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# Rational Solutions to the Boussinesq Equation

**Pierre Gaillard**

*Université de Bourgogne, Institut de mathématiques de Bourgogne 9 avenue Alain Savary, Dijon Cedex, France*

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## Abstract

Rational solutions to the Boussinesq equation are constructed as a quotient of two polynomials in $x$ and $t$. For each positive integer $N$, the numerator is a polynomial of degree $N(N+1)-2$ in $x$ and $t$, while the denominator is a polynomial of degree $N(N+1)$ in $x$ and $t$. So we obtain a hierarchy of rational solutions depending on an integer $N$ called the order of the solution. We construct explicit expressions of these rational solutions for $N=1$ to 4.

## 1. Introduction

We consider the Boussinesq equation (B) which can be written in the form

$$u_{tt}-u_{xx}+(u^2)_{xx}+\frac{1}{3}u_{xxxx}=0, \tag{1.1}$$

where the subscripts $x$ and $t$ denote partial derivatives.
This equation first appears first in 1871, in a paper written by Boussinesq [1, 2]. It is well known that the Boussinesq equation (1.1) is an equation solvable by inverse scattering [3, 4]. It gives the description of the propagation of long waves surfaces in shallow water. It appears in several physical applications as one-dimensional nonlinear lattice-waves [5], vibrations in a nonlinear string [6] and ion sound waves in plasma [7].
The first solutions were founded in 1977 by Hirota [8] by using Bäcklund transformations. Among the various works concerning this equation, we can mention the following studies. Ablowitz and Satsuma constructed non-singular rational solutions in 1978 by using the Hirota bilinear method [9]. Freemann and Nimmo expressed solutions in terms of wronskians in 1983 [10]. An algebra-geometrical method using trigonal curve was given by Matveev et al. in 1987 [11]. The same author constructed other types of solutions using Darboux transformation [12]. Bogdanov and Zakharov in 2002 constructed solutions by the $\overline{\partial}$ dressing method [13]. In $2008-2010$, Clarkson obtained solutions in terms of the generalized Okamoto, generalized Hermite or Yablonski Vorob'ev polynomials [14, 15].
Recently, in 2017, Clarkson et al. constructed new solutions as second derivatives of polynomials of degree $n(n+1)$ in $x$ and $t$ in [16].

In this paper, we study rational solutions of the Boussinesq equation. We present rational solutions as a quotient of two polynomials in $x$ and $t$. These following solutions belong to an infinite hierarchy of rational solutions written in terms of polynomials for each positive integer $N$. The study here is limited to the simplest cases where $N=1,2,3,4$.

## 2. First order rational solutions

We consider the Boussinesq equation

$$u_{tt}-u_{xx}+(u^2)_{xx}+\frac{1}{3}u_{xxxx}=0,$$

We have the following result at order $N=1$ :

***Email address and ORCID number***: **Pierre.Gaillard@u-bourgogne.fr, 0000-0002-7073-8284, (P. Gaillard)**

**Theorem 2.1.** *The function $v$ defined by*

$$v(x,t)=\frac{-2}{(-x+t+a_1)^2},$$

*is a solution to the Boussinesq equation (1.1) with $a_1$ an arbitrarily real parameter.*

**Proof** It is straightforward.
□

The parameter $a_1$ is only a translation parameter; it is not crucial. In the following solutions, we will omit it.

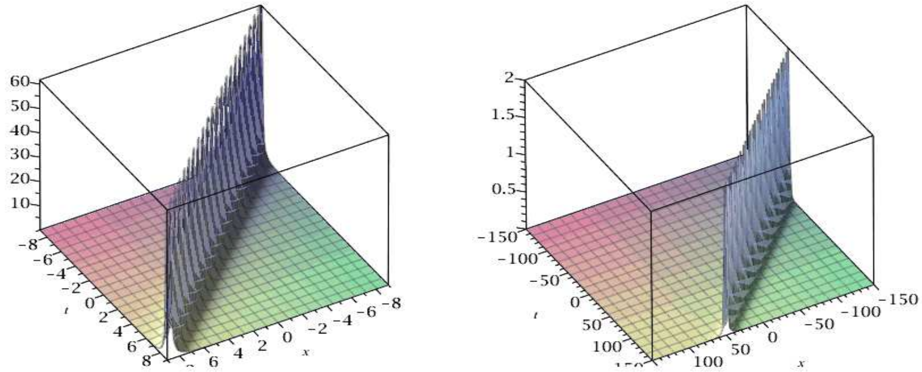


**Figure 1.** Solution of order 1 to (1.1), on the left $a_1=0$; on the right $a_1=100$.

In Figures 1., the singularity lines of respective equations $t=x$ and $t=x+a_1$ are clearly shown.

## 3. Second order rational solutions

The Boussinesq equation defined by (1.1) is always considered. We obtain the following solutions :

**Theorem 3.1.** *The function $v$ defined by*

$$v(x,t)=-2\,\frac{n(x,t)}{d(x,t)^{(2)}},\tag{3.1}$$

*with*

$$n(x,t)=3x^4+(-12t-4)x^3+(18t^2+2+12t)x^2+(-12t^2+8t-12t^3)x-4t+4t^3-10t^2+3t^4$$

*and*

$$d(x,t)=-x^3+(3t+1)x^2+(-3t^2-2t)x+t^3+t^2+2t$$

*is a rational solution to the Boussinesq equation (1.1), a quotient of two polynomials with the numerator of order 4 in $x$ and $t$, the denominator of degree 6 in $x$ and $t$.*

**Proof** It is sufficient to replace the expression of the solution given by (3.1) and check that (1.1) is verified.
□

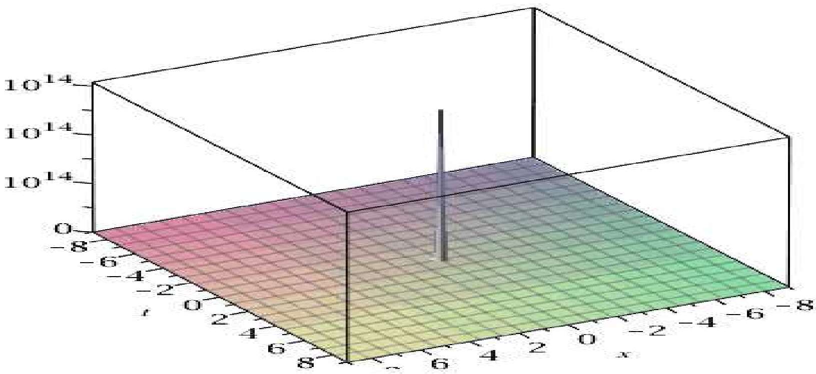


**Figure 2.** Solution of order 2 to (1.1).

This Figure 2. shows clearly the singularity in $(0;0)$.
The previous solution (3.1) can be rewritten as

$$-2\,\frac{3\,(t-x)^4+4\,(t-x)^3-4\,(t-x)^2-6t^2+6x^2-4t}{((t-x)^3+(t-x)^2+2t)^2}.$$

So, with this expression, it is obvious to show that $(0;0)$ is a singularity as it can be seen in figure (2).

## 4. Rational solutions of order three

We obtain the following rational solutions to the Boussinesq equation defined by (1.1) :

**Theorem 4.1.** *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^{(2)}},$$

*with*

$n(x,t) = 6x^{10} + (-40 - 60t)x^9 + (270t^2 + 110 + 360t)x^8 + (-1440t^2 - 720t^3 - 160 - 880t)x^7 + (1260t^4 + 100 + 3080t^2 + 1120t + 3360t^3)x^6 + (-740t - 1512t^5 - 5040t^4 - 3360t^2 - 6160t^3)x^5 + (200t + 5040t^5 + 3100t^2 + 1260t^6 + 5600t^3 + 7700t^4)x^4 + (-6160t^5 - 720t^7 - 3360t^6 - 7000t^3 - 3200t^2 - 5600t^4)x^3 + (2000t^2 + 1440t^7 + 3080t^6 + 270t^8 + 8300t^4 + 8400t^3 + 3360t^5)x^2 + (-880t^7 - 5200t^3 - 8000t^4 - 60t^9 - 360t^8 - 4900t^5 - 1120t^6)x + 3200t^4 + 2600t^5 + 800t^3 + 160t^7 + 6t^{10} + 40t^9 + 110t^8 + 1140t^6$

*and*

$d(x,t) = x^6 + (-6t - 4)x^5 + (15t^2 + 20t + 5)x^4 + (-20t^3 - 40t^2 - 30t)x^3 + (15t^4 + 40t^3 + 60t^2 + 20t)x^2 + (-6t^5 - 20t^4 - 50t^3 - 40t^2)x + t^6 + 4t^5 + 15t^4 + 20t^3 - 20t^2$

*is a rational solution to the Boussinesq equation (1.1), quotient of two polynomials with numerator of order* 10 *in x and t, denominator of degree* 12 *in x and t.*

**Proof** Replacing the expression of the solution given by (3.1), we check that the relation (1.1) is verified.
□

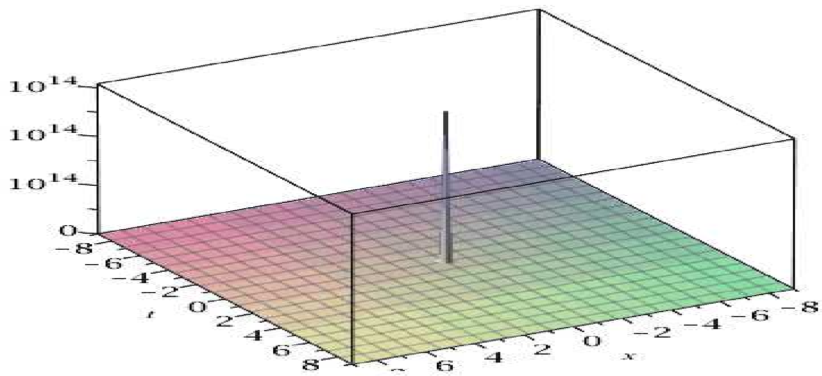


**Figure 3.** Solution of order 3 to (1.1).

The figure 3 clearly shows the singularity in $(0;0)$.

## 5. Rational solutions of fourth order

The following solutions of order 4 to the Boussinesq equation defined by (1.1) are obtained :

