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# Numerical Solution of Multi-Order Fractional Differential Equations Using Generalized Sine-Cosine Wavelets

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**Abstract**

In this work, we propose a numerical method based on the generalized sine-cosine wavelets for solving multi-order fractional differential equations. After introducing generalized sine-cosine wavelets, the operational matrix of Riemann-Liouville fractional integration is constructed using the properties of the block-pulse functions. The fractional derivative in the problem is considered in the Caputo sense. This method reduces the considered problem to the problem of solving a system of nonlinear algebraic equations. Finally, some examples are included to demonstrate the applicability of the new approach.

## 1. Introduction

The notion of the fractional differential equations (FDEs) was first developed as a pure mathematical theory in the middle of the 19th century [1]. A history of the development of the fractional differential operators can be found in [2, 3]. It has been revealed that many mathematical modelings contain FDEs. To mention a few, fractional derivatives are used in viscoelastic systems [4], economics [5], continuum and statistical mechanics [6], solid mechanics [7], electrochemistry [8], biology [9] and acoustics [10]. An important issue to shed light on is the fact that most of the FDEs do not have exact analytic solutions. Consequently, emphasis of efforts is on the importance of seeking numerical solutions for these equations. As a result, several numerical methods have been given to solve problems including FDEs, such as Adomian decomposition method [11], variational iteration method [12], fractional differential transform method [13], operational matrix method [14], homotopy analysis method [15], power series method [16] and modified homotopy perturbation method [17]. Also, there can be some classical solution techniques to be fond, e.g. Laplace transform method [18].

One way to solve equations numerically is to use wavelets. The basic idea of wavelets (both: translation and dilation) goes back to the early 1960's [19]. There are developments concerning the multiresolution analysis (MRA) algorithm based on wavelets [20] and the construction of compactly supported orthonormal wavelet bases [21]. Wavelets constitute unconditional (Riesz) bases for $L^2(\mathbb{R})$, the space of all square integrable functions on the real line. In other words, a function in $L^2(\mathbb{R})$ can be decomposed and stably reconstructed in terms of wavelets [22]. To illustrate, some wavelets which have been constructed and used for solving FDEs include B-spline wavelets [23], Haar wavelets [24], Chebyshev wavelets [25], Legendre wavelets [26] and Bernoulli wavelets [27].

Sine-cosine wavelet (SCW) has been used and showed efficient to solve various problems. To indicate this, we can refer to some works. Razzaghi and Yousefi in [28] have employed SCW to solve variational problems. Tavassoli Kajani et al. [29] have proposed a method based on SCW for solving integro-differential equations. They also applied this method to solve Fredholm integral equations in [30]. A numerical evaluation of Hankel transform for seismology has been given in [31] using SCWs approach. The present work introduces the generalized sine-cosine wavelets (GSCWs) operational matrix of fractional integration which can be used to solve fractional problems.

The organization of this paper is as follows: Section 2 gives a brief preliminaries of fractional calculus followed by orthonormal basis of GSCWs and their properties in Section 3. Section 4 is devoted to block-pulse functions and their basic properties. Section 5 introduces the fractional order of operational matrix of integration for GSCWs. A numerical method based on the GSCWs and block-pulse functions in order to solve multi-order FDEs is given in Section 6. Some examples are included in Section 7 to show the applicability and efficiency of this method followed by concluding remarks in Section 8.

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## 2. Preliminaries of fractional calculus

In this section, we briefly give some preliminaries and notations of fractional calculus. Two most important definitions for fractional integral and derivative operators are Riemann-Liouville integral and Caputo derivative. The Riemann-Liouville fractional integral operator $I^\alpha$ of order $\alpha \geq 0$ is defined as follows [32]:

$$I^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, & \alpha > 0, \\ u(t), & \alpha = 0, \end{cases}$$

where $\Gamma(\alpha)$ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Also, the Caputo fractional derivative operator $D^\alpha$ of order $\alpha$ is defined as follows [32]:

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N},$$

where $n = \lceil \alpha \rceil$ is the smallest integer greater than or equal to $\alpha$.

The following properties are satisfied for the Riemann-Liouville integral operator and Caputo derivative:

$$I^\alpha t^v = \frac{\Gamma(v+1)}{\Gamma(v+1+\alpha)} t^{v+\alpha}, \qquad v > -1,$$

$$I^\alpha (D^\alpha u(t)) = u(t) - \sum_{i=0}^{\lceil \alpha \rceil - 1} u^{(i)}(0) \frac{t^i}{i!}, \tag{2.1}$$

$$I^{\alpha-\beta}(D^\alpha u(t)) = D^\beta u(t) - \sum_{i=\lceil \beta \rceil}^{\lceil \alpha \rceil - 1} u^{(i)}(0) \frac{t^{i-\beta}}{\Gamma(i-\beta+1)}, \quad 0 \leq \beta < \alpha. \tag{2.2}$$

## 3. Generalized sine-cosine wavelets

### 3.1. Definition and function approximation

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\varphi(t)$ which is called the mother wavelet. When the dilation parameter and the translation parameter vary continuously, we have the following family of continuous wavelets as [19, 33, 34]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \varphi\left(\frac{t-b}{a}\right), \quad a,b \in \mathbb{R}, \quad a \neq 0,$$

where $a$ and $b$ are the dilation and translation parameters, respectively. If the parameters $a$ and $b$ are restricted to take values $a = a_0^{-k}$ and $b = nb_0 a_0^{-k}$, where $a_0 > 1$, $b_0 > 0$ and $n$, and $k$ are positive integers, a family of discrete wavelets which forms a wavelet basis for $L^2(\mathbb{R})$ is obtained as

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \varphi\left(a_0^k t - nb_0\right).$$

Especially, if $a_0 = 2$ and $b_0 = 1$, then the set $\{\psi_{k,n}(t)\}$ forms an orthonormal basis.

SCWs are usually defined on the interval $[0,1)$. Here, we replace the interval $[0,1)$ by $[0,T)$ where $T > 0$ and define GSCWs as

$$\psi_{n,m}(t) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{T}} f_m(2^k t - nT), & \frac{n}{2^k} T \leq t < \frac{n+1}{2^k} T, \\ 0, & \text{otherwise}, \end{cases}$$

with

$$f_m(t) = \begin{cases} \frac{1}{\sqrt{2}}, & m = 0, \\ \cos(\frac{2m\pi t}{T}), & m = 1,2,\ldots,L, \\ \sin(\frac{2(m-L)\pi t}{T}), & m = L+1, L+2, \ldots, 2L, \end{cases}$$

where $L$ is any positive integer, $n = 0,1,2,\ldots,2^k-1$ and $k = 0,1,2,\cdots$. The set of GSCWs forms an orthonormal basis for the space $L^2[0,T]$. Therefore, a function $u(t)$ in this space may be expanded in a series of GSCWs as

$$u(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{2^k-1} c_{n,m} \psi_{n,m}(t), \tag{3.1}$$

where

$$c_{n,m} = \langle u(t), \psi_{n,m}(t) \rangle = \int_0^T u(t)\psi_{n,m}(t)dt,$$

in which $\langle .,. \rangle$ denotes the inner product. If the infinite series in (3.1) is truncated, then an approximation of the function $u(t)$ is obtained as

$$u(t) \simeq \sum_{m=0}^{2L} \sum_{n=0}^{2^k-1} c_{n,m}\psi_{n,m}(t) = C^T\Psi_\omega(t), \tag{3.2}$$

where $\omega = 2^k(2L+1)$, and $C$ and $\Psi(t)$ are $2^k(2L+1) \times 1$ matrices given by

$$C = [c_{0,0}, c_{0,1}, \ldots, c_{0,2L}, c_{1,0}, c_{1,1}, \ldots, c_{1,2L}, \ldots, c_{2^k-1,0}, c_{2^k-1,1}, \ldots, c_{2^k-1,2L}]^T,$$

$$\Psi_\omega(t) = [\psi_{0,0}(t), \psi_{0,1}(t), \ldots, \psi_{0,2L}(t), \psi_{1,0}(t), \psi_{1,1}(t), \ldots, \psi_{1,2L}(t),$$
$$\ldots, \psi_{2^k-1,0}(t), \psi_{2^k-1,1}(t), \ldots, \psi_{2^k-1,2L}(t)]^T.$$

### 3.2. Convergence analysis

In this section, we get the convergence of the GSCW approximation of a function for all level of resolution $k$.

**Theorem 3.1.** *Let $L \to \infty$, then the series solution (3.2) converges to $u(t)$.*

**Proof**. Let $S_{k,M}(t)$ be a sequence of partial sums of $c_{n,m}\psi_{n,m}(t)$ as

$$S_{k,M}(t) = \sum_{m=0}^{M} \sum_{n=0}^{2^k-1} c_{n,m}\psi_{n,m}(t),$$

where $M = 2L$. We prove that $S_{k,M}$ is a Cauchy sequence in Hilbert space $L^2[0,T]$ and then we show that $S_{k,M}$ converges to $u(t)$, when $M \to \infty$. In order to reach the first aim, let $\hat{M} = 2\hat{L}$ with $L > \hat{L}$, then

$$\|S_{k,M} - S_{k,\hat{M}}\|^2 = \| \sum_{m=\hat{M}+1}^{M} \sum_{n=0}^{2^k-1} c_{n,m}\psi_{n,m}(t)\|^2$$

$$= \left\langle \sum_{m=\hat{M}+1}^{M} \sum_{n=0}^{2^k-1} c_{n,m}\psi_{n,m}(t), \sum_{m=\hat{M}+1}^{M} \sum_{n=0}^{2^k-1} c_{n,m}\psi_{n,m}(t) \right\rangle$$

$$= \sum_{m=\hat{M}+1}^{M} \sum_{n=0}^{2^k-1} \sum_{l=\hat{M}+1}^{M} \sum_{r=0}^{2^k-1} c_{n,m}c_{r,l} \left\langle \psi_{n,m}(t), \psi_{r,l}(t) \right\rangle$$

$$= \sum_{m=\hat{M}+1}^{M} \sum_{n=0}^{2^k-1} |c_{n,m}|^2.$$

From Bessel's inequality, we have $\sum_{m=0}^{\infty} \sum_{n=0}^{2^k-1} |c_{n,m}|^2$ is convergent. So

$$\|S_{k,M} - S_{k,\hat{M}}\|^2 \to 0 \text{ as } L \to \infty.$$

This suggests that $S_{k,M}$ is a Cauchy sequence and hence it converges to a function in $L^2[0,T]$, say, $f(t)$. We need to show that $f(t) = u(t)$,

$$\langle f(t) - u(t), \psi_{n,m}(t) \rangle = \langle f(t), \psi_{n,m}(t) \rangle - \langle u(t), \psi_{n,m}(t) \rangle$$
$$= \lim_{L \to \infty} \left\langle S_{k,M}(t), \psi_{n,m}(t) \right\rangle - c_{n,m}$$
$$= c_{n,m} - c_{n,m}$$
$$= 0.$$

Therefore $\sum_{m=0}^{2L} \sum_{n=0}^{2^k-1} c_{n,m}\psi_{n,m}(t)$ converges to $u(t)$ as $L \to \infty$.

**Remark 3.2.** *An error bound for the SCWs approximation of a function $u(t) \in L^2[0,1]$ can be found in [35].*

## 4. Block-pulse functions

Consider the interval $[0,T)$ and divide it into $\omega$ subintervals $[(i-1)h, ih)$, $i = 1, 2, \ldots, \omega$ with $h = \frac{T}{\omega}$. Then the block-pulse functions are defined by [36]

$$b_i(t) = \begin{cases} 1, & (i-1)h \leq t < ih, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear from the block-pulse functions' definition that the disjointness property for these functions is satisfied as follows:

$$b_i(t)b_j(t) = \begin{cases} b_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \qquad i, j = 1, 2, \ldots, \omega.$$

Furthermore, we have the orthogonality property as

$$\int_0^T b_i(t)b_j(t)dt = \begin{cases} h, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i, j = 1, 2, \ldots, \omega.$$

The block-pulse functions consist a complete orthogonal basis for the space $L^2[0,T)$. Therefore, every real bounded function $u(t)$ which is square integrable on the interval $[0,T)$ can be approximated using the block-pulse functions as

$$u(t) \simeq \sum_{i=1}^{\omega} u_i b_i(t) = U^T B_\omega(t), \tag{4.1}$$

where

$$B_\omega(t) = [b_1(t), b_2(t), \ldots, b_\omega(t)]^T,$$

$$U = [u_1, u_2, \ldots, u_\omega]^T,$$

and

$$u_i = \frac{1}{h} \int_0^t u(t)b_i(t)dt = \frac{1}{h} \int_{(i-1)h}^{ih} u(t)dt.$$

For the block-pulse vector $B_\omega(t)$ and the vector $U$, we have

$$B_\omega(t)B_\omega^T(t)U \simeq diag(U)B_\omega(t), \tag{4.2}$$

where $diag(U)$ is the following diagonal matrix

$$diag(U) = \begin{bmatrix} u_1 & 0 & \ldots & 0 \\ 0 & u_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_\omega \end{bmatrix}.$$

In [36], the authors have introduced the operational matrix of fractional integration of the block-pulse functions. They proved that

$$I^\alpha B_\omega(t) \simeq F_{\omega \times \omega}^\alpha B_\omega(t), \tag{4.3}$$

where

$$F_{\omega \times \omega}^\alpha = \left(\frac{T}{\omega}\right)^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_2 & \xi_3 & \xi_4 & \ldots & \xi_m \\ 0 & 1 & \xi_2 & \xi_3 & \ldots & \xi_{m-1} \\ 0 & 0 & 1 & \xi_2 & \ldots & \xi_{m-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & \xi_2 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}.$$

with $\xi_k = k^{\alpha+1} - 2(k-1)^{\alpha+1} + (k-2)^{\alpha+1}$.

## 5. Operational matrix of fractional integration

In this section, we introduce the fractional order operational matrix of integration for the GSCWs. To this aim, first we look for a matrix $Q_{\omega \times \omega}$ such that

$$\Psi_{\omega}(t) \simeq Q_{\omega \times \omega} B_{\omega}(t), \tag{5.1}$$

where $\omega = 2^k(2L+1)$. Using (4.1), we have

$$\psi_{n,m}(t) = \sum_{i=1}^{\omega} c_i^{n,m} b_i(t),$$

with

$$c_i^{n,m} = \frac{\omega}{T} \int_{\frac{(i-1)}{\omega}T}^{\frac{i}{\omega}T} \psi_{n,m}(t) dt.$$

Using the definition of the GSCWs, $c_i^{n,m}$ could be nonzero if

$$\frac{n}{2^k}T \leq \frac{i-1}{\omega}T < \frac{i}{\omega}T \leq \frac{n+1}{2^k}T.$$

This implies to have

$$n(2L+1)+1 \leq i \leq (n+1)(2L+1). \tag{5.2}$$

Taking (4.1) and (5.2) into consideration, we get

$$\psi_{n,m}(t) = \sum_{i=n(2L+1)+1}^{(n+1)(2L+1)} c_i^{n,m} b_i(t).$$

When $m = 0$, we have

$$c_i^{n,0} = \frac{\omega}{T} \int_{\frac{(i-1)}{\omega}T}^{\frac{i}{\omega}T} \frac{2^{\frac{k}{2}}}{\sqrt{T}} dt = \frac{2^{\frac{k}{2}}}{\sqrt{T}}.$$

For $m = 1, 2, \ldots, L$, we obtain

$$\begin{aligned}
c_i^{n,m} &= \frac{\omega}{T} \int_{\frac{(i-1)}{\omega}T}^{\frac{i}{\omega}T} \frac{2^{\frac{k+1}{2}}}{\sqrt{T}} \cos\left(\frac{2m\pi}{T}(2^k t - nT)\right) dt \\
&= \frac{2^{\frac{k-1}{2}}(2L+1)}{m\pi\sqrt{T}} \left[\sin\left(2\pi m\left(2^k \frac{i}{\omega} - n\right)\right) - \sin\left(2\pi m\left(2^k \frac{i-1}{\omega} - n\right)\right)\right],
\end{aligned}$$

and for $m = L+1, L+2, \ldots, 2L$, we get

$$\begin{aligned}
c_i^{n,m} &= \frac{\omega}{T} \int_{\frac{(i-1)}{\omega}T}^{\frac{i}{\omega}T} \frac{2^{\frac{k+1}{2}}}{\sqrt{T}} \sin\left(\frac{2(m-L)\pi}{T}(2^k t - nT)\right) dt \\
&= \frac{2^{\frac{k-1}{2}}(2L+1)}{(m-L)\pi\sqrt{T}} \left[\cos\left(2\pi(m-L)\left(2^k \frac{i-1}{\omega} - n\right)\right) \right. \\
&\qquad\qquad \left. - \cos\left(2\pi(m-L)\left(2^k \frac{i}{\omega} - n\right)\right)\right].
\end{aligned}$$

Hence, the matrix $Q_{\omega \times \omega}$ in (5.1) is obtained as

$$Q_{\omega \times \omega} = \begin{bmatrix}
Q_0 & O & O & \ldots & O \\
O & Q_1 & O & \ldots & O \\
O & O & Q_2 & \ldots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & O & \ldots & Q_{2^k-1}
\end{bmatrix},$$

where $O$ is the zero matrix of dimension $(2L+1) \times (2L+1)$ and $Q_n$, $n = 0, 1, 2, \ldots, 2^k - 1$, are $(2L+1) \times (2L+1)$ matrices as

$$Q_n = \left[a_{m,i}^n\right], \quad m = 0, 1, 2, \ldots, 2L, \quad i = 1, 2, 3, \ldots, 2L+1,$$

with $a_{m,i}^n = c_i^{n,m}$.
The matrix $Q_{\omega \times \omega}$ is an invertible matrix, so we have

$$B_{\omega}(t) \simeq Q_{\omega \times \omega}^{-1} \Psi_{\omega}(t). \tag{5.3}$$

Applying the Riemann-Liouville integral operator of order $\alpha$ to (5.1) and then utilizing (4.3) and (5.3), yield

$$I^{\alpha}\Psi_{\omega}(t) \simeq Q_{\omega\times\omega}I^{\alpha}B_{\omega}(t) \simeq Q_{\omega\times\omega}F^{\alpha}_{\omega\times\omega}B_{\omega}(t) \simeq Q_{\omega\times\omega}F^{\alpha}_{\omega\times\omega}Q^{-1}\Psi_{\omega}(t).$$

Therefore we have

$$I^{\alpha}\Psi_{\omega}(t) \simeq P^{\alpha}_{\omega\times\omega}\Psi_{\omega}(t), \tag{5.4}$$

with

$$P^{\alpha}_{\omega\times\omega} = Q_{\omega\times\omega}F^{\alpha}_{\omega\times\omega}Q^{-1}_{\omega\times\omega}.$$

In particular, for $T = 1$, $k = 1$, $L = 1$ and $\alpha = 0.5$, the GSCWs operational matrix of fractional order integration $P^{\alpha}_{\omega\times\omega}$ is given by

$$P^{0.5}_{6\times 6} = \begin{bmatrix} 0.5319 & -0.0253 & -0.2073 & 0.4407 & 0.0218 & 0.0993 \\ -0.0173 & 0.1651 & 0.0991 & 0.0149 & 0.0061 & 0.0148 \\ 0.1418 & -0.0991 & 0.2243 & -0.0679 & -0.0148 & -0.0449 \\ 0. & 0. & 0. & 0.5319 & -0.0253 & -0.2073 \\ 0. & 0. & 0. & -0.0173 & 0.1651 & 0.0991 \\ 0. & 0. & 0. & 0.1418 & -0.0991 & 0.2243 \end{bmatrix}.$$

## 6. Numerical method

In this section, we use the properties of the GSCWs together with the block-pulse functions to solve a class of nonlinear multi-order FDEs. Consider the following FDE

$$D^{\alpha}u(t) = \sum_{k=1}^{r} a_k(t)D^{\beta_k}u(t) + a_0(t)u(t) + a(t)[u(t)]^m + f(t), \tag{6.1}$$

with initial conditions

$$u^{(s)}(0) = u_0^s, \quad s = 0, 1, \ldots, \lceil\alpha\rceil - 1,$$

where $\alpha > \beta_1 > \beta_2 > \ldots > \beta_r$, $D^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha$, $a(t)$, $a_k(t)$, $k = 0, 1, 2, \ldots, r$ and $f(t)$ are given known functions, $\lceil.\rceil$ is the ceiling function and $u(t)$ is the unknown function to be determined. In order to obtain a numerical solution for (6.1), we suppose that

$$D^{\alpha}u(t) \simeq U^T\Psi_{\omega}(t), \tag{6.2}$$

then using (2.1), (5.4) and (6.2), we have

$$\begin{aligned} u(t) &\simeq U^T P^{\alpha}_{\omega\times\omega}\Psi_{\omega}(t) + \sum_{s=0}^{\lceil\alpha\rceil-1} u_0^s \frac{t^s}{s!} \\ &\simeq (U^T P^{\alpha}_{\omega\times\omega} + U_0^T)\Psi_{\omega}(t) \\ &= \Lambda_0^T\Psi_{\omega}(t), \end{aligned} \tag{6.3}$$

where we have used

$$\sum_{s=0}^{\lceil\alpha\rceil-1} u_0^s \frac{t^s}{s!} \simeq U_0^T\Psi_{\omega}(t),$$

and

$$\Lambda_0 = (U^T P^{\alpha}_{\omega\times\omega} + U_0^T)^T.$$

Also, taking into consideration (2.2), (5.4) and (6.2), we have

$$D^{\beta_k}u(t) \simeq (U^T P^{\alpha-\beta_k}_{\omega\times\omega} + U_k^T)\Psi_{\omega}(t) = \Lambda_k^T\Psi_{\omega}(t), \tag{6.4}$$

where

$$\sum_{s=\lceil\beta_k\rceil}^{\lceil\alpha\rceil-1} u_0^s \frac{t^{s-\beta_k}}{\Gamma(s-\beta_k+1)} \simeq U_k^T\Psi_{\omega}(t),$$

and

$$\Lambda_k = (U^T P^{\alpha-\beta_k}_{\omega\times\omega} + U_k^T)^T.$$

Now, suppose that

$$\begin{aligned} a_k(t) &\simeq A_k^T\Psi_{\omega}(t), \quad k = 0, 1, 2, \ldots, r, \\ a(t) &\simeq A^T\Psi_{\omega}(t), \\ f(t) &\simeq F^T\Psi_{\omega}(t). \end{aligned} \tag{6.5}$$

Substituting approximations (6.3)-(6.5) into (6.1) yields

$$U^T \Psi_\omega(t) = \sum_{k=0}^{r} A_k^T \Psi_\omega(t) \Psi_\omega^T(t) \Lambda_k + A^T \Psi_\omega(t) \left[ \Lambda_0^T \Psi_\omega(t) \right]^m + F^T \Psi_\omega(t). \tag{6.6}$$

By employing (4.2) and (5.1), we get

$$A_k^T \Psi_\omega(t) \Psi_\omega^T(t) \Lambda_k \simeq A_k^T Q_{\omega \times \omega} B_\omega(t) B_\omega^T(t) Q_{\omega \times \omega}^T \Lambda_k$$
$$\simeq A_k^T Q_{\omega \times \omega} diag(Q_{\omega \times \omega}^T \Lambda_k) B_\omega(t) \tag{6.7}$$

In a similar way, we obtain

$$A^T \Psi_\omega(t) \left[ \Lambda_0^T \Psi_\omega(t) \right]^m \simeq A^T Q_{\omega \times \omega} \left[ diag(Q_{\omega \times \omega}^T \Lambda_0) \right]^m B_\omega(t) \tag{6.8}$$

At the end, taking consideration (6.7) and (6.8) into (6.6), we get

$$U^T Q_{\omega \times \omega} - \sum_{k=0}^{r} A_k^T Q_{\omega \times \omega} diag(Q_{\omega \times \omega}^T \Lambda_k) - A^T Q_{\omega \times \omega} \left[ diag(Q_{\omega \times \omega}^T \Lambda_0) \right]^m - F^T Q_{\omega \times \omega} = 0,$$

which is a system of nonlinear algebraic equations that can be solved using iterative methods. By solving this system, we obtain the approximate solution $u(t)$ as

$$u(t) \simeq U^T P_{\omega \times \omega}^\alpha \Psi_\omega(t) + \sum_{s=0}^{\lceil \alpha \rceil - 1} u_0^s \frac{t^s}{s!}. \tag{6.9}$$

**Remark 6.1.** *In the linear case of the equation (6.1) with constant coefficients, i.e.*

$$D^\alpha u(t) = \sum_{k=1}^{r} a_k D^{\beta_k} u(t) + a_0 u(t) + f(t),$$

*the following linear system is resulted from employing our method*

$$U^T - \sum_{k=0}^{r} a_k \Lambda_k^T - a_0 \Lambda_0^T - F^T = 0.$$

# 7. Illustrative examples

In this section we present four examples and apply the method presented in the previous section for solving them. The function "FindRoot" in "Mathematica" software has been employed for solving the final nonlinear systems obtained by the method.

**Example 7.1.** *Consider the Bagley-Torvik equation [37, 38]*

$$aD^2 u(t) = -bD^{3/2} u(t) - cu(t) + c(1+t), \quad t \in [0,1], \tag{7.1}$$

*subject to initial conditions $u(0) = u'(0) = 1$.*

The exact solution of this problem is $u(t) = 1 + t$. By considering $k = 0$ and $L = 1$, we employ the present method for this problem with $a = 1$, $b = 0.5$ and $c = 0.5$. In this case, the basis functions are given by

$$\psi_{0,0}(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_{0,1}(t) = \begin{cases} \sqrt{2}\cos(2\pi t), & 0 \le t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_{0,2}(t) = \begin{cases} \sqrt{2}\sin(2\pi t), & 0 \le t < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Suppose that

$$D^2 u(t) \simeq u_{0,0} \psi_{0,0}(t) + u_{0,1} \psi_{0,1}(t) + u_{0,2} \psi_{0,2}(t) = U^T \Psi_3(t), \tag{7.2}$$

then using the initial conditions of the problem, we get

$$D^{3/2} u(t) \simeq U^T P_{3 \times 3}^{\frac{1}{2}} \Psi_3(t), \tag{7.3}$$

$$u(t) \simeq U^T P_{3 \times 3}^2 \Psi_3(t) + 1 + t \simeq (U^T P_{3 \times 3}^2 + U_0^T) \Psi_3(t), \tag{7.4}$$

where $U_0$ is obtained by approximating the function $1 + t$ as

$$U_0 = \left[ \frac{3}{2}, 0, -\frac{1}{\sqrt{2}\pi} \right]^T,$$

and $P^{\frac{1}{2}}_{3\times3}$ and $P^{2}_{3\times3}$ are given, respectively, by

$$P^{\frac{1}{2}}_{3\times3} = \begin{bmatrix} \frac{4}{3\sqrt{\pi}} & \frac{4}{27}(\sqrt{2}+\sqrt{6}-4)\sqrt{\pi} & \frac{4}{81}(-9\sqrt{2}+4\sqrt{3}+\sqrt{6})\sqrt{\pi} \\ \frac{\sqrt{2}+\sqrt{6}-4}{\pi^{3/2}} & \frac{2(-4\sqrt{2}+\sqrt{3}+5)}{3\sqrt{3}\pi} & \frac{2(8\sqrt{2}-3\sqrt{3}-5)}{9\sqrt{\pi}} \\ \frac{9\sqrt{2}-4\sqrt{3}-\sqrt{6}}{3\pi^{3/2}} & -\frac{2(8\sqrt{2}-3\sqrt{3}-5)}{9\sqrt{\pi}} & \frac{2(\sqrt{3}+4\sqrt{6}-9)}{9\sqrt{\pi}} \end{bmatrix},$$

$$P^{2}_{3\times3} = \begin{bmatrix} \frac{1}{6} & \frac{1}{81}\sqrt{\frac{2}{3}}\pi & -\frac{1}{27}\left(\sqrt{2}\pi\right) \\ \frac{1}{6\sqrt{6}\pi} & -\frac{1}{54} & 0 \\ \frac{1}{2\sqrt{2}\pi} & 0 & -\frac{5}{54} \end{bmatrix}.$$

By substituting (7.2)–(7.4) into (7.1), we obtain

$$U^T\Psi_3(t) = -0.5U^T P^{\frac{1}{2}}_{3\times3}\Psi_3(t) - 0.5(U^T P^{2}_{3\times3} + U^T_0)\Psi_3(t) + 0.5U^T_0\Psi_3(t),$$

which leads us to have

$$U^T = -0.5U^T P^{\frac{1}{2}}_{3\times3} - 0.5U^T P^{2}_{3\times3}.$$

By solving this linear system, the unknown parameters are computed as

$$u_{0,0} = u_{0,1} = u_{0,2} = 0.$$

Thus using (6.9), we get

$$u(t) = 1 + t,$$

which is the exact solution.

**Example 7.2.** *Consider the following multi-order FDE [37, 39]:*

$$D^3 u(t) = -D^{5/2}u(t) - [u(t)]^2 + t^4,$$

*subject to initial conditions $u(0) = u'(0) = 0$ and $u''(0) = 2$.*

The exact solution of this problem is $u(t) = t^2$. The absolute error of the numerical solutions obtained by the present method in this paper is given in Table 1 and Figure 7.1. We have displayed the numerical results for $T = 1$ using the GSCWs with $L = 1$ and $k = 2, 4, 6, 8$ in Table 1. In Figure 7.1, plot of the absolute error obtained by $L = 1$ and different values of $k$ are shown. It is seen from Table 1 and Figure 7.1 that the absolute error decreases as the level of resolution increases.

**Table 1:** Absolute error at some selected point with $L = 1$ and different values of $k$ for Example 7.2.

| $t$ | k=2 | k=4 | k=6 | k=8 |
|-----|-----|-----|-----|-----|
| 0.0 | $2.06 \times 10^{-7}$ | $1.41 \times 10^{-11}$ | $9.22 \times 10^{-16}$ | $5.81 \times 10^{-20}$ |
| 0.1 | $3.71 \times 10^{-8}$ | $2.68 \times 10^{-10}$ | $6.23 \times 10^{-12}$ | $4.52 \times 10^{-13}$ |
| 0.2 | $3.83 \times 10^{-7}$ | $2.82 \times 10^{-9}$ | $2.26 \times 10^{-10}$ | $1.31 \times 10^{-11}$ |
| 0.3 | $8.76 \times 10^{-8}$ | $2.99 \times 10^{-8}$ | $1.51 \times 10^{-9}$ | $9.87 \times 10^{-11}$ |
| 0.4 | $3.87 \times 10^{-6}$ | $9.13 \times 10^{-8}$ | $6.63 \times 10^{-9}$ | $3.97 \times 10^{-10}$ |
| 0.5 | $2.21 \times 10^{-5}$ | $4.28 \times 10^{-7}$ | $2.06 \times 10^{-8}$ | $1.21 \times 10^{-9}$ |
| 0.6 | $9.09 \times 10^{-6}$ | $8.18 \times 10^{-7}$ | $4.54 \times 10^{-8}$ | $2.91 \times 10^{-9}$ |
| 0.7 | $3.57 \times 10^{-5}$ | $1.48 \times 10^{-6}$ | $9.99 \times 10^{-8}$ | $6.12 \times 10^{-9}$ |
| 0.8 | $3.98 \times 10^{-5}$ | $3.19 \times 10^{-6}$ | $1.85 \times 10^{-7}$ | $1.18 \times 10^{-8}$ |
| 0.9 | $1.15 \times 10^{-4}$ | $5.05 \times 10^{-6}$ | $3.38 \times 10^{-7}$ | $2.07 \times 10^{-8}$ |
| 1.0 | $9.37 \times 10^{-5}$ | $7.72 \times 10^{-6}$ | $5.35 \times 10^{-7}$ | $3.44 \times 10^{-8}$ |

**Example 7.3.** *Consider the following multi-order FDE [37, 39]:*

$$D^4 u(t) = -D^{3.5}u(t) - [u(t)]^3 + t^9,$$

*subject to initial conditions $u(0) = u'(0) = u''(0) = 0$ and $u'''(0) = 6$.*

The exact solution of this problem is $u(t) = t^3$. The absolute error of the numerical solutions obtained by the present method is given in Table 2 and Figure 7.2. The numerical results for $T = 1$ using the GSCWs with $L = 1$ and $k = 2, 4, 6, 8$ are displayed in Table 2. Plot of the absolute error obtained by $L = 1$ and different values of $k$ are shown in Figure 7.2. The results here confirm the convergence of the numerical solution to the exact solution of this problem.

