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THE MESHLESS KERNEL-BASED METHOD OF LINES FOR SOLVING THE DISSIPATIVE GENERALIZED SRLW EQUATIONS WITH DAMPING TERM

Bahar KARAMAN, Yılmaz DERELİ*

Mathematics Department, Eskişehir Technical University, Eskişehir, Türkiye

ABSTRACT

In this study, we evaluate the numerical solutions of the dissipative generalized symmetric regularized long wave equations with damping term. The problem is a nonlinear partial differential equations system. Numerical solutions of the problem are evaluated by using the meshless kernel-based method of lines for known initial-boundary conditions on the given solution domain. This used numerical method is known to be a truly meshless approximation. Radial basis functions are used as kernel functions on the meshless method. The performance of this meshless method is illustrated on many standard test problems. Numerical computations are performed by using Gaussian and Wendland's radial basis functions. Error comparisons for computed numerical results are made in the sense of L_∞ error norm. Graphs of wave simulations for test problems are plotted in this study. The results show that the used meshless method is suitable to solve numerically this specific type of nonlinear equations.

Keywords: Meshless Method, Method of Lines, Damping term, Dissipative Symmetric RLW Equation

1. INTRODUCTION

The generalized symmetric regularized long wave (SRLW) equation has the following first-order equations system form:

$$\begin{aligned} u_{xxt} - u_t &= \rho_x + \frac{1}{p}(u^p)_x \\ \rho_t + u_x &= 0 \end{aligned} \tag{1}$$

where $x \in [x_L, x_R]$, $t \in [0, T]$, $p \geq 2$ and ρ and u are the dimensionless electron charge density and the fluid velocity, respectively.

While $p = 2$ in Equation (1) symmetric regularized long-wave equation is obtained. SRLW equation was first described by Seyler and Fenstermacher [1] as a model of the propagation of weakly nonlinear ion-acoustic and space-charge waves.

Guo [2] presented the existence, uniqueness and regularity of the numerical solutions for the periodic initial value problem of generalized SRLW equation. Also, solitary wave solutions of Equation (1) are evaluated as follows by Duan et. al in [3]

$$u(x,t) = \left[\frac{p(p+1)(c^2-1)}{2c}\right]^{\frac{1}{p-1}} \sec h^{2/(p-1)} \ \frac{p-1}{2c}\sqrt{c^2-1}\ (x-ct) \tag{2}$$

and

$$\rho(x,t) = \frac{1}{c}\left[\frac{p(p+1)(c^2-1)}{2c}\right]^{\frac{1}{p-1}} \sec h^{2/(p-1)} \ \frac{p-1}{2c}\sqrt{c^2-1}\ (x-ct) \tag{3}$$

*Corresponding Author: ydereli@eskisehir.edu.tr
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where $p \geq 2$ is positive integer, c is velocity and $c^2 > 1$. Some articles about the SRLW equation are given in references $[3 - 7]$.

In Equation (1), some physical phenomena such as the effect of gravity, resistance of the propagation medium and friction of air are neglected. However, these effects should be considered in the study of the movements of nonlinear waves. The dissipative generalized SRLW equation with damping terms is obtained by adding dissipative and damping terms to Equation (1). This equation has been defined in the following form by Zhou in [8]:

$$u_{xxt} - u_t + \mu u_{xx} = \rho_x + \frac{1}{p}(u^p)_x$$
$$\rho_t + u_x + \gamma\rho = 0 \tag{4}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad x \in [x_L, x_R] \tag{5}$$

and boundary conditions

$$u(x_L,t) = u(x_R,t) = 0, \quad \rho(x_L,t) = \rho_0(x_R,t), \quad t \in [0,T] \tag{6}$$

where μ and γ are positive constants. μ is the dissipative coefficient and γ is the damping term. In this study, numerical treatments will be performed for different values of p.

While $p = 2$ in Equation (4) dissipative symmetric RLW equation with damping term is obtained. If $p \geq 3$ generalized forms of the equation is obtained. Some studies about the equation are given in the references [8-12].

The governing equation (4) is a nonlinear partial differential equations system. Since it is very difficult to find an analytical solution for the general case. Therefore, numerical solutions of the governing equation can be evaluated by using a suitable numerical technique.