**Theorem 5.1.** *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^{(2)}}, \tag{5.1}$$

*with*

$n(x,t) = 10x^{18} + (-180t - 180)x^{17} + (1460 + 3060t + 1530t^2)x^{16} + (-23600t - 8160t^3 - 6960 - 24480t^2)x^{15} + (30600t^4 + 21200 + 108000t + 122400t^3 + 178800t^2)x^{14} + (-781200t^2 - 842800t^3 - 428400t^4 - 321300t - 41300 - 85680t^5)x^{13} + (1113840t^5 + 2254000t^2 + 48300 + 2766400t^4 + 632800t + 3494400t^3 + 185640t^6)x^{12} + (-9703400t^3 - 4447800t^2 - 10810800t^4 - 318240t^7 - 805000t - 2227680t^6 - 29400 - 6704880t^5)x^{11} + (18972800t^3 + 28644000t^4 + 3500640t^7 + 24504480t^5 + 630000t + 12412400t^6 + 6013000t^2 + 437580t^8 + 7350)x^{10} + (-4375800t^8 - 17903600t^7 - 5467000t^2 - 26383000t^3 - 42042000t^6 - 54785500t^4 - 61345900t^5 - 294000t - 486200t^9)x^9 + (98313600t^6 + 20334600t^8 + 113097600t^5 + 24822000t^3 + 4375800t^9 + 55598400t^7 + 3228750t^2 + 73500t + 75778500t^4 + 437580t^{10})x^8 + (-318240t^{11} - 3500640t^{10} - 18246800t^9 - 57142800t^8 - 1176000t^2 - 150603600t^5 - 12544000t^3 - 67662000t^4 - 119790000t^7 - 171771600t^6)x^7 + (-882000t^3 + 45645600t^9 + 2227680t^{11} + 185640t^{12} + 119128800t^5 + 213150000t^6 + 111526800t^8 + 12892880t^{10} + 294000t^2 + 194409600t^7 + 19379500t^4)x^6 + (-78963500t^9 - 217182000t^7 - 85680t^{13} + 3920000t^3 - 1113840t^{12} - 7098000t^{11} - 140238000t^6 - 28108080t^{10} + 32928000t^4 - 164033100t^8 + 1528800t^5)x^5 + (13104000t^{11} + 41857200t^{10} + 158560500t^8 - 39690000t^4 - 980000t^3 + 30600t^{14} + 111132000t^7 + 101948000t^9 + 428400t^{13} + 2984800t^{12} - 115395000t^5 - 49808500t^6)x^4 + (-58107000t^8 - 45383800t^{10} + 19600000t^4 + 78400000t^7 - 122400t^{14} - 4477200t^{12} + 186984000t^6 - 16109800t^{11} + 113680000t^5 - 926800t^{13} - 81081000t^9 - 8160t^{15})x^3 + (-146510000t^6 - 52920000t^5 + 13708800t^{11} + 1530t^{16} + 1058400t^{13} - 59057250t^8 + 4256000t^{12} + 27617800t^{10} + 18942000t^9 + 200400t^{14} - 4900000t^4 + 24480t^{15} - 161994000t^7)x^2 + (89376000t^7 + 7840000t^5 - 690900t^{13} - 3389400t^{10} - 154800t^{14} - 180t^{17} + 50960000t^6 - 2519300t^{12} + 72912000t^8 - 26960t^{15} + 22778000t^9 - 3060t^{16} - 5635000t^{11})x - 16660000t^7 - 980000t^6 - 21070000t^8 - 13450500t^9 + 10t^{18} - 1960000t^5 + 180t^{17} + 1700t^{16} + 10560t^{15} + 52000t^{14} + 212800t^{13} + 521500t^{12} + 238000t^{11} - 3618650t^{10}$

*and*

$d(x,t) = x^{10} + (-10t - 10)x^9 + (45t^2 + 90t + 40)x^8 + (-120t^3 - 360t^2 - 350t - 70)x^7 + (210t^4 + 840t^3 + 1330t^2 + 700t + 35)x^6 +$

$(-252t^5-1260t^4-2870t^3-2730t^2-700t)x^5+(210t^6+1260t^5+3850t^4+5600t^3+2975t^2+350t)x^4+(-120t^7-840t^6-3290t^5-6650t^4-5600t^3-1400t^2)x^3+(45t^8+360t^7+1750t^6+4620t^5+5425t^4+2100t^3+700t^2)x^2+(-10t^9-90t^8-530t^7-1750t^6-2660t^5-1400t^4-2800t^3)x+t^{10}+10t^9+70t^8+280t^7+525t^6+350t^5+2100t^4+1400t^3$
*is a rational solution to the Boussinesq equation (1.1), quotient of two polynomials with numerator of order* 18 *in x and t, denominator of degree* 20 *in x and t.*

**Proof** We have to check that the relation (1.1) is verified when we replace the expression of the solution given by (5.1).
□

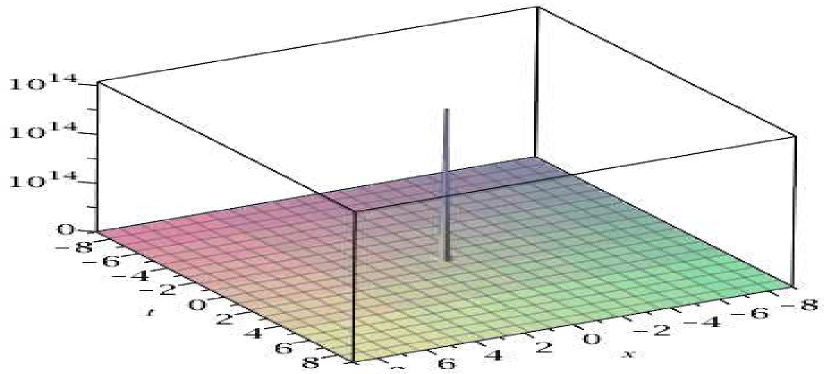


**Figure 4.** Solution of order 4 to (1.1).

As in the preceding cases, the figure 4 clearly shows the singularity in $(0;0)$.

## 6. Conclusion

Rational solutions to the Boussinesq equation of order 1, 2, 3, 4 have been constructed here. The following asymptotic behavior has been highlighted : $\lim_{t\to\infty} v(x,t) = 0$, $\lim_{x\to\pm\infty} v(x,t) = 0$.
It will relevant to construct rational solutions to the Boussinesq equation at order $N$ and to give a representation of these solutions in terms of determinants. Namely, for every integer $N$, these solutions can be written as a quotient of determinants of order $N$, where the numerator is a polynomial of degree $N(N+1)-2$ in $x$, $t$, and the denominator is a polynomial of degree $N(N+1)$ in $x$, $t$.

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# Extended Semi-Local Convergence of Newton's Method using the Center Lipschitz Condition and the Restricted Convergence Domain

**Ioannis K. Argyros**[1] **and Santhosh George**[2*]

[1]*Department of Mathematical Sciences, Cameron University Lawton, OK 73505, USA*
[2]*Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-757 025*
[*]*Corresponding author E-mail: sgeorge@nitk.ac.in*

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**Abstract**

The objective of this study is to extend the usage of Newton's method for Banach space valued operators. We use our new idea of restricted convergence domain in combination with the center Lipschitz hypothesis on the Fréchet-derivatives where the center is not necessarily the initial point. This way our semi-local convergence analysis is tighter than in earlier works (since the new majorizing function is at least as tight as the ones used before) leading to weaker criteria, better error bounds more precise information on the solution. These improvements are obtained under the same computational effort.

## 1. Introduction

Let $X, Y$ denote Banach spaces and $\Omega \subseteq X$ be a convex set. Numerous problems in diverse areas are written as an equation like

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \longrightarrow Y$ is a twice continuously Fréchet-differentiable operator. One wishes that a solution $x_*$ of equation (1.1) can be found in closed form [1]-[10]. However, this is done only in special cases. This is why most researchers use iterative procedures to generate a sequence $\{x_n\}$ approximating $x_*$. The most popular iterative procedure is undoubtedly Newton's method defined for some given initial point $x_0 \in \Omega$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n),$$

for each $n = 0, 1, 2, \ldots$. There is literature on convergence results for Newton's method, see [3, 8, 9, 10] and the references therein. The convergence domain of Newton's method is small in general under generalized-type Lipschitz conditions. This fact limits the applicability of Newton's method. Therefore, techniques that will enlarge the convergence domain without additional hypotheses are useful. In particular, we are motivated by the work of Ezquerro and Hernandez in [5, 6], where the center-Lipschitz on the second Fréchet-derivative was used but the center is not necessarily the starting point for Newton's method. This idea has also been used but on the first Fréchet-derivative. Using this technique in connection to majorizing functions and sequences a semi-local convergence analysis was given in [6] for the special case, when $X = Y = \mathbb{R}^m$, where $m$ is a positive integer. The choice of a point other than $x_0$ in the center-Lipschitz condition allows more flexibility in the choice of majorizing functions and sequences. Moreover, the convergence domain may be extended in some cases as it was shown in [6] for a certain class of nonlinear integral equations.

In the present study we also use the center Lipschitz condition at $x_0$ as well as at a point other than $x_0$. This way we locate a smaller domain $\Omega$ where the iterates $\{x_n\}$ are located. Then, the majorizing function related to the smaller domain $U_0$ is always at least as small as the majorizing function in [5, 6] derived using the set $\Omega$. We then provide a semi-local convergence analysis along the lines of the work in [5, 6] but with the center-Lipschitz condition on the first derivative instead of the second leading to tighter error estimates on $\|F'(x_n)^{-1}\|$. This

***Email address and ORCID number***: **ioannisa@cameron.edu, https://orcid.org/0000-0002-9189-9298 (I. K. Argyros),sgeorge@nitk.ac.in, https://orcid.org/0000-0002-3530-5539 (S. George)**

modification together with the usage of the new majorant function instead of the old one leads to an at least tighter semi-local convergence. Some of the advantages include weaker sufficient semi-local convergence criteria (i.e., larger convergence domain than before, tighter error estimates on the distances $\|x_{n+1}-x_n\|$, $\|x_n-x_*\|$ and more precise information on the location of the solution $x_*$. The interesting part of this new technique is the fact that no additional conditions are utilized since the computation of the old majorant function requires the computation of the new majorant function as a special case. Our idea can be extended in the case $F^{(i)}$ is center-Lipschitz continuous where $i \geq 2$ [2]-[4].
The lay out of the rest of the paper contains: The semi-local convergence of Newton's method in Section 2. Section 3 has the examples on which the theoretical results are tested.

## 2. Semi-local convergence

Let $\gamma \geq 0$. Define $R = \sup\{t \geq \gamma : U(x_0,t) \subseteq \Omega\}$. Throughout this paper $U(x_0,r), \bar{U}(x_0,r)$, stand respectively for the open and closed balls in $X$ with center at $x_0$ and radius $r$. We base the semi-local convergence analysis of Newton's method on the conditions $(\mathscr{A})$:

$(\mathscr{A}_0)$ Operator $F : \Omega \subseteq X \longrightarrow Y$ is twice Fréchet differentiable in the Fréchet sense.
$(\mathscr{A}_1)$ Let $x_0 \in \Omega$. There exist $z \in D$, and $\delta \geq 0$ such that $\|x_0 - z\| = \delta$. Set $t_0 = \gamma + \delta$.
$(\mathscr{A}_2)$ There exist operator $\Gamma_0 = F'(x_0)^{-1} \in L(Y,X)$, $b_1 > 0$ such that $\|\Gamma_0\| \leq b_1$ and a function $g_1 : [\gamma,+\infty) \longrightarrow [0,+\infty)$ continuous and nondecreasing such that

$$b_1\|F'(x_0)-F'(x)\| \leq g_1(\|x_0-x\|)$$

for each $x \in U(x_0,R-t_0)$. Equation $g_1(t-t_0)-1=0$ has positive solutions $t \geq t_0$. Denote by $\rho_1$ the smallest such solution.
Or

$(\mathscr{A}_2')$ there exist operator $\Delta = F'(z)^{-1} \in L(Y,X), b_2 > 0$ such that $\|\Delta\| \leq b_2$ and a function $g_2 : [\gamma,+\infty) \longrightarrow [0,+\infty)$ continuous and nondecreasing such that

$$b_2\|F'(z)-F'(x)\| \leq g_2(\|z-x\|)$$

for each $x \in U(x_0,R-t_0)$. Equation $b_2 g_2(t-\gamma)-1=0$ has a minimal solutions $\rho_2 \geq \gamma$. Notice that if $g_1$ or $g_2$ are strictly increasing, then $\rho_1 = g_1^{-1}(\frac{1}{b_1})+t_0$ and $\rho_2 = g_2(\frac{1}{b_2})+\gamma$.
$(\mathscr{A}_3)$ There exists $f : [\gamma,+\infty) \longrightarrow [0,+\infty)$ twice continuously differentiable such that

$$\|F''(z)\| \leq f''(\gamma)$$

and

$$\|\Gamma_0 F(x_0)\| \leq -\frac{f(t_0)}{f'(t_0)}.$$

$(\mathscr{A}_4)$ $\frac{b_1}{1-b_1 g_1(t)} \leq -\frac{1}{f'(t)}$ for all $t \in [t_0,\rho_1]$.
or

$(\mathscr{A}_4')$ $\frac{b_2}{1-b_2 g_2(t)} \leq -\frac{1}{f'(t)}$ for all $t \in [\gamma,\rho_2]$.
$(\mathscr{A}_5)$ $\|F''(x)-F''(z)\| \leq f''(t)-f''(\gamma)$ for all $x \in U_0, t \in [\gamma,R)$, where $U_0 = \Omega \cap U(x_0,\rho_1-t_0)$.
$(\mathscr{A}_5')$ $\|F''(x)-F''(z)\| \leq f''(t)-f''(\gamma)$ for all $x \in U_1, t \in [\gamma,R)$, where $U_1 = \Omega \cap U(x_0,\rho_2-\gamma)$.
$(\mathscr{A}_6)$ $b_1 \leq -\frac{1}{f'(t_0)}$ or
$(\mathscr{A}_6')$ $b_2 \leq -\frac{1}{f'(t_0)}$.