**Example 7.4.** *As the last example, consider the following linear multi-order FDE [37, 40]:*

$$D^2 u(t) = 2Du(t) - D^{0.5}u(t) - u(t) + t^3 - 6t^2 + 6t + \frac{16}{5\sqrt{\pi}}t^{2.5},$$

*subject to initial conditions $u(0) = u'(0) = 0$.*

The exact solution is $u(t) = t^3$. Numerical results for this example are presented in Table 3 and Figure 7.3. The absolute errors at some selected points on the interval $[0, 1]$ using the GSCWs with $L = 1$ and $k = 2, 4, 6, 8$ are given in Table 3. In Figure 7.3, the exact solution and numerical solution obtained by $L = 1$ and different values of $k$ are displayed. The absolute error reported in Table 3 and Figure 7.3 show the convergence of the numerical solution to the exact solution.

**Figure 7.1:** Plot of the absolute error with $L = 1$ and $k = 2, 4, 6, 8$ for Example 7.2.

**Table 2:** Absolute error at some selected point with $L = 1$ and different values of $k$ for Example 7.3.

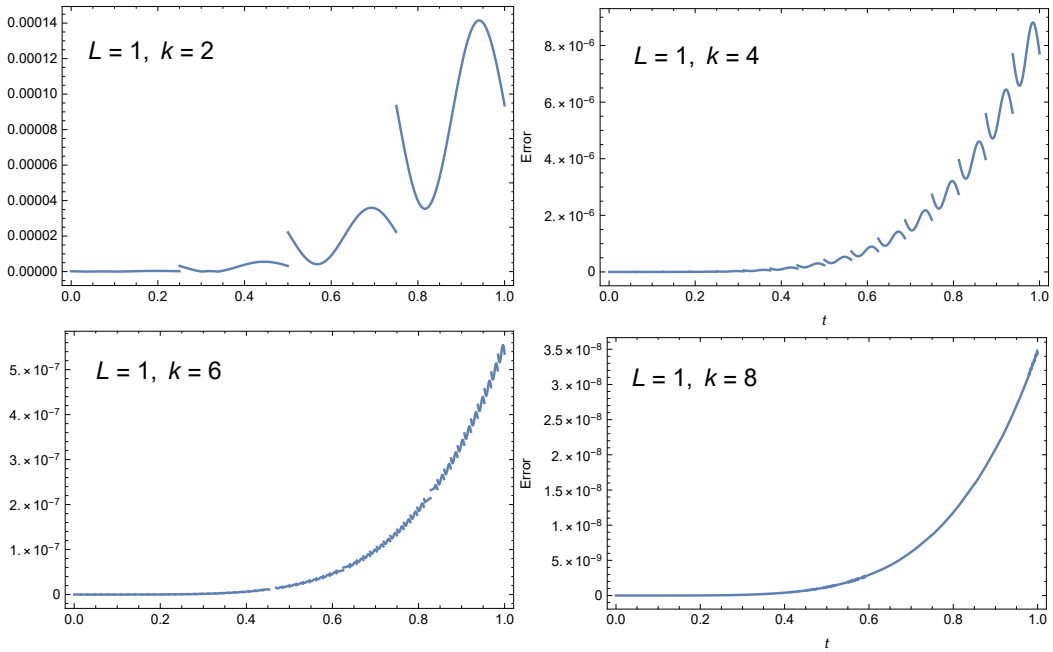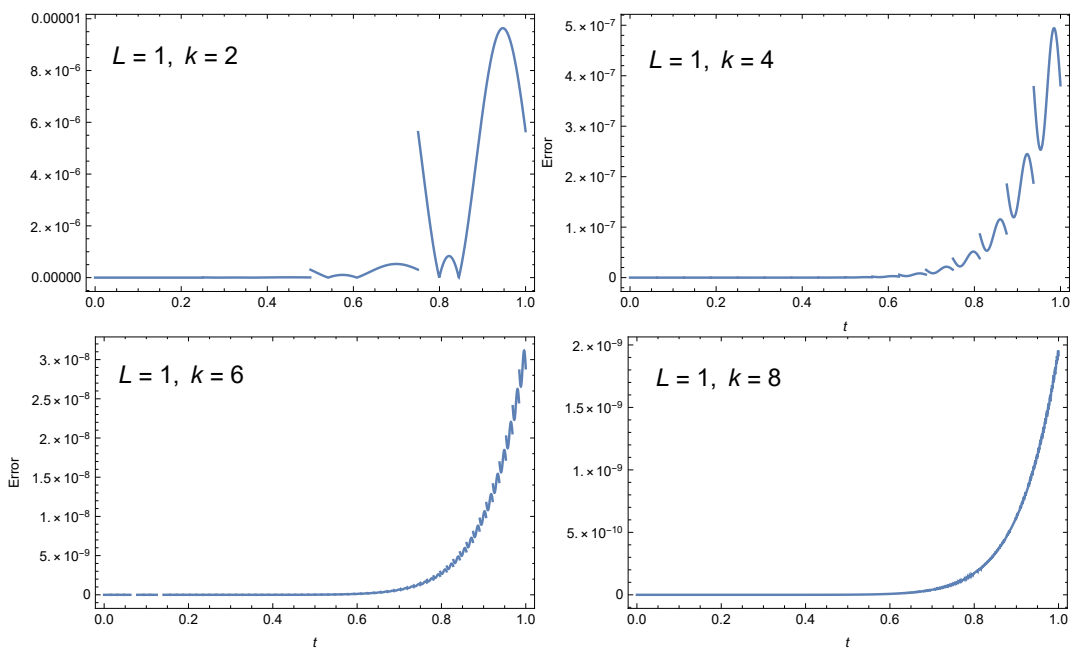| $t$ | k=2 | k=4 | k=6 | k=8 |
|-----|-----|-----|-----|-----|
| 0.0 | $1.34 \times 10^{-11}$ | $2.22 \times 10^{-19}$ | $3.51 \times 10^{-27}$ | $5.39 \times 10^{-35}$ |
| 0.1 | $4.63 \times 10^{-12}$ | $1.03 \times 10^{-16}$ | $2.16 \times 10^{-19}$ | 0.00 |
| 0.2 | $2.31 \times 10^{-11}$ | $8.25 \times 10^{-16}$ | $8.51 \times 10^{-16}$ | $4.51 \times 10^{-17}$ |
| 0.3 | $1.88 \times 10^{-9}$ | $1.68 \times 10^{-12}$ | $5.74 \times 10^{-14}$ | $3.94 \times 10^{-15}$ |
| 0.4 | $6.26 \times 10^{-9}$ | $1.78 \times 10^{-11}$ | $1.55 \times 10^{-12}$ | $8.80 \times 10^{-14}$ |
| 0.5 | $3.06 \times 10^{-7}$ | $6.39 \times 10^{-10}$ | $1.97 \times 10^{-11}$ | $1.07 \times 10^{-12}$ |
| 0.6 | $4.30 \times 10^{-8}$ | $2.47 \times 10^{-9}$ | $1.14 \times 10^{-10}$ | $7.58 \times 10^{-12}$ |
| 0.7 | $5.27 \times 10^{-7}$ | $9.09 \times 10^{-9}$ | $6.67 \times 10^{-10}$ | $3.98 \times 10^{-11}$ |
| 0.8 | $4.17 \times 10^{-8}$ | $5.11 \times 10^{-8}$ | $2.67 \times 10^{-9}$ | $1.73 \times 10^{-10}$ |
| 0.9 | $6.36 \times 10^{-6}$ | $1.42 \times 10^{-7}$ | $1.02 \times 10^{-8}$ | $6.16 \times 10^{-10}$ |
| 1.0 | $5.66 \times 10^{-5}$ | $3.81 \times 10^{-7}$ | $2.88 \times 10^{-8}$ | $1.91 \times 10^{-9}$ |



**Figure 7.2:** Plot of the absolute error with $L = 1$ and $k = 2, 4, 6, 8$ for Example 7.3.

**Table 3:** Absolute error at some selected point with $L = 1$ and different values of $k$ for Example 7.4.

| $t$ | k=2 | k=4 | k=6 | k=8 |
|-----|-----|-----|-----|-----|
| 0.0 | $7.57 \times 10^{-3}$ | $1.17 \times 10^{-4}$ | $1.84 \times 10^{-6}$ | $2.87 \times 10^{-8}$ |
| 0.1 | $2.14 \times 10^{-3}$ | $3.84 \times 10^{-4}$ | $7.92 \times 10^{-5}$ | $2.04 \times 10^{-5}$ |
| 0.2 | $5.85 \times 10^{-3}$ | $1.08 \times 10^{-3}$ | $2.68 \times 10^{-4}$ | $6.68 \times 10^{-5}$ |
| 0.3 | $1.08 \times 10^{-2}$ | $2.45 \times 10^{-3}$ | $6.01 \times 10^{-4}$ | $1.50 \times 10^{-4}$ |
| 0.4 | $2.49 \times 10^{-2}$ | $5.04 \times 10^{-3}$ | $1.31 \times 10^{-3}$ | $3.24 \times 10^{-4}$ |
| 0.5 | $1.48 \times 10^{-1}$ | $2.64 \times 10^{-2}$ | $6.04 \times 10^{-3}$ | $1.47 \times 10^{-3}$ |
| 0.6 | $4.36 \times 10^{-2}$ | $1.20 \times 10^{-2}$ | $2.91 \times 10^{-3}$ | $7.33 \times 10^{-4}$ |
| 0.7 | $5.55 \times 10^{-2}$ | $1.30 \times 10^{-2}$ | $3.28 \times 10^{-3}$ | $8.18 \times 10^{-4}$ |
| 0.8 | $6.78 \times 10^{-2}$ | $1.72 \times 10^{-2}$ | $4.27 \times 10^{-3}$ | $1.07 \times 10^{-3}$ |
| 0.9 | $1.14 \times 10^{-1}$ | $2.58 \times 10^{-2}$ | $6.62 \times 10^{-3}$ | $1.64 \times 10^{-3}$ |
| 1.0 | $2.88 \times 10^{-1}$ | $8.79 \times 10^{-2}$ | $2.31 \times 10^{-2}$ | $5.83 \times 10^{-3}$ |
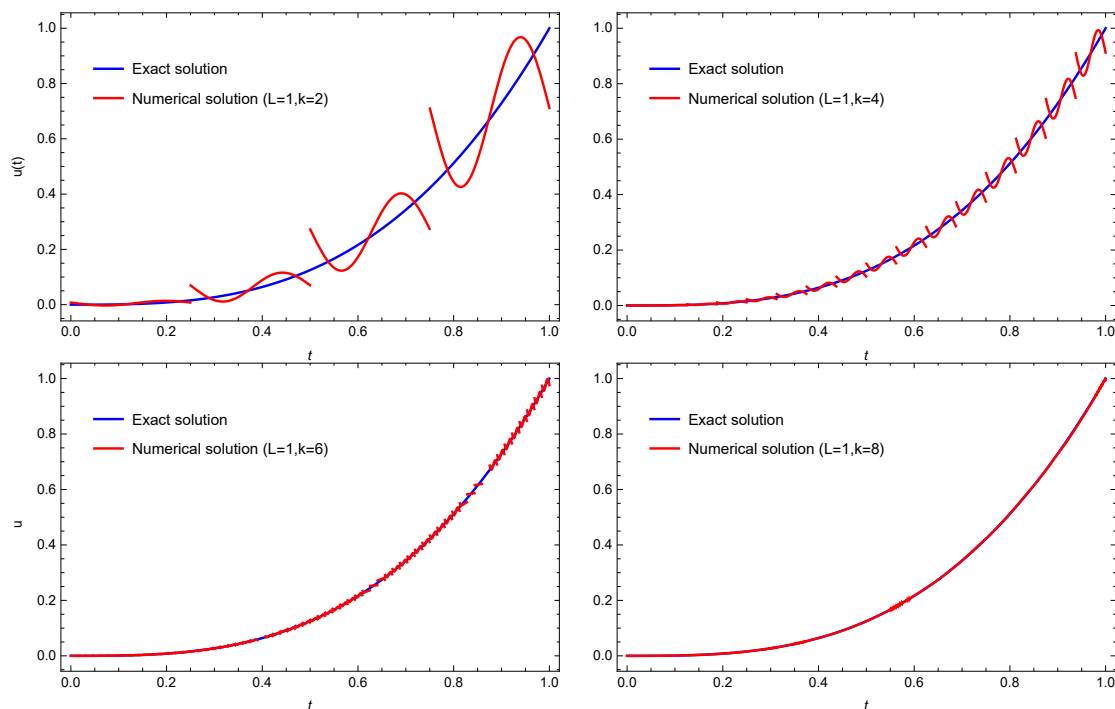


**Figure 7.3:** Plot of the exact solution and numerical solutions with $L = 1$ and $k = 2, 4, 6, 8$ for Example 7.4.

# 8. Concluding remarks

This work is devoted to the numerical solution of the multi-order fractional differential equations using the generalized sine-cosine wavelets. The fractional order operational matrix of integration has been introduced using the properties of the block-pulse functions and generalized sine-cosine wavelets. Using the properties of sine-cosine wavelets and block-pulse functions, the considered problem is reduced to a system of nonlinear algebraic equations which can be solved using iterative methods. The numerical results of four examples show that the proposed method gives high accuracy approximations of the solutions. As it is seen from the tables and figures, the absolute error decreases as the level of resolution, $k$, increases.

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# Sasakian Statistical Manifolds with Semi-Symmetric Metric Connection

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## Article Info

## Abstract

In the present paper, firstly we express the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the torsion-free connection $\nabla$ and obtain the relation between the curvature tensors $\tilde{R}$ of $\tilde{\nabla}$ and $R$ of $\nabla$. After, we obtain these relations for $\tilde{\nabla}$ and the dual connection $\nabla^*$. Also, we give the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors $R$ and $R^*$ of the connections $\nabla$ and $\nabla^*$ on Sasakian statistical manifolds, respectively. We obtain the relations between the Ricci tensor (and scalar curvature) of semi-symmetric metric connection $\tilde{\nabla}$ and the Ricci tensors (and scalar curvatures) of the connections $\nabla$ and $\nabla^*$. Finally, we construct an example of a 3-dimensional Sasakian manifold with statistical structure admitting the semi-symmetric metric connection in order to verify our results.

## 1. Introduction

The theory of statistical manifolds, the so called information geometry, has started with a paper of Rao in 1945 [1] and after that, the information geometry, which is typically deals with the study of various geometric structures on a statistical manifold, has begun as a study of the geometric structures possessed by a statistical model of probability distributions. Nowadays, the information geometry has an important application area, such as, information theory, stochastic processes, dynamical systems and times series, statistical physics, quantum systems and the mathematical theory of neural networks [2], [3].

In 1985, the notion of dual connection (or conjugate connection) in affine geometry, has been first introduced into statistics by Amari [4]. A statistical model equipped with a Riemannian metric together with a pair of dual affine connections is called a *statistical manifold*. For more information about statistical manifolds and information geometry, we refer to [5], [6], [7], [8], [9], [10] and etc.

Also, if $\Phi$ is a tensor field of type $(1,1)$, $\eta$ is a 1-form and $\xi$ is a vector field on a $(2n+1)$-dimensional differentiable manifold $M$, then almost contact structure $(\Phi, \eta, \xi)$ which is related to almost complex structures and satisfies the conditions $\Phi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$ has been determined by Sasaki in 1960 [11]. With the aid of this definition, different types of this manifold such as Sasakian manifold, Kenmotsu manifold, trans-Sasakian manifold and etc. have been defined and studied by many mathematicians [11], [12], [13] and etc.

According to these notions, the differential geometry of statistical manifolds are being studying by geometers by adding different geometric structures to these manifolds. For instance, in [14] quaternionic Kähler-like statistical manifold have been studied and in [15], the authors have introduced the notion of Sasakian statistical structure and obtained the condition for a real hypersurface in a holomorphic statistical manifold to admit such a structure. In [16], the notion of a Kenmotsu statistical manifold is introduced and they have showed that, a Kenmotsu statistical manifold of constant $\Phi$-sectional curvature is constructed from a special Kahler manifold, which is an important example of holomorphic statistical manifold. Also, the projection of a dualistic structure has been defined on a twisted product manifold induces dualistic structures on the base and the fiber manifolds, and conversely in [3].

This paper is organized as follows:

In Section 2, we recall some basic notions about statistical structures and semi-symmetric metric connection. After Preliminaries, by expressing the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the torsion-free connection $\nabla$, we obtain the relation between the curvature tensors $\tilde{R}$ of $\tilde{\nabla}$ and $R$ of $\nabla$ in Section 3 and then, we obtain these relations for $\tilde{\nabla}$ and the dual connection $\nabla^*$. In Section 4,

we give the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors $R$ and $R^*$ of the connections $\nabla$ and $\nabla^*$ on Sasakian statistical manifolds, respectively. Also, we obtain the relations between the Ricci tensor (and scalar curvature) of semi-symmetric metric connection $\tilde{\nabla}$ and the Ricci tensors (and scalar curvatures) of the connections $\nabla$ and $\nabla^*$. At the end of this section, we construct an example of a 3-dimensional Sasakian manifold with statistical structure admitting the semi-symmetric metric connection in order to verify our results.

## 2. Preliminaries

In this section, we recall some notions about statistical structures and semi-symmetric metric connection, respectively. Throughout this paper, we assume that $M$ is a $(2n+1)$-dimensional manifold, $g$ is a Riemannian metric, $\hat{\nabla}$ is the Levi-Civita connection associated with $g$ and $\Gamma(TM^{(p,q)})$ means the set of tensor fields of type $(p,q)$ on $M$.

A pair $(\nabla, g)$ is called a *statistical structure* on $M$, if $\nabla$ is torsion-free and

$$(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z), \ \forall X,Y,Z \in \Gamma(TM) \tag{2.1}$$

holds, where the equation (2.1) is generally called *Codazzi equation*. In this case, $(M, \nabla, g)$ is called a *statistical manifold*.

Let $(\nabla, g)$ be a statistical structure on $M$. Then the connection $\nabla^*$ which is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z)$$

is called *conjugate or dual connection* of $\nabla$ with respect to $g$. If $(\nabla, g)$ is a statistical structure on $M$, then $(\nabla^*, g)$ is a statistical structure on $M$, too.

For a statistical structure $(\nabla, g)$, one can define the *difference tensor field* $K \in \Gamma(TM^{(1,2)})$ as

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y, \ \forall X,Y \in \Gamma(TM), \tag{2.2}$$

where $K$ satisfies

$$K(X,Y) = K(Y,X),$$
$$g(K(X,Y),Z) = g(Y,K(X,Z)).$$

Furthermore, we have

$$K = \hat{\nabla} - \nabla^* = \frac{1}{2}(\nabla - \nabla^*). \tag{2.3}$$

For a more detailed treatment, we refer to [7], [15] and [17].

On the other hand in [18], Hayden introduced a metric connection with a non-zero torsion on a Riemannian manifold and this connection is called a Hayden connection. In [19], the authors have introduced the semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = w(Y)X - w(X)Y,$$

where the 1-form $w$ is defined by

$$w(X) = g(X,U),$$

for vector fields $X$, $Y$ and $U$ on $M$. Also, a semi-symmetric connection $\tilde{\nabla}$ is called a semi-symmetric metric connection if it further satisfies $\tilde{\nabla}g = 0$. If $\hat{\nabla}$ is the Levi-Civita connection of a Riemannian manifold $M$, then the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and $\hat{\nabla}$ is

$$\tilde{\nabla}_X Y = \hat{\nabla}_X Y + w(Y)X - g(X,Y)U, \tag{2.4}$$

where $w(Y) = g(Y,U)$.

## 3. Curvature of semi-symetric metric connection on statistical manifolds

In this section, firstly we'll express the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the torsion-free connection $\nabla$ and obtain the relation between the curvature tensors $\tilde{R}$ of $\tilde{\nabla}$ and $R$ of $\nabla$. After, we'll obtain these relations for $\tilde{\nabla}$ and the dual connection $\nabla^*$.

Let $M$ be an $n$-dimensional Riemannian manifold and $(\nabla, g)$ be a statistical structure on $M$.

From (2.2) and (2.4), we obtain the relation between the connections $\tilde{\nabla}$ and $\nabla$ as

$$\tilde{\nabla}_X Y = \nabla_X Y + w(Y)X - g(X,Y)U - K(X,Y). \tag{3.1}$$

The Riemannian curvature tensor $\tilde{R}$ of $M$ with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z, \tag{3.2}$$

for all $X,Y,Z \in \Gamma(TM)$. From (3.1), we have

$$\begin{aligned}
\tilde{\nabla}_X \tilde{\nabla}_Y Z = & \ \nabla_X \nabla_Y Z + w(\nabla_Y Z)X + w(\nabla_X Z)Y - w(K(X,Z))Y - w(K(Y,Z))X + w(Z)\nabla_X Y - w(Z)K(X,Y) + w(Y)w(Z)X \\
& - g(X,Y)w(Z)U - g(Y,Z)w(U)X + g(Y,Z)w(X)U \\
& - g(X,\nabla_Y Z)U + g(Z,\nabla_X U)Y - g(\nabla_X Y,Z)U - g(Y,\nabla_X Z)U - g(Y,Z)\nabla_X U \\
& - g(Z,K(X,U))Y + g(K(X,Y),Z)U + g(Y,K(X,Z))U + g(X,K(Y,Z))U \\
& + g(Y,Z)K(X,U) - K(X,\nabla_Y Z) - \nabla_X K(Y,Z) + K(X,K(Y,Z))
\end{aligned} \tag{3.3}$$

and

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + w(Z)\nabla_X Y - w(Z)\nabla_Y X - g(\nabla_X Y, Z)U + g(\nabla_Y X, Z)U - K(\nabla_X Y, Z) + K(\nabla_Y X, Z). \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), we obtain the Riemannian curvature tensor $\tilde{R}$ of $M$ with respect to the semi-symmetric metric connection $\tilde{\nabla}$ as

$$\begin{aligned}
\tilde{R}(X,Y)Z = {}& R(X,Y)Z + \{w(X)U - w(U)X - \nabla_X U + K(X,U)\}g(Y,Z) - \{w(Y)U - w(U)Y - \nabla_Y U + K(Y,U)\}g(X,Z) \\
& - g(w(X)U - \nabla_X U + K(X,U), Z)Y + g(w(Y)U - \nabla_Y U + K(Y,U), Z)X \\
& - (\nabla_X K)(Y,Z) + (\nabla_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)).
\end{aligned}$$

Here, $R$ is the Riemannian curvature tensor of $M$ with respect to the torsion-free connection $\nabla$ which is defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$.

Similarly, from (2.3) and (2.4), we obtain the relation between the connections $\tilde{\nabla}$ and $\nabla^*$ as

$$\tilde{\nabla}_X Y = \nabla^*_X Y + w(Y)X - g(X,Y)U + K(X,Y). \tag{3.5}$$

for all $X, Y, Z \in \Gamma(TM)$. From (3.5), we have

$$\begin{aligned}
\tilde{\nabla}_X \tilde{\nabla}_Y Z = {}& \nabla^*_X \nabla^*_Y Z + w(\nabla^*_Y X)X + w(\nabla^*_Y Z)Y + w(K(X,Z))Y + w(K(Y,Z))X + w(Z)\nabla^*_X Y + w(Z)K(X,Y) + w(Y)w(Z)X \\
& - g(X,Y)w(Z)U - g(Y,Z)w(U)X + g(Y,Z)w(X)U \\
& - g(X, \nabla^*_Y Z)U + g(Z, \nabla^*_X U)Y - g(\nabla^*_X Y, Z)U - g(Y, \nabla^*_X Z)U - g(Y,Z)\nabla^*_X U \\
& + g(Z, K(X,U))Y - g(K(X,Y), Z)U - g(Y, K(X,Z))U - g(X, K(Y,Z))U \\
& - g(Y,Z)K(X,U) + K(X, \nabla^*_Y Z) + \nabla^*_X K(Y,Z) + K(X, K(Y,Z))
\end{aligned} \tag{3.6}$$

and

$$\tilde{\nabla}_{[X,Y]}Z = \nabla^*_{[X,Y]}Z + w(Z)\nabla^*_X Y - w(Z)\nabla^*_Y X - g(\nabla^*_X Y, Z)U + g(\nabla^*_Y X, Z)U + K(\nabla^*_X Y, Z) - K(\nabla^*_Y X, Z). \tag{3.7}$$

Using (3.6) and (3.7) in (3.2), we obtain the Riemannian curvature tensor $\tilde{R}$ of $M$ with respect to the semi-symmetric metric connection $\tilde{\nabla}$ as

$$\begin{aligned}
\tilde{R}(X,Y)Z = {}& R^*(X,Y)Z + \{w(X)U - w(U)X - \nabla^*_X U - K(X,U)\}g(Y,Z) - \{w(Y)U - w(U)Y - \nabla^*_Y U - K(Y,U)\}g(X,Z) \\
& - g(w(X)U - \nabla^*_X U - K(X,U), Z)Y + g(w(Y)U - \nabla^*_Y U - K(Y,U), Z)X \\
& + (\nabla^*_X K)(Y,Z) - (\nabla^*_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)).
\end{aligned}$$

Here, $R^*$ is the Riemannian curvature tensor of $M$ with respect to the dual connection $\nabla^*$ which is defined by $R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]}Z$.

Hence, we can give the following Proposition:

**Proposition 3.1.** *Let $(\nabla, g)$ be a statistical structure on a Riemannian manifold $M$. Then, the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors $R$ and $R^*$ of the connections $\nabla$ and $\nabla^*$, respectively, are*

$$\begin{aligned}
\tilde{R}(X,Y)Z = {}& R(X,Y)Z + \{w(X)U - w(U)X - \nabla_X U + K(X,U)\}g(Y,Z) - \{w(Y)U - w(U)Y - \nabla_Y U + K(Y,U)\}g(X,Z) \\
& - g(w(X)U - \nabla_X U + K(X,U), Z)Y + g(w(Y)U - \nabla_Y U + K(Y,U), Z)X \\
& - (\nabla_X K)(Y,Z) + (\nabla_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)) \\
= {}& R^*(X,Y)Z + \{w(X)U - w(U)X - \nabla^*_X U - K(X,U)\}g(Y,Z) - \{w(Y)U - w(U)Y - \nabla^*_Y U - K(Y,U)\}g(X,Z) \\
& - g(w(X)U - \nabla^*_X U - K(X,U), Z)Y + g(w(Y)U - \nabla^*_Y U - K(Y,U), Z)X \\
& + (\nabla^*_X K)(Y,Z) - (\nabla^*_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)),
\end{aligned} \tag{3.8, 3.9}$$

*for all vector fields $X, Y$ and $Z$ on $M$.*

## 4. Semi-symmetric metric connection on Sasakian statistical manifolds

A $(2n+1)$-dimensional differentiable manifold $M$ is said to admit an *almost contact Riemannian structure* $(\Phi, \eta, \xi, g)$, where $\Phi$ is a $(1,1)$-tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a Riemannian metric on $M$ such that

$$\begin{aligned}
& \Phi\xi = 0, \ \eta(\xi) = 1, \ g(\xi, X) = \eta(X), \\
& \Phi^2 X = -X + \eta(X)\xi, \\
& g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y),
\end{aligned} \tag{4.1}$$

for any vector fields $X, Y$ on $M$. In addition, if $(\Phi, \eta, \xi, g)$ satisfy the equations

$$\begin{aligned}
& d\eta = 0, \ \hat{\nabla}_X \xi = \Phi X, \\
& (\hat{\nabla}_X \Phi)Y = \eta(Y)X - g(X,Y)\xi,
\end{aligned} \tag{4.2}$$

then $M$ is called a *Sasakian manifold* (for detail, see [15] and [20]).

Also in [15], the authors have defined the notion of Sasakian statistical structure and have obtained the necessary and sufficient conditions for a statistical structure on an almost contact metric manifold to be a Sasakian statistical structure as follows:

**Definition 4.1.** *A quadruple* $(\nabla, g, \Phi, \xi)$ *is called a Sasakian statistical structure on M, if* $(\nabla, g)$ *is a statistical structure,* $(g, \Phi, \xi)$ *is a Sasakian structure on M and the formula* $K(X, \Phi Y) + \Phi K(X, Y) = 0$ *holds for any vector fields X and Y on M.*

**Theorem 4.2.** *Let* $(\nabla, g)$ *be a statistical structure and* $(g, \Phi, \xi)$ *an almost contact metric structure on M.* $(\nabla, g, \Phi, \xi)$ *is a Sasakian statistical structure if and only if the following formulas hold:*

$$\nabla_X \Phi Y - \Phi \nabla_X^* Y = g(Y, \xi)X - g(Y, X)\xi \tag{4.3}$$

$$\nabla_X \xi = \Phi X + g(\nabla_X \xi, \xi)\xi. \tag{4.4}$$

So, we can give the following Example:

**Example 4.3.** *Let* $(\Phi, \eta, \xi, g)$ *be an almost contact Riemannian structure on M. Set the connection* $\check{\nabla}$ *as*

$$\check{\nabla}_X Y = \hat{\nabla}_X Y + 3\eta(X)\eta(Y)\xi, \tag{4.5}$$

*for any* $X, Y \in \Gamma(TM)$. *Then,* $\check{\nabla}$ *is torsion-free and satisfies the Codazzi equation (2.1). So,* $(\check{\nabla}, g)$ *is a statistical structure on the almost contact Riemannian manifold* $(M, \Phi, \eta, \xi, g)$.
*Also, from (2.2), (2.3) and (4.5) we have* $K(X, Y) = 3\eta(X)\eta(Y)\xi$ *and* $\check{\nabla}_X^* Y = \hat{\nabla}_X Y - 3\eta(X)\eta(Y)\xi$. *So, the equations (4.3) and (4.4) hold for the connection* $\check{\nabla}$. *Hence* $(\check{\nabla}, g, \Phi, \eta, \xi)$ *is a Sasakian statistical structure on M.*

Now, firstly we'll give the relations between the curvature tensor $\tilde{R}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensors $R$ and $R^*$ of the connections $\nabla$ and $\nabla^*$ on Sasakian statistical manifolds with the aid of Proposition 3.1. For this, we use the equation

$$\tilde{\nabla}_X Y = \hat{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi, \tag{4.6}$$

which has been obtained by Yano [21] on almost contact manifolds. Here, $\tilde{\nabla}$ is the semi-symmetric metric connection and $\hat{\nabla}$ is the Levi-Civita connection on $(M, g)$, $\eta$ is a 1-form and $\xi$ is a vector field defined by $w(X) = g(X, \xi)$. If we write $\eta$ instead of $w$ and $\xi$ instead of $U$ in the equations (3.8) and (3.9) and use (2.2), (2.3), (4.1) and (4.2), then we have the following Theorem:

**Theorem 4.4.** *Let* $(M, \nabla, g, \Phi, \eta, \xi)$ *be a* $(2n+1)$*-dimensional Sasakian statistical manifold. Then, the relations between the curvature tensor* $\tilde{R}$ *of semi-symmetric metric connection* $\tilde{\nabla}$ *and the curvature tensors R and* $R^*$ *of the connections* $\nabla$ *and* $\nabla^*$, *respectively, are*

$$\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + \{\Phi^2 X - \Phi X\}g(Y,Z) - \{\Phi^2 Y - \Phi Y\}g(X,Z) \\
&\quad + g(\Phi X, Z)Y - g(\Phi Y, Z)X - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X \\
&\quad - (\nabla_X K)(Y,Z) + (\nabla_Y K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)) \\
&= R^*(X,Y)Z + \{\Phi^2 X - \Phi X\}g(Y,Z) - \{\Phi^2 Y - \Phi Y\}g(X,Z) \\
&\quad + g(\Phi X, Z)Y - g(\Phi Y, Z)X - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X \\
&\quad + (\nabla_X^* K)(Y,Z) - (\nabla_Y^* K)(X,Z) + K(X, K(Y,Z)) - K(Y, K(X,Z)),
\end{aligned} \tag{4.7}$$

$$\tag{4.8}$$

*for all vector fields X, Y and Z on M.*

**Corollary 4.5.** *Let* $(M, \nabla, g, \Phi, \eta, \xi)$ *be a* $(2n+1)$*-dimensional Sasakian statistical manifold. Then, we have*

$$\begin{aligned}
\tilde{R}(X,Y)\xi &= R(X,Y)\xi + \eta(X)\Phi Y - \eta(Y)\Phi X - (\nabla_X K)(Y,\xi) + (\nabla_Y K)(X,\xi) \tag{4.9} \\
&= R^*(X,Y)\xi + \eta(X)\Phi Y - \eta(Y)\Phi X + (\nabla_X^* K)(Y,\xi) - (\nabla_Y^* K)(X,\xi) \tag{4.10}
\end{aligned}$$

*and*

$$\begin{aligned}
\tilde{R}(\xi,X)Y &= R(\xi,X)Y + \eta(Y)\Phi X - g(\Phi X, Y)\xi - (\nabla_\xi K)(X,Y) + (\nabla_X K)(\xi,Y) + K(\xi, K(X,Y)) - K(X, K(\xi,Y)) \tag{4.11} \\
&= R^*(\xi,X)Y + \eta(Y)\Phi X - g(\Phi X, Y)\xi + (\nabla_\xi^* K)(X,Y) - (\nabla_X^* K)(\xi,Y) + K(\xi, K(X,Y)) - K(X, K(\xi,Y)), \tag{4.12}
\end{aligned}$$

*for all vector fields X and Y on M.*

*Proof.* We know that [15], on a Sasakian statistical manifold, the equation $\nabla_X \xi = \Phi X + \eta(\nabla_X \xi)\xi$ holds. So, from (2.2) we get $K(X, \xi) = \eta(\nabla_X \xi)\xi$. Using this, we have $K(X, K(Y, \xi)) = \eta(\nabla_X \xi)\eta(\nabla_Y \xi)\xi$ and so, we obtain that

$$K(X, K(Y, \xi)) = K(Y, K(X, \xi)). \tag{4.13}$$

Using (4.1) and (4.13) in (4.7) and (4.8), we reach the equations (4.9)-(4.12) and the proof completes. □

Now, let us give the relations between the Ricci tensor $\tilde{S}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the Ricci tensors $S$ and $S^*$ of the connections $\nabla$ and $\nabla^*$, respectively.