To our knowledge, Equation (4) is not solved by using the meshless kernel-based method of lines. That's why this method, this method is used in this paper to solve numerically the mentioned nonlinear equation. In algorithms, Gaussian and Wendland's compactly supported radial basis functions are used as kernel functions in the meshless method. For different values of p, numerical solutions will be obtained by using these kernel functions.

The damping does not have an effect and dissipative is no appear on waveform for Equation (4) in the initial time. So the same initial conditions with generalized SRLW equation (1) are used for the numerical solution of Equation (4).

2. NUMERICAL METHOD

The main purpose of this study is to obtain the numerical values of the unknown functions $u(x,t)$ and $\rho(x,t)$ in the solution domain $[x_L, x_R] \times [0,T]$. For this purpose, the meshless method will be used with different radial functions. Approximation to the unknown functions $u(x,t)$ and $\rho(x,t)$ as defined in [13] as follows:

$$u(x,t)=\sum_{j=1}^{n} v_j(x)\alpha_j(t),\quad \rho(x,t)=\sum_{j=1}^{n} w_j(x)\beta_j(t) \tag{7}$$

where $\alpha_j(t),\ \beta_j(t)$ are unknown time-dependent functions to be determined each time level as column vectors and $v_j(x)$, $w_j(x)$ are defined by any well-known radial basis functions. Derivatives in Equation (7) with respect to time and space variables can be described as:

$$u_t(x,t)=\sum_{j=1}^{n} v_j(x)\alpha_j{}'(t),\quad \rho_t(x,t)=\sum_{j=1}^{n} w_j(x)\beta_j{}'(t) \tag{8}$$

$$u_x(x,t)=\sum_{j=1}^{n} v_j{}'(x)\alpha_j(t),\quad \rho_x(x,t)=\sum_{j=1}^{n} w_j{}'(x)\beta_j(t) \tag{9}$$

$$u_{xxt}(x,t)=\sum_{j=1}^{n} v_j{}''(x)\alpha_j{}'(t) \tag{10}$$

The formulas of the used basis functions are defined as follows:

Gaussian radial basis function is an infinitely smooth function and it is defined in the following form:

$$G:\quad \phi(r)=\exp\left(-\frac{r^2}{\varepsilon^2}\right) \tag{11}$$

where r is the Euclidean distance between collocation points and ε is a shape parameter.

Wendland's functions [14] are a class of compactly supported radial basis function and have the following general form:

$$\phi_{l,k}(r)=(1-r)_+^n\, p_{l,k}(r) \tag{12}$$

with following conditions:

$$(1-r)_+^n=\begin{cases}(1-r)^n, & if\ 0\le r<1\\ 0, & if\ r\ge 1\end{cases} \tag{13}$$

where p is a prescribed polynomial for $k\ge 1$ and l is the dimension number. In our calculations, following form of Wendland's function is used:

$$\phi_{7,5}(r)=(1-r)_+^{12}\ (9+108r+566r^2+1644r^3+2697r^4+2048r^5) \tag{14}$$

For ease of notation in tables $\phi_{l,k}(r)$ will be used as W. In approximation, the unknown functions' computed values will be written in initial conditions (5) to find the numerical solutions of functions u and ρ. To apply the numerical scheme, derivatives of unknown functions $\alpha_j(t)$ and $\beta_j(t)$ are also required. The derivatives of unknown functions are taken with respect to time and spatial variables.

Substituting Equations $(7)-(10)$ derivatives into the main equations system (4) and rearrange the following equations system is obtained:

$$\sum_{j=1}^{n} v_j''(x)\,\alpha_j'(t) - \sum_{j=1}^{n} v_j(x)\alpha_j'(t) + \mu \sum_{j=1}^{n} v_j''(x)\,\alpha_j(t) = \sum_{j=1}^{n} w_j'(x)\beta_j(t) + \left(\sum_{j=1}^{N} v_j(x)\alpha_j(t)\right)^{p-1} \sum_{j=1}^{N} v_j'(x)\alpha_j(t) \quad (15)$$
$$\sum_{j=1}^{n} w(x)\beta_j'(t) + \sum_{j=1}^{n} v_j'(x)\alpha_j(t) + \gamma \sum_{j=1}^{n} w(x)\beta_j(t) = 0$$