We shall use the majorizing Newton iteration function $f$ defined by,

$$t_n = N_f(t_{n-1}) = t_{n-1} - \frac{f(t_{n-1})}{f'(t_{n-1})} \text{ for all } n = 1,2,\ldots, \tag{2.1}$$

where $t_0$ is given. Conditions $(\mathscr{A}_0)$, $(\mathscr{A}_1)$–$(\mathscr{A}_6)$ or conditions $(\mathscr{A}_0)$, $(\mathscr{A}_1)$, $(\mathscr{A}_2')$, $(\mathscr{A}_3)$, $(\mathscr{A}_4')$, $(\mathscr{A}_5')$ and $(\mathscr{A}_6')$ shall be called the conditions $(\mathscr{A})$.

**Remark 2.1.** *The following conditions were used in [5]-[10] for the special case $X = Y = \mathbb{R}^m$:*

$(\mathscr{C}_1)$ *There exists $z \in \Omega$ and $\delta \geq 0$ such that $\|x_0 - z\| = \delta$ and $\|F''(z)\| \leq f_1''(\gamma)$.*
$(\mathscr{C}_2)$ *There exists the operator $\Gamma_0 = F'(x_0)^{-1} \in L(\mathbb{R}^m,\mathbb{R}^m)$ with $\|\Gamma_0\| \leq -\frac{1}{f'(t_0)}$ and $\|\Gamma_0 F(x_0)\| \leq -\frac{f_1(t_0)}{f_1'(t_0)}$.*
$(\mathscr{C}_3)$ *$\|F''(x)-F''(z)\| \leq f_1''(t)-f_1''(\gamma)$ for $\|x-z\| \leq t-\gamma$, $x \in \Omega$ and $t \in [\gamma,+\infty)$ and the majorizing Newton sequence is defined by $\bar{t}_0$ given,*

$$\bar{t}_n = N_{f_1}(\bar{t}_{n-1}) = \bar{t}_{n-1} - \frac{f_1(\bar{t}_{n-1})}{f_1'(\bar{t}_{n-1})}$$

*for each $n = 1,2,3,\ldots$ and $\bar{t}_0 = t_0$.*

*Notice that $U_0 \subseteq \Omega$ and $U_1 \subseteq \Omega$. Therefore $f$ is at least as tight as $f_1$, i.e.,*

$$f(t) \leq f_1(t)$$

$$-\frac{1}{f'(t)} \leq -\frac{1}{f_1'(t)}$$

*and*

$$f''(t)-f''(\gamma) \leq f_1''(t)-f_1''(\gamma).$$

Next, we state a well known result [9, 10].

**Lemma 2.2.** *Suppose that there exists a nonnegative scalar sequence $\{t_n\}$ majorizing a sequence $\{x_n\} \subseteq \Omega$. Moreover, suppose that $\lim_{n\longrightarrow+\infty} t_n = t_*$ for some $t_* \geq 0$. Then, there exists $x_* \in \Omega$ such that $\lim_{n\longrightarrow\infty} x_n = x_*$ and $\|x_* - x_n\| \leq t_* - t_n$ for each $n = 0,1,2,3,\ldots$.*

The proof of the next two results are skipped, since these are immediately obtained from the ones in [5, 6] by using function $f$, iteration $\{t_n\}$, condition $(\mathscr{A})$ instead of function $f_1$, iterate $\{\bar{t}_n\}$ and conditions $(\mathscr{C}_1)$–$(\mathscr{C}_3)$, respectively. Moreover, these results involve solutions of scalar equations related to Newton's sequence $\{t_n\}$.

**Proposition 2.3.** *Assume that there exists a twice continuously differentiable function $f : [\gamma,+\infty) \longrightarrow \mathbb{R}$ with $\gamma \in \mathbb{R}$ such that the $(\mathscr{A})$ conditions are satisfied.*

*(1) If there exists a solution $\delta \in (\gamma,+\infty)$ of equation $f'(t) = 0$, then $\delta$ is the minimum value of $f$ in $[\gamma,+\infty)$ and $f$ is non-increasing in $[t_0,\delta)$.*
*(2) If $f(\delta) \leq 0$, then the equation $f(t) = 0$ has a unique solution $t_*$ in $(\gamma,\delta)$ satisfying $t_0 < t_* < \delta$.*

**Proposition 2.4.** *Assume that there exists a twice continuously differentiable function $f : [\gamma,+\infty) \longrightarrow \mathbb{R}$ with $\gamma \in \mathbb{R}$ such that condition $(\mathscr{A})$ are satisfied. If there exist a solution $\delta \in [\gamma,+\infty)$ of equation $f'(t) = 0$ satisfying $f(\delta) \leq 0$, then the scalar sequence $\{t_n\}$ given by (2.1) is nondecreasing and converges to the minimal solution $t_*$ of $f(t) = 0$.*

Next, the semi-local convergence of Newton's method follows.

**Theorem 2.5.** *Let $F : \Omega \subseteq X \longrightarrow Y$ be a twice continuously differentiable operator in the Fréchet sense. Assume that there exist a function $f : [\gamma,+\infty) \longrightarrow \mathbb{R}$ twice continuously differentiable with $\gamma \in \mathbb{R}$ such that conditions $(\mathscr{A})$ are satisfied and a solution $\delta \in (\gamma,+\infty)$ of equation $f'(t) = 0$ satisfying $f(\delta) \leq 0$ and $U(x_0,t_* - t_0) \subset \Omega$. Then, the sequence $\{x_n\}$ generated by Newton's method is well defined stays in $\bar{U}(x_0,t_* - t_0)$, and converges to a solution $x_* \in \bar{U}(x_0,t_* - t_0)$ of equation $F(x) = 0$, so that*

$$\|x_* - x_n\| \leq t_* - t_n \text{ for each } n = 0,1,2,\ldots$$

*where sequence $\{t_n\}$ is given in (2.1).*

*Proof.* We use $(\mathscr{A}_2)$ instead of $(\mathscr{C}_3)$ used in [5, 6] to obtain

$$\|F'(x_i)^{-1}\| \leq \frac{b_1}{1 - b_1 g_1(t_i)}$$

or under $(\mathscr{A}_2')$

$$\|F'(x_i)^{-1}\| \leq \frac{b_2}{1 - b_2 g_2(t_i)}$$

instead of

$$\|F'(x_i)^{-1}\| \leq -\frac{1}{f'(t_i)}.$$

Then, by $(\mathscr{A}_4)$ or $(\mathscr{A}_4')$ we get that the preceding estimate also holds in our setting. Using this modification, the rest of proof follows as in [5, 6] with sequence $\{t_n\}$ replacing $\{\bar{t}_n\}$.

□ □

The next result provides information about the location of the solution.

**Proposition 2.6.** *Assume that the condition $(\mathscr{A})$ are satisfied. If the equation $f(t) = 0$ has two solutions such that $t_0 < t_* \leq t_{**}$, then $x_*$ is unique in $U(x_0,t_{**} - t_0) \cap \Omega$ provided that $t_* < t_{**}$ or in $\bar{U}(x_0,t_* - t_0)$, provided that $t_* = t_{**}$.*

*Proof.* Simply replace $f_1,\bar{t}_*,\bar{t}_{**},\mathbb{R}^m,\mathbb{R}^m$ by $f,t_*,t_{**},X,Y$, respectively in [5, 6, Theorem 7].

□ □

The following error bounds are also available:

**Proposition 2.7.** *Assume that the hypotheses of Proposition 2.6 are satisfied.*

*(1) For $t_* < t_{**}$, suppose there exist $a_1 > 0$ and $b_1 > 0$ such that $a_1 \leq \min\{\varphi(t) : t \in [t_0,t_*]\}$ and $b_1 \geq \max\{\varphi(t) : t \in [t_0,t_*]\}$, then*

$$\frac{(t_{**} - t_*)\tau^{2^n}}{a_1 - \tau^{2^n}} \leq t_* - t_n \leq (t_{**} - t_*)c^{2^n}$$

*for all $n = 0,1,2,\ldots$ where $\varphi(t) = \frac{(t_{**}-t)h'(t)-h(t)}{(t_*-t)h'(t)-h(t)}$, $f(t) = (t - t_*)(t - t_{**})h(t), h(t_*) \neq 0$, $h(t_{**}) = 0$ and $\tau = \frac{t_*}{t_{**}}a_1$, provided that $\tau < 1$ and $c < 1$.*
*(2) For $t_* = t_{**}$, suppose there exists $b_3 > 0$ such that $b_3 \leq \min\{\psi(t) : t \in [t_0,t_*]\}$, then*

$$a_2^n t_* \leq t_* - t_n \leq b_3^n t_*$$

*for all $n = 0,1,2,\ldots$ provided that $a_2 < 1$ and $b_3 < 1$, where $\psi(t) = \frac{(t_*-t)h'(t)-h(t)}{(t_*-t)h'(t)-2h(t)}$.*

*Proof.* Simply replace $f_1,\bar{t}_n,\bar{t}_*,\bar{t}_{**}$ by $f,t_n,t_*,t_{**}$ in [5, 6, Theorem 8]. □

**Remark 2.8.** *(i) It follows from Proposition 2.7 that the convergence order is quadratic for $t_* < t_{**}$, and linear, for $t_* = t_{**}$.*

*(ii)* *The uniqueness of the solution $x_*$ is more precise under the new conditions. Notice that $f(\bar{t}_*) \leq f_1(\bar{t}_*) = 0$ so $t_* \leq \bar{t}_*$. Let us suppose that for $t_0 = 0$ [3, 5]-[10]*

$$f(t) = \frac{p}{2}t^2 - \frac{t}{b} + \frac{\eta}{b},$$

$$f_1(t) = \frac{q}{2}t^2 - \frac{t}{b} + \frac{\eta}{b},$$

*then $0 < p \leq q$ (provided that $2bq\eta \leq 1$), we have that $t_* \leq \bar{t}_*$ and $t_{**} \leq \bar{t}_{**}$. Hence, the uniqueness of the solution $x_*$ is improved. Similar favorable comparisons are given for the lower and upper bounds given in Proposition 2.7.*

*(iii)* *The construction of function $f$ defined on $U_0$ as identical to the construction of function $f_1$ on $\Omega$ in [5, 6] is omitted. See also preceding case (ii) and the example in the next Section.*

## 3. Numerical example

We present an example where our results apply to solve an equation but not earlier ones [5, 6].

**Example 3.1.** *Let $X = Y = \mathbb{R}, \Omega = \bar{U}(x_0, 1-p)$, $x_0 = 1$, $p \in I_0 = [2-\sqrt{3}, \frac{1}{2}]$ and $z = x_0$. Define function $F$ on $\Omega$ by*

$$F(x) = \frac{x^3}{3} - px + \frac{2p}{3}.$$

*Under the approach in [5, 6],*

$$\|F''(x)\| = 2\|x\| \leq 2(\|x-x_0\| + \|x_0\|) \leq 2(1-p+1) = 2(2-p), \tag{3.1}$$

$$\|F''(x_0) - F''(x)\| = 2\|x_0 - x\| \leq 2, \tag{3.2}$$

$$b = \|F'(x_0)^{-1}\| = 1 \tag{3.3}$$

*and*

$$\|F'(x_0)^{-1}F(x_0)\| = \frac{1}{3}(1-p) = \eta. \tag{3.4}$$

*If polynomial $f$ (or $f_{old}$) satisfies $f(\mu_2) \leq 0$ (or $f_{old}(\mu_4) \leq 0$), then it has a negative solution and two positive solutions. In view of (3.1)–(3.4), the old function $f_{old}$ satisfying the conditions of Theorem 13 in [5, 6] is given by*

$$f_{old}(t) = \frac{1}{3}t^3 + (2-p)t^2 - t + \frac{1}{3}(1-p) \tag{3.5}$$

*Polynomial in (3.5) has a maximum at $t = \mu_3 = \frac{1}{2-p-\sqrt{(2-p)^2+1}} < 0$ and a minimum at $t = \mu_4 = \frac{1}{2-p+\sqrt{(2-p)^2+1}} > 0$ and $f_{old}(\mu_4) >$* 0, for all $p \in I_0$.