**Theorem 4.6.** *Let* $(M, \nabla, g, \Phi, \eta, \xi)$ *be a* $(2n+1)$*-dimensional Sasakian statistical manifold. Then, the relations between the Ricci tensors of semi-symmetric metric connection* $\tilde{\nabla}$ *and the connections* $\nabla$ *and* $\nabla^*$, *respectively, are*

$$\begin{aligned}
\tilde{S}(X,Y) &= S(X,Y) - (2n-1)g(\Phi X, \Phi Y + Y) \tag{4.14} \\
&\quad - \sum_{i=1}^{2n+1} g((\nabla_X K)(e_i, e_i) - (\nabla_{e_i} K)(X, e_i) - K(X, K(e_i, e_i)) + K(e_i, K(X, e_i)), Y) \\
&= S^*(X,Y) - (2n-1)g(\Phi X, \Phi Y + Y) \tag{4.15} \\
&\quad + \sum_{i=1}^{2n+1} g((\nabla_X^* K)(e_i, e_i) - (\nabla_{e_i}^* K)(X, e_i) + K(X, K(e_i, e_i)) - K(e_i, K(X, e_i)), Y),
\end{aligned}$$

*for all vector fields X and Y on M.*

*Proof.* Using (4.1) and (4.7) in the equation $\tilde{S}(X,Y) = \sum_{i=1}^{2n+1} g(\tilde{R}(X,e_i)e_i,Y)$, we get (4.14). Similarly, using (4.1) and (4.8) in the equation

$\tilde{S}(X,Y) = \sum_{i=1}^{2n+1} g(\tilde{R}(X,e_i)e_i,Y)$, we get (4.15).                                                                                                    $\square$

Here, let us give the relations between the scalar curvature $\tilde{\tau}$ of semi-symmetric metric connection $\tilde{\nabla}$ and the scalar curvatures $\tau$ and $\tau^*$ of the connections $\nabla$ and $\nabla^*$.

**Theorem 4.7.** *Let $(M,\nabla,g,\Phi,\eta,\xi)$ be a $(2n+1)$-dimensional Sasakian statistical manifold. Then, the relations between the scalar curvature of semi-symmetric metric connection $\tilde{\nabla}$ and the connections $\nabla$ and $\nabla^*$, respectively, are*

$$\tilde{\tau} = \tau + 2n - 4n^2 - \sum_{i,j=1}^{2n+1} g((\nabla_{e_j}K)(e_i,e_i) - (\nabla_{e_i}K)(e_j,e_i) - K(e_j,K(e_i,e_i)) + K(e_i,K(e_j,e_i)),e_j) \tag{4.16}$$

$$= \tau^* + 2n - 4n^2 + \sum_{i,j=1}^{2n+1} g((\nabla^*_{e_j}K)(e_i,e_i) - (\nabla^*_{e_i}K)(e_j,e_i) + K(e_j,K(e_i,e_i)) - K(e_i,K(e_j,e_i)),e_j). \tag{4.17}$$

*Proof.* Using (4.14) and (4.15) in the equation $\tilde{\tau} = \sum_{j=1}^{2n+1} \tilde{S}(e_j,e_j)$, we get (4.16) and (4.17), respectively.                           $\square$

Finally, let us construct an example of a 3-dimensional Sasakian manifold with statistical structure admitting the semi-symmetric metric connection in order to verify our results.

**Example 4.8.** *Let us consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where $(x,y,z)$ are the standart coordinates in $\mathbb{R}^3$. We choose the vector fields $\{e_1,e_2,e_3\}$ as*

$$e_1 = \frac{\partial}{\partial x}, \; e_2 = -x\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right) + \frac{\partial}{\partial y}, \; e_3 = \frac{1}{2}\frac{\partial}{\partial z},$$

*which are linearly independent at each point of M.*
*Let g be the Riemannian metric defined by $g(e_i,e_j) = 0$, $i \neq j$, $i,j = 1,2,3$ and $g(e_k,e_k) = 1$, $k = 1,2,3$.*
*Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z,e_3)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M.*
*Let $\phi$ be the $(1,1)$-tensor field defined by*

$$\phi e_1 = -e_2, \; \phi e_2 = e_1, \; \phi e_3 = 0. \tag{4.18}$$

*Using the linearity of $\phi$ and g, we have $\eta(e_3) = 1$, $\phi^2 Z = -Z + \eta(Z)e_3$ and $g(\phi Z,\phi U) = g(Z,U) - \eta(Z)\eta(U)$, for any $U,Z \in \chi(M)$. Thus, for $e_3 = \xi$, $(\phi,\xi,\eta,g)$ defines an almost contact metric structure on M.*
*Now, we have*

$$[e_1,e_2] = -e_1 + 2e_3, \; [e_1,e_3] = 0, \; [e_2,e_3] = 0.$$

*The Levi-Civita connection $\hat{\nabla}$ of the metric tensor g is given by Koszul's formula which is defined as*

$$2g(\hat{\nabla}_X Y,Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).$$

*Taking $e_3 = \xi$ and using Koszul's formula, we get the following*

$$\begin{aligned}
\hat{\nabla}_{e_1}e_1 &= e_2, \; \hat{\nabla}_{e_1}e_2 = -e_1 + e_3, \; \hat{\nabla}_{e_1}e_3 = -e_2, \\
\hat{\nabla}_{e_2}e_1 &= -e_3, \; \hat{\nabla}_{e_2}e_2 = 0, \; \hat{\nabla}_{e_2}e_3 = e_1, \\
\hat{\nabla}_{e_3}e_1 &= -e_2, \; \hat{\nabla}_{e_3}e_2 = e_1, \; \hat{\nabla}_{e_3}e_3 = 0.
\end{aligned} \tag{4.19}$$

*From the above, it can be easily seen that $(\phi,\xi,\eta,g)$ is a Sasakian structure on M. Consequently, $(M,\phi,\xi,\eta,g)$ is a 3-dimensional Sasakian manifold.*
*Now, the components of the curvature tensors, Ricci tensors and scalar curvature with respect to the Levi-Civita connection $\hat{\nabla}$ are obtained by*

$$\begin{aligned}
\hat{R}(e_1,e_2)e_1 &= 4e_2, \; \hat{R}(e_1,e_2)e_2 = -4e_1, \; \hat{R}(e_1,e_2)e_3 = 0, \\
\hat{R}(e_1,e_3)e_1 &= -e_3, \; \hat{R}(e_1,e_3)e_2 = 0, \; \hat{R}(e_1,e_3)e_3 = e_1, \\
\hat{R}(e_2,e_3)e_1 &= 0, \; \hat{R}(e_2,e_3)e_2 = -e_3, \; \hat{R}(e_2,e_3)e_3 = e_2,
\end{aligned}$$

$$\begin{aligned}
\hat{S}(e_1,e_1) &= -3, \; \hat{S}(e_1,e_2) = 0, \; \hat{S}(e_1,e_3) = 0, \\
\hat{S}(e_2,e_1) &= 0, \; \hat{S}(e_2,e_2) = -3, \; \hat{S}(e_2,e_3) = 0, \\
\hat{S}(e_3,e_1) &= 0, \; \hat{S}(e_3,e_2) = 0, \; \hat{S}(e_3,e_3) = 2
\end{aligned}$$

*and*

$$\hat{\tau} = -4,$$

*respectively.*

*Here, let us add a statistical structure to this Sasakian manifold. From (2.2) and (4.19), we have*

$$\nabla_{e_1}e_1 = e_2 + K(e_1,e_1), \ \nabla_{e_1}e_2 = -e_1 + e_3 + K(e_1,e_2), \ \nabla_{e_1}e_3 = -e_2 + K(e_1,e_3),$$
$$\nabla_{e_2}e_1 = -e_3 + K(e_2,e_1), \ \nabla_{e_2}e_2 = K(e_2,e_2), \ \nabla_{e_2}e_3 = e_1 + K(e_2,e_3), \tag{4.20}$$
$$\nabla_{e_3}e_1 = -e_2 + K(e_3,e_1), \ \nabla_{e_3}e_2 = e_1 + K(e_3,e_2), \ \nabla_{e_3}e_3 = K(e_3,e_3).$$

*So, the components of the curvature tensors, Ricci tensors and scalar curvature with respect to the torsion-free connection $\nabla$ are obtained by*

$$R(e_1,e_2)e_1 = 4e_2 + K(e_1,e_1) - 3K(e_1,e_3) - K(e_2,e_2) + \nabla_{e_1}K(e_2,e_1) - \nabla_{e_2}K(e_1,e_1),$$
$$R(e_1,e_2)e_2 = -4e_1 + 2K(e_1,e_2) - 3K(e_2,e_3) + \nabla_{e_1}K(e_2,e_2) - \nabla_{e_2}K(e_1,e_2),$$
$$R(e_1,e_2)e_3 = K(e_1,e_1) + K(e_1,e_3) + K(e_2,e_2) - 2K(e_3,e_3) + \nabla_{e_1}K(e_2,e_3) - \nabla_{e_2}K(e_1,e_3),$$
$$R(e_1,e_3)e_1 = -e_3 - K(e_1,e_2) - K(e_2,e_3) + \nabla_{e_1}K(e_3,e_1) - \nabla_{e_3}K(e_1,e_1),$$
$$R(e_1,e_3)e_2 = K(e_1,e_1) + K(e_1,e_3) - K(e_3,e_3) + \nabla_{e_1}K(e_3,e_2) - \nabla_{e_3}K(e_1,e_2), \tag{4.21}$$
$$R(e_1,e_3)e_3 = e_1 + K(e_2,e_3) + \nabla_{e_1}K(e_3,e_3) - \nabla_{e_3}K(e_1,e_3),$$
$$R(e_2,e_3)e_1 = -K(e_2,e_2) + K(e_3,e_3) + \nabla_{e_2}K(e_3,e_1) - \nabla_{e_3}K(e_2,e_1),$$
$$R(e_2,e_3)e_2 = -e_3 + K(e_2,e_1) + \nabla_{e_2}K(e_3,e_2) - \nabla_{e_3}K(e_2,e_2),$$
$$R(e_2,e_3)e_3 = e_2 - K(e_3,e_1) + \nabla_{e_2}K(e_3,e_3) - \nabla_{e_3}K(e_2,e_3),$$

$$S(e_1,e_1) = -3 + g(2K(e_1,e_2) - 2K(e_2,e_3) + \nabla_{e_1}K(e_2,e_2) - \nabla_{e_2}K(e_1,e_2) + \nabla_{e_1}K(e_3,e_3) - \nabla_{e_3}K(e_1,e_3),e_1),$$
$$S(e_1,e_2) = g(2K(e_1,e_2) - 2K(e_2,e_3) + \nabla_{e_1}K(e_2,e_2) - \nabla_{e_2}K(e_1,e_2) + \nabla_{e_1}K(e_3,e_3) - \nabla_{e_3}K(e_1,e_3),e_2),$$
$$S(e_1,e_3) = g(2K(e_1,e_2) - 2K(e_2,e_3) + \nabla_{e_1}K(e_2,e_2) - \nabla_{e_2}K(e_1,e_2) + \nabla_{e_1}K(e_3,e_3) - \nabla_{e_3}K(e_1,e_3),e_3),$$
$$S(e_2,e_1) = g(-K(e_1,e_1) + 2K(e_1,e_3) + K(e_2,e_2) - \nabla_{e_1}K(e_2,e_1) + \nabla_{e_2}K(e_1,e_1) + \nabla_{e_2}K(e_3,e_3) - \nabla_{e_3}K(e_2,e_3),e_1),$$
$$S(e_2,e_2) = -3 + g(-K(e_1,e_1) + 2K(e_1,e_3) + K(e_2,e_2) - \nabla_{e_1}K(e_2,e_1) + \nabla_{e_2}K(e_1,e_1) + \nabla_{e_2}K(e_3,e_3) - \nabla_{e_3}K(e_2,e_3),e_2), \tag{4.22}$$
$$S(e_2,e_3) = g(-K(e_1,e_1) + 2K(e_1,e_3) + K(e_2,e_2) - \nabla_{e_1}K(e_2,e_1) + \nabla_{e_2}K(e_1,e_1) + \nabla_{e_2}K(e_3,e_3) - \nabla_{e_3}K(e_2,e_3),e_3),$$
$$S(e_3,e_1) = g(K(e_3,e_2) - \nabla_{e_1}K(e_3,e_1) - \nabla_{e_2}K(e_3,e_2) + \nabla_{e_3}K(e_1,e_1) + \nabla_{e_3}K(e_2,e_2),e_1),$$
$$S(e_3,e_2) = g(K(e_3,e_2) - \nabla_{e_1}K(e_3,e_1) - \nabla_{e_2}K(e_3,e_2) + \nabla_{e_3}K(e_1,e_1) + \nabla_{e_3}K(e_2,e_2),e_2),$$
$$S(e_3,e_3) = 2 + g(K(e_3,e_2) - \nabla_{e_1}K(e_3,e_1) - \nabla_{e_2}K(e_3,e_2) + \nabla_{e_3}K(e_1,e_1) + \nabla_{e_3}K(e_2,e_2),e_3)$$

*and*

$$\tau = -4 + g(K(K(e_1,e_2),e_2) - K(e_1,K(e_2,e_2)) + K(e_3,K(e_1,e_3)) - K(e_1,K(e_3,e_3))$$
$$+ (\nabla_{e_1}K)(e_2,e_2) - (\nabla_{e_2}K)(e_1,e_2) + (\nabla_{e_1}K)(e_3,e_3) - (\nabla_{e_3}K)(e_1,e_3),e_1)$$
$$+ g(K(e_1,K(e_2,e_1)) - K(e_2,K(e_1,e_1)) - K(e_2,K(e_3,e_3)) + K(e_3,K(e_2,e_3))$$
$$- (\nabla_{e_1}K)(e_2,e_1) + (\nabla_{e_2}K)(e_1,e_1) + (\nabla_{e_2}K)(e_3,e_3) - (\nabla_{e_3}K)(e_2,e_3),e_2)$$
$$+ g(K(e_1,K(e_3,e_1)) + K(e_2,K(e_3,e_2)) - K(e_3,K(e_1,e_1)) - K(e_3,K(e_2,e_2))$$
$$- (\nabla_{e_1}K)(e_3,e_1) - (\nabla_{e_2}K)(e_3,e_2) + (\nabla_{e_3}K)(e_1,e_1) + (\nabla_{e_3}K)(e_2,e_2),e_3), \tag{4.23}$$

*respectively. (Similarly, the above equations can be obtained for the dual connection $\nabla^*$.)*

*Finally, from (4.6) (or from (3.1) for $w = \eta$ and $U = \xi$) and (4.19), we have*

$$\tilde{\nabla}_{e_1}e_1 = e_2 - e_3, \ \tilde{\nabla}_{e_1}e_2 = -e_1 + e_3, \ \tilde{\nabla}_{e_1}e_3 = -e_2 + e_1,$$
$$\tilde{\nabla}_{e_2}e_1 = -e_3, \ \tilde{\nabla}_{e_2}e_2 = -e_3, \ \tilde{\nabla}_{e_2}e_3 = e_1 + e_2,$$
$$\tilde{\nabla}_{e_3}e_1 = -e_2, \ \tilde{\nabla}_{e_3}e_2 = e_1, \ \tilde{\nabla}_{e_3}e_3 = 0.$$

*and the curvature tensors, Ricci tensors and scalar curvature with respect to the semi-symmetric metric connection $\tilde{\nabla}$ are obtained as follows, respectively:*

$$\tilde{R}(e_1,e_2)e_1 = 5e_2, \ \tilde{R}(e_1,e_2)e_2 = -5e_1, \ \tilde{R}(e_1,e_2)e_3 = 0,$$
$$\tilde{R}(e_1,e_3)e_1 = -e_3, \ \tilde{R}(e_1,e_3)e_2 = -e_3, \ \tilde{R}(e_1,e_3)e_3 = e_1 + e_2, \tag{4.24}$$
$$\tilde{R}(e_2,e_3)e_1 = e_3, \ \tilde{R}(e_2,e_3)e_2 = -e_3, \ \tilde{R}(e_2,e_3)e_3 = e_2 - e_1,$$

$$\tilde{S}(e_1,e_1) = -4, \ \tilde{S}(e_1,e_2) = 1, \ \tilde{S}(e_1,e_3) = 0,$$
$$\tilde{S}(e_2,e_1) = -1, \ \tilde{S}(e_2,e_2) = -4, \ \tilde{S}(e_2,e_3) = 0, \tag{4.25}$$
$$\tilde{S}(e_3,e_1) = 0, \ \tilde{S}(e_3,e_2) = 0, \ \tilde{S}(e_3,e_3) = 2$$

*and*

$$\tilde{\tau} = -6. \tag{4.26}$$

*Hence, one can easily see that, from (4.1), (4.18), (4.21) and (4.24), the equation (4.7) in Theorem 4.4 is verified; from (4.1), (4.18), (4.22) and (4.25), the equation (4.14) in Theorem 4.6 is verified and from (4.23) and (4.26), the equation (4.16) in Theorem 4.7 is verified for $n = 1$. Similarly, obtaining the above equations for dual connection $\nabla^*$, one can easily see that, the equations (4.8), (4.15) and (4.17) are verified, too.*

Now, by choosing the difference tensor field $K$ as in Example 4.3, we'll obtain the equations (4.20)-(4.23) in the following Example.

**Example 4.9.** *If we choose the difference tensor field $K$ as $K(X,Y) = 3\eta(X)\eta(Y)\xi$, then the equations (4.20)-(4.23) are obtained as*

$$\nabla_{e_1}e_1 = e_2, \ \nabla_{e_1}e_2 = -e_1 + e_3, \ \nabla_{e_1}e_3 = -e_2,$$
$$\nabla_{e_2}e_1 = -e_3, \ \nabla_{e_2}e_2 = 0, \ \nabla_{e_2}e_3 = e_1,$$
$$\nabla_{e_3}e_1 = -e_2, \ \nabla_{e_3}e_2 = e_1, \ \nabla_{e_3}e_3 = 3e_3,$$

$$R(e_1,e_2)e_1 = 4e_2, \ R(e_1,e_2)e_2 = -4e_1, \ R(e_1,e_2)e_3 = -6e_3,$$
$$R(e_1,e_3)e_1 = -e_3, \ R(e_1,e_3)e_2 = -3e_3, \ R(e_1,e_3)e_3 = e_1 - 3e_2, \quad\quad (4.27)$$
$$R(e_2,e_3)e_1 = 3e_3, \ R(e_2,e_3)e_2 = -e_3, \ R(e_2,e_3)e_3 = e_2 + 3e_1,$$

$$S(e_1,e_1) = -3, \ S(e_1,e_2) = -3, \ S(e_1,e_3) = 0,$$
$$S(e_2,e_1) = 3, \ S(e_2,e_2) = -3, \ S(e_2,e_3) = 0, \quad\quad (4.28)$$
$$S(e_3,e_1) = 0, \ S(e_3,e_2) = 0, \ S(e_3,e_3) = 2$$

*and*

$$\tau = -4. \quad\quad (4.29)$$

*Here, one can easily see that, from (4.1), (4.18), (4.24)-(4.26) and (4.27)-(4.29), the Theorems 4.4, 4.6 and 4.7 are verified.*

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# Different Computational Approach for Sumudu Integral Transform by Using Differential Transform Method

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**Abstract**

In this work, we present a different technique for calculation of Sumudu Integral Transform (SIT) by considering differential transform method (DTM). By means of our technique, Sumudu Transform of functions is obtained easily without complicated integration procedures.

## 1. Introduction

In mathematical calculus, integral transforms are a specific branch that has used various applied area. In 1993, Watugala introduced a new integral transform called Sumudu Integral Transform (SIT) to solve differential equations and engineering problems [1]. The Sumudu Transform of the function $f(t)$ is defined over the set of $A$ (seen [1], [2], [3], [4])

$$A = \{f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_i}}, if \quad t \in (-1)^i \times [0, \infty)\}$$

as below formula

$$F(u) = S[f(t) : u] = \int_0^\infty f(ut)e^{-t}dt \quad u \in (-\tau_1, \tau_2) \tag{1.1}$$

Also, modified version of (1.1) is presented as

$$F(u) = \int_0^\infty \frac{f(t)e^{-\frac{t}{u}}}{u}dt \quad u \in (-\tau_1, \tau_2) \tag{1.2}$$

by Watugala [1], [4] and Belgacem [2],[3]. Hereafter, many authors consider the Sumudu Integral Transform to investigate properties, applications and relations with other transforms [1]-[10].

In recent time, homotopy perturbation, differential transform and adomian decomposition methods are applied to find Laplace transform as seen [11],[12],[13] respectively. Furthermore, homotopy perturbation method is also applied to Sumudu transform [8].

The goal of this paper is to present a different approach to obtain Sumudu transform of functions. In order to do this, we use the differential transform method (DTM) and first order initial value problem which has a solution that corresponds to Sumudu transform of desired functions. DTM is very famous and powerful analytic technique and it does not required complex integration process. So, very accurate and efficient results are obtained easily.

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| Functions | Transformed form of functions |
|---|---|
| $v(t) = e^{at}$ | $V(k) = \frac{a^k}{k!}$ |
| $v(t) = \cos(at)$ | $V(k) = \begin{cases} \frac{a^k(-1)^{\frac{k}{2}}}{k!}, & if \quad k \quad even \\ 0, & if \quad k \quad odd \end{cases}$ |
| $v(t) = \sin(at)$ | $V(k) = \begin{cases} \frac{a^k(-1)^{\frac{k-1}{2}}}{k!}, & if \quad k \quad odd \\ 0, & if \quad k \quad even \end{cases}$ |
| $v(t) = \cosh(at)$ | $V(k) = \begin{cases} \frac{a^k}{k!}, & if \quad k \quad even \\ 0, & if \quad k \quad odd \end{cases}$ |
| $v(t) = \sinh(at)$ | $V(k) = \begin{cases} \frac{a^k}{k!}, & if \quad k \quad odd \\ 0, & if \quad k \quad even \end{cases}$ |

**Table 1:** Basic transformations of DTM for some functions.

## 2. Basic idea of DTM

The differential transform of the analytical $v(t)$ function is defined (seen [14], [15], [16], [17]) as

$$V(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} v(t) \right]_{t=0} \tag{2.1}$$

where $V(k)$ is the transformed function of $v(t)$ which is called spectrum function. And the inverse transform of $V(k)$ is defined (seen [14], [15], [16], [17]) as

$$v(t) = \sum_{k=0}^{\infty} V(k)t^k \tag{2.2}$$

Combining (2.1) and (2.2), we obtain the DTM solution of $v(t)$ as follow

$$v(t) = \sum_{k=0}^{n} \frac{1}{k!} \left[ \frac{d^k}{dt^k} v(t) \right]_{t=0} t^k + R_{n+1}(t) \tag{2.3}$$

Here $R_{n+1}(t) = \sum_{k=n+1}^{\infty} V(k)t^k$ are remaining terms of solution series. Some of the transformed functions are presented in Table 1.

## 3. Results by using DTM

**Theorem 3.1.** *Let $v(t)$ is an analytic function and $r$ is positive constant. Also, we consider the linear initial value problem as follow*

$$v(t)' = \frac{1}{r}v(t) + \frac{1}{r}q(t) \tag{3.1}$$
$$v(0) = 0$$

*Then, the Sumudu transform of $q(t)$ is*

$$S[q(t)] = \left[ e^{-\frac{t}{r}} \sum_{i=0}^{\infty} V(i)t^i \right]_{t=0}^{t=\infty}$$

*Here, $V(i)$ is differential transform of $v(t)$.*

*Proof.* First of all, we can write the solution of (3.1) as

$$v(t) = e^{\frac{t}{r}} \left( \int \frac{q(t)e^{-\frac{t}{r}}}{r} dt \right) \tag{3.2}$$

and by rewriting two side of (3.2) from zero to infinity, we obtain the relation between (3.2) and Sumudu transform as follow

$$\left[ v(t)e^{-\frac{t}{r}} \right]_{t=0}^{t=\infty} = \left( \int_0^{\infty} \frac{q(t)e^{-\frac{t}{r}}}{r} dt \right) \tag{3.3}$$

It is clearly seen that right hand side of (3.3) is the definition of Sumudu transform of $q(t)$ as seen in (1.2).
In order to find Sumudu transform of $q(t)$, we construct the differential transformed form of (3.1) as

$$V(i+1) = \frac{1}{r}\frac{V(i)}{i+1} + \frac{1}{r}\frac{Q(i)}{i+1}$$

$$V(0) = 0$$

(3.4)

where $V(i)$, $Q(i)$ are differential transformed functions of $v(t)$ and $q(t)$ respectively. Then, by using the inverse differential transform as in (2.2), (2.3), we obtain the DTM solution of $v(t)$ as

$$v(t) = \sum_{i=0}^{\infty} V(i)t^i$$

(3.5)

Finally, put (3.5) into (3.3), we find the sumudu transform of $q(t)$ as

$$S[q(t)] = \left( \int_0^{\infty} \frac{q(t)e^{-\frac{t}{r}}}{r}dt \right) = \left[ e^{-\frac{t}{r}} \sum_{i=0}^{\infty} V(i)t^i \right]_{t=0}^{t=\infty}$$

(3.6)

This completes the proof.

□

The following illustrations are given to show accuracy, efficiency and easy applicability of our approach to find Sumudu transform of functions.

