Y^V = Z. \\ Ker(J_1 + Id_{TM}) &= \{Z \in T_{(x,u)}TM, \quad J_1 Z = -Z\} = \mathscr{H}_{(x,u)}. \\ Ker(J_1 - Id_{TM}) &= \{Z \in T_{(x,u)}TM, \quad J_1 Z = Z\} = \mathscr{V}_{(x,u)}. \end{aligned}$$

■

Lemma 14. *If g_f^S is the Mus-Sasaki metric on TM, then for any vector fields $\widetilde{X}, \widetilde{Y} \in \Gamma(TTM)$, we have*

$$g_f^S(J_1\widetilde{X}, \widetilde{Y}) = g_f^S(\widetilde{X}, J_1\widetilde{Y}),$$

i.e. g_f^S is pure with respect to J_1.

From Lemma 14, we obtain the following theorem:

Theorem 15. *Let (M,g) be a Riemannian manifold. If J_1 denote the almost paracomplex structure defined by (5), then (TM, J_1, g_f^S) is an almost paracomplex Norden manifold.*

Theorem 16. *Let (M,g) be a Riemannian manifold and J_1 be the almost paracomplex structure defined by (5). For all $X,Y,Z \in \Gamma(TM)$, $x \in M$ and $u \in T_xM$, we have*

1. $(\phi^{J_1} g_f^S)(X^H, Y^H, Z^H) = 0.$
2. $(\phi^{J_1} g_f^S)(X^H, Y^H, Z^V) = -2g_f^S\big((R(X,Y)u)^V, Z^V\big).$
3. $(\phi^{J_1} g_f^S)(X^H, Y^V, Z^H) = -2g_f^S\big((R(X,Z)u)^V, Y^V\big).$
4. $(\phi^{J_1} g_f^S)(X^H, Y^V, Z^V) = -2X(\alpha)\dfrac{\partial \ln F}{\partial s} g_f^S(Y^V, Z^V).$
5. $(\phi^{J_1} g_f^S)(X^V, Y^H, Z^H) = 0.$
6. $(\phi^{J_1} g_f^S)(X^V, Y^V, Z^H) = 0.$
7. $(\phi^{J_1} g_f^S)(X^V, Y^H, Z^V) = 0.$
8. $(\phi^{J_1} g_f^S)(X^V, Y^V, Z^V) = 0.$

Proof. Using the definition of Tachibana operator ϕ^J (formula (4)), for $J = J_1$ we obtain

$$\begin{aligned} 1.\,(\phi^J g_f^S)(X^H, Y^H, Z^H) &= (JX^H) g_f^S(Y^H, Z^H) - X^H g_f^S(JY^H, Z^H) \\ &\quad + g_f^S\big((L_{Y^H}J)X^H, Z^H\big) + g_f^S\big(Y^H, (L_{Z^H}J)X^H\big) \\ &= -X^H g_f^S(Y^H, Z^H) + X^H g_f^S(Y^H, Z^H) \\ &\quad + g_f^S\big(L_{Y^H}JX^H - J(L_{Y^H}X^H), Z^H\big) \\ &\quad + g_f^S\big(Y^H, L_{Z^H}JX^H - J(L_{Z^H}X^H)\big) \\ &= -g_f^S\big([Y^H, X^H], Z^H\big) - g_f^S\big(J[Y^H, X^H], Z^H\big) \\ &\quad - g_f^S\big(Y^H, [Z^H, X^H]\big) - g_f^S\big(Y^H, J[Z^H, X^H]\big) \\ &= 0. \end{aligned}$$

$$\begin{aligned} 2.\,(\phi^J g_f^S)(X^H, Y^H, Z^V) &= (JX^H) g_f^S(Y^H, Z^V) - X^H g_f^S(JY^H, Z^V) \\ &\quad + g_f^S\big((L_{Y^H}J)X^H, Z^V\big) + g_f^S\big(Y^H, (L_{Z^V}J)X^H\big) \\ &= -g_f^S\big([Y^H, X^H], Z^V\big) - g_f^S\big(J[Y^H, X^H], Z^V\big) \\ &\quad - g_f^S\big(Y^H, [Z^V, X^H]\big) - g_f^S\big(Y^H, J[Z^V, X^H]\big) \\ &= -2g_f^S\big([Y^H, X^H], Z^V\big) \\ &= -2g_f^S\big((R(X,Y)u)^V, Z^V\big). \end{aligned}$$