*Hence, the old results cannot guarantee that $\lim_{n \longrightarrow +\infty} x_n = x_*$. Under the new approach, since $g_1(t) = (3-p)t$, $U_0 = \Omega \cap U(x_0, \frac{1}{3-p}) = U(x_0, \frac{1}{3-p}), \rho_1 = \frac{1}{3-p}$, so $U_0$ is a strict subset of $\Omega$ and,*

$$\|F''(x)\| = 2\|x\| \leq 2[\|x-x_0\| + \|x_0\|] \leq 2(\frac{1}{3-p} + 1).$$

*Then, the new function $f$ is defined by*

$$f(t) = \frac{1}{3}t^3 + \frac{4-p}{3-p}t^2 - t + \frac{1}{3}(1-p). \tag{3.6}$$

*Polynomial given in (3.6) has a maximum at $t = \mu_1 = \frac{1}{\frac{4-p}{3-p} - \sqrt{(\frac{4-p}{3-p})^2+1}} < 0$ and a minimum at $t = \mu_2 = \frac{1}{\frac{4-p}{3-p} + \sqrt{(\frac{4-p}{3-p})^2+1}} > 0$.*

*Notice that $(\mathscr{A}_4)$ holds, if $p \in I_0, t \geq 0$ since it reduces to*

$$\frac{1}{1-(3-p)t} \leq -\frac{1}{t^2 + 2(\frac{4-p}{3-p})t - 1}$$

*or*

$$\frac{p^2 - 4p + 1}{3-p} \leq t$$

*or*

$$p^2 - 4p + 1 \leq 0,$$

*which is true for $p \in I_0$. Moreover, we have that*

$$f(\mu_2) \leq 0, \text{ for all } p \in I_0.$$

*Therefore, under our approach* $\lim_{n\longrightarrow\infty} x_n = x_*$. *Furthermore, although the old results do not apply, we also have that for each* $t, \bar{t} \in [0, \rho_1]$ *with* $t < \bar{t}$, $f(t) \leq f_{old}(\bar{t})$ *and* $f'(t) \leq f'_{old}(\bar{t}) < 0$ *so*

$$-\frac{f(t)}{f'(t)} \leq -\frac{f_{old}(\bar{t})}{f'_{old}(\bar{t})}$$

*leading to*

$$t_n \leq \bar{t}_n,$$

$$t_{n+1} - t_n \leq \bar{t}_{n+1} - \bar{t}_n$$

*and*

$$t_* \leq \bar{t}_*.$$

*Hence, the error bounds on* $\|x_{n+1} - x_n\|, \|x_n - x_*\|$ *are improved as well as the location of the solution.*

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# Order-Preserving Variants of the Basic Principles of Functional Analysis

**A. A. Zaitov**

*Tashkent institute of architecture and civil engineering, Tashkent, Uzbekistan*

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**Abstract**

We will establish order-preserving versions of the basic principles of functional analysis such as Hahn-Banach, Banach-Steinhaus, open mapping, and Banach-Alaoglu theorems.

## 1. Introduction

Recently researches in the field of idempotent mathematics and also Choquet integrals intensively develop. Since its introduction in 1974 by Sugeno, the concept of fuzzy measure has been often used in multicriteria decision making. Later in [1], the authors explained the methodology of using the Choquet integral in multicriteria decision making. The notion of idempotent measure (Maslov integral) finds important applications in different part of mathematics, fuzzy topology, mathematical physics, and economics (see the article [2] and the bibliography therein). As well known idempotent measures and Choquet integrals are weakly additive, order-preserving functionals. But for this functionals there not establish yet the basic principles (analogous principles of Functional Analysis, see, for example, [3]). In the present paper, we will establish order-preserving versions of the basic principles of Functional Analysis such as the Hahn-Banach, Banach-Steinhaus, open mapping and Banach-Alaoglu theorems. Recently, in [4] the uniform boundedness principle for nonlinear operators on cones of functions was investigate. In works [5]-[7] by the author announced open mapping theorem for order-preserving operators and Banach-Alaoglu theorem for order-preserving functionals, in particular case, on the function spaces.
Remind that *partially ordered vector space* is a pair $(L, \leqslant)$ where $L$ is a vector space over the field $\mathbb{R}$ of real numbers, $\leqslant$ is an order satisfying the following conditions:
1) if $x \leqslant y$, then $x+u \leqslant y+u$ for all $x,\ y,\ u \in L$;
2) if $x \leqslant y$, then $\lambda x \leqslant \lambda y$ for all $x,\ y \in L$ and $\lambda \in \mathbb{R}_+$.
If the conditions 1) and 2) hold then they say that $\leqslant$ is a linear order. A formation of a vector space $L$ with a linear order $\leqslant$ over $\mathbb{R}$ is equivalent to indicate a set $L_+ \subset L$ called a *positive cone* in $L$ and owning the properties:

$$L_+ + L_+ \subset L_+; \qquad \lambda L_+ \subset L_+, \quad \lambda \in \mathbb{R}_+; \qquad L_+ \cap (-L_+) = \{0\}.$$

In this case, the order $\leqslant$ and the positive cone $L_+$ are connected by a relation

$$x \leqslant y \Leftrightarrow y - x \in L_+, \qquad x,\ y \in L.$$

Elements of $L_+$ is called *positive vectors* of $L$.
Let $(L, L_+)$ be a partially ordered vector space. We say [8] that $L_+$ *is full* (or that $L_+$ *is a full cone*) if $L = L_+ - L_+$.
Let $L$ be a partially ordered vector space over the field $\mathbb{R}$ of real numbers, and $L_+$ be a full cone in it. Let $x_1, x_2 \in L$ be arbitrary various points. The set $[x_1, x_2] = \{\alpha x_1 + (1-\alpha)x_2 \colon \alpha \in [0, 1]\}$ is called *a segment* connecting points $x_1$ and $x_2$. A point $x \in [x_1, x_2]$ is *an inner point* of the segment $[x_1, x_2]$ if $x_1 \neq x \neq x_2$.

***Email addresses and ORCID numbers***: **adilbek_zaitov@mail.ru, 0000-0002-2248-0442, (A. A. Zaitov)**

Let $x \in L_+$. The point $x$ is said to be *an inner point* of the cone $L_+$ if for any segment $[x_1, x_2]$ containing $x$ as an inner point, the segment $[x_1, x_2] \cap L_+$ also contains it as an inner point. The set of all inner points of the cone $L_+$ is called *an interior* of this cone, and it denotes as $\operatorname{Int} L_+$.
Fix an inner point $x_0 \in L_+$. For a $\delta > 0$ we determine a $\delta$*-neighbourhood* (with respect to the cone $L_+$ and the point $x_0$) of zero $0 \in L$ as following:

$$\langle 0; \delta \rangle = \{x \in L\colon \ (\delta x_0 \pm x) \in \operatorname{Int} L_+\}. \tag{1.1}$$

It is easy to see that a family of the sets of the view (1.1) forms a base of neighbourhoods of zero. A neighbourhood of an arbitrary point $z \in L$ can be defined by the shifts of the neighbourhoods of zero:

$$\langle z; \delta \rangle = \langle 0; \delta \rangle + z = \{x + z \in L\colon \ x \in \langle 0; \delta \rangle\} = \{x + z \in L\colon \ (\delta x_0 \pm x) \in \operatorname{Int} L_+\} = \{y \in L\colon \ (\delta x_0 \pm (y - z)) \in \operatorname{Int} L_+\}.$$

**Proposition 1.1.** *A collection*

$$\{\langle z; \delta \rangle\colon \ z \in L, \ \delta > 0\}$$

*forms a base of a Hausdorff topology on L. Further, L equipped with this topology becomes a topological vector space.*

*Proof.* The proof consists of direct checking. □

An element $1 \in L$ of a partially ordered vector space $L$ is called *(strongly) order unit* if $L = \bigcup_{n=1}^{\infty} [-n1, n1]$. This is equivalent to what for every $x \in L$ there exists $\lambda > 0$ such that $-\lambda 1 \leqslant x \leqslant \lambda 1$. Let $x \in L$. A partially ordered vector space $L$ is called *Archimedean* if the inequality $nx \leqslant 1$ executed for all $n = 1, \ 2, \ldots,$ implies $x \leqslant 0$. In this case on $L$ one can define a norm by the equality

$$\|x\| = \inf\{\lambda > 0\colon \ \ -\lambda 1 \leqslant x \leqslant \lambda 1\}.$$

The obtained norm is said to be *an order norm*. A partially ordered vector space $L$ is called *a vector space with an order unit* if $L$ has an order unit and $L$ is an Archimedean space. A topology on $L$ generated by the norm (1.2) is called *order (vector)* topology. For a subset $X \subset L$ by $\operatorname{Int} X$ we denote the interior of $X$ according to the order topology on $L$. We accept the following agreement

$$x < y \Leftrightarrow y - x \in \operatorname{Int} L_+.$$

A set $U(0_E, \varepsilon) = \{x \in E : -\varepsilon 1_E < x < \varepsilon 1_E\}$ is an open neighbourhood of zero $0_E$ concerning to the order topology. As vector topology is translation invariance then for every point $x \in E$ a set $U(x, \varepsilon) = \{y \in E : -\varepsilon 1_E < y - x < \varepsilon 1_E\}$ is an open neighbourhood of $x$ with respect to the order topology.

**Proposition 1.2.** *The order topology and the topology introduced by Proposition 1.1 on a vector space with an order unit coincide.*

*Proof.* Proof is trivial. □

## 2. Extensions of order-preserving functionals

In this section, we will prove the order-preserving functional's variant of the Hahn-Banach theorem, one of the basic principles of functional analysis.

**Definition 2.1.** *A subset B of a partially ordered vector space L is said to be an A-subspace concerning to a fixed inner point $x_0$ of a full cone $L_+ \subset L$ if $0 \in B$, and $x \in B$ implies $(x + \lambda x_0) \in B$ for each $\lambda \in \mathbb{R}$.*

The following assertion is evident.

**Lemma 2.2.** *A subspace B of the partially ordered vector space L is an A-subspace according to $x_0$ iff it contains $x_0$.*

Note that the space $L$ and its subspace $\{\lambda x_0\colon \ \lambda \in \mathbb{R}\}$ are *trivial* $A$-subspaces. As distinct from the linear case the set $\{0\}$ is not $A$-subspace. It is easy to see that an intersection of any collection of $A$-subspaces is a $A$-subspace. In particular, an intersection of all $A$-subspaces containing a given set $X$ is the minimal $A$-subspace, containing $X$; this $A$-subspace we call as *a weakly additive span* of $X$, and designate through $A(X)$. The following statement describes a structure of the weakly additive span of a given set.

**Proposition 2.3.** *A weakly additive span $A(X)$ of a subset X of a partially ordered linear space L consists of a (set-theoretic) union of $\{\lambda x_0\colon \ \lambda \in \mathbb{R}\}$ and the collection of all sums of the look $x + \lambda x_0$, $x \in X$, $\lambda \in \mathbb{R}$, i.e.*

$$A(X) = \{\lambda x_0\colon \ \lambda \in \mathbb{R}\} \cup \bigcup_{\substack{x \in X, \\ \lambda \in \mathbb{R}}} \{x + \lambda x_0\} = \bigcup_{\substack{x \in X \cup \{0\}, \\ \lambda \in \mathbb{R}}} \{x + \lambda x_0\},$$

*in particular, if $x_0 \in X$ then*

$$A(X) = \bigcup_{\substack{x \in X, \\ \lambda \in \mathbb{R}}} \{x + \lambda x_0\}.$$

*Proof.* The proof is obvious. □

Let's denote

$$\Lambda = \{\lambda x_0;\ \lambda \in \mathbb{R}\}.$$

Then we have

$$A(X) = \bigcup_{x \in X \cup \{0\}} (x + \Lambda).$$

The last equality explains the name '$A$-subspace'. Every $A$-subspace $A(X)$ consists of the union of one-dimensional subspace $\Lambda \subset L$ and affine subsets $x + \Lambda \subset L$, $x \in X$.