**Case 1:** In the Theorem 3.1, let $q(t) = e^{at}$. Then considering (3.4) and transformed form of $q(t) = e^{at}$, we can write

$$V(i+1) = \frac{1}{r}\frac{V(i)}{i+1} + \frac{1}{r(i+1)}\frac{a^i}{i!}$$

$$V(0) = 0$$

Some of the $V(i)$ are obtained as

$$V(1) = \frac{1}{r} \quad V(2) = \frac{1+ar}{2!r^2} \quad V(3) = \frac{1+ar+a^2r^2}{3!r^3}$$

$$V(4) = \frac{1+ar+a^2r^2+a^3r^3}{4!r^4} \quad V(5) = \frac{1+ar+a^2r^2+a^3r^3+a^4r^4}{5!r^5}$$

(3.7)

$$V(6) = \frac{1+ar+a^2r^2+a^3r^3+a^4r^4+a^5r^5}{6!r^6}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

From (3.5) and from (3.7), we have

$$\sum_{i=0}^{\infty} V(i)t^i = \left( \frac{t}{r} + \frac{t^2}{2!r^2} + \frac{t^3}{3!r^3} + \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \cdots \right) + ar\left( \frac{t^2}{2!r^2} + \frac{t^3}{3!r^3} + \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \frac{t^6}{6!r^6} + \cdots \right)$$

(3.8)

$$+ a^2r^2\left( \frac{t^3}{3!r^3} + \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \frac{t^6}{6!r^6} + \frac{t^7}{7!r^7} + \cdots \right) + a^3r^3\left( \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \frac{t^6}{6!r^6} + \frac{t^7}{7!r^7} + \frac{t^8}{8!r^8} + \cdots \right) + \cdots$$

And the equation (3.8) can be written as equivalently following

$$\sum_{i=0}^{\infty} V(i)t^i = \left( e^{\frac{t}{r}} - 1 \right) + ar\left( e^{\frac{t}{r}} - 1 - \frac{t}{r} \right) + a^2r^2\left( e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} \right)$$

(3.9)

$$+ a^3r^3\left( e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} \right) + a^4r^4\left( e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4} \right) + \cdots$$

Finally, by using (3.6) and (3.9) we find the Sumudu transform of $e^{at}$

$$S\left[e^{at}\right] = \left\{ e^{-\frac{t}{r}}\left( e^{\frac{t}{r}} - 1 \right) + ar \times e^{-\frac{t}{r}}\left( e^{\frac{t}{r}} - 1 - \frac{t}{r} \right) + a^2r^2 \times e^{-\frac{t}{r}}\left( e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} \right) + \cdots \right\}_{t=0}^{t=\infty}$$

$$= 1 + ar + a^2r^2 + a^3r^3 + a^4r^4 + a^5r^5 + a^6r^6 + \cdots$$

$$= \frac{1}{1-ar}$$

**Case 2:** In the Theorem 3.1, let $q(t) = \cos(at)$. Then, from (3.4) and Table 1 we can write

$$V(i+1) = \frac{1}{r}\frac{V(i)}{i+1} + \frac{1}{r(i+1)}\left\{ \begin{array}{ll} \frac{a^i(-1)^{\frac{i}{2}}}{i!}, & i \quad even \\ 0, & i \quad odd \end{array}\right.$$

$$V(0) = 0$$

Thus, $V(i)$ are obtained as

$$V(1) = \frac{1}{r} \quad V(2) = \frac{1+ar}{2!r^2} \quad V(3) = \frac{1-a^2r^2}{3!r^3} \quad V(4) = \frac{1-a^2r^2}{4!r^4}$$

$$V(5) = \frac{1-a^2r^2+a^4r^4}{5!r^5} \quad V(6) = \frac{1-a^2r^2+a^4r^4}{6!r^6}$$  (3.10)

$$V(7) = \frac{1-a^2r^2+a^4r^4-a^6r^6}{7!r^7} \quad V(8) = \frac{1-a^2r^2+a^4r^4-a^6r^6}{8!r^8}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Using the (3.5) and (3.10), we have

$$\sum_{i=0}^{\infty} V(i)t^i = \left(\frac{t}{r} + \frac{t^2}{2!r^2} + \frac{t^3}{3!r^3} + \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \cdots\right) - a^2r^2\left(\frac{t^3}{3!r^3} + \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \frac{t^6}{6!r^6} + \cdots\right)$$  (3.11)

$$+ a^4r^4\left(\frac{t^5}{5!r^5} + \frac{t^6}{6!r^6} + \frac{t^7}{7!r^7} + \frac{t^8}{8!r^8} + \cdots\right) - a^6r^6\left(\frac{t^7}{7!r^7} + \frac{t^8}{8!r^8} + \frac{t^9}{9!r^9} + \frac{t^{10}}{10!r^{10}} + \cdots\right) \pm \cdots$$

The (3.11) can be rewritten as follow

$$\sum_{i=0}^{\infty} V(i)t^i = \left(e^{\frac{t}{r}} - 1\right) - a^2r^2\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2}\right) + a^4r^4\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4}\right)$$

$$- a^6r^6\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4} - \frac{t^5}{5!r^5} - \frac{t^6}{6!r^6}\right) \pm \cdots$$

At the end, the Sumudu transform of $\cos(at)$

$$S[\cos(at)] = \left(1 - \frac{1}{e^{\frac{t}{r}}}\right)_{t=0}^{t=\infty} - a^2r^2\left(1 - \frac{1}{e^{\frac{t}{r}}} - \frac{t}{re^{\frac{t}{r}}} - \frac{t^2}{2r^2e^{\frac{t}{r}}}\right)_{t=0}^{t=\infty} + a^4r^4\left(1 - \frac{1}{e^{\frac{t}{r}}} - \frac{t}{re^{\frac{t}{r}}} - \frac{t^2}{2r^2e^{\frac{t}{r}}} - \frac{t^3}{3!r^3e^{\frac{t}{r}}} - \frac{t^4}{4!r^4e^{\frac{t}{r}}}\right)_{t=0}^{t=\infty}$$

$$- a^6r^6\left(1 - \frac{1}{e^{\frac{t}{r}}} - \frac{t}{re^{\frac{t}{r}}} - \frac{t^2}{2r^2e^{\frac{t}{r}}} - \frac{t^3}{3!r^3e^{\frac{t}{r}}} - \frac{t^4}{4!r^4e^{\frac{t}{r}}} - \frac{t^5}{5!r^5e^{\frac{t}{r}}} - \frac{t^6}{6!r^6e^{\frac{t}{r}}}\right)_{t=0}^{t=\infty} \pm \cdots$$

$$= 1 - a^2r^2 + a^4r^4 - a^6r^6 + a^8r^8 - a^{10}r^{10} \pm \cdots$$

$$= \frac{1}{1+a^2r^2}$$

**Case 3:** In the Theorem 3.1, let $q(t) = \frac{\sin(at)}{t}$. Again, by considering (3.4) and Table 1 we can write

$$V(i+1) = \frac{1}{r}\frac{V(i)}{i+1} + \frac{1}{r(i+1)}\left\{ \begin{array}{ll} \frac{a^i(-1)^{\frac{i}{2}}}{(i+1)!}, & i \quad even \\ 0, & i \quad odd \end{array}\right.$$  (3.12)

$$V(0) = 0$$

In that case, we can write some of $V(i)$ as

$$V(1) = \frac{1}{r} \quad V(2) = \frac{1}{2!r^2} \quad V(3) = \frac{3-a^2r^2}{3 \times 3!r^3} \quad V(4) = \frac{3-a^2r^2}{3 \times 4!r^4}$$

$$V(5) = \frac{15-5a^2r^2+3a^4r^4}{15 \times 5!r^5} \quad V(6) = \frac{15-5a^2r^2+3a^4r^4}{15 \times 6!r^6}$$  (3.13)

$$V(7) = \frac{105-35a^2r^2+21a^4r^4-15a^6r^6}{105 \times 7!r^7}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

By considering (3.5) in the Theorem 3.1 and using (3.13), we have

$$\sum_{i=0}^{\infty} V(i)t^i = a\left(\frac{t}{ar} + \frac{t^2}{2!ar^2} + \frac{t^3}{3!ar^3} + \frac{t^4}{4!ar^4} + \frac{t^5}{5!ar^5} + \cdots\right) - \frac{1}{3}a^3r^2\left(\frac{t^3}{3!ar^3} + \frac{t^4}{4!ar^4} + \frac{t^5}{5!ar^5} + \frac{t^6}{6!ar^6} + \cdots\right) \tag{3.14}$$
$$+ \frac{1}{5}a^5r^4\left(\frac{t^5}{5!ar^5} + \frac{t^6}{6!ar^6} + \frac{t^7}{7!ar^7} + \frac{t^8}{8!ar^8} + \cdots\right) - \frac{1}{7}a^7r^6\left(\frac{t^7}{7!ar^7} + \frac{t^8}{8!ar^8} + \frac{t^9}{9!ar^9} + \frac{t^{10}}{10!ar^{10}} + \cdots\right) \pm \cdots$$

The (3.14) can be rewritten equally as below

$$\sum_{i=0}^{\infty} V(i)t^i = a\left[\frac{1}{a}\left(e^{\frac{t}{r}} - 1\right)\right] - \frac{a^3r^2}{3}\left[\frac{1}{a}\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2}\right)\right] + \frac{a^5r^4}{5}\left[\frac{1}{a}\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4}\right)\right]$$
$$- \frac{a^7r^6}{7}\left[\frac{1}{a}\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4} - \frac{t^5}{5!r^5} - \frac{t^6}{6!r^6}\right)\right] \pm \cdots$$

Finally, by using (3.6) we obtain the Sumudu transform of $\frac{\sin(at)}{t}$

$$S\left[\frac{\sin(at)}{t}\right] = \left[e^{-\frac{t}{r}}\sum_{i=0}^{\infty} V(i)t^i\right]_{t=0}^{t=\infty} = a - \frac{a^3r^2}{3} + \frac{a^5r^4}{5} - \frac{a^7r^6}{7} \pm \cdots = \frac{\tan^{-1}(ar)}{r}$$

**Case 4:** In the Theorem 3.1, let $q(t) = \sinh(at)$. Then, considering (3.4) and Table 1 we can write

$$V(i+1) = \frac{1}{r}\frac{V(i)}{i+1} + \frac{1}{r(i+1)}\left\{\begin{array}{ll}\frac{a^i}{i!}, & i \quad odd \\ 0, & i \quad even\end{array}\right. \tag{3.15}$$
$$V(0) = 0$$

By means of (3.15), $V(i)$ are obtained following

$$V(1) = 0 \quad V(2) = \frac{a}{2!r} \quad V(3) = \frac{a}{3!r^2} \quad V(4) = \frac{a(1+a^2r^2)}{4!r^3}$$
$$V(5) = \frac{a(1+a^2r^2)}{5!r^4} \quad V(6) = \frac{a(1+a^2r^2+a^4r^4)}{6!r^5} \tag{3.16}$$
$$V(7) = \frac{a(1+a^2r^2+a^4r^4)}{7!r^6} \quad V(8) = \frac{a(1+a^2r^2+a^4r^4+a^6r^6)}{8!r^7}$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

From (3.5) in the Theorem 3.1 and from (3.16), we have

$$\sum_{i=0}^{\infty} V(i)t^i = ar\left(\frac{t^2}{2!r^2} + \frac{t^3}{3!r^3} + \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \cdots\right) + a^3r^3\left(\frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \frac{t^6}{6!r^6} + \cdots\right) \tag{3.17}$$
$$+ a^5r^5\left(\frac{t^6}{6!r^6} + \frac{t^7}{7!r^7} + \frac{t^8}{8!r^8} + \cdots\right) + a^7r^7\left(\frac{t^8}{8!r^8} + \frac{t^9}{9!r^9} + \frac{t^{10}}{10!r^{10}} + \cdots\right) + \cdots$$

And the equation (3.17) can be written as equivalently below

$$\sum_{i=0}^{\infty} V(i)t^i = ar\left(e^{\frac{t}{r}} - 1 - \frac{t}{r}\right) + a^3r^3\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3}\right) + a^5r^5\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4} - \frac{t^5}{5!r^5}\right)$$
$$+ a^7r^7\left(e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4} - \frac{t^5}{5!r^5} - \frac{t^6}{6!r^6} - \frac{t^7}{7!r^7}\right) + \cdots$$

As a results, we find the Sumudu transform of $\sinh(at)$

$$S[\sinh(at)] = \left[e^{-\frac{t}{r}}\sum_{i=0}^{\infty} V(i)t^i\right]_{t=0}^{t=\infty} = ar + a^3r^3 + a^5r^5 + a^7r^7 + a^9r^9 + \cdots = \frac{ar}{1-a^2r^2}$$

**Case 5:** In the Theorem 3.1, let $q(t) = \cosh(at)$. Then, considering (3.4) and Table 1 we can write

$$V(i+1) = \frac{1}{r}\frac{V(i)}{i+1} + \frac{1}{r(i+1)}\left\{\begin{array}{ll}\frac{a^i}{i!}, & i \quad even \\ 0, & i \quad odd\end{array}\right.$$
$$V(0) = 0$$

Hence, some of the $V(i)$ are obtained as

$$V(1) = \frac{1}{r} \quad V(2) = \frac{1}{2!r^2} \quad V(3) = \frac{1+a^2r^2}{3!r^3} \quad V(4) = \frac{1+a^2r^2}{4!r^4}$$
$$V(5) = \frac{1+a^2r^2+a^4r^4}{5!r^5} \quad V(6) = \frac{1+a^2r^2+a^4r^4}{6!r^6} \tag{3.18}$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Once again, by using (3.5) in the Theorem 3.1 and from (3.18), we have

$$\sum_{i=0}^{\infty} V(i)t^i = \left( \frac{t}{r} + \frac{t^2}{2!r^2} + \frac{t^3}{3!r^3} + \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \cdots \right) + a^2r^2 \left( \frac{t^3}{3!r^3} + \frac{t^4}{4!r^4} + \frac{t^5}{5!r^5} + \frac{t^6}{6!r^6} + \cdots \right)$$
$$+ a^4r^4 \left( \frac{t^5}{5!r^5} + \frac{t^6}{6!r^6} + \frac{t^7}{7!r^7} + \frac{t^8}{8!r^8} + \cdots \right) + a^6r^6 \left( \frac{t^7}{7!r^7} + \frac{t^8}{8!r^8} + \frac{t^9}{9!r^9} + \frac{t^{10}}{10!r^{10}} + \cdots \right) + \cdots \tag{3.19}$$

The (3.19) can be rewritten equally as follow

$$\sum_{i=0}^{\infty} V(i)t^i = \left( e^{\frac{t}{r}} - 1 \right) + a^2r^2 \left( e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} \right)$$
$$+ a^4r^4 \left( e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4} \right) \tag{3.20}$$
$$+ a^6r^6 \left( e^{\frac{t}{r}} - 1 - \frac{t}{r} - \frac{t^2}{2r^2} - \frac{t^3}{3!r^3} - \frac{t^4}{4!r^4} - \frac{t^5}{5!r^5} - \frac{t^6}{6!r^6} \right) + \cdots$$

Finally, we find the Sumudu transform of $\cosh(at)$

$$S[\sinh(at)] = \left[ e^{-\frac{t}{r}} \sum_{i=0}^{\infty} V(i)t^i \right]_{t=0}^{t=\infty} = 1 + a^2r^2 + a^4r^4 + a^6r^6 + a^8r^8 + \cdots = \frac{1}{1-a^2r^2}$$

# 4. Conclusion

As a result, we use the differential transform method (DTM) to find Sumudu Transform of functions as a different way. Moreover, contrary to the literature we obtain the Sumudu transform of functions easily without complex integration and long calculations.

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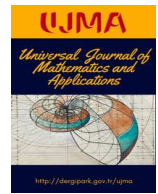
# Holditch-Type Theorem for Non-Linear Points in Generalized Complex Plane $\mathbb{C}_p$

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## Abstract

The generalized complex number system and generalized complex plane were studied by Yaglom [1, 2] and Harkin [3]. Moreover, Holditch-type theorem for linear points in $\mathbb{C}_p$ were given by Erişir et al. [4]. The aim of this paper is to find the answers of the questions "How is the polar moments of inertia calculated for trajectories drawn by non-linear points in $\mathbb{C}_p$?", "How is Holditch-type theorem expressed for these points in $\mathbb{C}_p$?" and finally "Is this paper a new generalization of [4]?".

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## 1. Introduction and preliminaries

H. Holditch expressed the Holditch theorem in the article entitled "Geometrical Theorem" in 1858. Holditch theorem is stated that "If the end points of a chord, with constant length $a+b$, draw any closed curve, any point on this chord draw different closed curve. So, the area between these curves is always $\pi ab$", [5]. The most important point of this classic Holditch theorem in Euclidean plane is that the area between these curves is independent of the selection of the curves. Thus, this theorem has attracted a lot of attention and been generalized with various methods and different perspectives. Then, Steiner calculated the area formula of the trajectory in a moving plane drawn by a point in the fixed plane in terms of Steiner points, [6].

Blaschke and Müller considered trajectories drawn by three points and generalized the Holditch theorem in Euclidean plane, [7]. Then, Hering expressed the Holditch theorem with respect to the length of the envelope curve with the aid of non-linear three points, [8]. Considering the above studies, there are many studies concerned with the Holditch theorem, [9, 10, 11].

The polar moment of inertia instead of area in Holditch theorem can be calculated by similar processes. Holditch theorem expressed in terms of the polar moment of inertia is called as "Holditch-type theorem".

Müller calculated the polar moment of inertia of the trajectory drawn by a point in the Euclidean plane. Moreover, Müller gave a conclusion that the geometric locus of all fixed points on the moving plane which has same polar moment of inertia is the circle with center which is Steiner point, [12]. Then, considering the study [12], there are lots of studies related to Holditch-type theorem, [13, 14, 15, 16, 17].

In the Euclidean plane, the Cauchy formula of the closed envelope of a family of the straight lines $g$ and the length of the envelope of trajectories of straight lines were given by Blaschke and Müller, [7]. In the Lorentzian plane, the Cauchy formula for the envelope of a family of lines was given by Yüce and Kuruoğlu. Moreover, they proved the length of the envelope of trajectories of non-null lines and gave the Holditch theorem for the length of the envelope of trajectories for Lorentzian motion, [18].

The generalized complex number system is defined as

$$\mathbb{C}_p = \left\{ x + iy \ : \ x, y \in \mathbb{R}, \ i^2 = p \in \mathbb{R} \right\}$$

and expressed by Yaglom and Harkins, [1, 2, 3]. This system involves in complex ($p = -1$), dual ($p = 0$) and hyperbolic ($p = +1$) number systems and also different planes for other values of p.

Considering the studies given by Yaglom and Harkins, some studies were done in the generalized complex plane. Gürses and Yüce considered the one parameter planar motion in Affine-Cayley Klein planes and p-complex plane $\mathbb{C}_J = \left\{ x + Jy \ : \ x, y \in \mathbb{R}, \ J^2 = p, \ p \in \{-1, 0, 1\} \right\} \subset \mathbb{C}_p$, [19, 20]. Moreover, Erişir et. al. calculated the Steiner area formula and proved Holditch theorem in the generalized complex plane $\mathbb{C}_p$,

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[21]. Then, they calculated the polar moment of inertia of trajectories under the one-parameter planar motion and proved Holditch-type theorem in $\mathbb{C}_p$, [4]. Moreover, Erişir and Güngör gave the Cauchy-length formula and proved Holditch theorem for non-linear points in $\mathbb{C}_p$, [22].

Now, using the above studies, we give some operations on this system.

The addition, substraction and product on this generalized complex plane $\mathbb{C}_p$ are

$$Z_1 \pm Z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = x_1 \pm x_2 + i(y_1 \pm y_2)$$

and

$$M^p(Z_1, Z_2) = (x_1 x_2 + p y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

where $Z_1 = (x_1 + iy_1)$, $Z_2 = (x_2 + iy_2) \in \mathbb{C}_p$, [2, 3]. In addition, the p−magnitude of $Z = x + iy \in \mathbb{C}_p$ is

$$|Z|_p = \sqrt{|M^p(Z, \bar{Z})|} = \sqrt{|x^2 - py^2|}.$$

The unit circle in $\mathbb{C}_p$ is the set of points in the form $|Z|_p = 1$. So, now we consider the special values of p in $\mathbb{C}_p$ as follows.

**1)** Let us consider $p < 0$. Thus, the generalized complex number system matches up with the elliptical complex number system. For $p = -1$, the unit circle in $\mathbb{C}_p$ corresponds to the Euclidean unit circle and the plane $\mathbb{C}_{-1}$ matches up with Euclidean plane.

**2)** If we consider $p = 0$, the plane $\mathbb{C}_0$ matches up with Gallilean plane. The unit circle in $\mathbb{C}_p$ corresponds to Gallilean circle.

**3)** We take $p > 0$. In this case, the generalized complex number system is equal to the hyperbolic complex number system. If we take $p = 1$, the plane $\mathbb{C}_1$ corresponds to the Lorentzian plane, (Figure 1.1), [3].
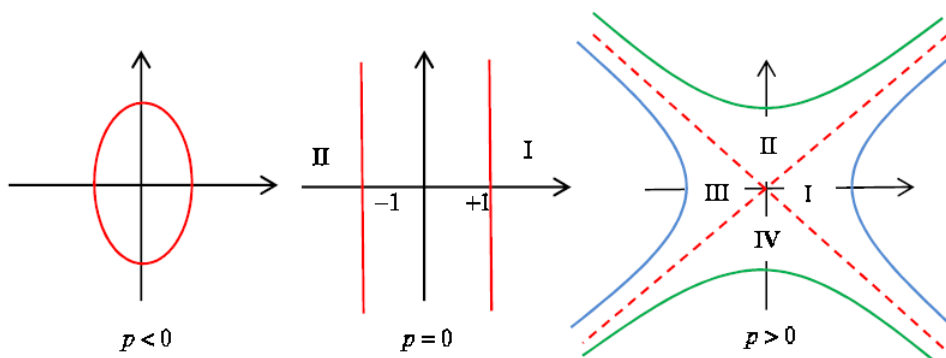


**Figure 1.1:** Unit Circles in $\mathbb{C}_p$

So, we can give the following definition.

**Definition 1.1.** *Let us consider a circle in the generalized complex plane $\mathbb{C}_p$. This circle has the center $M(a, b)$ and the radius r. So, the equation of this circle is*

$$\left| (x - a)^2 - p(y - b)^2 \right| = r^2$$

*where $i^2 = p \in \mathbb{R}$, [3].*

Now, we mention the angle in $\mathbb{C}_p$. Let us consider $\sigma \equiv y/x$ and $Z = x + iy$. So, we can write

$$\tan p\theta_p = \frac{\sin p\theta_p}{\cos p\theta_p},$$

[3]. In addition, the generalized Euler formula

$$e^{i\theta_p} = \cos p\theta_p + i \sin p\theta_p$$

where $i^2 = p$ in $\mathbb{C}_p$. Thus, the polar and exponential forms of the generalized complex number $Z$ is

$$z = r_p(\cos p\theta_p + i \sin p\theta_p) = r_p e^{i\theta_p}$$

where $\theta_p$ and $r_p = |Z|_p$ are p−argument and p−magnitude of generalized complex number Z, respectively. The p−rotation matrix obtained by $e^{i\theta_p}$ is

$$A(\theta_p) = \begin{bmatrix} \cos p\theta_p & p \sin p\theta_p \\ \sin p\theta_p & \cos p\theta_p \end{bmatrix}.$$

Moreover, the derivatives of the p−trigonometric functions $\cos p$ and $\sin p$ can be written by

$$\frac{d}{d\alpha}(\cos p\alpha) = p\sin p\alpha, \qquad \frac{d}{d\alpha}(\sin p\alpha) = \cos p\alpha,$$

[3].

Throughout this study, we consider one-parameter planar motion $\mathbb{K}_p/\mathbb{K}'_p$ in generalized complex plane $\mathbb{C}_p$. Moreover, we study in the branch I of $\mathbb{C}_p$.

Now, we mention Cauchy formula in $\mathbb{C}_p$ which is used in this study. This formula in $\mathbb{C}_p$ was studied by Erişir and Güngör in [22].

Let $g$ be a line in the branch I of $\mathbb{C}_p$. So, the Hesse form of this line $g$ in $\mathbb{C}_p$ is written by

$$h = x_1 \cos p\psi_p - px_2 \sin p\psi_p$$

where $(h, \psi_p)$ is the Hesse coordinates in $\mathbb{C}_p$ and $h = h(\psi_p)$ is the distance to the origin $O$ from the right line and the point $X(x_1, x_2)$ is the contact point of the line $g$ with the envelope curve $(g)$. Moreover, the Cauchy-length formula in $\mathbb{C}_p$ is written by

$$L = \frac{1}{\sqrt{|p|}} \int_{t_0}^{t_1} |ph - \ddot{h}| \, d\psi_p.$$

Similarly, we give the length of the enveloping curve $(g)$ according to the fixed generalized complex plane $\mathbb{K}'_p$. So, we can write the Hesse form of the line $g$ according to the fixed generalized complex plane $\mathbb{K}'_p$ as

$$h' = x'_1 \cos p\psi'_p - px'_2 \sin p\psi'_p$$

where $h'$ is the distance to the origin $O'$ from the right line $g$. If the necessary operations are considered, it is obtained that

$$h' = h - u_1 \cos p\psi_p + pu_2 \sin p\psi_p.$$

So, we obtain that

$$L' = \frac{1}{\sqrt{|p|}} \left| ph\delta_p - A\cos p\psi_p + pB\sin p\psi_p \right|$$

where $A = \int_{t_0}^{t_1} (pu_1 - \ddot{u}_1)d\theta_p$ and $B = \int_{t_0}^{t_1} (pu_2 - \ddot{u}_2)d\theta_p$.

Moreover, we know that

$$L' = \sqrt{|p|} \left( \int_{t_0}^{t_1} \bar{q}d\theta_p + L_Q^g \right)$$

where $L_Q^g = q_2 \cos p\psi_p - q_1 \sin p\psi_p \Big|_{t_0}^{t_1}$ is the length of orthogonal projection of the line segment $Q_1Q_2$ of the moving pole curve $(Q)$ on the line $g$. Moreover, $\bar{q} = h - q_1 \cos p\psi_p + pq_2 \sin p\psi_p$ is distance of the pole point $Q$ to the line $g$ in the generalized complex plane in $\mathbb{C}_p$, [22]. In addition, the following theorem can be given.

**Theorem 1.2.** *All the fixed lines with Hesse coordinates $(h, \psi_p)$ of the generalized moving complex plane $\mathbb{K}_p$ whose envelope of trajectories have the same length $L' = c$ are tangent to the cycles with center $S_G = \left( \frac{A}{p\delta_p}, \frac{B}{p\delta_p} \right)$ and radius $\frac{c}{\sqrt{|p|\delta_p}}$ in the generalized moving plane $\mathbb{K}_p$, [22].*

## 2. Main theorems and proofs

In this section, we prove the Holditch-type theorem for non-linear points in the generalized complex plane $\mathbb{C}_p$ for one-parameter planar motion with $S = S_G$. We firstly express and prove following theorem.

**Theorem 2.1.** *Let the non-linear points $X = (0,0)$, $Y = (a+b,0)$ and $Z = (a,c)$ be fixed on the generalized moving plane $\mathbb{K}_p$ in $\mathbb{C}_p$. In addition, the points $X$, $Y$ and $Z$ move along the trajectories $k_X$, $k_Y$ and $k_Z$ on $\mathbb{K}'_p$ with moments $T_X$, $T_Y$ and $T_Z$, respectively. So, the relationship between the polar moments of inertia $T_X$, $T_Y$ and $T_Z$ is*

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_p \left( pc^2 + ab \right) - 2\sqrt{|p|}cL_{XY}$$

*where $L_{XY}$ is the length of the enveloping curve of $(XY)$.*

*Proof.* Let the points $X$, $Y$ and $Z$ be non-linear points. Moreover, we consider that these points $X = (0,0)$, $Y = (a+b,0)$ and $Z = (a,c)$. We know that the polar moments of inertia of any point $X$ in $\mathbb{C}_p$ is given

$$T_Z = T_0 + \delta_p \left( x_1{}^2 - px_2{}^2 - 2x_1s_1 + 2px_2s_2 \right)$$

in [4]. So, if we use this formula for the points $X$, $Y$ and $Z$, we find that

$$T_X = T_0 \tag{2.1}$$

$$T_Y = T_X + \delta_p \left( (a+b)^2 - 2(a+b)s_1 \right) \tag{2.2}$$

$$T_Z = T_X + \delta_p \left( a^2 - pc^2 - 2as_1 + 2pcs_2 \right). \tag{2.3}$$

From the equations (2.1) and (2.2), we have

$$s_1 = \frac{a+b}{2} + \frac{T_X - T_Y}{2\delta_p(a+b)}. \tag{2.4}$$

Moreover, from the equations (2.3) and (2.4), we find that

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_p \left( pc^2 + ab \right) + 2p\delta_p cs_2.$$

The other hand, from $S = S_G$ we know that

$$s_2 = \frac{B}{p\delta_p}.$$

Finally, if $L'$ is written for $X = (0,0)$, $Y = (a+b,0)$ and $Z = (a,c)$ we obtain that

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_p \left( pc^2 + ab \right) - 2\sqrt{|p|}cL_{XY} \tag{2.5}$$

$$\square$$

So, the following conclusion can be given.

**Conclusion 2.2.** *Let us take that $X,Y$ and $Z$ are linear points during the motion with $S = S_G$ in $\mathbb{C}_p$. Namely, we have $c = 0$. From the equation* (2.5) *the relation between the polar moments of inertia of trajectory drawn by the points $X$, $Y$ and $Z$ is*

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_p ab.$$

*This formula is the formula given relationship between polar moments of inertia for the linear three points in [4]. So, the formula* (2.5) *is generalization of the formula in [4].*

*Note:* For the value $p = 0$, the formula (2.5) is obtain that

$$T_Z = \frac{aT_Y + bT_X}{a+b} - \delta_p ab. \tag{2.6}$$

This formula is also the formula between polar moments of inertia for the linear three points in [4]. Namely, for $p = 0$, the formula of polar moment of inertia for linear three points is same the formula of moment for non-linear three points. The reason of this is the metric in the plane $C_0$. From the definition of metric in $C_0$ ($p = 0$) the distance between the points $X$ and $R$ (the orthogonal projection of the point $Z$ on the line segment $XY$), $(a)$, is same the distance between the points $X$ and $Z$. Similarly, the distance between the points $Y$ and $R$, $(b)$, is same the distance between the points $Y$ and $Z$. So, for $p = 0$ the equation (2.6) is valid the polar moments of inertia for both linear three points and non-linear three points.
In addition, we give the following conclusions.

**Conclusion 2.3.** *If the points $X$ and $Y$ move along the same trajectories $k_X$ with moment $T_X$, the formula* (2.5) *is obtained that*

$$T_Z = T_X - \delta_p \left( pc^2 + ab \right) - 2\sqrt{|p|}cL_{XY}.$$

**Conclusion 2.4.** *The relationship between the length of envelope curve $(g)$ and the length of the enveloping curve of $(XY)$ is*

$$L' = \sqrt{|p|} \left( h\delta_p + \left( \frac{T_Y - T_X}{2\delta_p(a+b)} - \frac{a+b}{2} \right) \delta_p \cos p\psi_p - \sqrt{|p|}L_{XY} \sin p\psi_p \right)$$

Finally, we can give the main theorem from the equation (2.5).

**Theorem 2.5. Main Theorem (Holditch-Type Theorem):** *Let us consider motion with $S = S_G$ and the points $X = (0,0)$, $Y = (a+b,0)$ and the point $Z = (a,c)$ non-linear with $X$ and $Y$ fixed on $\mathbb{K}_p$. In a specific time interval, while the points $X$ and $Y$ move along the same trajectories $k_X$ with moment $T_X$, the point $Z$ non-linear with the points $X$ and $Y$ draws different trajectory $k_Z$ with the moment $T_Z$. The moment of section between the curves $k_X$ ($k_Y$) and $k_Z$ depends on the distances of the point $R$ to the endpoints $X$ and $Y$, the distance of the point $Z$ to the line $\overline{XY}$, the length of the enveloping curve and the rotation angle of the motion. This moment is independent of the choice of curves.*

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# Universal Journal of Mathematics and Applications

# Homotopy Analysis Aboodh Transform Method for Nonlinear System of Partial Differential Equations

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## Article Info

## Abstract

In this paper, a combined form of homotopy analysis method with Aboodh transform method is proposed to solve nonlinear system of partial differential equations. This method is called the homotopy analysis Aboodh transform method (HAATM). The homotopy analysis Aboodh transform method can easily be applied to many problems of nonlinear system, and is capable of reducing the size of computational work.

## 1. Introduction

The nonlinear evolution equations have attracted the attention of many researchers because of their wide applications in various fields such as physics, fluid mechanics, bio-mathematics, chemical physics and other areas of science and engineering. The investigation of exact solutions for the nonlinear evolution equations is a particularly hot topic [1]. So we find that a lot of researchers are working to develop new methods to solve this kind of equations. These efforts have strengthened this area of research through many methods, among them we find, homotopy analysis method (HAM). This method was developed in 1992 by Liao Shijun ([2], [3], [4], [5]), and was used by many researchers to solven nonlinear differential equations ([6], [7], [8]). Then, a new option emerged recently, includes the composition of Laplace transform, Sumudu transform, Natural transform or Aboodh transform with this method to solve nonlinear differential equations. Among which are the homotopy analysis method coupled with Laplace transform ([9], [10], [11]), homotopy analysis Sumudu transform method ([12], [13], [14]), homotopy Natural transform method ([15], [16]) and homotopy analysis Aboodh transform method [17].

The aim of this study is to combine homotopy analysis method and Aboodh transform method in order to obtain a more effective method, characterized by speed in solution and accuracy in the results obtained. The modified method is called homotopy analysis Aboodh transform method (HAATM). Three examples of nonlinear partial differential equations are given to re-confirm the strength and effectiveness of this modified method.

The present paper has been organized as follows: In Section 2 Some basic definitions and properties of the Aboodh transform method. In section 3 We give an analysis of the proposed method. In section 4 We present three examples explaining how to apply the proposed method. Finally, the conclusion follows.

## 2. Definitions and properties of the Aboodh transform

In this section, we give some basic definitions and properties of Aboodh transform which are used further in this paper.
A new transform called the Aboodh transform defined for function of exponential order, we consider functions in the set $\bar{A}$, defined by [18]:

$$\bar{A} = \left\{ f(t) : \exists M, \; k_1, k_2 > 0, \; |f(t)| < M e^{-vt} \right\}.$$

For given function in the set $\bar{A}$, the constant $M$ must be finite number, $k_1, k_2$ my be finite or infinite.