The equations system (12) is solved by using an ode solver in the MATLAB. The system is written in the symbolic form as:

$$(V_{xx} - V) * \alpha'(t) = -\mu\big(V_{xx} * \alpha(t)\big) + (W_x * \beta(t)) + \big(V * \alpha(t)\big)^{p-1} \big(V_x * \alpha(t)\big) \quad (16)$$
$$\big(V_x * \alpha(t)\big) + \big(W * \beta'^{(t)}\big) = -\,\gamma(W * \beta(t))$$

where the symbol $*$ means the pointwise product. Also, V, V_x, V_{xx}, W and W_x are invertible matrices [15] consisted of $v_j\ (x), w_j(x),\ \alpha(t), \alpha'(t), \beta(t)$ and $\beta'(t)$ are vectors consisted of $\alpha_j(t)$ and its derivatives with respect to t. So following system of differential equations is obtained:

$$\alpha'(t) = (V_{xx} - V)^{-1} * \Big(-\mu(V_{xx} * \alpha(t)) + (W_x * \beta(t)) + \big(V * \alpha(t)\big)^{p-1} \big(V_x * \alpha(t)\big)\Big) \quad (17)$$
$$\beta'(t) = -W^{-1} * (\,\gamma(W * \beta(t)) + \big(V_x * \alpha(t)\big)$$

Obviously the system (14) can be written in the following system:

$$\frac{d}{dt}\alpha(t) = F_1\big(t, \alpha(t), \beta(t)\big)$$
$$\frac{d}{dt}\beta(t) = F_2\big(t, \alpha(t), \beta(t)\big) \quad (18)$$

where

$$F_1\big(t, \alpha(t), \beta(t)\big) = (V_{xx} - V)^{-1} * \Big(-\mu(V_{xx} * \alpha(t)) + (W_x * \beta(t)) + \big(V * \alpha(t)\big)^{p-1} \big(V_x * \alpha(t)\big)\Big) \quad (19)$$
$$F_2\big(t, \alpha(t), \beta(t)\big) = -W^{-1} * (\,\gamma(W * \beta(t)) + \big(V_x * \alpha(t)\big)$$

The system (15) is written in the vector form as follows

$$\frac{d}{dt}X(t) = F(t, X(t)) \quad (20)$$

This first-order differential equation system is suitable to be solved by MATLAB solver. In our calculations, we used ode solver ode113. This is known as the Adams-Bashforth-Moulton method. Here, it can be used for any different method.

3. NUMERICAL SIMULATIONS

In the initial time, there isn't any effect of damping and dissipation on the wave propagation. Thus, the governing equation (4) and generalized SRLW equation (1) have the same initial conditions. In other words, initial conditions for Equations (1) and (4) are the same to begin evaluations at $t = 0$. There is no exact solution of Equation (4) in the literature. Therefore, an error estimates method is used as in previous studies $[8-11]$. In this estimation method, the evaluated numerical solution for the selected fixed values of time step Δt and space step Δx is considered as a reference solution. We consider the solution on mesh $\Delta x=\Delta t=0.01$ as the reference solution. Computed numerical solutions for larger time step and space step are compared with these reference solutions. Error comparisons will be made in the sense of L_∞ error norm which can be calculated as follows:

$$L_\infty = \max_{1\le j\le N} |u_j^{\{exact\}} - u_j^{\{numerical\}}|$$

In comparisons, we used step sizes for space and time as $\Delta x=\Delta t=0.2$, $\Delta x=\Delta t=0.1$, $\Delta x=\Delta t=0.05$ and $\Delta x=\Delta t=0.02$. Numerical experiments will be performed for $\mu=\gamma=1$, $\mu=\gamma=0.5$ and $c=1.5$ in the solution domain $[-20,20]$ up to time $T=5$.

3.1. Test Problems

In this part, we present some numerical examples for different values of $\boldsymbol{p}$.