**Definition 2.4.** *A functional $f\colon L \to \mathbb{R}$ is called:*
1) *weakly additive (according to the point $x_0$) if*

$$f(x + \lambda x_0) = f(x) + \lambda f(x_0), \qquad x \in L, \quad \lambda \in \mathbb{R};$$

2) *order-preserving (concerning to the cone $L_+$) if for every pair $x$, $y \in L$ belonging $y - x \in L_+$ implies the inequality*

$$f(x) \leqslant f(y);$$

3) *normed (with respect to the point $x_0$) if $f(x_0) = 1$.*

From the definition immediately follows that weakly additive functional is linear on the one-dimensional subspace $\{\lambda x_0;\ \lambda \in \mathbb{R}\}$ of $L$. From here we have $f(0) = 0$.
Let $(L, L_+)$ be a partially ordered real vector space. A functional $f\colon L \to \mathbb{R}$ is called *positive* if $f(L_+) \subseteq [0, +\infty)$. Each weakly additive, order-preserving functional is positive. Really, let $x \in L_+$. Then $x - 0 \in L_+$. Since $f$ is order-preserving functional, then $f(x) \geqslant f(0)$. Consequently, $f(x) \geqslant 0$. There exists a function which is weakly additive, positive but does not order-preserving.

**Example 2.5.** Let $L = \mathbb{R}^2 = \{(x_1, x_2)\colon\ x_i \in \mathbb{R},\ i = 1,\ 2\}$ be a partially ordered vector space with respect to the usual linear operations '$\cdot$' – the multiplication by real numbers, '$+$' – the sum of elements of $L$, and to the pointwise order $\leqslant$ on $L$, which defines as $(x_1, x_2) \leqslant (y_1, y_2) \Leftrightarrow x_1 \leqslant y_1$ and $x_2 \leqslant y_2$. The set $L_+ = \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_i \geqslant 0, i = 1,\ 2\}$ is a full positive cone in $L$. Fix an inner point $\mathbf{1} = (1, 1) \in L_+$ and define a functional $f\colon \mathbb{R}^2 \to \mathbb{R}$ by the rule

$$f(x_1, x_2) = \frac{1}{2}\left(x_1 + x_2 + \sqrt{|x_2 - x_1|}\right), \qquad (x_1, x_2) \in \mathbb{R}^2.$$

It is clear that $f$ is a positive functional. Moreover, for every $(x_1, x_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ we have

$$f(x_1 + \lambda, x_2 + \lambda) = \frac{1}{2}\left((x_1 + \lambda) + (x_2 + \lambda) + \sqrt{|(x_2 + \lambda) - (x_1 + \lambda)|}\right) = \\ = \frac{1}{2}\left(x_1 + x_2 + \sqrt{|x_2 - x_1|}\right) + \lambda = f(x_1, x_2) + \lambda f(1, 1),$$

i.e. $f$ is a weakly additive (according to the inner point $(1, 1)$) functional. But we have $f\left(\frac{1}{2}, \frac{1}{2}\right) < f\left(\frac{1}{4}, \frac{1}{2}\right)$ though $\left(\frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{4}, \frac{1}{2}\right) \in L_+$. Thus, $f$ is weakly additive, positive but does not order-preserving functional.

**Proposition 2.6.** *If an order-preserving, weakly additive functional $f\colon L \to \mathbb{R}$ is continuous at zero 0 then it is continuous on the whole $L$.*

*Proof.* Let for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x)| < \varepsilon$ for all $x \in \langle 0;\, \delta\rangle \subset L$. Let $y \in L$ be an arbitrary nonzero element. Consider a neighbourhood

$$\left\langle y; \frac{\delta}{2}\right\rangle = \left\{z \in L\colon\ \left(\frac{\delta}{2}x_0 \pm (z - y)\right) \in \operatorname{Int} L_+\right\}.$$

For every $z \in \left\langle y; \frac{\delta}{2}\right\rangle$ we have:
1) $f(\frac{\delta}{2}x_0) + f(y) \geqslant f(z)$ since $\left(\frac{\delta}{2}x_0 + y\right) - z \in \operatorname{Int} L_+$;
2) $f(\frac{\delta}{2}x_0) + f(z) \geqslant f(y)$ since $\left(\frac{\delta}{2}x_0 + z\right) - y \in \operatorname{Int} L_+$.
From here follows that

$$|f(z) - f(y)| \leqslant f\left(\frac{\delta}{2}x_0\right). \tag{2.1}$$

Now we take $x \in L$ such that $x - \frac{\delta}{2}x_0 \in L_+$ and $\frac{3\delta}{4}x_0 - x \in L_+$. Then $f(\frac{\delta}{2}x_0) \leqslant f(x)$ and $f(x) \leqslant f(\frac{3\delta}{4}x_0)$. On the other hand $f(\frac{3\delta}{4}x_0) < \varepsilon$ so far as $\frac{3\delta}{4}x_0 \in \langle 0;\, \delta\rangle$. Consequently $|f(z) - f(y)| < \varepsilon$ for each $z \in \left\langle y; \frac{\delta}{2}\right\rangle$. So $f$ is continuous at $y \in L$. Thus $f$ is continuous on the whole $L$ owing to arbitrariness of $y \in L$.

□

A weakly additive, order-preserving functional $f\colon L \to \mathbb{R}$ is called *bounded* if $\sup\{|f(x)|\colon\ x \in \langle 0;\, 1\rangle\} < \infty$.

**Proposition 2.7.** *A weakly additive, order-preserving functional is bounded if and only if it is continuous.*

*Proof.* Let $f\colon L \to \mathbb{R}$ be weakly additive, order-preserving bounded functional. Let $f(x_0) = a < \infty$ and $\delta x_0 \pm (z-y) \in \operatorname{Int} L_+$. Then similarly to (2.1) one can show that $|f(z) - f(y)| < \delta f(x_0) = \delta a$, and consequently $f$ is continuous.
Conversely, let a weakly additive, order-preserving functional $f\colon L \to \mathbb{R}$ be continuous. Then there exists $\delta > 0$ such that $|f(x)| < 1$ at all $x \in \langle 0;\delta\rangle$. In particular, $\left|\frac{\delta}{2} f(x_0)\right| < 1$ so far as $\frac{\delta}{2}x_0 \in \langle 0;\delta\rangle$. Hence, $|f(x_0)| < \frac{2}{\delta} < \infty$, i.e. $\sup\{|f(x)|\colon\ x \in \langle 0;1\rangle\} < \frac{2}{\delta} < \infty$.

□

**Corollary 2.8.** *A weakly additive, order-preserving, normed functional is continuous (or, the same, bounded).*

The following statement is an analog of Hahn-Banach theorem for weakly additive, order-preserving functionals.

**Theorem 2.9.** *Let $B$ be an $A$-subspace of the space $L$. Then for every weakly additive, order-preserving functional $f\colon B \to \mathbb{R}$ there exists a weakly additive, order-preserving functional $f_0\colon L \to \mathbb{R}$ such that $f_0|_B = f$.*

*Proof.* Let $y \in L \setminus B$. Put $B' = B \cup \{y + \lambda x_0\colon\ \lambda \in \mathbb{R}\}$. Obviously that $B'$ is an $A$-subspace of $L$. Put

$$B^+ = \{z \in B\colon\ z - y \in B\} \qquad \text{and} \qquad B^- = \{z \in B\colon\ y - z \in B\}.$$

The obtained sets $B^+$ and $B^-$ are not empty. Indeed, take $\lambda > 0$ such that $y \in \langle 0;\lambda\rangle$. Then evidently that $2\lambda x_0 \in B^+$ and $-2\lambda x_0 \in B^-$. Put

$$p^+ = \inf\{f(z)\colon\ z \in B^+\}, \qquad p^- = \sup\{f(z)\colon\ z \in B^-\}.$$

We have $p^- \leqslant p^+$. Indeed, $y - z_1 \in L_+$ if provided $z_1 \in B^-$, and $z_2 - y \in L_+$ if provided $z_2 \in B^+$. From here we get $z_2 - z_1 \in L_+$. Consequently $f(z_1) \leqslant f(z_2)$ for all $z_1 \in B^-$ and $z_2 \in B^+$, i.e. $p^- \leqslant p^+$. Take a number $p$ such that $p^- \leqslant p \leqslant p^+$ and put

$$f'(y + \lambda x_0) = p + \lambda f(x_0).$$

In such a way we define an extension $f'$ of $f$ from $B$ on $B'$. From the definition directly implies that $f'$ is a weakly additive functional. We will show that $f'$ is order-preserving. It is order-preserving on $B$ owing to $f'|B = f$. Besides it is evident that $f'$ is order-preserving on $\{y + \lambda x_0\colon\ \lambda \in \mathbb{R}\}$. Let now $z - (y + \lambda x_0) \in L_+$, where $z \in B$. Then $(z - \lambda x_0) - y \in L_+$, i.e. $(z - \lambda x_0) \in B^+$. That is why

$$f'(z - \lambda x_0) = f(z - \lambda x_0) = f(z) - \lambda f(x_0) \geqslant p^+ \geqslant p = f'(y),$$

i.e. $f'(z) \geqslant f'(y + \lambda x_0)$. In the case when $(y + \lambda x_0) - z \in L_+$ one can similarly show that $f'(z) \leqslant f'(y + \lambda x_0)$.
Thus, a weakly additive, order-preserving continuous functional $f\colon B \to \mathbb{R}$ defining on an $A$-subspace $B$ can be extended to a weakly additive, order-preserving continuous functional $f'\colon B' \to \mathbb{R}$ on a wider $A$-subspace $B'$ of $L$. At the same time the equality $f'(x_0) = f(x_0)$ holds.
Consider the set of all pairs $(B', f')$ such that $B \subset B' \subset L$ where $B'$ is an $A$-subspace, $f'\colon B' \to \mathbb{R}$ is a weakly additive, order-preserving continuous extension of $f$. The relation $(B', f') \leqslant (B'', f'')$ meaning that $f''\colon B'' \to \mathbb{R}$ is a weakly additive, order-preserving continuous extension of $f'$ on a subspace $B''$, $B' \subset B'' \subset L$, turns this set into a partially ordered set in which all chains are bounded. By Zorn's lemma, there is the maximal element $(B_0, f_0)$ of this set. We will show that $B_0 = L$.
Suppose that $B_0 \neq L$. Take any point $y \in L \setminus B_0$ and put $B_1 = B_0 \cup \{y + \lambda x_0\colon\ \lambda \in \mathbb{R}\}$. Then $f_0$ can be extended to $f_1\colon B_1 \to \mathbb{R}$, and consequently, $(B_0, f_0) \leqslant (B_1, f_1)$. We got a contradiction with maximality of $B_0$. So, $B_0 = L$.

□

## 3. Uniform boundedness principle for order-preserving operators

Let $(E, \leqslant)$ and $(F, \leqslant)$ be partially ordered vector spaces.

**Definition 3.1.** *A map $T\colon E \to F$ is said to be an order-preserving operator if for arbitrary points $x, y \in E$ the inequality $x \leqslant y$ implies $T(x) \leqslant T(y)$.*

Let $(E, \leqslant)$ be a partially ordered vector space with an order unit $1_E$ and $(F, \leqslant)$ be a partially ordered vector space.

**Definition 3.2.** *A map $T\colon E \to F$ is said to be a weakly additive operator if $T(x + \lambda 1_E) = T(x) + \lambda T(1_E)$ takes place for each $x \in E$ and $\lambda \in \mathbb{R}$.*

The last definition immediately implies $T(0_E) = T(1_E - 1_E) = T(1_E) - T(1_E) = 0_F$, i.e. $T(0_E) = 0_F$ for a weakly additive operator $T\colon E \to F$.
The following statement shows weakly additive, order-preserving operators of vector spaces with an order unit are automatical continuous.