$$
\begin{aligned}
4.\,(\phi^J g_f^S)(X^H,Y^V,Z^V) &= (JX^H)g_f^S(Y^V,Z^V) - X^H g_f^S(JY^V,Z^V) \\
&\quad + g_f^S\big((L_{Y^V}J)X^H,Z^V\big) + g_f^S\big(Y^V,(L_{Z^V}J)X^H\big) \\
&= -2X^H(fg(Y,Z)) - 2g_f^S\big([Y^V,X^H],Z^V\big) - 2g_f^S\big(Y^V,[Z^V,X^H]\big) \\
&= -2\big\{X(\alpha)\frac{\partial F}{\partial s}g(Y,Z) + fXg(Y,Z)\big\} + 2g_f^S\big((\nabla_X Y)^V,Z^V\big) \\
&\quad + 2g_f^S\big(Y^V,(\nabla_X Z)^V\big) \\
&= -2\big\{X(\alpha)\frac{\partial \ln F}{\partial s}g_f^S(Y^V,Z^V) + fXg(Y,Z)\big\} \\
&\quad + 2fg(\nabla_X Y,Z\big) + 2fg(Y,\nabla_X Z) \\
&= -2X(\alpha)\frac{\partial \ln F}{\partial s}g_f^S(Y^V,Z^V).
\end{aligned}
$$

■

From Theorem 16 we deduce the following result:

Theorem 17. *Let (M,g) be a Riemannian manifold and J_1 be the almost paracomplex structure defined by (5). Then (TM,J_1,g_f^S) is para-Kähler-Norden manifold if and only if M is flat and α is constant.*

Proof. Let $X,Y,Z \in \Gamma(TM)$, $k,h,l \in \{H,V\}$, $u \in T_xM$ and $x \in M$. We have:

$$
(\phi_J g_f^S)(X^h,Y^k,Z^l) = 0 \quad \Leftrightarrow \quad \begin{cases} R(X,Y)u & = 0 \\ R(X,Z)u & = 0 \\ X(\alpha)\dfrac{\partial \ln F}{\partial s}(\alpha(x),\beta(r)) & = 0 \end{cases}
$$

$$
\Leftrightarrow \quad \begin{cases} R & = 0 \\ X(\alpha) & = 0 \ or\ \dfrac{\partial \ln F}{\partial s}(\alpha(x),\beta(r)) = 0 \end{cases}
$$

$$
\Leftrightarrow \quad \begin{cases} R & = 0 \\ \alpha & = constant \end{cases}
$$

■

Theorem 18. *Let (M,g) be a Riemannian manifold. If $\widehat{J}$ denote the structure defined by*

$$
\begin{cases} \widehat{J}(X^H) & = X^H \\ \widehat{J}(X^V) & = -X^V \end{cases} \tag{6}
$$

then $\widehat{J}$ is an almost paracomplex structure and $(TM,\widehat{J},g_f^S)$ is para-Kähler-Norden manifold if and only if M is flat and α is constant.

Theorem 19. *Let (M,g) be a Riemannian manifold. If $\overline{J}$ denote the structure defined by*

$$
\begin{cases} \overline{J}(X^H) & = X^V \\ \overline{J}(X^V) & = X^H \end{cases} \tag{7}
$$

then $\overline{J}$ is an almost paracomplex structure and $(TM,\overline{J},g_f^S)$ is an almost paracomplex Norden manifold if and only if $f = 1$.

Using the definition of Mus-Sasaki metric g_f^S (Definition 9) and the formulae (6) and (7), we deduce the following theorem:

Theorem 20. *Let (M,g) be a Riemannian manifold. If $\overline{J}$ denote the structure defined by (7), then the pseudo-metric*

$$
H_f^S(\widetilde{X},\widetilde{Y}) = g_f^S(\bar{J}\widetilde{X},\widetilde{Y}) + g_f^S(\widetilde{X},\bar{J}\widetilde{Y})
$$

satisfies the following

$$
\begin{cases} H_f^S(X^H,Y^H) & = H_f^S(X^V,Y^V) = 0 \\ H_f^S(X^H,Y^V) & = H_f^S(X^V,Y^H) = (1+f)g(X,Y) \\ H_f^S(\widehat{J}X^k,Y^h) & = -H_f^S(X^k,\widehat{J}Y^h), \quad \forall h,k \in \{V,H\} \end{cases} \tag{8}
$$

and $(TM, \widehat{J}, H_f^S)$ *is an almost para-Hermitian manifold.*

Theorem 21. *Let* $(TM, \widehat{J}, H_f)$ *be an almost para-Hermitian manifold defined in Theorem 20. Then the para-Kähler form*

$$\Omega(\widetilde{X}, \widetilde{Y}) = H_f^S(\widetilde{X}, \widehat{J}\widetilde{Y})$$

satisfies the following:

1. $\Omega(X^H, Y^H) = \Omega(X^V, Y^V) = 0.$
2. $\Omega(X^V, Y^H) = (1+f)g(X,Y) = -\Omega(X^H, Y^V).$
3. $d\Omega(X^H, Y^H, Z^H) = 0.$
4. $d\Omega(X^H, Y^H, Z^V) = \frac{\partial F}{\partial s}\big[Y(\alpha)g(Z,X) - X(\alpha)g(Y,Z)\big].$
5. $d\Omega(X^H, Y^V, Z^H) = \frac{\partial F}{\partial s}\big[X(\alpha)g(Y,Z) - Z(\alpha)g(X,Y)\big].$
6. $d\Omega(X^V, Y^H, Z^H) = \frac{\partial F}{\partial s}\big[Z(\alpha)g(X,Y) - Y(\alpha)g(Z,X)\big].$
7. $d\Omega(X^V, Y^H, Z^V) = 2\beta'\frac{\partial F}{\partial t}\big[g(Z,u)g(X,Y) - g(X,u)g(Y,Z)\big].$
8. $d\Omega(X^H, Y^V, Z^V) = 2\beta'\frac{\partial F}{\partial t}\big[g(Y,u)g(Z,X) - g(Z,u)g(X,Y)\big].$
9. $d\Omega(X^V, Y^V, Z^H) = 2\beta'\frac{\partial F}{\partial t}\big[g(X,u)g(Y,Z) - g(Y,u)g(Z,X)\big].$
10. $d\Omega(X^V, Y^V, Z^V) = 0.$

Proof. The proof follows from Definition 9, Lemma 11, Theorem 20 and the next identity:

$$\begin{aligned} d\Omega(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) &= \widetilde{X}\Omega(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y}\Omega(\widetilde{Z}, \widetilde{X}) + \widetilde{Z}\Omega(\widetilde{X}, \widetilde{Y}) \\ &\quad -\Omega([\widetilde{X}, \widetilde{Y}], \widetilde{Z}) - \Omega([\widetilde{Y}, \widetilde{Z}], \widetilde{X}) - \Omega([\widetilde{Z}, \widetilde{X}], \widetilde{Y}). \end{aligned}$$

For $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned} 1.\, d\Omega(X^H, Y^H, Z^H) &= X^H\Omega(Y^H, Z^H) + Y^H\Omega(Z^H, X^H) + Z^H\Omega(X^H, Y^H) \\ &\quad -\Omega([X^H, Y^H], Z^H) - \Omega([Y^H, Z^H], X^H) - \Omega([Z^H, X^H], Y^H) \\ &= \Omega((R(X,Y)u)^V, Z^H) + \Omega((R(Y,Z)u)^V, X^H) \\ &\quad +\Omega((R(Z,X)u)^V, Y^H) \\ &= (1+f)\big[g(R(X,Y)u, Z) + g(R(Y,Z)u, X) + g(R(Z,X)u, Y)\big] \\ &= -(1+f)g(R(X,Y)Z + R(Y,Z)X + R(Z,X)Y, u) \\ &= 0. \end{aligned}$$

$$\begin{aligned} 2.\, d\Omega(X^H, Y^H, Z^V) &= X^H\Omega(Y^H, Z^V) + Y^H\Omega(Z^V, X^H) + Z^V\Omega(X^H, Y^H) \\ &\quad -\Omega([X^H, Y^H], Z^V) - \Omega([Y^H, Z^V], X^H) - \Omega([Z^V, X^H], Y^H) \\ &= X^H\big[-(1+f)g(Y,Z)\big] + Y^H\big[(1+f)g(Z,X)\big] \\ &\quad -\Omega([X,Y]^H, Z^V) - \Omega((\nabla_Y Z)^V, X^H) + \Omega((\nabla_X Z)^V, Y^H) \\ &= -X(\alpha)\frac{\partial F}{\partial s}g(Y,Z) - (1+f)\big[g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\big] \\ &\quad +Y(\alpha)\frac{\partial F}{\partial s}g(Z,X) + (1+f)\big[g(\nabla_Y Z, X) + g(Z, \nabla_Y X)\big] \\ &\quad +(1+f)g([X,Y], Z) - (1+f)g(\nabla_Y Z, X) + (1+f)g(\nabla_X Z, Y) \\ &= \frac{\partial F}{\partial s}\big[Y(\alpha)g(Z,X) - X(\alpha)g(Y,Z)\big]. \end{aligned}$$