The Aboodh transform denoted by the operator $A(\cdot)$ defined by the integral equation:

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt, \ t \geqslant 0, k_1 \leqslant v \leqslant k_2.$$

We will summarize here some results of simple functions related to Aboodh transform in the following table [18]:

| $f(t)$ | $A[f(t)]$ | $f(t)$ | $A[f(t)]$ |
|:---:|:---:|:---:|:---:|
| 1 | $\frac{1}{v^2}$ | $\sin at$ | $\frac{a}{v(v^2+a^2)}$ |
| $t$ | $\frac{1}{v^3}$ | $\cos at$ | $\frac{1}{v^2+a^2}$ |
| $t^n$ | $\frac{n!}{v^{n+2}}$ | $\sinh at$ | $\frac{a}{v(v^2-a^2)}$ |
| $e^{at}$ | $\frac{1}{v^2-av}$ | $\cosh at$ | $\frac{1}{v^2-a^2}$ |

**Theorem 2.1.** *Let $K(v)$ is the Aboodh transform of $f(t)$, then one has:*

$$A\left[f'(t)\right] = vK(v) - \frac{f(0)}{v},$$

$$A\left[f''(t)\right] = v^2K(v) - \frac{f'(0)}{v} - f(0),$$

$$A\left[f^{(n)}(t)\right] = v^nK(v) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}}.$$

*Proof.* (see [18]). □

Aboodh transform of partial derivative: To obtain Aboodh transform of partial derivative, we use integration by parts, and then we have:

$$A\left[\frac{\partial u(x,t)}{\partial t}\right] = vK(x,v) - \frac{u(x,0)}{v},$$

$$A\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = v^2K(x,v) - \frac{1}{v}\frac{\partial u(x,0)}{\partial t} - u(x,0),$$

For the proof of these formulas, you can see [19].

**Theorem 2.2.** *Let $K(x,v)$ is the Aboodh transform of $u(x,t)$, then one has:*

$$A\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] = v^nK(x,v) - \sum_{k=0}^{n-1}\frac{1}{v^{2-n+k}}\frac{\partial^k u(x,0)}{\partial t^k}.$$

*Proof.* (see [17]). □

## 3. Homotopy analysis Aboodh transform method (HAATM)

To illustrate the basic idea of this method, we consider a general non-homogeneous, nonlinear partial diffrential equation

$$L_t[V(x,t)] + R[V(x,t)] + N[V(x,t)] = f(x,t), \tag{3.1}$$

where $L_t$ denotes a first-order partial diffrential operator, $R$ is the general linear operators, $N$ is the nonlinear operator and $f(x,t)$ is the source terms.
Taking the Aboodh transform on both sides of (3.1), we get

$$A(L_t[V(x,t)]) + A(R[V(x,t)] + N[V(x,t)]) = A[f(x,t)]$$

Using the property of the Aboodh transform, we have

$$A[V(x,t)] - \frac{1}{v^2}V(x,0) + \frac{1}{v}(A[R(V(x,t)) + N(V(x,t)) - f(x,t)]) = 0$$

Define the nonlinear operators

$$N[\phi(x,t;p)] = A[\phi(x,t;p)] - \frac{1}{v^2}V(x,0;p) + \frac{1}{v}(A[R(\phi(x,t;p)) + N(\phi(x,t;p)) - f(x,t;p)])$$

By means of homotopy analysis method ([2], [3], [4], [5]), we construct the so-called the zero-order deformation equation

$$(1-q)A[\phi(x,t;p)-V_0(x,t)] = phH(x,t)N[\phi(x,t;p)], \tag{3.2}$$

where $p$ is an embedding parameter and $p \in [0, 1]$, $H(x,t) \neq 0$ is an auxiliary function, $h \neq 0$ is an auxiliary parameter, $A$ is an auxiliary linear Aboodh operator. When $p = 0$ and $p = 1$, we have

$$\begin{cases} \phi(x,t;0) = V_0(x,t), \\ \phi(x,t;1) = V(x,t). \end{cases}$$

When $P$ increases from 0 to 1, the $\phi(x,t,p)$ various from $V_0(x,t)$ to $V(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to $p$, we have

$$\phi(x,t;p) = V_0(x,t) + \sum_{m=1}^{+\infty} V_m(x,t)p^m, \tag{3.3}$$

where

$$V_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\Big|_{p=0}$$

When $p = 1$, the formula (3.3) becomes

$$V(x,t) = V_0(x,t) + \sum_{m=1}^{+\infty} V_m(x,t).$$

Define the vectors

$$\overrightarrow{V} = \{V_0(x,t), V_1(x,t), V_2(x,t), \ldots, V_m(x,t)\}.$$

Differentiating (3.2) $m-$times with respect to $p$, then setting $p = 0$ and finally dividing them by $m!$, we obtain the so-called $m^{th}$ order deformation equation

$$A[V_m(x,t) - \chi_m V_{m-1}(x,t)] = hH(x,t)\Re_m(\overrightarrow{V}_{m-1}(x,t)), \tag{3.4}$$

where

$$\Re_m(\overrightarrow{V}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N(x,t;p)}{\partial p^{m-1}}\Big|_{p=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases}$$

Applying the inverse Aboodh transform on both sides of (3.4), we can obtain

$$V_m(x,t) = \chi_m V_{m-1}(x,t) + hA^{-1}\left[H(x,t)\Re_m(\overrightarrow{V}_{m-1}(x,t))\right]. \tag{3.5}$$

The $m^{th}$ deformation equation (3.5) is a linear which can be easily solved. So, the solution of (3.1) can be written into the following form

$$V(x,t) = \sum_{m=0}^{N} V_m(x,t),$$

when $N \to \infty$, we can obtain an accurate approximation solution of (3.1).
For the proof of the convergence of the homotopy analysis method see [3].

# 4. Application of this method

In this section, we apply the homotopy analysis method (HAM) coupled with Aboodh transform method for solving system of nonlinear partial differential equations.

**Example 4.1.** *We consider the following system of nonlinear coupled Burgers partial differential equations*

$$
\begin{cases}
U_t - U_{xx} - 2UU_x + (UV)_x = 0 \\
V_t - V_{xx} - 2VV_x + (UV)_x = 0
\end{cases}, \tag{4.1}
$$

*with the initial conditions*

$$
U(\varkappa,0) = \sin x, \ V(x,0) = \sin x.
$$

*The nonlinear operators are*

$$
\begin{cases}
N[\phi(x,t,p)] = A\left[\phi(x,t;p)\right] - \frac{1}{v^2}\sin x \\
+ \frac{1}{v}A\left[-\phi_{xx}(x,t;p) - 2\phi(x,t;p)\phi_x(x,t;p) + (\phi(x,t;p)\varphi(x,t;p))_x\right] \\
N[\varphi(x,t,p)] = A\left[\varphi(x,t;p)\right] - \frac{1}{v^2}\sin x \\
+ \frac{1}{v}A\left[-\varphi_{xx}(x,t;p) - 2\varphi(x,t;p)\varphi_x(x,t;p) + (\phi(x,t;p)\varphi(x,t;p))_x\right]
\end{cases}.
$$

*Thus, we obtain the $m^{th}$ order deformation equations given by*

$$
\begin{cases}
U_m(x,t) = \chi_m U_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{U}_{m-1}(x,t))] \\
V_m(x,t) = \chi_m V_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{V}_{m-1}(x,t))]
\end{cases}. \tag{4.2}
$$

*with*

$$
\begin{cases}
\Re_m(\overrightarrow{U}_{m-1}(x,t)) = A\left[U_{m-1}(x,t)\right] - \frac{1}{v^2}\left(1 - \chi_m\right)\sin x \\
+ \frac{1}{v}A\left[\sum_{i=0}^{m-1}(U_i V_{m-1-i})_\varkappa - 2\sum_{i=0}^{m-1}U_i(U_{m-1-i})_\varkappa - \sum_{i=0}^{m-1}(U_i)_{xx}\right], \\
\Re_m(\overrightarrow{V}_{m-1}(x,t)) = A\left[V_{m-1}(x,t)\right] - \frac{1}{v^2}\left(1 - \chi_m\right)\sin x \\
+ \frac{1}{v}A\left[\sum_{i=0}^{m-1}(U_i V_{m-1-i})_\varkappa - 2\sum_{i=0}^{m-1}V_i(V_{m-1-i})_\varkappa - \sum_{i=0}^{m-1}(V_i)_{xx}\right],
\end{cases} \tag{4.3}
$$

*and*

$$
\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases}
$$

*According to (4.2) and (4.3), the formulas of the first terms is given by*

$$
\begin{aligned}
U_1(x,t) &= hA^{-1}\left(\frac{1}{v}A\left[(U_0 V_0)_x - 2U_0(U_0)_x - (U_0)_{xx}\right]\right), \\
U_2(x,t) &= (1+h)U_1(x,t) \\
&+ hA^{-1}\left(\frac{1}{v}A\left[(U_0 V_1 + U_1 V_0)_x - 2\left(U_0 U_{1x} + U_1 U_{0x}\right) - (U_1)_{xx}\right]\right), \\
U_3(x,t) &= (1+h)U_2(x,t) \\
&+ hA^{-1}\left(\frac{1}{v}A\left[(U_0 V_2 + U_1 V_1 + U_2 V_0)_x - 2\left(2U_0 U_{2x} + U_1 U_{1x}\right) - (U_2)_{xx}\right]\right), \\
&\vdots
\end{aligned} \tag{4.4}
$$

*and*

$$
\begin{aligned}
V_1(x,t) &= hA^{-1}\left(\frac{1}{v}A\left[(U_0 V_0)_x - 2V_0(V_0)_x - (V_0)_{xx}\right]\right), \\
V_2(x,t) &= (1+h)V_1(x,t) \\
&+ hA^{-1}\left(\frac{1}{v}A\left[(U_0 V_1 + U_1 V_0)_x - 2\left(V_0 V_{1x} + V_1 V_{0x}\right) - (V_1)_{xx}\right]\right), \\
V_3(x,t) &= (1+h)V_2(x,t) \\
&+ hA^{-1}\left(\frac{1}{v}A\left[(U_0 V_2 + U_1 V_1 + U_2 V_0)_x - 2\left(2V_0 V_{2x} + V_1 V_{1x}\right) - (V_2)_{xx}\right]\right), \\
&\vdots
\end{aligned} \tag{4.5}
$$

*From the equations (4.4) and (4.5), the first solution terms of homotopy analysis Aboodh transform method of the system (4.1), is given by*

$$U_0(x,t) = \sin x,$$
$$V_0(x,t) = \sin x,$$
$$U_1(x,t) = (h)\sin(x)t,$$
$$V_1(x,t) = (h)\sin(x)t,$$
$$U_2(x,t) = (h)(1+h)\sin(x)t + (h^2)\sin(x)\frac{t^2}{2!},$$
$$V_2(x,t) = (h)(1+h)\sin(x)t + (h^2)\sin(x)\frac{t^2}{2!},$$
$$U_3(x,t) = (h)(1+h)^2\sin(x)t + 2(1+h)(h^2)\sin(x)\frac{t^2}{2!} + (h^3)\sin(x)\frac{t^3}{3!},$$
$$V_3(x,t) = (h)(1+h)^2\sin(x)t + 2(1+h)(h^2)\sin(x)\frac{t^2}{2!} + (h^3)\sin(x)\frac{t^3}{3!},$$
$$\vdots$$

*and so on.*

*The other components of the (HAATM) can be determined in a similar way. Finally, the approximate solution $(U,V)$ of the system (4.1) in a series form, is given by*

$$\begin{cases} U(x,t) = \sin x \left(1 + h(3+3h+h^2)t + (3+2h)h^2\frac{t^2}{2!} + h^3\frac{t^3}{3!} + \cdots\right) \\ V(x,t) = \sin x \left(1 + h(3+3h+h^2)t + (3+2h)h^2\frac{t^2}{2!} + h^3\frac{t^3}{3!} + \cdots\right) \end{cases}$$

*Substiting $h = -1$ in (??), the approximate solution of the system (4.1) is given as follows*

$$\begin{cases} U(x,t) = \sin x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \\ V(x,t) = \sin x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \end{cases}$$

*And in the closed form, the solution $(U,V)$ is given by*

$$\begin{cases} U(x,t) = \sin(x)e^{-t} \\ V(x,t) = \sin(x)e^{-t} \end{cases}.$$
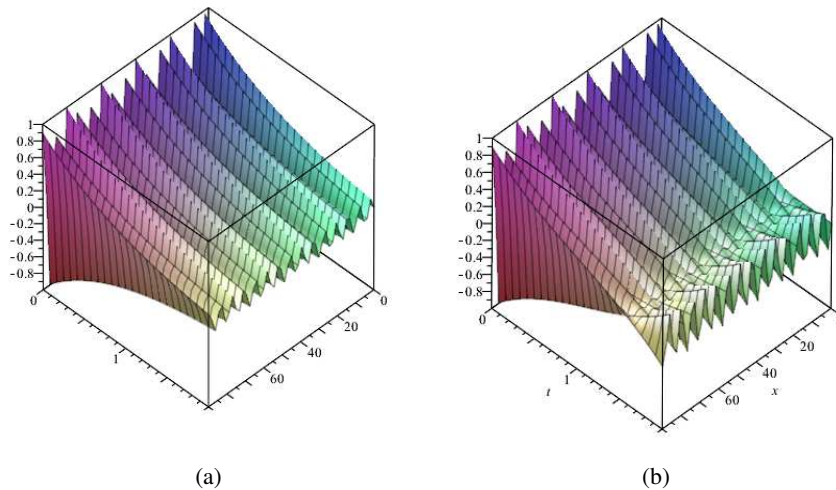


**Figure 4.1:** (a) Exact solution for $U(x,t)$ and $V(x,t)$, (b) Approximate solution $U(x,t)$ and $V(x,t)$ when $h \longrightarrow -0.99$.

**Example 4.2.** *Consider the nonlinear system of inhomogeneous partial differential equations [20]*

$$\begin{cases} U_t + U_x V + U = 1 \\ V_t - UV_x - V = 1 \end{cases}. \tag{4.6}$$

*with the initial conditions*

$$U(\varkappa,0) = e^x, \ V(x,0) = e^{-x}.$$

*The nonlinear operators are*

$$\begin{cases} N[\phi(x,t,p)] = A[\phi(x,t;p)] - \frac{1}{v^2}e^x + \frac{1}{v}A[\phi_x(x,t;p)\varphi(x,t;p) + \phi(x,t;p) - 1] \\ N[\varphi(x,t,p)] = A[\varphi(x,t;p)] - \frac{1}{v^2}e^{-x} + \frac{1}{v}A[-\phi(x,t;p)\varphi_x(x,t;p) - \varphi(x,t;p) - 1] \end{cases}$$

*Thus, we obtain the $m^{th}$ order deformation equations given by*

$$\begin{cases} U_m(x,t) = \chi_m U_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{U}_{m-1}(x,t))] \\ V_m(x,t) = \chi_m V_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{V}_{m-1}(x,t))] \end{cases} . \tag{4.7}$$

*with*

$$\begin{cases} \Re_m(\overrightarrow{U}_{m-1}(x,t)) = A[U_{m-1}(x,t)] - \frac{1}{v^2}(1-\chi_m)e^x \\ \quad + \frac{1}{v}A\left[\sum_{i=0}^{m-1}(U_i)_x V_{m-1-i} + \sum_{i=0}^{m-1} U_i - 1\right], \\ \Re_m(\overrightarrow{V}_{m-1}(x,t)) = A[V_{m-1}(x,t)] - \frac{1}{v^2}(1-\chi_m)e^{-x} \\ \quad + \frac{1}{v}A\left[-\sum_{i=0}^{m-1} U_i(V_{m-1-i})_x - \sum_{i=0}^{m-1} V_i - 1\right], \end{cases} \tag{4.8}$$

*and*

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases}$$

*According to (4.7) and (4.8), the formulas of the first terms is given by*

$$\begin{aligned} U_1(x,t) &= hA^{-1}\left(\frac{1}{v}A[(U_0)_x V_0 + U_0 - 1]\right), \\ U_2(x,t) &= (1+h)U_1(x,t) + hA^{-1}\left(\frac{1}{v}A[(U_0)_x V_1 + (U_1)_x V_0 + U_1]\right), \\ U_3(x,t) &= (1+h)U_2(x,t) \\ &\quad + hA^{-1}\left(\frac{1}{v}A[(U_0)_x V_2 + (U_1)_x V_1 + (U_2)_x V_0 + U_2]\right), \\ &\qquad\qquad \vdots \end{aligned} \tag{4.9}$$

*and*

$$\begin{aligned} V_1(x,t) &= hA^{-1}\left(\frac{1}{v}A[-U_0(V_0)_x - V_0 - 1]\right), \\ V_2(x,t) &= (1+h)V_1(x,t) + hA^{-1}\left(\frac{1}{v}A[-U_0(V_1)_x - U_1(V_0)_x - V_1]\right), \\ V_3(x,t) &= (1+h)V_2(x,t) \\ &\quad + hA^{-1}\left(\frac{1}{v}A[-U_0(V_2)_x - U_1(V_1)_x - U_2(V_0)_x - V_2]\right), \\ &\qquad\qquad \vdots \end{aligned} \tag{4.10}$$

*From the equations (4.9) and (4.10), the first solution terms of homotopy analysis Aboodh transform method of the system (4.6), is given by*

$$\begin{aligned} U_0(x,t) &= e^x, \\ V_0(x,t) &= e^{-x}, \\ U_1(x,t) &= (h)e^x t, \\ V_1(x,t) &= (-h)e^{-x}t, \\ U_2(x,t) &= (h)(1+h)e^x t + (h^2)e^x \frac{t^2}{2!}, \\ V_2(x,t) &= (-h)(1+h)e^{-x}t + (h^2)e^{-x}\frac{t^2}{2!}, \\ U_3(x,t) &= (h)(1+h)^2 e^x t + 2(1+h)(h^2)e^x \frac{t^2}{2!} + (h^3)e^x \frac{t^3}{3!}, \\ V_3(x,t) &= (-h)(1+h)^2 e^{-x}t + 2h^2(1+h)e^{-x}\frac{t^2}{2!} + (-h^3)e^{-x}\frac{t^3}{3!}, \\ &\qquad\qquad \vdots \end{aligned}$$

*and so on.*
*The other components of the (HAATM) can be determined in a similar way. Finally, the approximate solution $(U,V)$ of the system (4.6)in a series form, is given by*

$$\begin{cases} U(x,t) = e^x\left(1 + h(3+3h+h^2)t + (3+2h)h^2 \frac{t^2}{2!} + h^3 \frac{t^3}{3!} + \cdots\right) \\ V(x,t) = e^{-x}\left(1 + (-h)(3+3h+h^2)t + (3+2h)h^2 \frac{t^2}{2!} + (-h^3)\frac{t^3}{3!} + \cdots\right) \end{cases},$$

*and in the case $h = -1$, the approximate solution is given as follows*

$$\begin{cases} U(x,t) = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \\ V(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \end{cases}$$

*And in the closed form, the solution $(U,V)$ is given by*

$$\begin{cases} U(x,t) = e^{x-t} \\ V(x,t) = e^{-x+t} \end{cases}$$

(a)                                              (b)
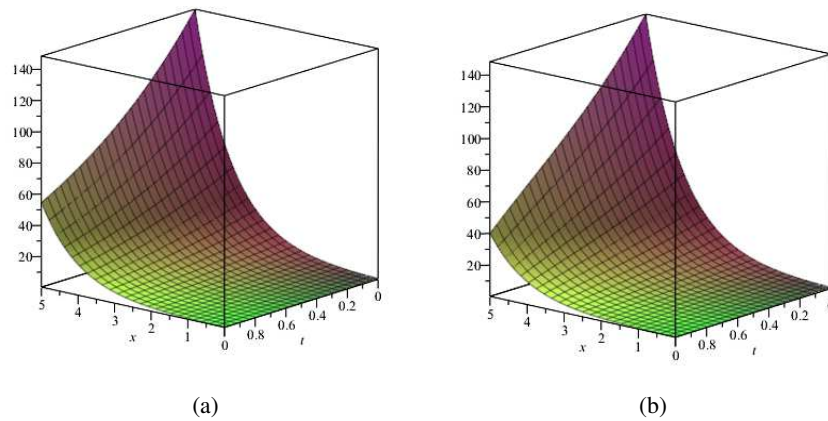
**Figure 4.2:** (a) Exact solution $U(x,t)$. (b) Approximate solution $U(x,t)$ when $h \longrightarrow -1.09$.



(c)                                              (d)
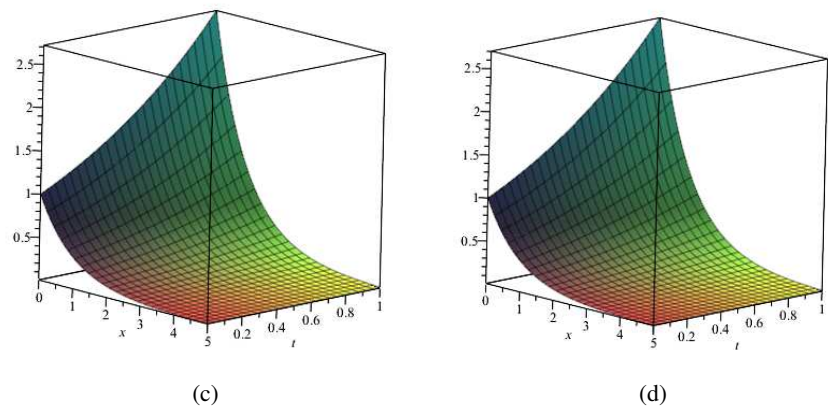
**Figure 4.3:** (c) Exact solution $V(x,t)$. (d) Approximate solution $V(x,t)$ when $h \longrightarrow -1.09$.

**Example 4.3.** *Consider the system of nonlinear coupled partial differential equations [21]*

$$\begin{cases} U_t(x,y,t) - V_x(x,y,t)W_y(x,,y,t) = 1 \\ V_t(x,y,t) - W_x(x,y,t)U_y(x,,y,t) = 5 \\ W_t(x,y,t) - U_x(x,y,t)V_y(x,,y,t) = 5 \end{cases} \quad , \tag{4.11}$$

*with the initial conditions*

$$U(\varkappa,y,0) = x + 2y, \ V(x,y,0) = x - 2y, \ W(x,y,0) = -x + 2y.$$

*The nonlinear operators are*

$$\begin{cases} N[\phi(x,t,p)] = A[\phi(x,t;p)] - \frac{1}{v^2}(x+2y) - \frac{1}{v}A\left[\varphi_x(x,t;p)\psi_y(x,t;p) + 1\right] \\ N[\varphi(x,t,p)] = A[\varphi(x,t;p)] - \frac{1}{v^2}(x-2y) - \frac{1}{v}A\left[\psi_x(x,t;p)\phi_y(x,t;p) + 5\right] \\ N[\psi(x,t,p)] = A[\psi(x,t;p)] - \frac{1}{v^2}(-x+2y) - \frac{1}{v}A\left[\phi_x(x,t;p)\varphi_y(x,t;p) + 5\right] \end{cases}$$

*Thus, we obtain the $m^{th}$ order deformation equations given by*

$$\begin{cases} U_m(x,t) = \chi_m U_{m-1}(x,t) + hA^{-1}[\Re_m(\vec{U}_{m-1}(x,t))] \\ V_m(x,t) = \chi_m V_{m-1}(x,t) + hA^{-1}[\Re_m(\vec{V}_{m-1}(x,t))] \\ W_m(x,t) = \chi_m W_{m-1}(x,t) + hA^{-1}[\Re_m(\vec{W}_{m-1}(x,t))] \end{cases} \tag{4.12}$$

*with,*

$$
\begin{cases}
\Re_m(\overrightarrow{U}_{m-1}(x,t)) = A\left[U_{m-1}(x,t)\right] - \frac{1}{v^2}\left(1 - \chi_m\right)(x+2y) \\
\qquad - \frac{1}{v}A\left[\sum_{i=0}^{m-1}(V_i)_x(W_{m-1-i})_y + 1\right], \\
\Re_m(\overrightarrow{V}_{m-1}(x,t)) = A\left[V_{m-1}(x,t)\right] - \frac{1}{v^2}\left(1 - \chi_m\right)(x-2y) \\
\qquad - \frac{1}{v}A\left[\sum_{i=0}^{m-1}(W_i)_x(U_{m-1-i})_y + 5\right], \\
\Re_m(\overrightarrow{W}_{m-1}(x,t)) = A\left[W_{m-1}(x,t)\right] - \frac{1}{v^2}\left(1 - \chi_m\right)(-x+2y) \\
\qquad - \frac{1}{v}A\left[\sum_{i=0}^{m-1}(U_i)_x(V_{m-1-i})_y + 5\right],
\end{cases}
\tag{4.13}
$$

$$
\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases}
$$

*According to (4.12) and (4.13), the formulas of the first terms is given by*

$$
U_1(x,t) = -hA^{-1}\left(\frac{1}{v}A\left[(V_0)_x(W_0)_y + 1\right]\right),
$$
$$
U_2(x,t) = (1+h)U_1(x,t) - hA^{-1}\left(\frac{1}{v}A\left[(V_0)_x(W_1)_y + (V_1)_x(W_0)_y\right]\right),
$$
$$
U_3(x,t) = (1+h)U_2(x,t)
$$
$$
-hA^{-1}\left(\frac{1}{v}A\left[(V_0)_x(W_2)_y + (V_1)_x(W_1)_y + (V_2)_x(W_0)_y\right]\right),
$$
$$
\vdots
$$

$$
V_1(x,t) = -hA^{-1}\left(\frac{1}{v}A\left[(W_0)_x(U_0)_y + 5\right]\right),
$$
$$
V_2(x,t) = (1+h)V_1(x,t) - hA^{-1}\left(\frac{1}{v}A\left[(W_0)_x(U_1)_y + (W_1)_x(U_0)_y\right]\right),
$$
$$
V_3(x,t) = (1+h)V_2(x,t)
$$
$$
-hA^{-1}\left(\frac{1}{v}A\left[(W_0)_x(U_2)_y + (W_1)_x(U_1)_y + (W_2)_x(U_0)_y\right]\right),
$$
$$
\vdots
$$

*and*

$$
W_1(x,t) = -hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x(V_0)_y + 5\right]\right),
$$
$$
W_2(x,t) = (1+h)W_1(x,t) - hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x(V_1)_y + (U_1)_x(V_0)_y\right]\right),
$$
$$
W_3(x,t) = (1+h)W_2(x,t)
$$
$$
-hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x(V_2)_y + (U_1)_x(V_1)_y + (U_2)_x(V_0)_y\right]\right),
$$
$$
\vdots
$$

*From the equations (4.4) and (4.5), the first solution terms of homotopy analysis Aboodh transform method of the system (4.1), is given by*

$$
U_0(\varkappa,y,t) = x+2y, \qquad V_0(x,y,t) = x-2y,
$$
$$
W_0(x,y,t) = -x+2y,
$$
$$
U_1(x,y,t) = -3(h)t, \qquad V_1(x,y,t) = -3(h)t,
$$
$$
W_1(x,y,t) = -3(h)t,
$$
$$
U_2(x,y,t) = -3h(1+h)t, \qquad V_2(x,y,t) = -3h(1+h)t,
$$
$$
W_2(x,y,t) = -3h(1+h)t,
$$
$$
U_3(x,t) = -3h(1+h)^2 t, \qquad V_3(x,t) = -3h(1+h)^2 t,
$$
$$
W_3(x,t) = -3h(1+h)^2 t,
$$
$$
\vdots
$$

*and so on.*
*The other components of the (HAATM) can be determined in a similar way. Finally, the approximate solution* $(U,V,W)$ *of the system (4.11)in a series form, is given by*

$$
\begin{cases}
U(x,y,t) = x+2y-3(h)t-3h(1+h)^2 t-3h(1+h)^3 t+\cdots \\
V(x,y,t) = x-2y-3(h)t-3h(1+h)^2 t-3h(1+h)^3 t+\cdots \\
W(x,y,t) = -x+2y-3(h)t-3h(1+h)^2 t-3h(1+h)^3 t+\cdots
\end{cases}
$$

*Substiting* $h = -1$ *in (??), the exact solution of the system (4.11) is given by*

$$
\begin{cases}
U(x,y,t) = x+2y+3t \\
V(x,y,t) = x-2y+3t \\
W(x,y,t) = -x+2y+3t
\end{cases}
\tag{4.14}
$$

(a)                                                                                    (b)
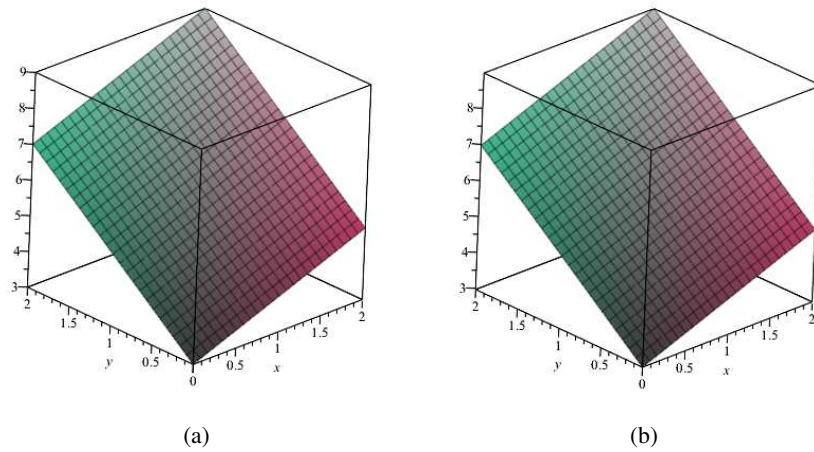
**Figure 4.4:** (a) Exact solution $U(x,y,t)$ at the moment $t = 1$. (b) Approximate solution $U(x,y,t)$ at the moment $t = 1$ when $h \longrightarrow -0.99$.



(c)                                                                                    (d)
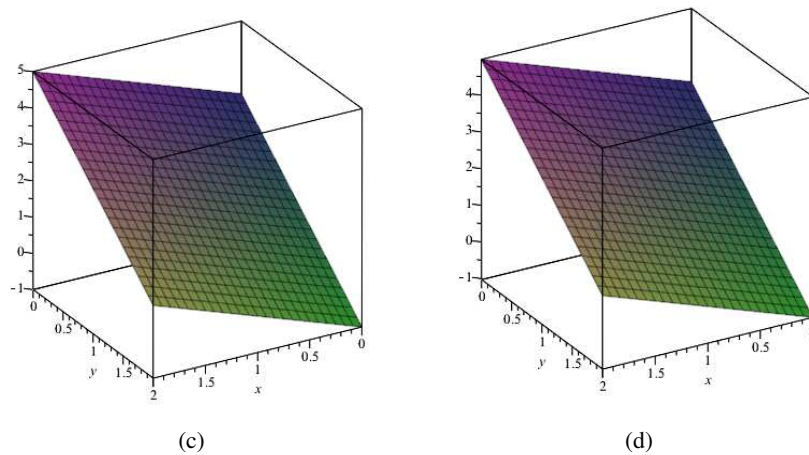
**Figure 4.5:** (c) Exact solution $V(x,y,t)$ at the moment $t = 1$. (d) Approximate solution $V(x,y,t)$ at the moment $t = 1$ when $h \longrightarrow -0.99$.



(e)                                                                                    (f)
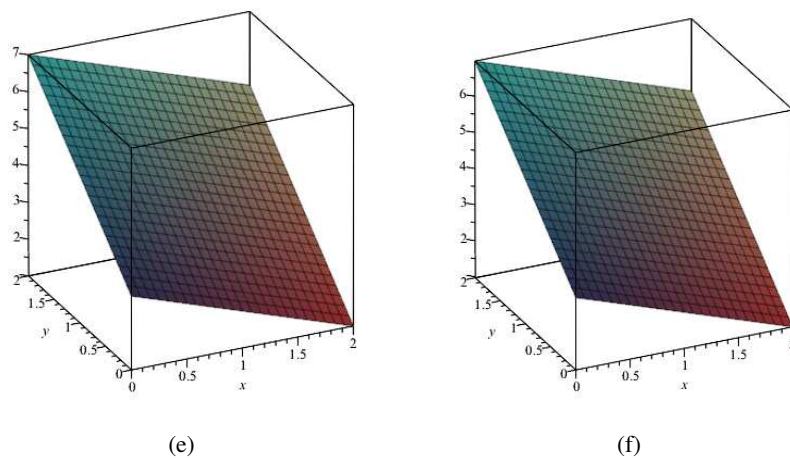
**Figure 4.6:** (e) Exact solution $W(x,y,t)$ at the moment $t = 1$. (f) Approximate solution $W(x,y,t)$ at the moment $t = 1$ when $h \longrightarrow -0.99$.