Case 1: When $p=2$, dissipative SRLW equations with damping term are obtained. Initial conditions from the solution functions (2) and (3) are taken as

$$u(x,0)=\frac{5}{2}sech^2\frac{\sqrt{5}}{6}x \tag{21}$$

$$\rho(x,0)=\frac{5}{3}sech^2\frac{\sqrt{5}}{6}x \tag{22}$$

Numerical values of $u(x,t)$ and $\rho(x,t)$ for different values of Δx and Δt are evaluated. The errors in the sense of L_∞ norm for u and ρ are presented in Tables 1 and 2. Wave motions for u and ρ are plotted in the Figures $(1-6)$ at different times. While time increases, it is seen that the height of wave functions u and ρ decreases with the effect of damping term and dissipation.

The value of the error norms decreases gradually as the value of $\Delta x=\Delta t$ approaches the value in the reference solution.

Table 1. Error norms of numerical solution $u(x,5)$ for $p=2$.

RBF	$\Delta x=\Delta t=0.2$	$\Delta x=\Delta t=0.1$	$\Delta x=\Delta t=0.05$	$\Delta x=\Delta t=0.02$
		$\mu=\gamma=1$		
G	1.00871e-03	5.70213e-04	2.78068e-04	7.36005e-05
W	8.20198e-03	5.59855e-04	2.98513e-04	3.80913e-05
		$\mu=\gamma=0.5$		
G	6.33803e-04	3.65172e-04	1.80020e-04	4.79804e-05
W	3.88077e-03	4.74889e-04	2.05207e-04	6.80983e-05

Table 2. Error norms of numerical solution $\rho(x,5)$ for $p=2$.

RBF	$\Delta x=\Delta t=0.2$	$\Delta x=\Delta t=0.1$	$\Delta x=\Delta t=0.05$	$\Delta x=\Delta t=0.02$
		$\mu=\gamma=1$		
G	3.35005e-04	1.89576e-04	9.24962e-05	2.50042e-05
W	1.42835e-03	1.84598e-04	9.85389e-05	2.25869e-05
		$\mu=\gamma=0.5$		
G	9.57927e-04	1.70524e-04	9.67215e-05	2.28051e-05
W	1.39325e-03	2.46834e-04	2.45140e-04	3.17313e-05