$$
\begin{aligned}
3.\, d\Omega(X^H,Y^V,Z^H) &= X^H\Omega(Y^V,Z^H)+Y^V\Omega(Z^H,X^H)+Z^H\Omega(X^H,Y^V)\\
&\quad -\Omega([X^H,Y^V],Z^H)-\Omega([Y^V,Z^H],X^H)-\Omega([Z^H,X^H],Y^V)\\
&= X^H\big[(1+f)g(Y,Z)\big]+Z^H\big[-(1+f)g(X,Y)\big]\\
&\quad -\Omega((\nabla_X Y)^V,Z^H)+\Omega((\nabla_Z Y)^V,X^H)-\Omega([Z,X]^H,Y^V)\\
&= X(\alpha)\frac{\partial F}{\partial s}g(Y,Z)+(1+f)\big[g(\nabla_X Y,Z)+g(Y,\nabla_X Z)\big]\\
&\quad -Z(\alpha)\frac{\partial F}{\partial s}g(X,Y)-(1+f)\big[g(\nabla_Z X,Y)+g(X,\nabla_Z Y)\big]\\
&\quad -(1+f)g(\nabla_X Y,Z)+(1+f)g(\nabla_Z Y,X)+(1+f)g([Z,X],Y)\\
&= \frac{\partial F}{\partial s}\big[X(\alpha)g(Y,Z)-Z(\alpha)g(X,Y)\big].
\end{aligned}
$$

$$
\begin{aligned}
4.\, d\Omega(X^V,Y^H,Z^H) &= X^V\Omega(Y^H,Z^H)+Y^H\Omega(Z^H,X^V)+Z^H\Omega(X^V,Y^H)\\
&\quad -\Omega([X^V,Y^H],Z^H)-\Omega([Y^H,Z^H],X^V)-\Omega([Z^H,X^V],Y^H)\\
&= Y^H\big[-(1+f)g(Z,X)\big]+Z^H\big[(1+f)g(X,Y)\big]\\
&\quad +\Omega((\nabla_Y X)^V,Z^H)-\Omega([Y,Z]^H,X^V)-\Omega((\nabla_Z X)^V,Y^H))\\
&= -Y(\alpha)\frac{\partial F}{\partial s}g(Z,X)-(1+f)\big[g(\nabla_Y Z,X)+g(Z,\nabla_Y X)\big]\\
&\quad +Z(\alpha)\frac{\partial F}{\partial s}g(X,Y)+(1+f)\big[g(\nabla_Z X,Y)+g(X,\nabla_Z Y)\big]\\
&\quad +(1+f)g(\nabla_Y X,Z))+(1+f)g([Y,Z],X)-(1+f)g(\nabla_Z X,Y)\\
&= \frac{\partial F}{\partial s}\big[Z(\alpha)g(X,Y)-Y(\alpha)g(Z,X)\big].
\end{aligned}
$$

$$
\begin{aligned}
5.\, d\Omega(X^V,Y^H,Z^V) &= X^V\Omega(Y^H,Z^V)+Y^H\Omega(Z^V,X^V)+Z^V\Omega(X^V,Y^H)\\
&\quad -\Omega([X^V,Y^H],Z^V)-\Omega([Y^H,Z^V],X^V)-\Omega([Z^V,X^V],Y^H)\\
&= X^V\big[-(1+f)g(Y,Z)\big]+Z^V\big[(1+f)g(X,Y)\big]\\
&= -2\beta' g(X,u)\frac{\partial F}{\partial t}g(Y,Z)+2\beta' g(Z,u)\frac{\partial F}{\partial t}g(X,Y)\\
&= 2\beta'\frac{\partial F}{\partial t}\big[g(Z,u)g(X,Y)-g(X,u)g(Y,Z)\big].
\end{aligned}
$$

$$
\begin{aligned}
6.\, d\Omega(X^H,Y^V,Z^V) &= X^H\Omega(Y^V,Z^V)+Y^V\Omega(Z^V,X^H)+Z^V\Omega(X^H,Y^V)\\
&\quad -\Omega([X^H,Y^V],Z^V)-\Omega([Y^V,Z^V],X^H)-\Omega([Z^V,X^H],Y^V)\\
&= Y^V\big[(1+f)g(Z,X)\big]+Z^V\big[-(1+f)g(X,Y)\big]\\
&= 2\beta' g(Y,u)\frac{\partial F}{\partial t}g(Z,X)-2\beta' g(Z,u)\frac{\partial F}{\partial t}g(X,Y)\\
&= 2\beta'\frac{\partial F}{\partial t}\big[g(Y,u)g(Z,X)-g(Z,u)g(X,Y)\big].
\end{aligned}
$$