## 5. Conclusion

In this paper, we have seen that the coupling of homotopy analysis method (HAM) and the Aboodh transform method, proved very effective to solve nonlinear system of partial differential equations. The proposed algorithm (HAATM) is suitable for such problems and is very user friendly. The advantage of this method is its ability to combine two powerful methods to obtain exact solutions of nonlinear system of

partial differential equations. The results obtained in the examples presented shows that this modified method is very powerful and efficient technique in finding exact solutions for wide classes of problems.

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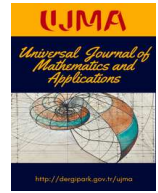
# Compact Totally Real Minimal Submanifolds in a Bochner-Kaehler Manifold

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[*]*Corresponding author*

## Abstract

In this paper, we establish the following results: Let $M$ be an $m-$dimensional compact totally real minimal submanifold immersed in a locally symmetric Bochner-Kaehler manifold $\tilde{M}$ with Ricci curvature bounded from below. Then either $M$ is a totally geodesic or

$$\inf r \leq \frac{1}{2} \left( \frac{1}{2} m(m-1)\tilde{k} - \frac{1}{3}(m+1)\tilde{c} \right),$$

where $r$ is the scalar curvature of $M$.

## 1. Introduction

The Bochner tensor was originally introduced in 1948 by S. Bochner as a Kaehler analogue of the Weyl conformal curvature tensor. Kaehler manifolds with vanishing Bochner tensor are known as Bochner-Kaehler manifolds, [1]. The Bochner tensor has interesting connections to several areas of mathematics and Bochner-Kaehler manifolds have been studied quite intensively in the last two decades, see for instance, [1, 2, 3].

In this work, we make us of Yau's [4] maximum principle to compact study totally real minimal submanifold with Ricci curvature bounded from below and obtain the following results:

**Main Theorem.** Let $M$ be an $m-$dimensional compact totally real minimal submanifold immersed in a locally symmetric Bochner-Kaehler manifold $\tilde{M}$ with Ricci curvature bounded from below. Then either $M$ is totally geodesic or $\inf r \leq \frac{1}{2} \left( \frac{1}{2} m(m-1)\tilde{k} - \frac{1}{3}(m+1)\tilde{c} \right)$ where $r$ is the scalar curvature of $M$.

We use the same notation and terminologies as in [5] unless otherwise stated.

Let $\tilde{M}$ be an $n-$dimensional Kaehler manifold and denote by $g_{AB}$, $F_{AB}$, $\tilde{K}_{ABCD}$ and $\tilde{K}$, the metric tensor, the complex structure tensor, the curvature tensor, the Ricci tensor and the scalar curvature of $\tilde{M}$, respectively. Suppose that the Boechner curvature tensor of $\tilde{M}$ vanishes, then we have

$$\begin{aligned} \tilde{K}_{ABCD} = & -g_{AD}L_{BC} + g_{BD}L_{AC} - L_{AD}g_{BC} + L_{BD}g_{AC} - F_{AD}M_{BC} \\ & + F_{BD}M_{AC} - M_{AD}F_{BC} + M_{BD}F_{AC} + 2(M_{AB}F_{CD} + F_{AB}M_{CD}), \end{aligned} \tag{1.1}$$

where

$$L_{BC} = \tilde{K}_{BC} / (2n+4) - \tilde{K}g_{BC}/2(2n+2)(2n+4), \ \tilde{K}_{BC} = g^{AD}\tilde{K}_{ABDC},$$

$$\tilde{K} = g^{BC}\tilde{K}_{BC}, \ M_{BC} = -L_{BD}F_C^D, \ F_C^D = g^{BD}F_{CB}.$$

$L_{BC}$ are components of a hybrid tensor of type $(0,2)$. That is

$$L_{BC}F_A^B F_D^C = L_{AD}.$$

In order to avoid repetitions it will be agreed that our indices have the following ranges throughout this paper:

$$A, B, C, D, ... = 1, 2, ..., m, 1^*, 2^*, ..., m^*,$$
$$i, j, k, l, ... = 1, 2, ..., m; \alpha, \beta, \gamma, ... = 1^*, 2^*, ..., m^*.$$

In the following sections, $\tilde{M}$ is always supposed to be a Bochner-Kaehler manifold, that is, $\tilde{M}$ is a Kaehler manifold with curvature tensor $\tilde{K}_{ABCD}$ given by (1.1).

## 2. Totally real submanifolds in $\tilde{M}$

We call $M$ as a totally real submanifold of $\tilde{M}$ if $M$ admits an isometric immersion into $\tilde{M}$ such that for all $x \in M$, $F(T_x(M)) \subset v_x$, where $T_x(M)$ denotes the tangent space of $M$ at $x$ and $F$ the complex structure of $\tilde{M}$. If the real dimension of $M$ is $m$, then $m \leq n$, $n$ is the complex dimension of $\tilde{M}$. We choose a local field of orthonormal frames

$$e_1,...,e_m,e_{m+1},...,e_n\ ; \qquad e_{1^*} = Fe_{1,...},e_{m^*} = Fe_{m,...,e_{n^*}} = Fe_n,$$

in $\tilde{M}$ in a such a way that, restricted to $M$, $e_1,...,e_m$ are tangents to $M$. With respect to this frame field, $F$ and $g$ have the components

$$(F_{AB}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad (g_{AB}) = (I_{2n}),$$

where $I_k$ denotes the identity matrix of degree $k$.

We consider the case $n = m$ only in this paper.

The equation of Gauss of $M$ in $\tilde{M}$ is written as

$$K_{ijkl} = \tilde{K}_{ijkl} + \sum_{\alpha} \left( h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha} \right). \tag{2.1}$$

$K_{ijkl}$ is the curvature tensor and $h_{ij}^{\alpha}$ is the second fundamental tensor of $M$. Since $M$ is a totally real submanifold in $\tilde{M}$, with respect to the above frame we have the relation $h_{jk}^{i^*} = h_{ik}^{j^*}$. Let $\tilde{K}$ be the curvature tensor field of $\tilde{M}$ so that $\tilde{K}_{ABCD} = g\left( \tilde{K}(e_C,e_D)e_B, e_A \right)$. Then (1.1) is equivalent to

$$\begin{aligned} \tilde{K}(X,Y)Z = {}& L(Y,Z)X - L(X,Z)Y + \langle Y,Z \rangle NX - \langle X,Z \rangle NY \\ & + M(Y,Z)FX - M(X,Z)FY + \langle FY,Z \rangle PX \\ & - \langle FX,Z \rangle PY - 2(M(X,Y)FZ + \langle FX,Y \rangle PZ), \end{aligned} \tag{2.2}$$

where $NX, PX$ are defined by $g(NX,Y) = L(X,Y)$, $g(PX,Y) = M(X,Y)$ and $\langle , \rangle$ denotes the inner product with respect to $g$. Let $\tilde{K}(X)$ be the holomorphic sectional curvature spanned by a unit vector $X$ and $FX$. By (1.1) or (2.2) we have

$$\tilde{K}(X) = \tilde{K}(X,FX,FX,X) = \langle \tilde{K}(X,FX)FX,X \rangle = 8L(X,X),$$

Let $\tilde{\rho}(X,Y)$ denote the sectional curvature of $\tilde{M}$ determined by section $\{X,Y\}$ spanned by two orthonormal vector $\{X,Y\}$. If $X,Y$ are both tangent to the totally real submanifold $M$ then we have

$$\tilde{\rho}(X,Y) = L(X,X) + L(Y,Y) = \frac{1}{8}\left( \tilde{K}(X) + \tilde{K}(Y) \right). \tag{2.3}$$

The equation of (2.3) has been obtained by Iwasaki and Ogitsu, [6].

Let $\rho(X,Y)$ denote the sectional curvature of $M$ determined by orthonormal tangent vectors $\{X,Y\}$ of $M$. Then the equation of Gauss (2.1) and (2.3) imply

$$\rho(X,Y) = \frac{1}{8}\left( \tilde{K}(X) + \tilde{K}(Y) \right) + \langle \sigma(X,X), \sigma(Y,Y) \rangle - \|\sigma(X,Y)\|^2,$$

where $\sigma$ is the second fundamental form which is related to $h_{ij}^{\alpha}$ by $g(\sigma(X,Y),\xi) = h_{jk}^{i^*} X^j Y^k \xi^{i^*}$ for any normal $\xi = \xi^{i^*} e_{i^*}$.

Let $S$ be the Ricci tensor of $M$ and $r$ the scalar curvature of $M$. Then

$$S(X,Y) = (m-2)L(X,Y) + \frac{1}{8}m\tilde{k}\langle X,Y \rangle - \sum_{\alpha} g(h_{\alpha}X, h_{\alpha}Y),$$

$$r = \frac{1}{4}m(m-1)\tilde{k} - \|\sigma\|^2.$$

Let $\tilde{M}$ is locally symmetric. Let $\Delta$ denote the Laplacian, $\bigtriangledown'$ denote the covariant differentiation with respect to connection in (tangent bundle) $\oplus$ (normal bundle) of $M$ in $\tilde{M}$. If $M$ is a minimal submanifold of $\tilde{M}$ the following holds (see [5] for example). Since $\tilde{M}$ is assumed to be locally symmetric:

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\bigtriangledown'\sigma\|^2 + \frac{1}{4}(m+1)\tilde{c}\|\sigma\|^2 + \sum tr\left( h_{i^*}h_{j^*} - h_{j^*}h_{i^*} \right)^2 - \sum tr\left( h_{i^*}h_{j^*} \right)^2, \tag{2.4}$$

where $\tilde{c}$ is a function on $M$ defined by $h_{lk}^{j^*} h_{ik}^{j^*} \tilde{K}_{li} = \frac{1}{2}(m+1)\tilde{c}\|\sigma\|^2$.

In order to prove the main theorem, we need the following lemmas.

**Lemma 2.1.** *Let $H_i$, $i \geq 2$ be symmetric $n \times n$ matrices, $S_i = trH_i^2$, $S = \sum_i S_i$. Then*

$$\sum_{i,j} tr\left(H_iH_j - H_jH_i\right)^2 - \sum_{i,j} tr\left(H_iH_j\right)^2 \geq -\frac{3}{2}\|\sigma\|^4, \tag{2.5}$$

*and the equality holds if and only if either all $H_i = 0$ or there exists two of $H_i$ different from zero. Moreover, if $H_1 \neq 0, H_2 \neq 0, H_i = 0, i \neq 1, 2$, then $S_1 = S_2$ and there exists an orthogonal $n \times n$ matrices $T$ such that*

$$TH_1^t T = \begin{pmatrix} \frac{\sqrt{S_1}}{2} & 0 & \dots & 0 \\ 0 & -\frac{\sqrt{S_1}}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad TH_2^t T = \begin{pmatrix} 0 & \frac{\sqrt{S_1}}{2} & \dots & 0 \\ \frac{\sqrt{S_1}}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

*[7, 8].*

**Lemma 2.2.** *Let $N$ be a complete Riemannian manifold with Ricci curvature bounded from below and let $f$ be a $C^2-$function bounded from above on $N$, then for all $\varepsilon > 0$, there exists a point $x \in N$ at which ,*
*i) $\sup f - \varepsilon < f(x)$,*
*ii) $\|\nabla f(x)\| < \varepsilon$,*
*iii) $\Delta f(x) < \varepsilon$, in [9].*

## 3. Proof of the main theorem

In this section, the method proof used by Ximin in [9] is applied totally real minimal submanifold immersed in a Bochner-Kaehler manifold. From (2.4) and (2.5), we obtain

$$\frac{1}{2}\Delta\|\sigma\|^2 \geq \|\sigma\|^2\left(\frac{1}{4}(m+1)\tilde{c} - \frac{3}{2}\|\sigma\|^2\right). \tag{3.1}$$

We know that $\|\sigma\|^2 = \frac{1}{4}m(m-1)\tilde{k} - r$. By the condition of the theorem, we conclude that $\|\sigma\|^2$ is bounded. We define $f = \|\sigma\|^2$ and $F = (f+a)^{\frac{1}{2}}$ (where $a > 0$ is any positive constant number). $F$ is bounded. We have

$$dF = \frac{1}{2}(f+a)^{-\frac{1}{2}}df,$$

$$\Delta F = \frac{1}{2}\left(-\frac{1}{2}(f+a)^{-\frac{3}{2}}\|df\|^2 + (f+a)^{-\frac{1}{2}}\Delta f\right),$$
$$= \frac{1}{2}\left(-2\|dF\|^2 + \Delta f\right)(f+a)^{-\frac{1}{2}},$$

i.e.,

$$\Delta F = \frac{1}{2}\left(-2\|dF\|^2 + \Delta f\right).$$

Hence, $F\Delta F = -\|dF\|^2 + \frac{1}{2}\Delta f$ or $\frac{1}{2}\Delta f = F\Delta F + \|dF\|^2$. Applying Lemma 2.2 to $F$, we have for all $\varepsilon > 0$, there exists a point $x \in M$ such that at $x$

$$\|dF(x)\| \leq \varepsilon, \tag{3.2}$$

$$\Delta F(x) < \varepsilon, \tag{3.3}$$

$$F(x) > \sup F - \varepsilon. \tag{3.4}$$

From (3.2),(3.3) and (3.4), we have

$$\frac{1}{2}\Delta f < \varepsilon^2 + F\varepsilon = \varepsilon(\varepsilon + F). \tag{3.5}$$

We take a sequence $\{e_n\}$ such that $\varepsilon_n \to 0\,(n \to \infty)$ and for all $n$, there exists a point $x_n \in M$ such that (3.2), (3.3) and (3.4) hold. Therefore, $\varepsilon_n(\varepsilon_n + F(x_n)) \to 0\,(n \to \infty)$ (Because $F$ is bounded). From (3.4), we have $F(x_n) > \sup F - \varepsilon_n$. Because $\{F(x_n)\}$ is a bounded sequence. So we get $F(x_n) \to F_0$ (If necessary, we can choose a subsequence). Hence, $F_0 \geq \sup F$. So we have

$$F_0 = \sup F.$$

From the definition of $F$, we get

$$f(x_n) \to f = \sup f.$$

(3.1) and (3.5) imply that

$$f\left(\frac{1}{4}(m+1)\tilde{c} - \frac{3}{2}f\right) \leq \frac{1}{2}\Delta f \leq \varepsilon(\varepsilon + F),$$

and

$$f(x_n)\left(\frac{1}{4}(m+1)\tilde{c} - \frac{3}{2}f(x_n)\right) < \varepsilon_n^2 + \varepsilon_n F(x_n) \leq \varepsilon_n^2 + \varepsilon_n F_0,$$

let $n \to \infty$, then $\varepsilon_n \to 0$ and $f(x_n) \to f_0$. Hence,

$$f_0\left(\frac{1}{4}(m+1)\tilde{c} - \frac{3}{2}f_0\right) \leq 0.$$

*i*) If $f_0 = 0$, we have $f = \|\sigma\|^2 = 0$. Hence $M$ is a totally geodesic.
*ii*) If $f_0 > 0$, we have $\frac{1}{4}(m+1)\tilde{c} - \frac{3}{2}f_0 \leq 0$ and

$$f_0 \geq \frac{1}{6}(m+1)\tilde{c},$$

that is, $\sup\|\sigma\|^2 \geq \frac{1}{6}(m+1)\tilde{c}$. Therefore,

$$\inf r \leq \frac{1}{2}\left(\frac{1}{2}m(m-1)\tilde{k} - \frac{1}{3}(m+1)\tilde{c}\right).$$

This completes the proof.

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# Finite Element Method for the Solution of a Time-Dependent Heat-Like Lane-Emden Equation

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## Article Info

## Abstract

In this study, finite element method (FEM) with Galerkin Formula is applied to find the numerical solution of a time-dependent heat-like Lane-Emden equation. An example is solved to assess the accuracy of the method. The numerical results are obtained for different values (n) of equation. The results indicate that Galerkin method is effectively implemented. It is seen that results are compatible with exact solutions and consistent with other existing numerical methods.

## 1. Introduction

In this paper, we consider heat-type equation for physical problems

$$u_{xx} + \frac{r}{x}u_x + ag(x,t)u + h(x,t) = u_t, \tag{1.1}$$

for $0 < x \leq L$, $0 < t < T$, $r > 0$, $a \in Z$, subject to the boundary conditions

$$u(0,t) = v(t), \quad u_x(0,t) = 0.$$

where $g(x,t)y(u) + h(x,t)$ is nonlinear heat source, $u(x,t)$ is the temperature, and $t$ is the dimensionless time variable.

Some researchers dealt with this type of models. The analytic solutions to several forms of the above problem were presented by [1], Wazwaz used the Adomian decomposition method [2]. Chowdhury, He and Noorani solved these problems using homotopy-perturbation and variational iteration methods, Momani applied the method to the time fractional heat-like equation with variable coefficient, Ucar applied non-polynomial spline method to this equation [3, 4, 5, 6, 7].

In this study, we construct so-called finite element approximations to solutions to time-dependent heat-like equations. The term "finite element method" has come to be associated with using piecewise polynomials in one, two, and three dimensions together with so-called Rayleigh-Ritz method and its more general counter part, the Galerkin method, to approximate solutions to operator equations. In this study, we concentrate on Galerkin method with splines.

The paper is organized as follows: Galerkin method is described and solution of equation (1.1) is presented in Section 2 briefly. In Section 3 some numerical results that are illustrated using MATLAB programme are given to clarify the method. Concluding remarks are given in Section 4.

## 2. Galerkin method

A usual scalar product for two real valued functions $u(x)$ and $v(x)$ is defined by $< u, v >= \int_0^T u(x)v(x)dx$, $u(x)$ and $v(x)$ are orthogonal if $< u, v >= 0$. And a norm associated with this scalar product is defined by

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$\|u\| = \sqrt{<u, u>} = (\int_0^T |u(x)|^2 dx)^{\frac{1}{2}}$. Let

i. $T_h : 0 = x_0 < x_1 < ... < x_M < x_{M+1} = 1$ be a partition of $(0, 1)$, $h_j = x_j - x_{j-1}$
ii. $V_h = v : v$, continuous and piecewise linear function on $T_h$ with $v(0) = v(1) = 0$
iii. $\{\varphi_j\}$, $j = 1, ..M$ be a basis function for $V_h$ where

$$\varphi_j(x) = \left\{ \begin{array}{ll} \frac{x - x_{j-1}}{h_j}, & x_{j-1} \leq x \leq x_j \\ \frac{x_{j+1} - x}{h_{j+1}}, & x_j \leq x \leq x_{j+1} \\ 0 &, \quad \text{otherwise} \end{array} \right\}.$$

Firstly, we should modify the equation (1.1) by applying finite difference to the right-hand side of the equality to solve it by using Galerkin method:

$$u_i'' + \frac{r}{x} u_i' + ag(x_i, t)u_i + h(x_i, t) = \frac{u_i - f(x_i)}{k}$$

where $f(x_i) = u(x_i, 0)$. From algebraic manipulations we obtain

$$ku_i'' + \frac{rk}{x} u_i' + akg(x_i, t)u_i - u_i + kh(x_i, t) + f(x_i) = 0. \tag{2.1}$$

Now the Galerkin method for the equation (2.1) is formulated as follows:
Find the approximate solution $U(x) \in V_h$ such that

$$\int_0^1 W \left( kU'' + \frac{rk}{x} U' + akg(x, t)U - U + kh(x, t) + f(x) \right) dx = 0, \quad (\forall W(x) \in V_h)$$

so we get

$$\int_0^1 \left( kWU'' + \frac{rk}{x} WU' + (akg(x, t) - 1)WU \right) dx = -\int_0^1 (Wkh(x, t) + Wf(x)) dx.$$

$$\int_0^1 WU'' dx = \left[ [U'W] \right]_0^1 - \int_0^1 W'U' dx$$

and since $W(0) = W(1) = 0$ for $W(x) \in V_h$ we get

$$\int_0^1 \left( -kW'U' + \frac{rk}{x} WU' + (akg(x, t) - 1)WU \right) dx = -\int_0^1 (Wkh(x, t) + Wf(x)) dx. \tag{2.2}$$

We may find the approximate solution $U(x) \in V_h$ by using basis functions $\varphi_j(x)$ as

$$U(x) = \sum_{j=1}^M c_j \varphi_j(x), \quad U'(x) = \sum_{j=1}^M c_j \varphi_j'(x), \quad W(x) = \sum_{i=1}^M s_i \varphi_i(x).$$

If we use these identities in equation (2.2), then we get

$$\int_0^1 \left[ \sum_{i=1}^M s_i \varphi_i'(x) \sum_{j=1}^M k c_j \varphi_j'(x) + \sum_{i=1}^M s_i \varphi_i(x) \sum_{j=1}^M \frac{rk}{x} c_j \varphi_j'(x) + \sum_{i=1}^M s_i \varphi_i(x) \sum_{j=1}^M (akg(x, t) - 1)c_j \varphi_j(x) \right] dx$$

$$= -\int_0^1 \left[ \sum_{i=1}^M s_i \varphi_i(x)(f(x) + kh(x, t)) \right] dx$$

$$\sum_{i=1}^M s_i \int_0^1 \sum_{j=1}^M c_j \left( k\varphi_i'(x)\varphi_j'(x) + \frac{rk}{x} \varphi_i(x)\varphi_j'(x) + (akg(x, t) - 1)\varphi_i(x)\varphi_j(x) \right) dx = -\sum_{i=1}^M s_i \int_0^1 \varphi_i(x)(f(x) + kh(x, t)) dx.$$

For $|i - j| > 1$ we have $\int_0^1 \varphi_j' \varphi_i dx = 0$ and $\int_0^1 \varphi_j \varphi_i dx = 0$, since if so then we have that $\varphi_j$ and $\varphi_i$ have non-overlapping supports. The method is described in matrix form in the following way:
for $i = 2$, $j = 1, .., M$

$$\alpha_{21} = \int_h^{2h} \left( k\varphi_2'(x)\varphi_1'(x) + \frac{rk}{x} \varphi_2(x)\varphi_1'(x) + (akg(x, t) - 1)\varphi_2(x)\varphi_1(x) \right) dx$$

$$\alpha_{22} = \int_h^{3h} \left( k\varphi_2'(x)\varphi_2'(x) + \frac{rk}{x} \varphi_2(x)\varphi_2'(x) + (akg(x, t) - 1)\varphi_2(x)\varphi_2(x) \right) dx$$

$$\alpha_{23} = \int_{2h}^{3h} \left( k\varphi_2'(x)\varphi_3'(x) + \frac{rk}{x} \varphi_2(x)\varphi_3'(x) + (akg(x, t) - 1)\varphi_2(x)\varphi_3(x) \right) dx$$

for $i = n$, $j = 1, .., M$

$$\alpha_{n(n-1)} = \int_{(n-1)h}^{nh} \left( k\varphi_n'(x)\varphi_{n-1}'(x) + \frac{rk}{x} \varphi_n(x)\varphi_{n-1}'(x) + (akg(x, t) - 1)\varphi_n(x)\varphi_{n-1}(x) \right) dx$$

$$\alpha_{nn} = \int_{(n-1)h}^{(n+1)h} \left( k\varphi_n'(x)\varphi_n'(x) + \frac{rk}{x}\varphi_n(x)\varphi_n'(x) + (akg(x,t)-1)\varphi_n(x)\varphi_n(x) \right) dx$$

$$\alpha_{n(n+1)} = \int_{nh}^{(n+1)h} \left( k\varphi_n'(x)\varphi_{n+1}'(x) + \frac{rk}{x}\varphi_n(x)\varphi_{n+1}'(x) + (akg(x,t)-1)\varphi_n(x)\varphi_{n+1}(x) \right) dx$$

for $i = M-1,\ j = 1,..,M$

$$\alpha_{(M-1)(M-2)} = \int_{(M-2)h}^{(M-1)h} \left( k\varphi_{M-1}'(x)\varphi_{M-2}'(x) + \frac{rk}{x}\varphi_{M-1}(x)\varphi_{M-2}'(x) + (akg(x,t)-1)\varphi_{M-1}(x)\varphi_{M-2}(x) \right) dx$$

$$\alpha_{(M-1)(M-1)} = \int_{(M-2)h}^{(M)h} \left( k\varphi_{M-1}'(x)\varphi_{M-1}'(x) + \frac{rk}{x}\varphi_{M-1}(x)\varphi_{M-1}'(x) + (akg(x,t)-1)\varphi_{M-1}(x)\varphi_{M-1}(x) \right) dx$$

$$\alpha_{(M-1)M} = \int_{(M-1)h}^{Mh} \left( k\varphi_{M-1}'(x)\varphi_M'(x) + \frac{rk}{x}\varphi_{M-1}(x)\varphi_M'(x) + (akg(x,t)-1)\varphi_{M-1}(x)\varphi_M(x) \right) dx$$

so we get the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & \dots & 0 & 0 \\ 0 & \alpha_{32} & \alpha_{33} & \alpha_{34} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & \alpha_{(M-1)(M-2)} & \alpha_{(M-1)(M-1)} & \alpha_{(M-1)M} \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} u(0,t) = e^{\sin t} \\ \int_h^{3h} \varphi_2(x)(f(x)+kh(x,t)) \\ \int_{2h}^{4h} \varphi_3(x)(f(x)+kh(x,t)) \\ \cdot \\ \cdot \\ \cdot \\ \int_{(M-1)h}^{(M+1)h} \varphi_M(x)(f(x)+kh(x,t)) \\ u(1,t) = e^{1+\sin t} \end{bmatrix},$$

$$C = [c_1, c_2, ..., c_M]'.$$

$$AC = B. \tag{2.3}$$

Finally the approximate solution $U$ is obtained by solving $C$ from equation (2.3) using Matlab programme.

## 3. Numerical example

In this section, we test our scheme on an example. We consider the numerical results obtained by applying the scheme discussed above to the following equation

$$u'' + \frac{2}{x}u' - (6+4x^2 - \cos t)u = u_t \quad 0 < x < 1, \quad t > 0$$

with initial condition

$$u(x,0) = e^{x^2}$$

and boundary conditions

$$u(0,t) = e^{\sin t}, \quad u_x(0,t) = 0.$$

The exact solution of the above problem is $u(x,t) = e^{x^2+\sin t}$. The problem is solved by using the scheme above in this paper. The maximum absolute errors are listed in Table 1. Also, numerical results given by scheme are shown in Figure 1.

Table 1: Maximum absolute errors, $k = 0.01$

| $n$ | Spline method[3] | Galerkin method |
|---|---|---|
| 11 | 6.8863e-03 | 2.3842e-03 |
| 21 | 1.9090e-03 | 6.7890e-04 |
| 41 | 7.8276e-04 | 1.8533e-04 |
| 61 | 5.9650e-04 | 7.6553e-05 |
| 121 | 5.2565e-05 | 3.8863e-05 |

## 4. Conclusion

In this paper, finite element method with Galerkin formula is applied for the numerical solution of the heat-like time-dependent Lane-Emden equation and the maximum absolute errors have shown in Table 1, which shows that this method approximate the exact solution very well. The implementation of the present method is more computational than the existing methods.

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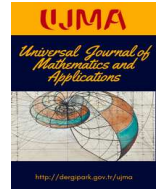
# An Arbitrary Order Differential Equations on Times Scale

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## Abstract

Here existence and stability results of $\psi$-Hilfer fractional differential equations on time scales is obtained. Here sufficient condition for existence and uniqueness of solution by using Schauder's fixed point theorem (FPT) and Banach FPT is produced. In addition, generalized Ulam stability of the proposed problem is also discussed. problem.

## 1. Introduction

In the past decade, fractional differential equations(FDEs) appeared as rich and beautiful field of research due to their applications to the physical and life sciences and it is witnessed by blossoming literature, for instance see [1]-[6].
Consider the dynamic equation on time scales with $\psi$-Hilfer fractional derivative (HFD) of the form

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(t) = \mathfrak{g}(\tau,\mathfrak{u}(\tau)), & \tau = [0,b] := J \subseteq \mathbb{T}, \\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(0) = \mathfrak{u}_0, & \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1.1}$$

where ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ is $\psi$-HFD defined on $\mathbb{T}$, $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\mathfrak{I}^{1-\gamma;\psi}$ is $\psi$-fractional interal of order $1 - \gamma(\gamma = \alpha + \beta - \alpha\beta)$. Let $\mathbb{T}$ be a time scale, that is nonempty subset of Banach space and $\mathfrak{g} : J \times \mathbb{T} \to R$ is a right-dense function.
Time scales calculus allows us to study the dynamic equations, which include both difference and differential equations, both of which are very important in implementing applications; for further information about the theoretical and potential applications of time scales, refer [7]-[9]
The dynamical behaviour of FDEs on time scales is currently undergoing active investigations. Several authors deliberate the existence and uniqueness solutions for problems involving classical fractional derivative [10, 11]. Motivated by the above works here we discuss the existence theory and stability criteria of FDEs on times scale. In order to solve the proposed problem $\psi$-HFD is utilized. The emergent and properties of $\psi$-HFD and the qualitative analysis is briefly studied in [12]-[14]. Further considerable attention paid to Ulam stability results for FDEs. For Ulam-Hyers stability theory of FDEs and its recent development, one can refer to [15]-[17]. Further the solution of generalized Ulam-Hyers-Rassias(UHR) is obtained.

## 2. Preliminaries

Throughout this study, let $C(J)$ be continuous function with norm

$$\|\mathfrak{u}\|_C = \max\{|\mathfrak{u}(\tau)| : \tau \in J\}.$$

We denote the space $C_\gamma(J)$ as follows

$$C_\gamma(J) := \left\{\mathfrak{g}(\tau) : J \to R | (\psi(\tau) - \psi(0))^\gamma \mathfrak{g}(\tau) \in C(J)\right\}, 0 \le \gamma < 1$$

the weighted space $C_\gamma(J)$ of the functions $\mathfrak{g}$ on the interval $J$. Thus, $C_\gamma(J)$ is the Banach space provided the norm

$$\|\mathfrak{g}\|_{C_\gamma} = \left\|(\psi(\tau) - \psi(0))^\gamma \mathfrak{g}(\tau)\right\|_C.$$

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**Definition 2.1.** *Let time scale be* $\mathbb{T}$*. The forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by* $\sigma(\tau) := \inf\{s \in \mathbb{T} : s > \tau\}$*, while the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by* $\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}$*.*

**Proposition 2.2.** *Suppose* $\mathbb{T}$ *is a time scale and* $[a,b] \subset \mathbb{T}$*,* $\mathfrak{g}$ *is increasing continuous function on* $[a,b]$*. If the extension of* $\mathfrak{g}$ *is given in the following form:*

$$\mathscr{G}(s) = \begin{cases} \mathfrak{g}(s); & s \in \mathbb{T} \\ \mathfrak{g}(\tau); & s \in (\tau, \sigma(\tau)) \notin \mathbb{T}. \end{cases}$$

*Then we have*

$$\int_a^b \mathfrak{g}(t)\Delta t \le \int_a^b \mathscr{G}(t)dt.$$

**Definition 2.3.** *Let* $\mathbb{T}$ *be a time scale,* $J \in \mathbb{T}$*. The left-sided R-L fractional integral of order* $\alpha \in R^+$ *of function* $\mathfrak{g}(\tau)$ *is defined by*

$$\left(^{\mathbb{T}}\mathfrak{J}^\alpha \mathfrak{g}\right)(\tau) = \int_0^\tau \psi'(s) \frac{(\psi(\tau) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathfrak{g}(s)\Delta s.$$

**Definition 2.4.** *Suppose* $\mathbb{T}$ *is a time scale,* $[0,b]$ *is an interval of* $\mathbb{T}$*. The R-L fractional derivative of order* $\alpha \in [n-1, n)$*,* $n \in \mathbb{Z}^+$ *of function* $\mathfrak{g}(\tau)$ *is defined by*

$$\left(^{\mathbb{T}}\Delta^\alpha \mathfrak{g}\right)(\tau) = \left(\frac{1}{\psi'(\tau)} \frac{d}{d\tau}\right)^n \int_0^\tau \psi'(s) \frac{(\psi(\tau) - \psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} \mathfrak{g}(s)\Delta s.$$

**Definition 2.5.** *[2] The* $\psi$*-HFD of order* $\alpha$ *and type* $\beta$ *of function* $\mathfrak{g}(\tau)$ *is defined by*

$$^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{g}(t) = \left(^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ ^{\mathbb{T}}\Delta(^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi}\mathfrak{g})\right)(\tau),$$

*where* $^{\mathbb{T}}\Delta := \frac{d}{d\tau}$*.*

**Remark 2.6.**    *1. Here* $^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ *is also written as*

$$^{\mathbb{T}}\Delta^{\alpha,\beta;\psi} = \ ^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ ^{\mathbb{T}}\Delta^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi} = \ ^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ ^{\mathbb{T}}\Delta^{\gamma;\psi}, \ \gamma = \alpha + \beta - \alpha\beta.$$

*2. Let* $\beta = 0$*, it transfers into R-L derivative given by* $^{\mathbb{T}}\Delta^\alpha := \ ^{\mathbb{T}}\Delta^{\alpha,0}$*.*
*3. Let* $\beta = 0$*, it turns to be Caputo fractional derivative given by* $^{\mathbb{T}}_c\Delta^\alpha := \ ^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\ ^{\mathbb{T}}\Delta$*.*

Next, we review some lemmas which will be used to extabilish our existence results.