**Proposition 3.3.** *If $E$ and $F$ are vector spaces with an order unit then each weakly additive, order-preserving operator $T\colon E \to F$ is continuous.*

*Proof.* We will show the operator $T$ is continuous at zero $0_E$. At first we note the following case. If $T(1_E) = 0_F$ then $T(x) = 0_F$ for all $x \in E$ since $T$ is a weakly additive and order-preserving operator, and for every $x$ there exists $\lambda > 0$ such that $-\lambda 1_E \leqslant x \leqslant \lambda 1_E$. So $T(E) \subset \{0_F\}$. This case we will not consider, i.e. suppose $T(1_E) \neq 0_F$. Then $\|T(1_E)\| \neq 0$.
Let $V(0_F, \varepsilon) = \{y \in F\colon\ -\varepsilon 1_F < y < \varepsilon 1_F\}$ be a neighbourhood of zero $0_F$ in $F$, where $\varepsilon > 0$. Take the neighbourhood $U\left(0_E, \frac{\varepsilon}{\|T(1_E)\|}\right)$ of zero $0_E$ in $E$. For each vector $x \in U$ we have $-\frac{\varepsilon}{\|T(1_E)\|} 1_E < x < \frac{\varepsilon}{\|T(1_E)\|} 1_E$. Then $-\frac{\varepsilon}{\|T(1_E)\|} T(1_E) < T(x) < \frac{\varepsilon}{\|T(1_E)\|} T(1_E)$ since $T$ is a weakly additive, order-preserving operator. From here we get $\|T(x)\| < \varepsilon$, i.e. $T(U) \subset V$. Thus $T$ is continuous at $0_E$.
The remaining part of the proof is similar to the Proof of Proposition 2.6.

□

**Remark 3.4.** It is obvious that each linear non-negative operator on spaces with an order unit is weakly additive and order-preserving. The converse, in general, is not true. But, nevertheless, such operators are linear on a one-dimensional subspace $\{\lambda 1_E : \lambda \in \mathbb{R}\} \subset E$. In this case the image of the subspace $\{\lambda 1_E : \lambda \in \mathbb{R}\}$ under the map $T$ is, as clearly, a one-dimensional subspace $\{\lambda T(1_E) : \lambda \in \mathbb{R}\} \subset F$. We have $T(1_E) \in F_+$ but it is optional $T(1_E) \in \operatorname{Int} F_+$. Therefore $T(1_E)$ is an order unit in $T(E)$ but it is optional to be an order unit in $F$. From here and Proposition 3.3 follows that for every weakly additive, order-preserving operator $T: E \to F$ on spaces $E$, $F$ with an order unit the inequality $\|T(1_E)\| < \infty$ takes place.

Remind the following notions. A set $A$ in a normed space $E$ is called *bounded* if there exists $R > 0$ such that $A$ can be placed into the ball $\{x \in E : \|x\| \leqslant R\}$. A map $T: E \to F$ of normed spaces is called *bounded* if it carries over a bounded set in $E$ to a bounded set in $F$. It is obvious that the boundedness of the map $T$ is equivalent to limitation of the set $\{\|T(x)\| : x \in E, \|x\| \leqslant R\}$ for every $R > 0$. In other words, $\sup\{\|T(x)\| : x \in E, \|x\| \leqslant R\} < \infty$ for every bounded map $T$ and for each $R > 0$.
The following statement shows weakly additive, order-preserving operators of vector spaces with an order unit are automatical bounded.

**Proposition 3.5.** *Each weakly additive, order-preserving operator $T: E \to F$ of spaces with an order unit is bounded.*

*Proof.* The proof follows from Remark 3.4. □

Let $E$ and $F$ be vector spaces with an order unit, $1_E$ and $1_F$, respectively. A collection $\mathscr{H}$ of weakly additive, order-preserving operators $T: E \to F$ is said to be *equicontinuous* if to every neighbourhood $V$ of zero in $F$ there corresponds a neighbourhood $U$ of zero in $E$ such that $T(U) \subset V$ for all $T \in \mathscr{H}$. If the collection $\mathscr{H}$ consists only one weakly additive, order-preserving operator $T$, then $\mathscr{H}$ is equicontinuous as $T$ is continuous, and $\mathscr{H}$ is uniformly bounded owing to boundedness of $T$. The following statement shows that each equicontinuous collection of weakly additive, order-preserving operators on vector spaces with an order unit is uniformly bounded.

**Proposition 3.6.** *Let $E$ and $F$ be vector spaces with an order unit, $\mathscr{H}$ an equicontinuous collection of weakly additive, order-preserving operators from $E$ into $F$, and $A$ a bounded subset of $E$. Then $F$ has a bounded subset $B$ such that $T(A) \subset B$ for every $T \in \mathscr{H}$.*

*Proof.* Put $B = \bigcup\limits_{T \in \mathscr{H}} T(A)$. Since the collection $\mathscr{H}$ is equicontinuous then for every neighbourhood $V = V(0_F, \varepsilon)$ of zero in $F$ there exists a neighbourhood $U = U(0_E, \delta)$ of zero in $E$ that $T(U) \subset V$ for all $T \in \mathscr{H}$. So far as $A$ is bounded for enough big $t \in \mathbb{R}$ we have $A \subset tU$. It is clear, that $T(A) \subset T(tU)$. Assume that $x \in tU$. Then $\|x\| < t\delta$, i.e. $-t\delta 1_E < x < t\delta 1_E$. As $T$ is weakly additive and order-preserving we have $-t\delta T(1_E) < T(x) < t\delta T(1_E)$, $\|T(x)\| < t\delta\|T(1_E)\|$, consequently, $\|\frac{1}{t}T(x)\| < \delta\|T(1_E)\| = \|T(\delta 1_E)\| \leqslant \varepsilon$. Hence, $T(tU) \subset tV$. Thus $T(A) \subset tV$ for all $T \in \mathscr{H}$. It means that $B \subset tV$, i.e. the set $B$ is bounded.

□

The following result is a weakly additive, order-preserving operators' variant of the Banach-Steinhaus theorem.

**Theorem 3.7.** *Let $E$ and $F$ be vector spaces with an order unit, $\mathscr{H}$ be a collection of weakly additive, order-preserving operators $T: E \to F$, and $A$ be a set consisting of all points $x \in E$ whose orbits $\mathscr{H}(x) = \{T(x) : T \in \mathscr{H}\}$ are bounded in $F$. If $A$ is a set of the second category then $A = E$ and the collection $\mathscr{H}$ is equicontinuous.*

*Proof.* Let $V = V(0_F, \varepsilon)$ and $W = W(0_F, \varepsilon')$ be neighbourhoods such that $\overline{V} + \overline{V} \subset W$ where $\overline{V}$ is the closure of $V$ with respect to order topology in $F$. Put $B = \bigcap\limits_{T \in \mathscr{H}} T^{-1}(\overline{V})$. Let $x \in A$. Then for some positive integer $n$ we have $\mathscr{H}(x) \subset nV$ by virtue of boundedness of $\mathscr{H}(x)$. Hence $T(x) \in nV$ or $x \in nT^{-1}(V)$ for all $T \in \mathscr{H}$. It means that $x \in nB$. Thus $A \subset \bigcup\limits_{n=1}^{\infty} nB$. Thence at least one of the sets $nB$ is the second category owing to $A$ is so. A map $x \mapsto nx$ is a homeomorphism $E$ onto itself. Consequently the set $B$ is the second category. Continuity of operators $T \in \mathscr{H}$ implies $B$ is closed in $E$. As $B$ is the second category set, it has an inner point. By the construction of $B$ one can see that $\delta 1_E$ lies in $B$ as an inner point for enough small $\delta \in \mathbb{R}$. Let $\delta 1_E$ be such an inner point in $B$. Then a set $B - \delta 1_E = \{x - \delta 1_E : x \in B\}$ contains some neighborhood $U = U(0_E, \delta')$ of zero and

$$T(U) \subset T(B - \delta 1_E) = \{T(x - \delta 1_E) : x \in B\} = \{T(x) - \delta T(1_E) : x \in B\} = T(B) - \delta T(1_E) \subset \overline{V} - \overline{V} \subset W$$

for all $T \in \mathscr{H}$. It means that $\mathscr{H}$ is a equicontinuous collection. Then $\mathscr{H}$ is uniform bounded by Proposition 3.6. That is why an orbit $\mathscr{H}(x)$ is bounded for each $x \in E$. Consequently, since $A$ consists of points of $E$ whose orbits $\mathscr{H}(x) = \{T(x) : T \in \mathscr{H}\}$ are bounded in $F$ we have $E \subset A$. Therefore $A = E$.

□

If a vector space with an order unit is a Banach space with respect to order norm then it said to be *a complete space with an order unit*. As each Banach space is a set of the second category then Theorem 3.7 directly implies

**Corollary 3.8.** *Let $E$ be a complete space with an order unit and $F$ a vector space with an order unit, $\mathscr{H}$ a collection of weakly additive, order-preserving operators $T: E \to F$, and a collection $\mathscr{H}(x) = \{T(x) : T \in \mathscr{H}\}$ bounded in $F$ for each $x \in E$. Then $\mathscr{H}$ is an equicontinuous collection.*

As Proposition 3.6 holds then Corollary 3.8 means that pointwise boundedness of an arbitrary collection weakly additive, order-preserving operators from a complete space with an order unit into a vector space with an order unit implies uniform boundedness of this collection.
Let $E$ and $F$ be vector spaces with an order unit, $\{T_n\}$ a sequence of weakly additive, order-preserving operators $T_n: E \to F$. If for every $x \in E$ there exists a limit $\lim\limits_{n \to \infty} T_n(x)$ then putting

$$T(x) = \lim_{n \to \infty} T_n(x), \qquad x \in E, \tag{3.1}$$

we have a weakly additive, order-preserving operator. Indeed,

$$T(x + \lambda 1_E) = \lim_{n \to \infty} T_n(x + \lambda 1_E) = \lim_{n \to \infty} (T_n(x) + \lambda T_n(1_E)) = T(x) + \lambda T(1_E),$$

and if $x \leqslant y$ then

$$T(x) = \lim_{n\to\infty} T_n(x) \leqslant \lim_{n\to\infty} T_n(y) = T(y).$$

**Corollary 3.9.** *Let $E$ and $F$ be vector spaces with an order unit, $\{T_n\}$ a sequence of weakly additive, order-preserving operators $T_n : E \to F$. If there exists a limit $\lim\limits_{n\to\infty} T_n(x)$, $x \in E$, then an operator $T : E \to F$ defined by (3.1) is also a weakly additive, order-preserving operator.*

## 4. Order-preserving variant of open mapping theorem

Remind that a map $f : X \to Y$ of topological spaces is called *open* at $x_0 \in X$ if for every open neighbourhood of $x_0$ in $X$ there exists an open neighbourhood $V$ of $f(x_0)$ in $Y$ such that $V \subset f(U)$. A map is *open* on a topological space $X$ if it is open at every point of $X$.