$$
\begin{aligned}
7.\, d\Omega(X^V,Y^V,Z^H) &= X^V\Omega(Y^V,Z^H)+Y^V\Omega(Z^H,X^V)+Z^H\Omega(X^V,Y^V)\\
&\quad -\Omega([X^V,Y^V],Z^H)-\Omega([Y^V,Z^H],X^V)-\Omega([Z^H,X^V],Y^V)\\
&= X^V\big[(1+f)g(Y,Z)\big]+Y^V\big[-(1+f)g(Z,X)\big]\\
&= 2\beta' g(X,u)\frac{\partial F}{\partial t}g(Y,Z)-2\beta' g(Y,u)\frac{\partial F}{\partial t}g(Z,X)\\
&= 2\beta'\frac{\partial F}{\partial t}\big[g(X,u)g(Y,Z)-g(Y,u)g(Y,X)\big].
\end{aligned}
$$

$$\begin{aligned} 8.\, d\Omega(X^V,Y^V,Z^V) &= X^V\Omega(Y^V,Z^V)+Y^V\Omega(Z^V,X^V)+Z^V\Omega(X^V,Y^V) \\ &\quad -\Omega([X^V,Y^V],Z^V)-\Omega([Y^V,Z^V],X^V)-\Omega([Z^V,X^V],Y^V) \\ &= 0. \end{aligned}$$

■

Theorem 22. *Let $(TM,\widehat{J},H_f)$ be an almost para-Hermitian manifold defined in Theorem 20. Then $(TM,\widehat{J},H_f)$ is an almost para-Kähler manifold if and only if $f = Const$.*

Proof. By the Theorem 21, we have

$$d\Omega = 0 \Leftrightarrow grad_M\alpha = 0 \quad and \quad \beta' = 0$$

then

$$d\Omega = 0 \Leftrightarrow f(x,r) = const.$$

■

Definition 23. *Let (M,g) be a Riemannian manifold, J be the almost paracomplex structure defined by formula (5) and $\widetilde{\nabla}$ be the Levi-Civita connection of (TM,g_f^S). We define a tensor field of type $(1,2)$ and a connection $\overline{\nabla}$ on TM by*

$$S(\widetilde{X},\widetilde{Y}) = \frac{1}{2}\left[(\widetilde{\nabla}_{J\widetilde{Y}}J)\widetilde{X}+J\big((\widetilde{\nabla}_{\widetilde{Y}}J)\widetilde{X}\big)-J\big((\widetilde{\nabla}_{\widetilde{X}}J)\widetilde{Y}\big)\right] \tag{9}$$

$$\overline{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}-S(\widetilde{X},\widetilde{Y}), \tag{10}$$

where $\widetilde{X},\widetilde{Y}\in\Gamma(TTM)$.

Lemma 24. *For $X,Y\in\Gamma(TM)$, $x\in M$ and $u\in T_xM$, we have*

1.) $S(X^H,Y^H) = -\frac{1}{2}(R(X,Y)u)^V.$

2. $S(X^H,Y^V) = \frac{f}{2}(R(u,Y)X)^H - X(\alpha)\frac{\partial \ln F}{\partial s}Y^V.$

3. $S(X^V,Y^H) = -f(R(u,X)Y)^H + \frac{1}{2}Y(\alpha)\frac{\partial \ln F}{\partial s}X^V.$

4. $S(X^V,Y^V) = -\frac{1}{2}g(X,Y)\frac{\partial \ln F}{\partial s}(grad_M\alpha)^H.$

The proof follows immediately from Theorem 12 and formula (9).

Theorem 25. *Let (M,g) be a Riemannian manifold, J be the almost paracomplex structure defined by formula (5). If $\overline{\nabla}$ denotes the connection defined by formula (10), then we have*

$$\begin{aligned} 1.\, \overline{\nabla}_{X^H}Y^H &= (\nabla_XY)^H, \\ 2.\, \overline{\nabla}_{X^H}Y^V &= (\nabla_XY)^V+\frac{3}{2}X(\alpha)\frac{\partial \ln F}{\partial s}Y^V, \\ 3.\, \overline{\nabla}_{X^V}Y^H &= \frac{3f}{2}(R(u,X)Y)^H, \\ 4.\, \overline{\nabla}_{X^V}Y^V &= \beta'\frac{\partial \ln F}{\partial t}\Big[g(Y,u)X^V+g(X,u)Y^V-g(X,Y)u^V\Big], \end{aligned}$$

where $X,Y\in\Gamma(TM)$, $x\in M$ and $u\in T_xM$.