**Lemma 2.7.** *If* $\alpha > 0$ *and* $\beta > 0$*, there exist*

$$\left[^{\mathbb{T}}\mathfrak{J}^\alpha (\psi(s) - \psi(0))^{\beta-1}\right](\tau) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(\tau) - \psi(0))^{\beta+\alpha-1}$$

**Lemma 2.8.** *Let* $\alpha \ge 0$*,* $\beta \ge 0$ *and* $\mathfrak{g} \in L^1(J)$*. Then*

$$^{\mathbb{T}}\mathfrak{J}^\alpha\ ^{\mathbb{T}}\mathfrak{J}^\beta \mathfrak{g}(\tau) \overset{a.e}{=} \ ^{\mathbb{T}}\mathfrak{J}^{\alpha+\beta}\mathfrak{g}(\tau).$$

**Lemma 2.9.** *If* $\mathfrak{g} \in C_\gamma(J)$ *and* $^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g} \in C_\gamma^1(J)$*, then*

$$^{\mathbb{T}}\mathfrak{J}^\alpha\ ^{\mathbb{T}}\Delta^\alpha \mathfrak{g}(\tau) = \mathfrak{g}(\tau) - \frac{\left(^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g}\right)(0)}{\Gamma(\alpha)} (\psi(\tau) - \psi(0))^{\alpha-1}.$$

**Lemma 2.10.** *Suppose* $\alpha > 0$*,* $a(\tau)$ *is a nonnegative function locally integrable on* $0 \le \tau < b$ *(some* $b \le \infty$*), and let* $g(\tau)$ *be a nonnegative, nondecreasing continuous function defined on* $0 \le \tau < b$*, such that* $g(\tau) \le K$ *for some constant K. Further let* $\mathfrak{u}(\tau)$ *be a nonnegative locally integrable on* $0 \le \tau < b$ *function with*

$$|\mathfrak{u}(\tau)| \le a(\tau) + g(\tau) \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} \mathfrak{u}(s)\Delta s,$$

*with some* $\alpha > 0$*. Then*

$$|\mathfrak{u}(\tau)| \le a(\tau) + \int_0^\tau \left[\sum_{n=1}^\infty \frac{(g(\tau)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \psi'(s) (\psi(\tau) - \psi(s))^{n\alpha-1}\right] \mathfrak{u}(s)\Delta s.$$

**Theorem 2.11.** *(Schauder FPT) Let* $\mathscr{E}$ *be a Banach space and* $\mathscr{Q}$ *be a nonempty bounded convex and closed subset of* $\mathscr{E}$ *and* $\mathscr{N} : \mathscr{Q} \to \mathscr{Q}$ *is compact, and continuous map. Then* $\mathscr{N}$ *has at least one fixed point in* $\mathscr{Q}$*.*

## 3. Existence results

**Lemma 3.1.** *Here* $\mathfrak{u}$ *is solution of* (1.1) *if and only if* $\mathfrak{u}$ *satisfies the following integral equation*

$$\mathfrak{u}(\tau) = \frac{\mathfrak{u}_0}{\Gamma(\gamma)}\left(\psi(\tau) - \psi(0)\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))\Delta s, \quad t > 0. \tag{3.1}$$

For further investigation, we give the following assumptions:

(H1)  The function $\mathfrak{g} : J \times R \to R$ is a rd-continuous.
(H2)  There exists a positive constants $L > 0$ such that

$$\left|\mathfrak{g}(\tau,\mathfrak{u}) - \mathfrak{g}(\tau,\mathfrak{v})\right| \le L\left|\mathfrak{u} - \mathfrak{v}\right|.$$

(H3)  There exists an increasing function $\varphi \in C_{1-\gamma}(J)$ and there exists $\lambda_\varphi > 0$ such that for any $\tau \in J$,

$$^{\mathbb{T}}\mathfrak{J}^\alpha \varphi(\tau) \le \lambda_\varphi \varphi(\tau).$$

**Theorem 3.2.** *Assume that (H1)-(H2) are fulfilled. Then, equation* (1.1) *has at least one solution.*

*Proof.* Consider the operator $\mathscr{P} : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$. The equivalent Volterra integral equation (3.1) which can be written in the operator form

$$(\mathscr{P}\mathfrak{u})(\tau) = \mathfrak{u}_0(\tau) + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))\Delta s$$

with

$$\mathfrak{u}_0(\tau) = \frac{\mathfrak{u}_0}{\Gamma(\gamma)}\left(\psi(\tau) - \psi(0)\right)^{\gamma-1}.$$

Define $B_r = \left\{\mathfrak{u} \in C_{1-\gamma,\psi}(J) : \|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \le r\right\}$.
Set $\tilde{\mathfrak{g}}(s) = \mathfrak{g}(s,0)$,

$$\sigma = \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b) - \psi(0)\right)^\alpha \|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}$$

and

$$\omega = \frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b) - \psi(0)\right)^\alpha.$$

To verify Theorem 2.11, we divide the proof into three steps.
**Step 1:** We check that $\mathscr{P}(B_r) \subset B_r$.

$$\left|\left(\psi(\tau) - \psi(0)\right)^{1-\gamma}(\mathscr{P}\mathfrak{u})(\tau)\right|$$

$$\le \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{\left(\psi(\tau) - \psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s))|\Delta s$$

$$\le \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{\left(\psi(\tau) - \psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s)) - \mathfrak{g}(s,0)|\Delta s$$

$$+ \frac{\left(\psi(\tau) - \psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,0)|\Delta s$$

$$\le \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{\left(\psi(\tau) - \psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1}L|\mathfrak{u}|\Delta s$$

$$+ \frac{\left(\psi(\tau) - \psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1}|\tilde{\mathfrak{g}}|\Delta s$$

$$\le \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b) - \psi(0)\right)^\alpha \|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}} + \frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b) - \psi(0)\right)^\alpha \|\mathfrak{u}\|_{C_{1-\gamma,\psi}}.$$

Hence

$$\|(\mathscr{P}\mathfrak{u})\| \le \sigma + \omega r \le r.$$

Which yields that $\mathscr{P}(B_r) \subset B_r$.
Next, the completely continuous of operator $\mathscr{P}$ is proved.
**Step 2:** The operator $\mathscr{P}$ is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $C_{1-\gamma,\psi}(J)$.

$$\left| (\psi(\tau) - \psi(0))^{1-\gamma} ((\mathscr{P}\mathfrak{u}_n)(t) - (\mathscr{P}\mathfrak{u})(\tau)) \right|$$

$$\leq \frac{(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} |\mathfrak{g}(s, \mathfrak{u}_n(s)) - \mathfrak{g}(s, \mathfrak{u}(s))| \Delta s$$

$$\leq \frac{(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} \sup_{s \in J} |\mathfrak{g}(s, \mathfrak{u}_n(s)) - \mathfrak{g}(s, \mathfrak{u}(s))| \Delta s$$

$$\leq \frac{(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} |\mathfrak{g}(s, \mathfrak{u}_n(s)) - \mathfrak{g}(s, \mathfrak{u}(s))| \, ds, \quad \text{(by Proposition 2.2)}$$

$$\leq \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} (\psi(b) - \psi(0))^\alpha \|\mathfrak{g}(\cdot, \mathfrak{u}_n(\cdot)) - \mathfrak{g}(\cdot, \mathfrak{u}(\cdot))\|_{C_{1-\gamma,\psi}},$$

Since $\mathfrak{g}$ is continuous, Lebesgue dominated convergence theorem implies

$$\|\mathscr{P}\mathfrak{u}_n - \mathscr{P}\mathfrak{u}\|_{C_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n \to \infty.$$

**Step 3:** $\mathscr{P}(B_r)$ is relatively compact.
Thus $\mathscr{P}(B_r)$ is uniformly bounded. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, then

$$\left| (\mathscr{P}\mathfrak{u})(\tau_2) (\psi(\tau_2) - \psi(0))^{1-\gamma} - (\mathscr{P}\mathfrak{u})(\tau_1) (\psi(\tau_1) - \psi(0))^{1-\gamma} \right|$$

$$\leq \left| \frac{(\psi(\tau_2) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^{\tau_2} \psi'(s) (\psi(\tau_2) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s)) \Delta s \right.$$

$$\left. - \frac{(\psi(\tau_1) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^{\tau_1} \psi'(s) (\psi(\tau_1) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s)) \Delta s \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \psi'(s) \left| (\psi(\tau_2) - \psi(0))^{1-\gamma} (\psi(\tau_2) - \psi(s))^{\alpha-1} - (\psi(\tau_1) - \psi(0))^{1-\gamma} (\psi(\tau_1) - \psi(s))^{\alpha-1} \right| |\mathfrak{g}(s, \mathfrak{u}(s))| \Delta s$$

$$+ \frac{(\psi(t_2) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \psi'(s) (\psi(\tau_2) - \psi(s))^{\alpha-1} |\mathfrak{g}(s, \mathfrak{u}(s))| \Delta s$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \psi'(s) \left| (\psi(\tau_2) - \psi(0))^{1-\gamma} (\psi(\tau_2) - \psi(s))^{\alpha-1} - (\psi(\tau_2) - \psi(0))^{1-\gamma} (\psi(\tau_1) - \psi(s))^{\alpha-1} \right| |\mathfrak{g}(s, \mathfrak{u}(s))| \, ds$$

$$+ \frac{(\psi(\tau_2) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} (\psi(\tau_2) - \psi(\tau_1))^{\alpha+\gamma-1} B(\gamma, \alpha) \|\mathfrak{g}\|_{C_{1-\gamma,\psi}}.$$

Thus, right-hand part tends to zero. Hence along with the Arzëla-Ascoli theorem and from Step 1-3, it is concluded that $\mathscr{P}$ is completely continuous. Thus the proposed problem has at least one solution. $\qquad\square$

**Lemma 3.3.** *Assume that (H1) and (H3) are fulfilled. If*

$$\left( \frac{LB(\gamma, \alpha)}{\Gamma(\alpha)} (\psi(b) - \psi(0))^\alpha \right) < 1 \tag{3.2}$$

*then there exists unique solution for Eq. (1.1).*

*Proof.* Define the operator $\mathscr{P} : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$.

$$(\mathscr{P}\mathfrak{u})(\tau) = \mathfrak{u}_0(\tau) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s)) \Delta s$$

with $\mathfrak{u}_0(\tau) = \frac{\mathfrak{u}_0}{\Gamma(\gamma)} (\psi(\tau) - \psi(0))^{\gamma-1}$.
Let $\mathfrak{u}_1, \mathfrak{u}_2 \in C_{1-\gamma,\psi}(J)$ and $\tau \in J$, then

$$\left| (\psi(\tau) - \psi(0))^{1-\gamma} ((\mathscr{P}\mathfrak{u}_1)(\tau) - (\mathscr{P}\mathfrak{u}_2)(\tau)) \right|$$

$$\leq \frac{(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} |\mathfrak{g}(s, \mathfrak{u}_1(s)) - \mathfrak{g}(s, \mathfrak{u}_2(s))| \Delta s$$

$$\leq \frac{(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} |\mathfrak{g}(s, \mathfrak{u}_1(s)) - \mathfrak{g}(s, \mathfrak{u}_2(s))| \, ds$$

$$\leq \frac{L(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} |\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| \, ds$$

$$\leq \frac{LB(\gamma, \alpha)}{\Gamma(\alpha)} (\psi(b) - \psi(0))^\alpha \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{C_{1-\gamma,\psi}}.$$

Then,

$$\|\mathscr{P}\mathfrak{u}_1 - \mathscr{P}\mathfrak{u}_2\|_{C_{1-\gamma,\psi}} \leq \frac{LB(\gamma, \alpha)}{\Gamma(\alpha)} (\psi(b) - \psi(0))^\alpha \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{C_{1-\gamma,\psi}}.$$

From (3.2), it follows that $\mathscr{P}$ has a unique fixed point which is solution of problem (1.1). $\qquad\square$

## 4. Stability analysis

Next, we shall give the definitions and the criteria generalized UHR stability.

**Definition 4.1.** *Equation* (1.1) *is generalized UHR stable with respect to* $\varphi \in C_{1-\gamma}(J)$ *if there exists a real number* $c_{\mathfrak{g},\varphi} > 0$ *such that for each solution* $\mathfrak{v} \in C_{1-\gamma}(J)$ *of the inequality*

$$\left| {}^{\mathbb{T}}\Delta^{\alpha,\beta}\mathfrak{v}(\tau) - \mathfrak{g}(\tau, \mathfrak{v}(\tau)) \right| \leq \varphi(t), \tag{4.1}$$

*there exists a solution* $\mathfrak{u} \in C^{\gamma}_{1-\gamma}(J)$ *of equation* (1.1) *with*

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq c_{\mathfrak{g},\varphi}\varphi(\tau).$$

**Theorem 4.2.** *Assume that (H1), (H3), (H4) and* (3.2) *are satisfied. Then, the problem* (1.1) *is generalized UHR stable.*

*Proof.* Let $\mathfrak{v} \in C_{1-\gamma}(J)$ be solution of the following inequality (4.1) and let $\mathfrak{u} \in C_{1-\gamma}(J)$ be the unique solution of the $\psi$-Hilfer type dynamics equation (1.1). By Lemma 3.1,

$$\mathfrak{u}(\tau) = \mathfrak{u}_0(\tau) + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}\mathfrak{g}(s, \mathfrak{u}(s))\Delta s.$$

By integration of (4.1) we obtain

$$\left| \mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}\mathfrak{g}(s, \mathfrak{v}(s))\Delta s \right| \leq \lambda_\varphi \varphi(\tau).$$

On the other hand, we have

$$\begin{aligned}
|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| &\leq \left| \mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}\mathfrak{g}(s, \mathfrak{v}(s))\Delta s \right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}|\mathfrak{g}(s, \mathfrak{v}(s)) - \mathfrak{g}(s, \mathfrak{u}(s))|\Delta s \\
&\leq \left| \mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}\mathfrak{g}(s, \mathfrak{v}(s))\Delta s \right| \\
&\quad + \frac{L}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}|\mathfrak{v}(s) - \mathfrak{u}(s)|\,ds \\
&\leq \lambda_\varphi\varphi(\tau) + \frac{L}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}|\mathfrak{v}(s) - \mathfrak{u}(s)|\,ds.
\end{aligned}$$

By applying Lemma 2.10, we obtain

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq \left[ (1 + \nu_1 L\lambda_\varphi)\lambda_\varphi \right]\varphi(\tau),$$

where $\nu_1 = \nu_1(\alpha)$ is a constant, then for any $\tau \in J$:

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq c_{\mathfrak{g}}\varepsilon\varphi(\tau),$$

Thus, the proof is complete. $\square$

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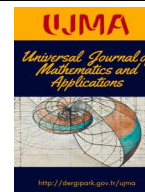
# Some New Cauchy Sequence Spaces

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### Article Info

### Abstract

In this paper, our goal is to introduce some new Cauchy sequence spaces. These spaces are defined by Cauchy transforms. We shall use notations $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ for these new sequence spaces. We prove that these new sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ are the $BK-$spaces and isomorphic to the spaces $l_\infty$, $c$ and $c_0$, respectively. Besides the bases of these spaces, $\alpha-$, $\beta-$ and $\gamma-$ duals of these spaces will be given. Finally, the matrix classes $(C(s,t):l_p)$ and $(C(s,t):c)$ have been characterized.

## 1. Preliminaries, background and notation

By $w$, we shall denote the space of all real or complex valued sequences. Any vector subspace of $w$ is called as a sequence space. We shall write $l_\infty$, $c$, $c_0$ and $l_p$ for the spaces of all bounded, convergent, null and absolutely $p-$summable sequences which are given by

$$l_\infty = \left\{ x = (x_k) \in w : \sup_{k \to \infty} |x_k| < \infty \right\},$$

$$c = \left\{ x = (x_k) \in w : \lim_{k \to \infty} x_k \text{ exists} \right\},$$

$$c_0 = \left\{ x = (x_k) \in w : \lim_{k \to \infty} x_k = 0 \right\}$$

and

$$l_p = \left\{ x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty, 1 \le p < \infty \right\}.$$

Also by $bs$, $cs$ and $l_1$, we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.

A sequence space $\lambda$ with a linear topology is called an $K-$ space provided of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the set of complex number and $\mathbb{N} = \{0, 1, 2, ...\}$. Let $\lambda$ be an $K-$ space. Then $\lambda$ is called an $FK-$ space provided $\lambda$ is a complete linear metric space. An $FK-$ space provided whose topology is normable is called a $BK-$ space [1].

Let $X$, $Y$ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers, where $n, k \in \mathbb{N}$. Then, we write $Ax = ((Ax)_n)$, the $A-$transform of $x$, if $A_n(x) = \sum_k a_{nk} x_k$ converges for each $n \in N$. If for every sequence $x = (x_k) \in X$, $A-$transform of $x$ sequence $Ax$ is in $Y$. Then we say that $A$ defines a matrix transformation from $X$ into $Y$ and denote it by $A : X \to Y$. By $(X : Y)$ we mean the class of all infinite matrices such that $A : X \to Y$.

Let $F$ denote the collection of all finite subsets on $\mathbb{N}$ and $K$, $\mathbb{N} \subset F$. The matrix domain $X_A$ of an infinite matrix $A$ in a sequence space $X$ is defined by

$$X_A = \{ x = (x_k) \in w : Ax \in X \} \tag{1.1}$$

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which is a sequence space.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been employed by many authors recently. They introduced the sequence spaces $(c_0)_{T^r} = t_0^r$ and $(c)_{T^r} = t_c^r$ in [2], $(c_0)_{E^r} = e_0^r$ and $(c)_{E^r} = e_c^r$ in [3], $(c_0)_C = \bar{c}_0$ and $c_C = \bar{c}$ in [3], $(l_p)_{E^r} = e_p^r$ in [4], $(l_\infty)_{R^t} = r_\infty^t$, $c_{R^t} = r_c^t$ and $(c_0)_{R^t} = r_0^t$ in [5], $(l_p)_C = X_p$ in [6] and $(l_p)_{N_q}$ in [7] where $T^r$, $E^r$, $C$, $R^t$ and $N_q$ denote the Taylor, Euler, Cesaro, Riesz and Nörlund means, respectively. In recent years, constructing a new sequence space by means of the domain of an infinite matrix was used by Candan [8, 9], Altay [10], Altay and Başar [11], Aydın and Başar [12], Başar [13, 14], Başar, Altay and Mursaleen [15], Polat and Başar [16].

Following [2]-[7], [17] by the same way, to introduce the new Cauchy sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ is the purpose of this paper.

## 2. The Cauchy matrix of inverse formula and Cauchy sequence spaces

Given two vectors $s$ and $t$ such that $s_i \neq -t_j$ for all $i$ and $j$, the $n \times n$ matrix $C = C(s,t)$ is a Cauchy (generalized Hilbert) matrix [18] where $C(s,t) = c_{ij} = [\frac{1}{s_i+t_j}]_{i,j=0}^{n-1}$. The inverse of Cauchy's Matrix [19] is given by

$$C^{-1}(s,t) = c_{ij}^{-1} = \frac{\prod\limits_{1 \leq k \leq n}(s_j+t_k)(s_k+t_i)}{(s_j+t_i)\left[\prod\limits_{\substack{1 \leq k \leq n \\ j \neq k}}(s_j-s_k)\right]\left[\prod\limits_{\substack{1 \leq k \leq n \\ i \neq k}}(t_i-t_k)\right]}. \tag{2.1}$$

$C(s,t)$ denotes the Cauchy mean defined by the matrix $C(s,t) = (c_{ij})$, $c_{ij} = [\frac{1}{s_i+t_j}]_{i,j=1}^n$ for each $n \in \mathbb{N}$.

We introduce the Cauchy sequence spaces,

$$C_\infty(s,t) = \left\{ x = (x_k) \in w : \sup_n \left| \sum_{k=1}^n \frac{1}{s_n+t_k} x_k \right| < \infty \right\},$$

$$C(s,t) = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{s_n+t_k} x_k \text{ exists} \right\}$$

and

$$C_0(s,t) = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{s_n+t_k} x_k = 0 \right\}.$$

By means of the notation (1.1), we may redefine the spaces $C_0(s,t)$ and $C(s,t)$ as follows:

$$C_0(s,t) = (c_0)_{C(s,t)} \text{ and } C(s,t) = (c)_{C(s,t)}. \tag{2.2}$$

If $\lambda$ is any arbitrary normed or paranormed sequence space, then we call the matrix domain $\lambda_{C(s,t)}$ as the Cauchy sequence space. We define the sequence $y = (y_k)$ which will be frequently used, as the $C(s,t)$ − transform of a sequence $x = (x_k)$ i.e.,

$$y_n = \sum_{k=1}^n \frac{1}{s_n+t_k} x_k. \tag{2.3}$$

It can be shown easily that $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ are linear and normed spaces by the following norm:

$$\|x\|_{C_0(s,t)} = \|C(s,t)\|_{C_\infty(s,t)} = \sup_n \left| \sum_{k=1}^n \frac{1}{s_n+t_k} x_k \right|. \tag{2.4}$$

**Theorem 2.1.** *The sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ are Banach spaces with the norm* (2.4).

*Proof.* Let $(x^p) = \left( x_0^{(p)}, x_1^{(p)}, x_2^{(p)}, ... \right)$ be a Cauchy sequence in $C_\infty(s,t)$ for all $p \in \mathbb{N}$. Then, there exists $n_0 = n_0(\varepsilon)$ for every $\varepsilon > 0$ such that $\|x^p - x^r\|_\infty < \varepsilon$ for all $p, r > n_0$. Hence, $|C(s,t)(x^p - x^r)| < \varepsilon$ for all $p, r > n_0$ and for each $k \in \mathbb{N}$.

Therefore, $(C(s,t)x_k^p) = ((C(s,t)x^0)_k, (C(s,t)x^1)_k, (C(s,t)x^2)_k, ...)$ is a Cauchy sequence in the set of complex numbers $\mathbb{C}$. Since $\mathbb{C}$ is complete, it is convergent we write $\lim\limits_{p \to \infty}(C(s,t)x^p)_k = (C(s,t)x)_k$ and $\lim\limits_{m \to \infty}(C(s,t)x^m)_k = (C(s,t)x)_k$ for each $k \in \mathbb{N}$. Hence, we have $\lim\limits_{m \to \infty}\left|C(s,t)x_k^p - x_k^m\right| = \left|C(s,t)(x_k^p - x_k) - C(s,t)(x_k^m - x_k)\right| \leq \varepsilon$ for all $n \geq n_0$. This implies that $\|x^p - x^m\| \to \infty$ for $p, m \to \infty$. Now, we should show that $x \in C_\infty(s,t)$. We have

$$\|x\|_\infty = \|C(s,t)x\|_\infty = \sup_n \left| \sum_{k=1}^n \frac{1}{s_n+t_k} x_k \right| = \sup_n \left| \sum_{k=1}^n \frac{1}{s_n+t_k}(x_k - x_k^p + x_k^p) \right|$$

$$\leq \sup_n \left|C(s,t)(x_k^p - x_k)\right| + \sup_n \left|C(s,t)x_k^p\right| \leq \|x^p - x\|_\infty + \left|C(s,t)x_k^p\right| < \infty$$

for $p, k \in \mathbb{N}$. This implies that $x = (x_k) \in C_\infty(s,t)$. Thus, $C_\infty(s,t)$ the space is a Banach space with the norm (2.4). $\qquad\square$

It can be shown that $C_0(s,t)$ and $C(s,t)$ are closed subspaces of $C_\infty(s,t)$ which leads us to the consequence that the spaces are also the Banach spaces with the norm (2.4). Furthermore, since $C_\infty(s,t)$ is a Banach space with continuous coordinates, i.e., $\left\| C(s,t)\left(x_k^p - x\right)\right\|_\infty \to 0$ imples $\left| C(s,t)\left(x_k^p - x_k\right)\right| \to 0$ for all $k \in N$, it is also a $BK-$ space.

**Theorem 2.2.** *The sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ are linearly isomorphic to the spaces $l_\infty$, $c$ and $c_0$ respectively, i.e., $C_\infty(s,t) \cong l_\infty$, $C(s,t) \cong c$ and $C_0(s,t) \cong c_0$.*

*Proof.* To prove the fact $C_0(s,t) \cong c_0$, we should show the existence of a linear bijection between the spaces $C_0(s,t)$ and $c_0$. Consider the transformation $F$ defined, with the notation (2.3), from $C_0(s,t)$ to $c_0$. The linearity of $F$ is clear. Further, it is trivial that $x = 0$ whenever $Fx = 0$ and hence $F$ is injective.

Let $y \in c_0$ and define the sequence $x = (x_k)$ by $x_k = \sum_{j=1}^{k} c_{kj}^{-1} y_j$ for each $k \in \mathbb{N}$. Wherein $c_{kj}^{-1}$ is as defined in (2.1). Then

$$\lim_{n\to\infty} (C(s,t)x)_n = \lim_{n\to\infty} \sum_{k=1}^{n} c_{nk}x_k = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{s_n + t_k} \sum_{j=1}^{k} c_{kj}^{-1} y_j = \lim_{n\to\infty} y_n = 0.$$

Thus, we have that $x \in C_0(s,t)$. In addition, note that

$$\|x\|_{C_0(s,t)} = \sup_{n\in\mathbb{N}} \left| \sum_{k=1}^{n} \frac{1}{s_n + t_k} \sum_{j=1}^{k} c_{kj}^{-1} y_j \right| = \sup_{n\in\mathbb{N}} |y_n| = \|y\|_{c_0} < \infty.$$

Consequently, $F$ is surjective and is norm preserving. Hence, $F$ is a linear bijection therefore we say that the spaces $C_0(s,t)$ to $c_0$ are linearly isomorphic. In the same way, it can be shown that $C(s,t)$ and $C_\infty(s,t)$ are linearly isomorphic to $c$ and $l_\infty$, respectively, and so we omit the detail. $\qquad\square$

**Theorem 2.3.** *The sequence space $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$ includes the sequence spaces $l_\infty$, $c$ and $c_0$ respectively i.e. $l_\infty \subset C_\infty(s,t)$, $c \subset C(s,t)$ and $c_0 \subset C_0(s,t)$.*

*Proof.* We only prove the conclusion $l_\infty \subset C_\infty(s,t)$ and the rest follows in a similar way. Let $x \in l_\infty$. Then, using (2.3) and (2.4), we obtain that

$$\|x\| = \|C(s,t)x\|_\infty = \sup_n \left| \sum_{k=1}^{n} \frac{1}{s_n + t_k} x_k \right|$$

$$\leq \sup_n |x_k| \sup_n |C(s,t)| = \|x\|_{C_\infty(s,t)}$$

it means that $x \in C_\infty(s,t)$.

$\qquad\square$

# 3. The basis for the spaces $C(s,t)$ and $C_0(s,t)$

Firstly, let us define the Schauder basis. A sequence $(b_n)_{n\in\mathbb{N}}$ in a normed sequence space $\lambda$ is called a Schauder basis (or briefly basis) [20], if for every $x \in \lambda$ there is a unique sequence $(\alpha_n)$ of scalars such that

$$\lim_{n\to\infty} \|x - (\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n)\| = 0.$$

In this section, we shall give the Schauder basis for the spaces $C(s,t)$ and $C_0(s,t)$.

**Theorem 3.1.** *Let $k \in \mathbb{N}$ be a fixed natural number and $b^{(k)} = \left\{ b_n^{(k)} \right\}_{n\in\mathbb{N}}$ where $b_n^{(k)} = \left[ c_{nk}^{-1} \right]_{k=1}^{n}$, $(n \in \mathbb{N})$. Wherein $c_{nk}^{-1}$ is as defined in (2.1). Then the following assertions are true:*

*i. The sequence $\left\{ b_n^{(k)} \right\}$ is a basis for the space $C_0(s,t)$ and every $x \in C_0(s,t)$ has a unique representation of the form $x = \sum_k \lambda_k b^{(k)}$ where $\lambda_k = (C(s,t)x)_k$ for all $k \in \mathbb{N}$. For simplicity, here and thereafter an unlimited sum symbol runs from zero to infinity.*

*ii. The set $\left\{ e, b^{(0)}, b^{(1)}, ..., b^{(k)}, ... \right\}$ is a basis for the space $C(s,t)$ and every $x \in C(s,t)$ has a unique representation of the form $x = le + \sum_k (\lambda_k - l) b^{(k)}$ where $l = \lim_{k\to\infty} (C(s,t)x)_k$ and $\lambda_k = (C(s,t)x)_k$ for all $k \in \mathbb{N}$.*

# 4. The $\alpha-$, $\beta-$ and $\gamma-$ Duals of the Spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$

In this section, we state and prove the theorems determining the $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence spaces $C_\infty(s,t)$, $C(s,t)$ and $C_0(s,t)$. For the sequence spaces $\lambda$ and $\mu$, we define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\}.$$

The $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence spaces $\lambda$, which are respectively denoted by $\lambda^\alpha$, $\lambda^\beta$ and $\lambda^\gamma$ are defined by Garling [21], by $\lambda^\alpha = S(\lambda, l_1)$, $\lambda^\beta = S(\lambda, cs)$ and $\lambda^\gamma = S(\lambda, bs)$. We shall begin with the lemmas due to Stieglitz and Tietz [22], which are needed in the proof of the Theorems 4.4-4.6.

**Lemma 4.1.** $A \in (c_0 : l_1) = (c : l_1)$ *if and only if, for* $(\alpha_k) \subset \mathbb{R}$

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty \tag{4.1}$$

**Lemma 4.2.** $A \in (c_0 : c)$ *if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty \tag{4.2}$$

*and*

$$\lim_{n \to \infty} a_{nk} = \alpha_k, \ (k \in \mathbb{N}).$$

**Lemma 4.3.** $A \in (c_0 : l_\infty)$ *if and only if* (4.2) *holds.*

In the following theorems, we denote by $K$ and $F$ finite subsets of $\mathbb{N}$.