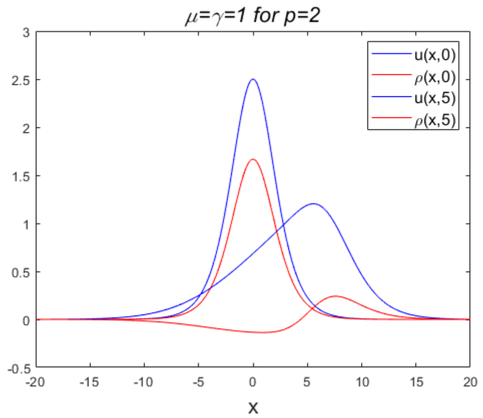

Figure 1. Wave forms at the initial and end time

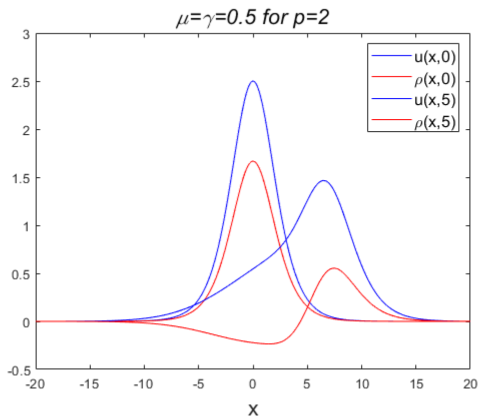

Figure 2. Wave forms at the initial and end time

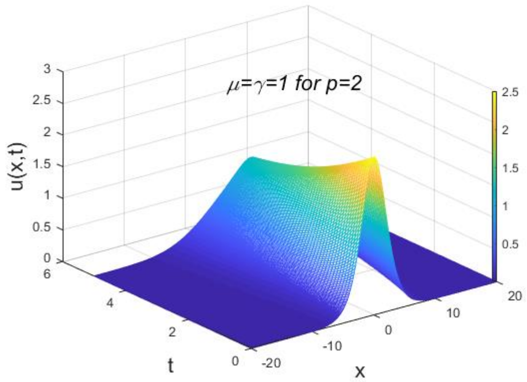

Figure 3. Simulation of the wave $u(x,t)$

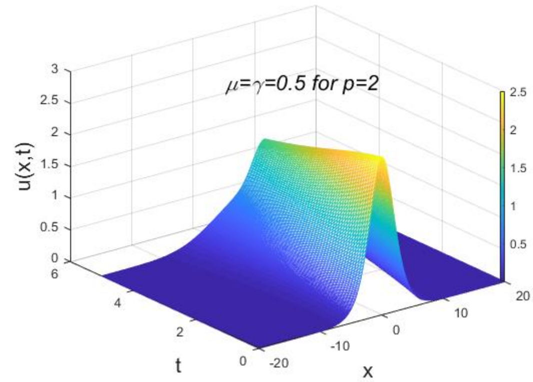

Figure 4. Simulation of the wave $u(x,t)$

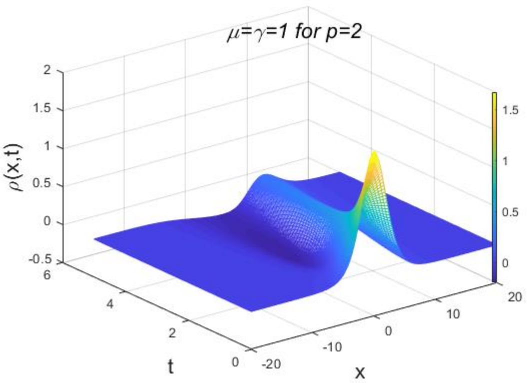

Figure 5. Simulation of the wave $\rho(x,t)$

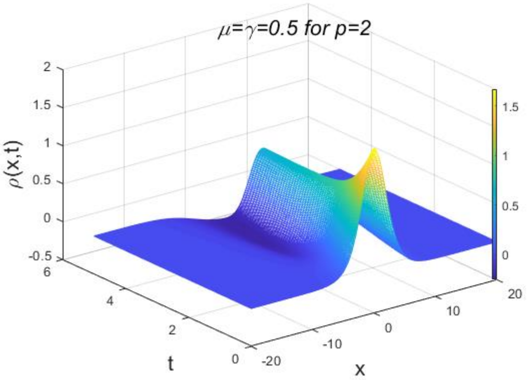

Figure 6. Simulation of the wave $\rho(x,t)$

Case 2 : When $p \geq 3$, dissipative generalized SRLW equations with damping term is obtained. In this case, we take $p = 3$, $\mu = \gamma = 1$ and $c = 1.5$. Initial conditions from Equations (2) and (3) are obtained as follows:

$$u(x,0) = \sqrt{5}sech\frac{\sqrt{5}}{3}x \tag{23}$$

$$\rho(x,0) = \frac{2\sqrt{5}}{3}sech\frac{\sqrt{5}}{3}x \tag{24}$$

Numerical experiments will be performed in the solution domain $[-20,20]$ up to time $T = 5.0$. The errors in the sense of L_∞ are compared with the reference solutions. The values of error norms are shown in Tables 3 and 4 depending on the selection of time and space step sizes.

As in the previous case, the error values are getting smaller. Profiles of waves are plotted in the Figures $(7-12)$. As time progresses, the waves are also smaller depending on the damping and dissipation.

Table 3. Error norms of numerical solution $u(x,5)$ for $p = 3$.

RBF	$\Delta x = \Delta t = 0.2$	$\Delta x = \Delta t = 0.1$	$\Delta x = \Delta t = 0.05$	$\Delta x = \Delta t = 0.02$
		$\mu = \gamma = 1$		
G	4.22019e-04	2.37890e-4	1.15819e-4	3.06233e-05
W	6.79394e-04	1.90328e-04	1.42247e-04	3.13538e-05
		$\mu = \gamma = 0.5$		
G	2.61658e-04	1.50345e-04	8.88174e-05	1.93906e-04
W	2.97575e-04	1.80756e-04	7.92676e-05	3.06567e-05

Table 4. Error norms of numerical solution $\rho(x,5)$ for $p=3$.