Proof.

$$\begin{aligned}
1.\,\overline{\nabla}_{X^H}Y^H &= \widetilde{\nabla}_{X^H}Y^H - S(X^H,Y^H)\\
&= (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V + \frac{1}{2}(R(X,Y)u)^V\\
&= (\nabla_X Y)^H.\\
2.\,\overline{\nabla}_{X^H}Y^V &= \widetilde{\nabla}_{X^H}Y^V - S(X^H,Y^V)\\
&= (\nabla_X Y)^V + \frac{f}{2}(R(u,Y)X)^H + \frac{1}{2}X(\alpha)\frac{\partial \ln F}{\partial s}Y^V\\
&\quad -\frac{f}{2}(R(u,Y)X)^H + X(\alpha)\frac{\partial \ln F}{\partial s}Y^V\\
&= (\nabla_X Y)^V + \frac{3}{2}X(\alpha)\frac{\partial \ln F}{\partial s}Y^V.\\
3.\,\overline{\nabla}_{X^V}Y^H &= \widetilde{\nabla}_{X^V}Y^H - S(X^V,Y^H)\\
&= \frac{f}{2}(R(u,X)Y))^H + \frac{1}{2}Y(\alpha)\frac{\partial \ln F}{\partial s}X^V + f(R(u,X)Y)^H\\
&\quad -\frac{1}{2}Y(\alpha)\frac{\partial \ln F}{\partial s}X^V\\
&= \frac{3f}{2}(R(u,X)Y)^H.\\
4.\,\overline{\nabla}_{X^V}Y^V &= \widetilde{\nabla}_{X^V}Y^V - S(X^V,Y^V)\\
&= \beta\frac{\partial \ln F}{\partial t}\Big[g(Y,U)X^V + g(X,U)Y^V - g(X,Y)U^V\Big]\\
&\quad -\frac{1}{2}g(X,Y)\frac{\partial F}{\partial s}(grad_M\alpha)^H + \frac{1}{2}g(X,Y)\frac{\partial \ln F}{\partial s}(grad_M\alpha)^H\\
&= \beta\frac{\partial \ln F}{\partial t}\Big[g(Y,U)X^V + g(X,U)Y^V - g(X,Y)U^V\Big]
\end{aligned}$$

where $U \in \Gamma(TM)$ such that $U_x = u$. ■

Lemma 26. *Let (M,g) be a Riemannian manifold and (TM,g_f^S) be its tangent bundle equipped with the twisted Sasaki metric. If $\overline{\nabla}$ denotes the connection defined by formula (10) and $\overline{T}$ the torsion tensor field of $\overline{\nabla}$, then we have*

1. $\overline{T}(X^H,Y^H) = (R(X,Y)u)^V.$
2. $\overline{T}(X^H,Y^V) = \dfrac{-3f}{2}(R(u,Y)X)^H + \dfrac{3}{2}X(\alpha)\dfrac{\partial \ln F}{\partial s}Y^V.$
3. $\overline{T}(X^V,Y^H) = \dfrac{3f}{2}(R(u,X)Y)^H - \dfrac{3}{2}Y(\alpha)\dfrac{\partial \ln F}{\partial s}X^V.$
4. $\overline{T}(X^V,Y^V) = 0.$

where $X,Y \in \Gamma(TM)$, $x \in M$ and $u \in T_xM$.

The proof of Lemma 26 follows from definition of torsion tensor field and Theorem 25.

Theorem 27. *Let (M,g) be a Riemannian manifold and (TM,g_f^S) be its tangent bundle equipped with the twisted Sasaki metric. If $\overline{\nabla}$ denotes the connection defined by formula (10), then $\overline{\nabla}$ is symmetric if and only if M is flat and α is constant.*

Proof. From Lemma 26, we have

$$\overline{T} = 0 \Leftrightarrow (R = 0 \quad and \quad grad_M\alpha = 0).$$

■

4. Conclusions

In this paper the geometry of some paracomplex structures on tangent fiber bundle TM equipped with a Mus-Sasaki metrics is studied.

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