**Theorem 4.4.** *Let* $a = (a_k) \in w$ *and define the matrix* $B = \left( c_{nk}^{-1} a_n \right)$ *for all* $k, n \in \mathbb{N}$. *The* $\alpha-$ *dual of the sequence spaces* $C_\infty(s,t)$, $C(s,t)$ *and* $C_0(s,t)$ *is the set* $D = \left\{ a = (a_k) \in w : \sup_{K \in F} \sum_n \left| \sum_{k \in K} c_{nk}^{-1} a_n \right| < \infty \right\}$. *Wherein* $c_{nk}^{-1}$ *is as defined in (2.1) for each* $k, n \in \mathbb{N}$.

*Proof.* Let $a = (a_n) \in w$ and consider the matrix $B$ whose rows are the products of the rows of the matrix $C^{-1}(s,t)$ and sequence $a = (a_n)$. Bearing in mind the relation (2.3), we immediately derive that

$$a_n x_n = \sum_{k=1}^{n} c_{nk}^{-1} a_n y_k = (By)_n, \ (n \in \mathbb{N}). \tag{4.3}$$

We therefore observe by (4.3) that $ax = (a_n x_n) \in l_1$ whenever $x \in C_\infty(s,t), C(s,t)$ and $C_0(s,t)$ if and only if $By \in l_1$ whenever $y \in l_\infty, c$, and $c_0$. Then, by means of Lemma 4.1, we get $\sup_{K \in F} \sum_n \left| \sum_{k \in K} c_{nk}^{-1} a_n \right| < \infty$ which yields the consequences that $\{C_\infty(s,t)\}^\alpha = \{C(s,t)\}^\alpha = \{C_0(s,t)\}^\alpha = D$. $\square$

**Theorem 4.5.** *Let us consider the sets* $B_1$, $B_2$, $B_3$ *and* $B_4$ *defined as follows:*

$$B_1 = \left\{ a = (a_k) \in w : \sup_n \sum_{k=1}^{n} \left| \sum_{j=k}^{n} c_{jk}^{-1} a_j \right| < \infty \right\},$$

$$B_2 = \left\{ a = (a_k) \in w : \sum_{j=k}^{\infty} c_{jk}^{-1} a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$B_3 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=1}^{n} \left| \sum_{j=k}^{n} c_{jk}^{-1} a_j \right| = \sum_{k=1}^{n} \left| \lim_{n \to \infty} \sum_{j=k}^{n} c_{jk}^{-1} a_j \right| \right\}$$

*and*

$$B_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=1}^{n} \sum_{j=k}^{n} c_{jk}^{-1} a_j \text{ exists} \right\}.$$

*Wherein* $c_{jk}^{-1}$ *is as defined in* (2.1) *for each* $j, k \in \mathbb{N}$. *Then* $\{C_0(s,t)\}^\beta = B_1 \cap B_2$, $\{C(s,t)\}^\beta = B_1 \cap B_2 \cap B_4$ *and* $\{C_\infty(s,t)\}^\beta = B_2 \cap B_3$.

*Proof.* We only give the proof for the space $C_0(s,t)$. Since the proof may give by a similar way for the spaces $C(s,t)$ and $C_\infty(s,t)$, we omit others. Consider the equation

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} \left[ \sum_{j=1}^{k} c_{kj}^{-1} y_j \right] a_k = \sum_{k=1}^{n} \left[ \sum_{j=k}^{n} c_{kj}^{-1} a_j \right] y_k = (By)_n,$$

where $B = (b_{nk})$ is defined by $b_{nk} = \sum_{j=k}^{n} c_{nj}^{-1} a_j$, $(n, k \in \mathbb{N})$. Thus, we deduce from Lemma 4.2 with (4.2) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k \in) C_0(s,t)$ if and only if $By \in c$ whenever $y = (y_k) \in c_0$. Therefore, we observe using relations (4.1) and (4.2), we conclude that $\lim_{n \to \infty} c_{nk}^{-1}$ exists for each $n, k \in \mathbb{N}$ and $\sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} c_{nk}^{-1} \right| < \infty$. Thus, we obtain $\{C_0(s,t)\}^\beta = B_1 \cap B_2$. $\square$

**Theorem 4.6.** *The* $\gamma-$ *dual of the sequence spaces* $C_\infty(s,t)$, $C(s,t)$ *and* $C_0(s,t)$ *is the set* $B_1$.

*Proof.* This theorem can be proved using the same technique as in the proof of Theorem 4.4 with Lemma 4.3 instead of Lemma 4.2. So, we omit the details. $\square$

# 5. Some matrix mappings related to Cauchy sequence spaces

**Lemma 5.1.** *[22, p. 57] The matrix mappings between $BK-$ spaces are continuous.*

**Lemma 5.2.** *[22, p. 128] $A \in (c : l_p)$ if and only if*

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} a_{nk} \right|^p < \infty, \ (1 \leq p < \infty) \tag{5.1}$$

**Theorem 5.3.** *$A \in (C(s,t) : l_p)$ if and only if the following conditions are satisfied*

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^n |g_{nk}| < \infty, \tag{5.2}$$

$$\lim_{n \to \infty} g_{nk} \ exists \ for \ all \ k \in \mathbb{N}, \tag{5.3}$$

$$\lim_{n \to \infty} \sum_{k=0}^n g_{nk} \ converges \ for \ all \ n \in \mathbb{N}, \tag{5.4}$$

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} g_{nk} \right|^p < \infty, (1 \leq p < \infty) \tag{5.5}$$

*and*

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^n |g_{nk}|^p < \infty, \ (p = \infty) \tag{5.6}$$

*where $g_{nk} = \sum_{j=k}^n c_{kj}^{-1} a_{nj}$ and $c_{kj}^{-1}$ is defined by (2.1).*

*Proof.* Let $1 \leq p < +\infty$. Assume that conditions (5.2)-(5.6) are satisfied and take any $x \in C(s,t)$. Then $(a_{nk}) \in (C(s,t))^\beta$ for all $k, n \in \mathbb{N}$, which implies that $Ax$ exists. We define the matrix $G = (g_{nk})$ for all $n, k \in \mathbb{N}$. Then, since condition (5.1) is satisfied for the matrix $G$, we have $G \in (c : l_p)$. Now consider the following equality obtained from the s th partial sum of the series $\sum_k a_{nk} x_k$:

$$\sum_{k=1}^s a_{nk} x_k = \sum_{k=1}^s \sum_{j=k}^s c_{jk}^{-1} a_{nj} y_k$$

$(s, n \in \mathbb{N})$. Therefore, we derive from that as $s \to \infty$ that

$$\sum_{k=1}^\infty a_{nk} x_k = \sum_{k=1}^\infty g_{nk} y_k \tag{5.7}$$

$(n \in \mathbb{N})$. Whence taking $l_p-$ norm we get

$$\|Ax\|_{l_p} = \|Gy\|_{l_p} < \infty.$$

This means that $A \in (C(s,t) : l_p)$. Now let $p = \infty$. Assume that conditions (5.2)-(5.6) are satisfied and take any $x \in C(s,t)$. Then $(a_{nk}) \in (C(s,t))^\beta$ for all $k, n \in \mathbb{N}$, which implies that $Ax$ exists. Whence taking $l_\infty-$ norm (5.7)

$$\|Ax\|_{l_\infty} = \sup_{n \in N} \left| \sum_k g_{nk} \right| \leq \|y\|_{l_\infty} \sup_{n \in N} \sum_k |g_{nk}| < \infty.$$

Then, we have $A \in (C(s,t) : l_\infty)$.
Conversely, assume that $A \in (C(s,t) : l_p)$. Then, since $C(s,t)$ and $l_p$ are $BK-$ spaces, it follows from Lemma 5.1 that there exists a real constant $K > 0$ such that

$$\|Ax\|_{l_p} = K \|x\|_{C(s,t)}$$

for all $x \in C(s,t)$. Since inequality **??** also holds for the sequence $x = (x_k) = \sum_{k \in F} b^{(k)} \in C(s,t)$ where $b_n^{(k)} = \left[ c_{nk}^{-1} \right]_{k=1}^n$, $(n \in \mathbb{N})$. Wherein $c_{nk}^{-1}$ is as defined in 2.1. We have $\|Ax\|_{l_p} = \left[ \sum_n |\sum_{k \in F} g_{nk}|^p \right]^{\frac{1}{p}} \leq K \|x\|_{C(s,t)} = K$ which shows the necessity of 5.5. $\square$

**Theorem 5.4.** *A ∈ (C(s,t) : c) if and only if conditions are satisfied*

$$g_{nk} \text{ exists for all } n, k \in \mathbb{N}, \tag{5.8}$$

$$\sup_n \sum_k |g_{nk}| < \infty \text{ for all } n, k \in \mathbb{N}, \tag{5.9}$$

$$\lim_n g_{nk} = \alpha_k \text{ for all } k \in \mathbb{N} \tag{5.10}$$

*and*

$$\lim_n \sum_k g_{nk} = \alpha \tag{5.11}$$

*where $g_{nk} = \sum_{j=k}^n c_{kj}^{-1} a_{nj}$ and $c_{kj}^{-1}$ is defined by* (2.1).

*Proof.* Assume that $A$ satisfies conditions (5.8)-(5.11). Let us take an arbitrary an $x = (x_k)$ in $C(s,t)$ such that $x_k \to l$ as $k \to \infty$. Then $Ax$ exists and it is trivial that the sequence $y = (y_k)$ associated with the sequence $x = (x_k)$ by relation (2.3) belongs to $c$ and is such that $y_k \to l$ as $k \to \infty$. At this stage, it follows from (5.4) and (5.6) that

$$\sum_{i=0}^k |\alpha_i| \le \sup_n \sum_i |g_{ni}| < \infty$$

for every $k \in \mathbb{N}$. This yield $\alpha_k \in l_1$. Considering $\sum_k a_{nk} x_k = \sum_k g_{nk} y_k$ we write

$$\sum_k a_{nk} x_k = \sum_k g_{nk}(y_k - l) + l \sum_k g_{nk} y_k \tag{5.12}$$

In this situation, letting $n \to \infty$ in (5.6), we establish that the first term on the right-hand side tends to $\sum_k \alpha_k(y_k - l)$ by (5.3) and (5.4) and the second term tends to $l\alpha$ by (5.12). Taking these facts into account, we deduce from (5.12) as $n \to \infty$ that $(Ax)_n = \sum_k \alpha_k(y_k - l) + l\alpha$ which shows that $A \in (C(s,t) : c)$.

Conversely, assume that $A \in (C(s,t) : c)$. Then, since the inclusion $c \subset l_\infty$ holds the necessity of (5.10), (5.12) is immediately obtained from $\sup_n \sum_k |b_{nk}| < \infty$. To prove the necessity of (5.11) consider the sequence $x = b^{(k)} = \left\{ b_n^{(k)} \right\}_{n \in \mathbb{N}}$ in $C(s,t)$ which defined above for every fixed $k \in \mathbb{N}$. Since $Ax$ exists and belongs to $c$ for every $x \in C(s,t)$, one can easily see that $Ab^{(k)} = \left\{ b_n^{(k)} \right\}_{n \in \mathbb{N}}$ for each $k \in \mathbb{N}$, which yields the necessity of (5.11). Similarly, by setting $x = e$ in (5.7), we obtain $Ax = \{\sum_k g_{nk}\}_{n \in \mathbb{N}}$, which belongs to the space $c$, and this shows the necessity of (5.12). Where $g_{nk} = \sum_{j=k}^n c_{kj}^{-1} a_{nj}$ and $c_{kj}^{-1}$ is defined by (2.1). This step conludes the proof. □

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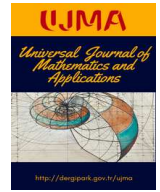
# Multiple Soliton Solutions of Some Nonlinear Partial Differential Equations

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## Article Info

## Abstract

In this paper, we implemented an improved tanh function method for multiple soliton solutions of new coupled Konno-Oono equation and extended (3+1)-dimensional KdV-type equation.

## 1. Introduction

Nonlinear partial differential equations (NPDEs) have an important place in applied mathematics and physics [1], [2]. Many analytical methods have been found in literature [3]-[11]. Besides these methods, there are many methods which reach to solution by using an auxiliary equation. Using these methods, partial differential equations are transformed into ordinary differential equations. These nonlinear partial differential equations are solved with the help of ordinary differential equations. These methods are given in [12]-[39].

We used the improved tanh function method to find the multiple soliton solutionsof new coupled Konno-Oono equation and extended (3+1)-dimensional KdV-type equationin this study. This method is presented by Chen and Zhang [15].

## 2. Analysis of method

Let's introduce the method briefly. Consider a general partial differential equation of two variables,

$$\varphi\left(v, v_t, v_x, v_{xx}, \ldots\right) = 0. \tag{2.1}$$

and transform equation (2.1) with

$$v(x,t) = v(\varnothing), \quad \varnothing = k(x - wt)$$

where $k$, $w$ are constants. With this conversion, we obtain a nonlinear ordinary differential equation for $v(\varnothing)$

$$\varphi'\left(v', v'', v''', \ldots\right) = 0. \tag{2.2}$$

We can express the solution of equation (2.2) as below,

$$v(\varnothing) = \sum_{i=0}^{n} a_i F^i(\varnothing),$$

here $n$ is a positive integer and is found as the result of balancing the highest order linear term and the highest order nonlinear term found in the equation.

If we write these solutions in equation (2.2), we obtain a system of algebraic equations for $F(\varnothing)$, $F^2(\varnothing),\dots,F^i(\varnothing)$, after, if the coefficients of $F(\varnothing)$, $F^2(\varnothing),\dots,F^i(\varnothing)$ are equal to zero, we can find the $k,w,a_0,a_1,\dots,a_n$ constants. The basic step of the method is to make full use of the Riccati equation satisfying the tanh function and to use $F(\varnothing)$, solutions. The Riccati equation required in this method is given below

$$F'(\varnothing) = A + BF(\varnothing) + CF^2(\varnothing)$$

where, $F'(\varnothing) = \frac{dF(\varnothing)}{d\varnothing}$ and $A$, $B$ and $C$ are constants. The authors expressed the solutions [15].

**Example 2.1.** *Example 1. Weconsider the new coupled Konno-Oono equation,*

$$v_t + 2uu_x = 0$$

$$u_{xt} - 2uv = 0. \tag{2.3}$$

*Using the wave variable $v(x,t) = v(\varnothing)$ and $u(x,t) = u(\varnothing)$, $\varnothing = k(x-wt)$, the equation (2.3) turns into an ordinary differential equation,*

$$-kwv' + 2kuu' = 0$$

$$-k^2wu'' - 2uv = 0. \tag{2.4}$$

*When balancing $v'$ with $uu'$ and $u''$ with $uv$ then $n_1 = 1$ and $n_2 = 2$ gives. The solution is as follows:*

$$u = a_0 + a_1 F(\varnothing)$$

$$v = b_0 + b_1 F(\varnothing) + b_2 F^2(\varnothing) \tag{2.5}$$

*(2.5) are substituted in equations (2.4), a system of algebraic equations for $k,w,a_0,a_1,b_0,b_1,b_2$ are obtained. The obtained systems of algebraic equations are as follows*

$$2Aka_0a_1 - Akwb_1 = 0,$$
$$2Bka_0a_1 + 2Aka_1^2 - Bkwb_1 - 2Akwb_2 = 0,$$
$$2Cka_0a_1 + 2Bka_1^2 - Ckwb_1 - 2Bkwb_2 = 0,$$
$$2Cka_1^2 - 2Ckwb_2 = 0 - ABk^2wa_1 - 2a_0b_0 = 0,$$
$$-B^2k^2wa_1 - 2ACk^2wa_1 - 2a_1b_0 - 2a_0b_1 = 0,$$
$$-3BCk^2wa_1 - 2a_1b_1 - 2a_0b_2 = 0,$$

$$-2C^2k^2wa_1 - 2a_1b_2 = 0.$$

*Solving the above system with the help of Mathematica, the coefficients are found as two cases:*
*Case 1:*

$$a_0 = 0, B = 0, b_1 = 0, A \neq 0, b_2 = \frac{Cb_0}{A}, k = \frac{ib_0}{Aa_1}, w = \frac{a_1^2}{b_2}, a_1 \neq 0, b_2 \neq 0.$$

*Case 2:*

$$a_0 = 0,\ B = 0,\ b_1 = 0,\ A = 0,\ b_2 \neq 0,\ k = \frac{ib_2}{Ca_1},\ w = \frac{a_1^2}{b_2},\ b_0 = 0,\ Ca_1 \neq 0.$$

*After these procedures, the solutions:*
*Solution 1:*

$$u_1(x,t) = a_1\left[Coth\left(\frac{2ib_0}{a_1}x + 2ia_1t\right) \pm Cosech\left(\frac{2ib_0}{a_1}x + 2ia_1t\right)\right]$$

$$v_1(x,t) = b_0 - b_0\left[Coth\left(\frac{2ib_0}{a_1}x + 2ia_1t\right) \pm Cosech\left(\frac{2ib_0}{a_1}x + 2ia_1t\right)\right]^2$$

$$u_2(x,t) = a_1\left[Tanh\left(\frac{2ib_0}{a_1}x + 2ia_1t\right) \pm iSech\left(\frac{2ib_0}{a_1}x + 2ia_1t\right)\right]$$

$$v_2(x,t) = b_0 - b_0 \left[ Tanh\left(\frac{2ib_0}{a_1}x + 2ia_1t\right) \pm iSech\left(\frac{2ib_0}{a_1}x + 2ia_1t\right) \right]^2$$

*Solution 2:*

$$u_3(x,t) = a_1 \left[ Sec\left(\frac{2ib_0}{a_1}x - 2ia_1t\right) \pm Tan\left(\frac{2ib_0}{a_1}x - 2ia_1t\right) \right]$$

$$v_3(x,t) = b_0 + b_0 \left[ Sec\left(\frac{2ib_0}{a_1}x - 2ia_1t\right) \pm Tan\left(\frac{2ib_0}{a_1}x - 2ia_1t\right) \right]^2$$

$$u_4(x,t) = a_1 \left[ Cosec\left(\frac{2ib_0}{a_1}x - 2ia_1t\right) \pm Cot\left(\frac{2ib_0}{a_1}x - 2ia_1t\right) \right]$$

$$v_4(x,t) = b_0 + b_0 \left[ Cosec\left(\frac{2ib_0}{a_1}x - 2ia_1t\right) \pm Cot\left(\frac{2ib_0}{a_1}x - 2ia_1t\right) \right]^2$$

$$u_5(x,t) = a_1 \left[ Sec\left(-\frac{2ib_0}{a_1}x + 2ia_1t\right) \pm Tan\left(-\frac{2ib_0}{a_1}x + 2ia_1t\right) \right]$$

$$v_5(x,t) = b_0 + b_0 \left[ Sec\left(-\frac{2ib_0}{a_1}x + 2ia_1t\right) \pm Tan\left(-\frac{2ib_0}{a_1}x + 2ia_1t\right) \right]^2$$

$$u_6(x,t) = a_1 \left[ Cosec\left(-\frac{2ib_0}{a_1}x + 2ia_1t\right) \pm Cot\left(-\frac{2ib_0}{a_1}x + 2ia_1t\right) \right]$$

$$v_6(x,t) = b_0 + b_0 \left[ Cosec\left(-\frac{2ib_0}{a_1}x + 2ia_1t\right) \pm Cot\left(-\frac{2ib_0}{a_1}x + 2ia_1t\right) \right]^2$$

*Solution 3:*

$$u_7(x,t) = a_1 \left[ Tanh\left(\frac{ib_0}{a_1}x + ia_1t\right) \right]$$

$$v_7(x,t) = b_0 - b_0 \left[ Tanh\left(\frac{ib_0}{a_1}x + ia_1t\right) \right]^2$$

$$u_8(x,t) = a_1 \left[ Coth\left(\frac{ib_0}{a_1}x + ia_1t\right) \right]$$

$$v_8(x,t) = b_0 - b_0 \left[ Coth\left(\frac{ib_0}{a_1}x + ia_1t\right) \right]^2$$

*Solution 4:*

$$u_9(x,t) = a_1 \left[ Tan\left(\frac{ib_0}{a_1}x - ia_1t\right) \right]$$

$$v_9(x,t) = b_0 + b_0 \left[ Tan\left(\frac{ib_0}{a_1}x - ia_1t\right) \right]^2$$

*Solution 5:*

$$u_{10}(x,t) = a_1 \left[ Cot\left(-\frac{ib_0}{a_1}x + ia_1t\right) \right]$$

$$v_{10}(x,t) = b_0 + b_0 \left[ Cot \left( -\frac{ib_0}{a_1}x + ia_1 t \right) \right]^2$$

**Solution 6:**

$$u_{11}(x,t) = -\frac{a_1}{\left( \frac{ib_2}{a_1}x - ia_1 t \right) + c_0}$$

$$v_{11}(x,t) = b_2 \left( -\frac{1}{\left( \frac{ib_2}{a_1}x - ia_1 t \right) + c_0} \right)^2$$

**Example 2.2.** *Now let's get the extended (3+1)-dimensional KdV-type equation,*

$$u_t + 6u_x u_y + u_{xxy} + u_{xxxxz} + 60u_x^2 u_z + 10u_{xxx}u_z + 20u_x u_{xxz} + 6u_x u_z + u_{xxz} = 0, \tag{2.6}$$

*Using the wave variable $u(x,y,z,t) = u(\varnothing)$ and, $\varnothing = k(x + \alpha y + \beta z - wt)$, the equation (2.6) turns into an ordinary differential equation,*

$$-wu' + 6k\alpha(u')^2 + k^2\alpha u''' + k^4\beta u^{(5)} + 60k^2\beta(u')^3 + 30k^3\beta u'u''' + 6k\beta(u')^2 + k^2\beta u''' = 0, \tag{2.7}$$

*When balancing $u^{(5)}$ with $u'u'''$ then $n = 1$ gives. The solution is as follows:*

$$u = a_0 + a_1 F(\varnothing) \tag{2.8}$$

*If (2.8) is substituted in equation (2.7), a system of algebraic equations for $k,w,\alpha,\beta,a_0,a_1$ can be obtained. The obtained systems of algebraic equations are as follows*

$$-Awa_1 + AB^2k^2\alpha a_1 + 2A^2Ck^2\alpha a_1 + AB^2k^2\beta a_1 + 2A^2Ck^2\beta a_1 + AB^4k^4\beta a_1 + 22A^2B^2Ck^4\beta a_1$$
$$+16A^3C^2k^4\beta a_1 + 6A^2k\alpha a_1^2 + 6A^2k\beta a_1^2 + 30A^2B^2k^3\beta a_1^2 + 60A^3Ck^3\beta a_1^2 + 60A^3k^2\beta a_1^3 = 0,$$

$$-Bwa_1 + B^3k^2\alpha a_1 + 8ABCk^2\alpha a_1 + B^3k^2\beta a_1 + 8ABCk^2\beta a_1 + B^5k^4\beta a_1 + 52AB^3Ck^4\beta a_1 + 136A^2BC^2k^4\beta a_1$$
$$+12ABk\alpha a_1^2 + 12ABk\beta a_1^2 + 60AB^3k^3\beta a_1^2 + 300A^2BCk^3\beta a_1^2 + 180A^2Bk^2\beta a_1^3 = 0,$$

$$-Cwa_1 + 7B^2Ck^2\alpha a_1 + 8AC^2k^2\alpha a_1 + 7B^2Ck^2\beta a_1 + 8AC^2k^2\beta a_1 + 31B^4Ck^4\beta a_1 +$$
$$292AB^2C^2k^4\beta a_1 + 136A^2C^3k^4\beta a_1 + 6B^2k\alpha a_1^2 + 12ACk\alpha a_1^2 + 6B^2k\beta a_1^2 + 12ACk\beta a_1^2 +$$
$$30B^4k^3\beta a_1^2 + 480AB^2Ck^3\beta a_1^2 + 300A^2C^2k^3\beta a_1^2 + 180AB^2k^2\beta a_1^3 + 180A^2Ck^2\beta a_1^3 = 0,$$

*If the system is solved, the coefficients are found as*

$$B = 0, \quad a_1 = \frac{1}{2}\sqrt{\frac{C}{A}}, \quad a_1 \neq 0, \quad A \neq 0, \quad k = -\frac{1}{4Aa_1}, \quad \alpha = -w, \quad \beta \neq 0.$$

*with the help of the Mathematica program. After these operations, the solutions of equation (2.6) as follow:*
**Solution 1:**

$$u_1(x,t) = \frac{i}{2}[Coth(ix - iwy + i\beta z - iwt) \pm Cosech(ix - iwy + i\beta z - iwt)]$$

$$u_2(x,t) = \frac{i}{2}[Tanh(ix - iwy + i\beta z - iwt) \pm iSech(ix - iwy + i\beta z - iwt)]$$

**Solution 2:**

$$u_3(x,t) = \frac{1}{2}[Sec(-x + wy - \beta z + wt) + Tan(-x + wy - \beta z + wt)] \tag{2.9}$$

$$u_4(x,t) = \frac{1}{2}[Cosec(-x + wy - \beta z + wt) - Cot(-x + wy - \beta z + wt)]$$

$$u_5(x,t) = \frac{1}{2}[Sec(x - wy + \beta z - wt) - Tan(x - wy + \beta z - wt)] \tag{2.10}$$

$$u_6(x,t) = \frac{1}{2}\left[Cosec\,(x - wy + \beta z - wt) + Cot\,(x - wy + \beta z - wt)\right] \tag{2.11}$$

*Solution 3:*

$$u_7(x,t) = \frac{i}{2}\left[Tanh\left(\frac{i}{2}x - \frac{i}{2}wy + \frac{i}{2}\beta z - \frac{i}{2}wt\right)\right] \tag{2.12}$$

$$u_8(x,t) = \frac{i}{2}\left[Coth\left(\frac{i}{2}x - \frac{i}{2}wy + \frac{i}{2}\beta z - \frac{i}{2}wt\right)\right] \tag{2.13}$$

*Solution 4:*

$$u_9(x,t) = \frac{1}{2}\left[Tan\left(-\frac{1}{2}x + \frac{1}{2}wy - \frac{1}{2}\beta z + \frac{1}{2}wt\right)\right] \tag{2.14}$$

*Solution 5:*

$$u_{10}(x,t) = \frac{1}{2}\left[Cot\left(\frac{1}{2}x - \frac{1}{2}wy + \frac{1}{2}\beta z - \frac{1}{2}wt\right)\right] \tag{2.15}$$

## 3. Explanations and graphical presentments of some of the solutions obtained

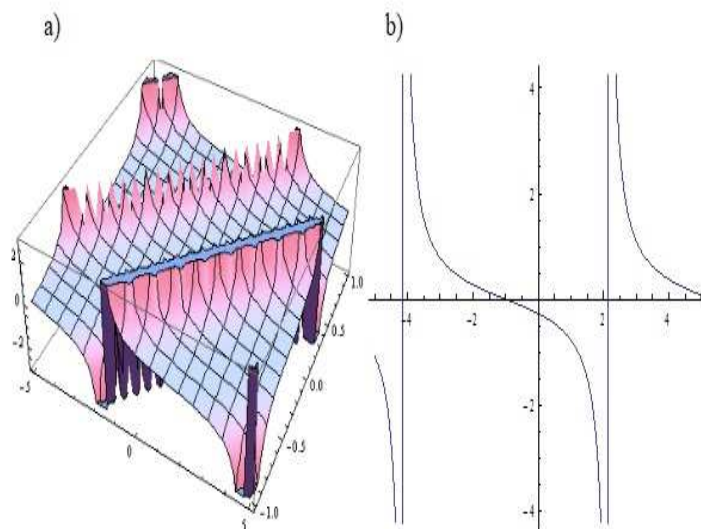The graphical demonstrations of some obtained solutions are shown in Figures 1-3.



**Figure 3.1:** a)The 3D surfaces of Eq.(2.9)for the values $y = 1$, $z = 0$ and $w = 5$ within the interval$-5 \le x \le 5$, $-1 \le t \le 1$. b) The 2D surfaces of Eq.(2.9)for thevalues $y = 1$, $z = 0$, $w = 5$ and $t = 1$ within the interval$-5 \le x \le 5$.
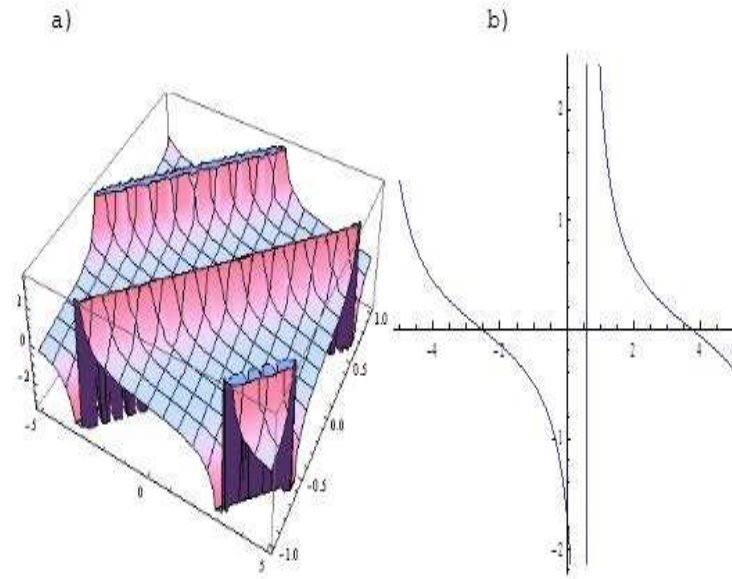
**Figure 3.2:** a) The 3D surfaces of Eq.(2.14)for the values $y = 1$, $z = 0$ and $w = 5$ within the interval $-5 \le x \le 5$, $-1 \le t \le 1$. b) The 2D surfaces of Eq.(2.14)for the values $y = 1$, $z = 0$, $w = 5$ and $t = 1$ within the interval$-5 \le x \le 5$.
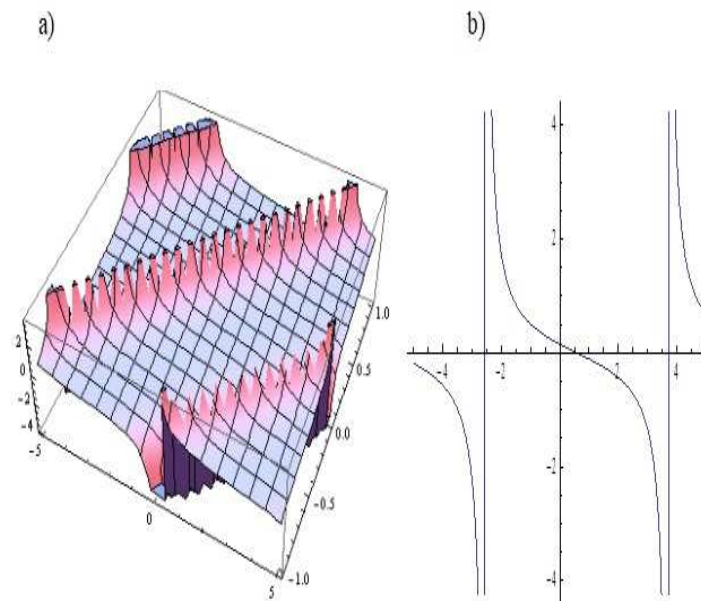


**Figure 3.3:** a)The 3D surfaces of Eq.(2.15)for the values and within the interval b) The 2D surfaces of Eq.(2.15) for the values and within the interval

## 4. Conclusion

We used the improved tanh function method to find the multiple soliton solutions of new coupled Konno-Oono equation and extended (3+1)-dimensional KdV-type equation. This method has been successfully applied to solve some nonlinear wave equations and can be used to many other nonlinear equations or coupled ones.

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