RBF	$\Delta x=\Delta t=0.2$	$\Delta x=\Delta t=0.1$	$\Delta x=\Delta t=0.05$	$\Delta x=\Delta t=0.02$
		$\mu=\gamma=1$		
G	1.41647e-4	7.99490e-5	3.89493e-5	1.05276e-5
W	1.84418e-04	6.25287e-05	5.50829e-05	4.75557e-05
		$\mu=\gamma=0.5$		
G	1.38461e-04	1.37569e-04	1.37588e-04	3.82329e-04
W	1.40798-04	8.56822e-05	6.47036e-05	4.64521e-05

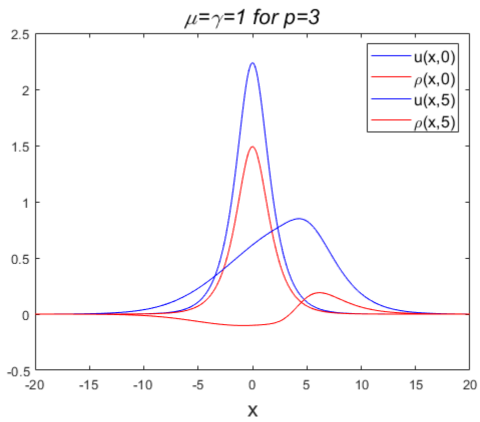

Figure 7. Wave forms at the initial and end time

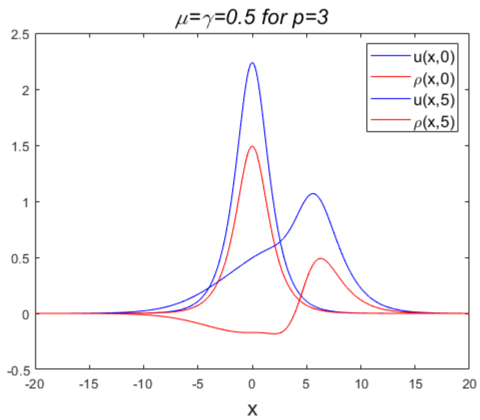

Figure 8. Wave forms at the initial and end time

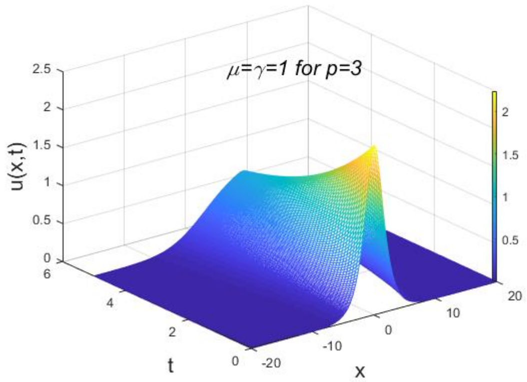

Figure 9. Simulation of the wave $u(x,t)$

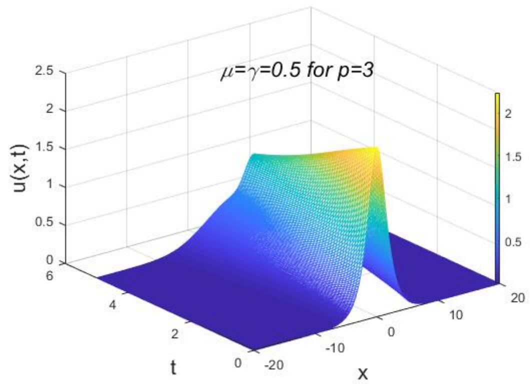

Figure 10. Simulation of the wave $u(x,t)$

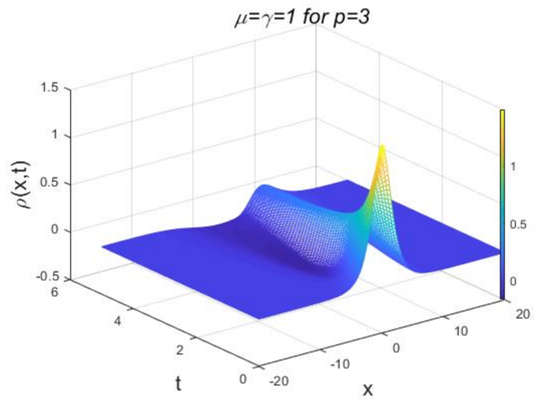

Figure 11. Simulation of the wave $\rho(x,t)$

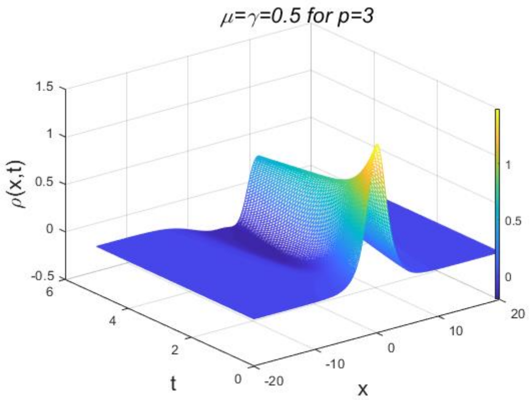

Figure 12. Simulation of the wave $\rho(x,t)$

Case 3: Now for Dissipative GRLW equations with Damping term, we take $p = 5, \mu = \gamma = 1$ and $c = 1.5$. The simulation is carried out over the same domain. The initial conditions are

$$u(x,0) = \sqrt[4]{\frac{25}{2}} \sqrt{sech \ \frac{2\sqrt{5}}{3} \ x} \tag{25}$$

$$\rho(x,0) = \frac{2}{3} \sqrt[4]{\frac{25}{2}} \sqrt{sech \ \frac{2\sqrt{5}}{3} \ x} \tag{26}$$

A comparison with the reference solution is shown in Tables 5 and 6. As in other cases, the decrease in the error values is associated with the selection of Δx and Δt. Simulations of two damped wave profiles are plotted in the Figures $(13 - 18)$. It is seen that wave moves to the right as damped when time is increasing.