**Lemma 4.1.** *Let $E$ and $F$ be vector spaces with an order unit, $T : E \to F$ a weakly additive, order-preserving onto operator. If $T$ is open at zero then it is open on all $E$.*

*Proof.* Let for every neighbourhood $U = U(0_E,\varepsilon)$ of $0_E$ its image $T(U) = \{T(x) : x \in U\}$ be open. We have $0_F \in T(U)$ as $T(0_E) = 0_F$. Thence there exists an open neighbourhood $V = V(0_F,\delta)$ of $0_F$ such that $V \subset T(U)$.
Now let $x_0 \in E$ be an arbitrary point and $U(x_0,\varepsilon)$ a neighbourhood of $x_0$ got by shifting $U(0_E,\varepsilon)$ on vector $x_0$. Besides let $V(T(x_0),\delta)$ be a neighbourhood of $T(x_0)$ got by shifting $V(0_F,\delta)$ on vector $T(x_0)$. The proof of the Lemma will finish if we show that the following diagram is true

$$\begin{array}{ccc} y \in V(T(x_0),\delta) & \overset{(1)}{\Longleftrightarrow} & y - T(x_0) \in V(0_F,\delta) \\ & & {\scriptstyle (2)}\Downarrow \\ y \in T(U(x_0,\varepsilon)) & \overset{(3)}{\Longleftrightarrow} & y - T(x_0) \in T(U(x_0,\varepsilon)). \end{array}$$

The equivalence of the double inequalities $-\delta 1_F < y - T(x_0) < \delta 1_F$ and $T(x_0) - \delta 1_F < y < \delta 1_F + T(x_0)$ implies $(1)$. Since $V \subset T(U)$ we have $(2)$. And the equivalence of the double inequalities $-\varepsilon T(1_E) < y - T(x_0) < \varepsilon T(1_E)$ and $T(x_0) - \varepsilon T(1_E) < y < \varepsilon T(1_E) + T(x_0)$ implies $(3)$.
Thus for an arbitrary point $x \in E$ and its arbitrary neighbourhood $U = U(x,\varepsilon)$ there exists open neighbourhood $V = V(T(x),\delta)$ such that $V \subset T(U)$.

□

Remind that a metric $d$ on a vector space $E$ is invariant concerning to a shift of points of $E$ if $d(x+z,y+z) = d(x,y)$ for all $x,y,z \in E$. Define an order metric by the rule

$$d(x,y) = \|y-x\| = \inf\{\lambda > 0 : \ -\lambda 1_E < y - x < \lambda 1_E\}.$$

It is easy to see that the following assertion holds.

**Lemma 4.2.** *The order metric on a vector space with an order unit is invariant according to a shift of points.*

Let $E$ and $F$ be vector spaces with an order unit. A product $E \times F$ over $(0_E, 0_F)$ becomes a vector space with an order unit if we will introduce to it coordinatewise operations of sum and multiplication by number

$$\alpha(x_1,x_2) + \beta(y_1,y_2) = (\alpha x_1 + \beta y_1, \ \alpha x_2 + \beta y_2),$$

and coordinatewise partially order

$$(x_1,x_2) \leqslant (y_1,y_2) \Leftrightarrow (x_1 \leqslant y_1 \text{ and } x_2 \leqslant y_2).$$

Order norm on $E \times F$ is defined by the rule

$$\|(x_1,x_2)\| = \inf\{\lambda > 0 : -\lambda(1_E,1_F) \leqslant (x_1,x_2) \leqslant \lambda(1_E,1_F)\}.$$

Here $(1_E,1_F)$ is one of inner points of $(E\times F)_+ = E_+ \times F_+$ that is why without losing generality we assume $(1_E,1_F)$ is an order unit in the product. Denote $1_{E\times F} = (1_E,1_F)$.
Let $T : E \to F$ be a weakly additive, order-preserving operator. The set of all pairs $(x,T(x))$, $x \in E$, is called *a graph* of $T$.

**Lemma 4.3.** *Let $E$ and $F$ be vector spaces with an order unit, $1_E$ an order unit in $E$, $T : E \to F$ a weakly additive, order-preserving operator. Then the graph $G$ of $T$ is an A-subspace of $E \times T(E)$ with an order unit $1_{E\times T(E)}$.*

*Proof.* We have $(0_E,0_F) \in G \subset E \times T(E)$ since $T(0_E) = 0_F$. Consider $(x_1,x_2) \in E \times T(E)$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}(x_1,x_2) + \lambda 1_{E\times T(E)} &= (x_1,T(x_1)) + (\lambda 1_E, \lambda 1_{T(E)}) = (x_1 + \lambda 1_E, T(x_1) + \lambda 1_{T(E)}) \\ &= (x_1 + \lambda 1_E, T(x_1) + \lambda T(1_E)) = (x_1 + \lambda 1_E, T(x_1 + \lambda 1_E)),\end{aligned}$$

i.e. $(x_1,x_2) + \lambda 1_{E\times T(E)} \in G$.

□

Lemma 4.3 and Remark 3.4 imply

**Corollary 4.4.** *Let $E$, $F$ be vector spaces with an order unit, $1_E$ an order unit in $E$, $T: E \to F$ a weakly additive, order-preserving operator. Then the image $T(E)$ is A-subspace of $F$ if and only if $T(1_E) \in \operatorname{Int} F_+$.*

**Remark 4.5.** Further, during the current section, without loss of generality, we will consider such weakly additive, order-preserving operators $T$ for which $T(1_E) \in \operatorname{Int} F_+$. Then we may assume that $T(1_E)$ is an order unit in $F$. Put $1_F = T(1_E)$.

At last, we will form a variant of the Open Mapping Theorem for weakly additive, order-preserving operators.

**Theorem 4.6.** *Let $E$ be a complete space with an order unit, $F$ a vector space with an order unit, and $T: E \to F$ a weakly additive, order-preserving operator such that $T(E) = F$ and $F$ is a set of the second category. Then*
*(i) the map $T$ is open;*
*(ii) $F$ is a complete space with an order unit.*

*Proof.* Let $U(0_E, \varepsilon)$ be an open neighbourhood. Then according to Remark 4.5 we have

$$T(U(0_E,\varepsilon)) = \{T(x) \in F : -\varepsilon 1_E < x < \varepsilon 1_E\} =$$
$$= \{T(x) \in F : -\varepsilon 1_F < T(x) < \varepsilon 1_F\} = \{y \in F : \text{ there exists } x \in U(0_E,\varepsilon) \text{ such that } y = T(x) \text{ and } -\varepsilon 1_F < y < \varepsilon 1_F\} = U(0_F,\varepsilon).$$

It reminds to show that $(ii)$ takes place.
Let $\{y_n\} \subset F$ be a fundamental (Cauchy) sequence. Then for every $\varepsilon > 0$ there exists a number $n$ such that at all $k \geqslant n$, $m \geqslant n$ the double inequalities

$$-\varepsilon 1_F < y_m - y_k < \varepsilon 1_F$$

hold. One may assume $\varepsilon = \frac{1}{n}$. Then $-\frac{1}{n}1_F < y_m - y_k < \frac{1}{n}1_F$. Since $T(U(0_E,\frac{1}{n})) = U(0_F,\frac{1}{n})$ there exists $x_m, x_k \in E$ such that $T(x_m) = y_m$, $T(x_k) = y_k$ and $-\frac{1}{n}1_E < x_m - x_k < \frac{1}{n}1_E$. So we have constructed a fundamental sequence $\{x_n\} \subset E$. By completeness of $E$ the sequence have a limit $x = \lim\limits_{n\to\infty} x_n$. As $T$ is continuous from Proposition 3.3 we have $T(x) = \lim\limits_{n\to\infty} T(x_n) = \lim\limits_{n\to\infty} y_n$. Then $\lim\limits_{n\to\infty} y_n \in T(E) = F$. Thus, $F$ is complete space with an order unit.

□

**Remark 4.7.** Note that open mapping principle for weakly additive, order-preserving operators it is impossible to form as the linear case. In the distinguishing from the linear case, weakly additivity and order-preserving of $T$, and being of $T(E)$ the second category set does not imply the equality $T(E) = F$. On the other hand $T(E)$ must not be open in $F$. At last, if $T$ is not onto in Lemma 4.1 then openness of $T$ at zero does not provide it openness on all the space.
Note that in linear topological spaces there is no open subspace different the whole space. But an $A$-subspace, distinguished from the subspace, may be open, closed or everywhere dense in the vector space with order unit.

**Example 4.8.** Let $L = \mathbb{R}^2 = \{(x_1,x_2): \ x_i \in \mathbb{R}, \ i = 1, \ 2\}$ be the vector space with an order unit considered in Example 2.5. Then $L_+ = \{(x_1,x_2) \in \mathbb{R}^2 : \ x_i \geqslant 0, i = 1, \ 2\}$ is a positive cone in $L$. Fix $\mathbf{1} = (1,1) \in \operatorname{Int}\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ as an order unit in it.
a) It is easy to see that the set $B = \{(x_1, x_1 + a) \in \mathbb{R}^2 : -1 < a < 1\}$ is an open (with respect to order topology) $A$-subspace, but $B \neq \mathbb{R}^2$.
b) Let $\mathbb{Q}$ be the set of rational numbers. Then $C = \{(x_1, x_1 + r) \in \mathbb{R}^2; r \in \mathbb{Q}\}$ is an everywhere dense $A$-subspace in $\mathbb{R}^2$.
c) The set $D = \{(x_1, x_1 + a) \in \mathbb{R}^2 : -1 \leqslant a \leqslant 1\}$ is a closed $A$-subspace in $\mathbb{R}^2$.
d) Define a map $T: \mathbb{R}^2 \to \mathbb{R}^2$ by the rule

$$T(x_1,x_2) = \begin{cases} (x_1, x_1 - 1), & \text{at} \quad x_2 \leqslant x_1 - 1, \\ (x_1, x_2), & \text{at} \quad x_1 - 1 < x_2 < x_1 + 1, \\ (x_1, x_1 + 1), & \text{at} \quad x_2 \geqslant x_1 + 1. \end{cases}$$

It is easy to check that $T$ is a weakly additive map. Let us show that the map $T$ is order-preserving. It clear that $T$ is order-preserving on $B$ by $T = id_B$.
Let $x_2 \geqslant x_1 + 1$. Take a vector $(y_1,y_2) \in \mathbb{R}^2$ such that $(x_1,x_2) \leqslant (y_1,y_2)$. The following three cases possible.
Case 1) $y_2 \geqslant y_1 + 1$. Then

$$T(x_1,x_2) = (x_1, x_1 + 1) \leqslant (y_1, y_1 + 1) = T(y_1,y_2).$$

Case 2) $y_1 - 1 \leqslant y_2 \leqslant y_1 + 1$. Then $x_1 + 1 \leqslant y_2$. That is why

$$T(x_1,x_2) = (x_1, x_1 + 1) \leqslant (y_1, y_2) = T(y_1,y_2).$$

Case 3) $y_2 \leqslant y_1 - 1$. Then $x_1 + 1 \leqslant y_1 - 1$. Censequently

$$T(x_1,x_2) = (x_1, x_1 + 1) \leqslant (y_1, y_1 - 1) = T(y_1,y_2).$$

Similarly, one may show that $T$ is order-preserving when $x_2 \leqslant x_1 - 1$. Thus $T$ is order-preserving on the whole $\mathbb{R}^2$.
We have $T(\mathbb{R}^2) = D \neq \mathbb{R}^2$ through the operator $T$ is weakly additive and order-preserving, and the image $T(\mathbb{R}^2)$ is the second category. Clearly the image $T(\mathbb{R}^2)$ is closed in $\mathbb{R}^2$ and it is not open. Moreover $T$ is open at zero but it is not open on $\mathbb{R}^2$. Really for the open neighbourhood $U((2,4),1) = \{(x_1,x_2) \in \mathbb{R}^2 : 1 < x_1 < 3, 3 < x_2 < 5\}$ of the point $(2,4) \in \mathbb{R}^2$ its image $T(U) = \{(x_1, x_1 + 1) : 1 < x_1 < 3\}$ is not open in $T(\mathbb{R}^2)$.

## 5. Order-preserving variant of Banach-Alaoglu theorem

Let $E$ be a vector space with an order unit. Fix $1_E$ as an order unit. By $E_+^W$ we denote the set of all weakly additive, order-preserving functionals $f\colon E \to \mathbb{R}$. On $E_+^W$ define algebraic operations pointwise. Then $E_+^W - E_+^W$ turns to a vector space with an order unit. Denote $E^W = E_+^W - E_+^W$. Put $E^O = \{f \in E_+^W : f(1_E) = 1\}$. Provide $E^W$ with the pointwise convergence topology. A collection of the sets of the view

$$\langle f; x_1, \dots, x_n; \varepsilon\rangle = \{g \in E^W : |f(x_i) - g(x_i)| < \varepsilon,\ i = 1, \dots, n\}$$

forms a base of open neighbourhoods of $f \in E^W$, where $\varepsilon > 0$, $x_i \in E$, $i = 1, \dots, n$.
The main result of the section is the following variant of the Banach-Alaoglu theorem for weakly additive, order-preserving functionals.