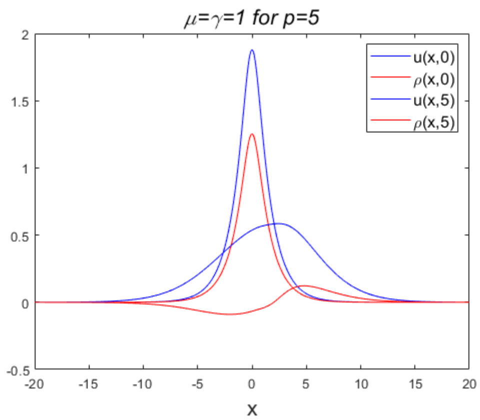

Figure 13. Wave forms at the initial and end time

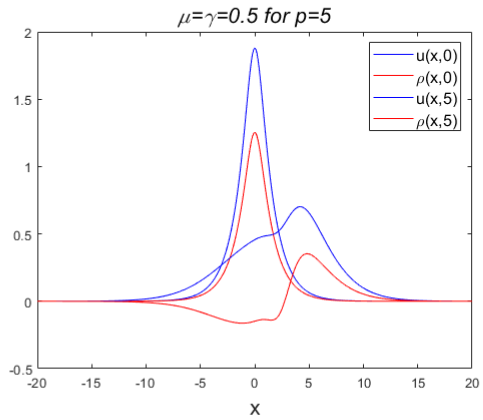

Figure 14. Wave forms at the initial and end time

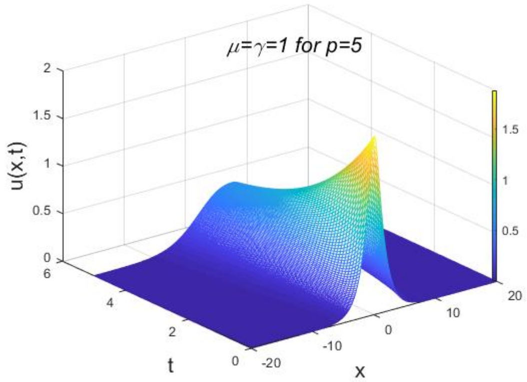

Figure 15. Simulation of the wave $u(x,t)$

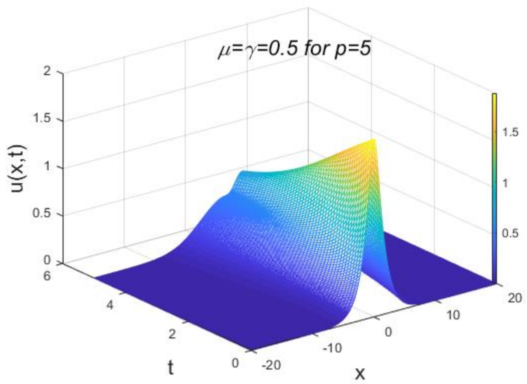

Figure 16. Simulation of the wave $u(x,t)$

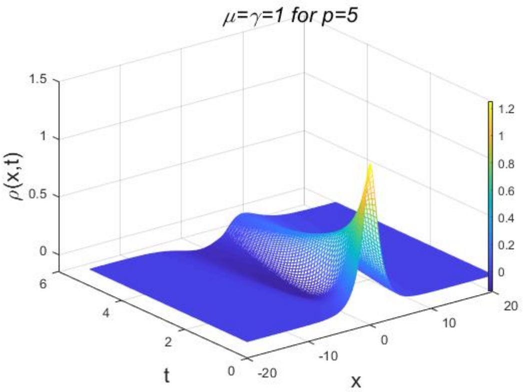

Figure 17. Simulation of the wave $\rho(x, t)$

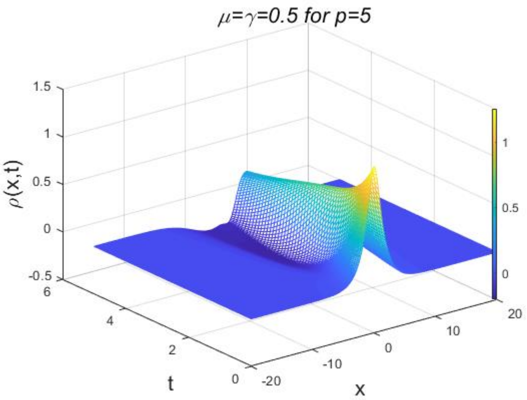

Figure 18. Simulation of the wave $\rho(x, t)$

Table 5. Error norms of numerical solution $u(x, 5)$ for $p = 5$.

RBF	$\Delta x = \Delta t = 0.2$	$\Delta x = \Delta t = 0.1$	$\Delta x = \Delta t = 0.05$	$\Delta x = \Delta t = 0.02$
		$\mu = \gamma = 1$		
G	2.24930e-04	1.265153e-04	6.15191e-05	1.62531e-05
W	2.10505-04	1.51494e-04	6.23328e-05	1.99231e-05
		$\mu = \gamma = 0.5$		
G	1.37951e-04	7.90442e-05	4.37893e-05	1.03305e-05
W	1.62994e-04	9.63008e-05	7.31232e-05	5.91318e-05

Table 6. Error norms of numerical solution $\rho(x,5)$ for $p=5$.

RBF	$\Delta x=\Delta t=0.2$	$\Delta x=\Delta t=0.1$	$\Delta x=\Delta t=0.05$	$\Delta x=\Delta t=0.02$
		$\mu=\gamma=1$		
G	7.58870e-05	4.27412e-05	2.07974e-05	5.62044e-06
W	7.04934e-05	5.09357e-01	2.08800e-05	1.40657e-05
		$\mu=\gamma=0.5$		
G	1.37951e-04	8.35726e-05	1.101632e-04	9.26203e-06
W	9.02156-05	9.16636e-05	1.35562e-05	1.34886e-05

4. CONCLUSION

The meshless kernel-based method of lines is implemented for the numerical solutions of the Dissipative Generalized Symmetric Regularized Long Wave (SRLW) Equations with Damping term. Since there isn't an analytical solution of the governing equation a reference solution is considered as an exact solution for small step size. The evaluated numerical results using the larger step values are compared with the reference solutions. In this way, the accuracy of the results is tested in the sense of L_∞ error norm. When Δx and Δt approach the used values in the reference solution, the value of error norms decreases.

In the figures, the motion of the wave was monitored. The height of wave crest decreases due to the effect of the damping term γ and the dissipation term μ. Also, it is observed that the values of dissipation term μ and damping term γ are chosen bigger the height of wave decrease faster.

As a result, it is said that the numerical calculations by using this meshless method are very successful and the method can be used as a powerful problem-solving method.

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