**Theorem 5.1.** *If $V$ is a neighbourhood of zero in $E$ then the set*

$$K = \{f \in E^O : |f(x)| \leqslant 1 \text{ for every } x \in V\}$$

*is a compact in the pointwise convergence topology.*

*Proof.* Since neighbourhoods of zero are absorbing sets, for every point $x \in E$ there exists $\gamma(x) \in \mathbb{R}_+$ such that $x \in \gamma(x)V$. That is why $|f(x)| \leqslant \gamma(x)$ for all $f \in E^W$ and $x \in E$. For every $x \in E$ denote $D_x = [-\gamma(x), \gamma(x)]$ and assume that $\tau$ is the Tychonoff topology in the product $P = \prod\limits_{x\in E} D_x$. It is well known that $P$ is a Hausdorff compact space. By the construction we have $K \subset P \cap E^W$. We will show that $K$ is closed in $P$. Let $f_0 \in P$ and $f_0 = f_0^+ - f_0^-$, where $f_0^+, f_0^- \in P \cap E_+^W$. Suppose $\{f_\alpha^+\} \subset P \cap E_+^W$ and $\{f_\theta^-\} \subset P \cap E_+^W$ are nets converging to $f_0^+$ and $f_0^-$ respectively. Then owing to Corollary 3.9 we have $f_0^+, f_0^- \in E^W$, and therefore $f_0 \in E^W$. On the other hand $|f_0(x)| = |(f_0^+(x) - f_0^-(x))| \leqslant \max\{|f_0^+(x)|, |f_0^-(x)|\} \leqslant \gamma(x)$ by $|f_\alpha^+(x)| \leqslant \gamma(x)$ and $|f_\theta^-(x)| \leqslant \gamma(x)$ for all $x \in E$, $\alpha$ and $\theta$. Therefore $|f_0^+(x)| \leqslant \gamma(x)$ for all $x \in E$ and $|f_0^+(x)| \leqslant 1$ so far as $x \in V$. It means that $f_0 \in K$.

□

**Corollary 5.2.** *$E^O$ is a compact in the pointwise convergence topology.*

If $E$ is a separable vector space with an order unit then Theorem 5.1 improves as

**Theorem 5.3.** *If $E$ is a separable vector space with an order unit, and $K$ is a compact (with respect to pointwise convergence topology) subspace of $E^W$ then $K$ is metrizable.*

*Proof.* Let $\{x_n\}$ be countable everywhere dense subset of $E$. For every $f \in E^W$ put $M_n(f) = f(x_n)$. By the definition of pointwise convergence topology, every $M_n$ is a continuous function on $E^W$. If $M_n(f) = M_n(f')$ for all $n$ then continuous functions $f$ and $f'$ coincide on everywhere dense subset. Thus $\{M_n\}$ is a countable family of continuous functions which separate points of the space $E^W$, in particular of $K$. Hence $K$ is metrizable as each Hausdorff compact space which has a countable sequence of real-valued functions separating its points is metrizable.

□

**Corollary 5.4.** *If $E$ is separable vector space with an order unit then $E^O$ is a metrizable compact in the pointwise convergence topology.*


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# On generalized $\Gamma$-hyperideals in ordered $\Gamma$-semihypergroups

**Abul Basar[1*], Mohammad Yahya Abbasi[1] and Satyanarayana Bhavanari[2]**

[1]*Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia, New Delhi-110 025, India*
[2]*Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, Guntur - 522 510, Andhra Pradesh, India*
[*]*Corresponding author E-mail: basar.jmi@gmail.com*

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## Abstract

In this article, we deal with ordered generalized $\Gamma$-hyperideals in ordered $\Gamma$-semihypergroups. In particular, we study $(m,n)$-regular ordered $\Gamma$-semihypergroups in terms of ordered $(m,n)$-$\Gamma$-hyperideals. Moreover, we obtain some ideal theoretic results in ordered $\Gamma$-semihypergroups.

## 1. Introduction

A semigroup is an algebraic structure together with a nonempty set and an associative binary operation. The systematic study of semigroups started in the early 20th century. Semigroups are important in different areas of Mathematics. The concept of hyperstructures was introduced in 1934 as a suitable generalization of classical algebraic structures by Marty [1]. He obtained various results on hypergroups and applied them in different areas, for instance, in algebraic rational fractions, functions, and noncommutative groups. Thereafter, many research papers have been published on this subject and has been studied recently by many algebraists such as: Prenowitz, Corsini, Jantosciak, Leoreanu, Heideri, Davvaz, Hila, Gutan, Griffiths and Halzen.

It is a well known fact that, in a semigroup, the composition of two elements is an element, while in a semihypergroup, the composition of two elements is a nonempty set. In fact, semihypergroups are the simplest algebraic hyperstructures with the properties of closure and associativity. They are very important in certain applications. Around the 1940s, the general notions of the theory and some applications in Geometry, Physics and Chemistry were studied. Various classical notions of semigroups have been extended to semihypergroups and $\Gamma$-semihypergroups and a lot of results on ordered $\Gamma$-semihypergroups are obtained by many algebraists all over the world.

The monograph on application of hyperstructures to various area of study has been written by Corsini et al. [2]. Prenowitz et al. investigated its applications in Geometry [3]. Davvaz et al. wrote a book beginning with some basic notions related to ring theory and algebraic hyperstructures [4]. Various types of hyperrings are introduced and discussed in this book. For application in Chemistry and Physics, we refer [5]-[12]. It describes various types of hyperstructures: e-hyperstructures and transposition hypergroups. Heideri et al. studied ordered hyperstructures [11]. For semihypergroups, we refer [6, 7, 8]. Hila and Davvaz studied quasi-hyperideals of ordered semihypergroups [13]. Corsini also studied hypergroup theory [14]- [15]. The notion of a $\Gamma$-hyperideal of a $\Gamma$-semihypergroup was introduced by Anvariyeh et al. [16]. Hila et al. studied the structure of $\Gamma$-semihypergroups [17]. Recently, Basar et.al. obtained various types of hyperideals in ordered semihypergroups, ordered LA-$\Gamma$-semigroups and LA-$\Gamma$-semihypergroups [18]- [20].

In the second part of this paper, we recollect some basic definitions and then, we define the concepts of $(m,n)$-$\Gamma$-hyperideal (resp. generalized $(m,n)$-$\Gamma$-hyperideal) and $(m,n)$-regular ordered $\Gamma$-semihypergroup, where $m,n$ are non-negative integers. In the third part of this paper, we study ordered generalized $\Gamma$-hyperideals in ordered $\Gamma$-semihypergroups. In particular, we study $(m,n)$-regular ordered $\Gamma$-semihypergroups in terms of ordered $(m,n)$-$\Gamma$-hyperideals and obtain some ideal theoretic results in ordered $\Gamma$-semihypergroups.

***Email addresses and ORCID numbers*: basar.jmi@gmail.com, https://orcid.org/0000-0003-2740-0322 (A. Basar), mabbasi@jmi.ac.in, https://orcid.org/ 0000-0002-3283-1410) (M. Y. Abbasi), bhavanari2002@yahoo.co.in, https://orcid.org/0000-0003-0928-0582) (S. Bhavanari)**

## 2. Basic definitions

Let $H$ be a nonempty set, then the mapping $\circ : H \times H \to H$ is called a hyperoperation or a join operation on $H$, where $P^{\star}(H) = P(H) \setminus \{0\}$ is the set of all nonempty subsets of $H$. Let $A$ and $B$ be two nonempty sets. Then, a hypergroupoid $(S, \circ)$ is called a $\Gamma$-semihypergroups if for every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$,

$$x \circ \alpha \circ (y \circ \beta \circ z) = (x \circ \alpha \circ y) \circ \beta \circ z,$$

i.e.,

$$\bigcup_{u \in y \circ \alpha \circ z} x \circ \alpha \circ u = \bigcup_{v \in x \circ \alpha \circ y} v \circ \beta \circ z.$$

A $\Gamma$-semihypergroup $(S, \circ)$ together with a partial order " $\leq$ " on $S$ that is compatible with $\Gamma$-semihypergroup operation such that for all $x, y, z \in S$, we have

$$x \leq y \Rightarrow z \circ \alpha \circ x \leq z \circ \beta \circ y \text{ and } \mathrm{x} \circ \alpha \circ \mathrm{z} \leq \mathrm{y} \circ \beta \circ \mathrm{z},$$

is called an ordered $\Gamma$-semihypergroup. For subsets $A, B$ of an ordered $\Gamma$-semihypergroup $S$, the product set $A \circ \Gamma \circ B$ of the pair $(A, B)$ relative to $S$ is defined as below:

$$A \circ \Gamma \circ B = \{a \circ \gamma \circ b : a \in A, b \in B, \gamma \in \Gamma\},$$

and for $A \subseteq S$, the product set $A \circ \Gamma \circ A$ relative to $S$ is defined as $A^2 = A \circ \Gamma \circ A$.

For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m$ for some $m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

Let $A \subseteq S$. Then, for a non-negative integer $m$, the power of $A$ is defined by $A^m = A \circ \Gamma \circ A \circ \Gamma \circ A \circ \Gamma \circ A \cdots$, where $A$ occurs $m$ times. Note that the power vanishes if $m = 0$. So, $A^0 \circ \Gamma \circ S = S = S \circ \Gamma \circ A^0$.

In what follows, we denote ordered $\Gamma$-semihypergroup $(S, \circ, \Gamma, \leq)$ by $S$ unless otherwise specified.

Suppose $S$ is an ordered $\Gamma$-semihypergroup and $I$ is a nonempty subset of $S$. Then, $I$ is called an ordered right (resp. left) $\Gamma$-hyperideal of $S$ if

(i) $I \circ \Gamma \circ S \subseteq I (resp.\ S \circ \Gamma \circ I \subseteq I)$,
(ii) $a \in I, b \leq a$ for $b \in S \Rightarrow b \in I$.

We now define the concepts of $(m,n)$-$\Gamma$-hyperideal (resp. generalized $(m,n)$-$\Gamma$-hyperideal) and $(m,n)$-regular ordered $\Gamma$-semihypergroup, where $m, n$ are non-negative integers.

**Definition 2.1.** Suppose $B$ is a sub-$\Gamma$-semihypergroup (resp. nonempty subset) of an ordered $\Gamma$-semihypergroup $S$. Then, $B$ is called an $(m,n)$-$\Gamma$-hyperideal (resp. generalized $(m,n)$-$\Gamma$-hyperideal) of $S$, where $m, n$ are non-negative integers if (i) $B^m \circ \Gamma \circ S \circ \Gamma \circ B^n \subseteq B$, and (ii) for $b \in B$, $s \in S$, $s \leq b \Rightarrow s \in B$.

Note that in the above Definition 2.1, if we set $m = n = 1$, then $B$ is called a (generalized) bi-$\Gamma$-hyperideal of $S$.

**Definition 2.2.** Suppose $(S, \Gamma, \circ, \leq)$ is an ordered $\Gamma$-semihypergroup and $m, n$ are non-negative integers. Then, $S$ is called $(m,n)$-regular if for any $s \in S$, there exists $x \in S$ such that $s \leq s^m \circ \gamma_1 \circ x \circ \gamma_2 \circ s^n$ for $\gamma_1, \gamma_2 \in \Gamma$. Equivalently: $(S, \Gamma, \circ, \leq)$ is $(m,n)$-regular if $s \in (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ for all $s \in S$.

## 3. Ordered $(m,n)$-$\Gamma$-hyperideals

In this part, some classical notions of semigroups and semihypergroups have been extended to ordered $\Gamma$-semihypergroups and some results on generalized ordeded $(m,n)$-$\Gamma$-hyperideals and $(m,n)$-regular ordered $\Gamma$-semihypergroups are obtained. The results concern with ordered $\Gamma$-semihypergroup theory which represent the most general algebraic context in which these results are studied. We begin with the following:

**Lemma 3.1.** Suppose $(S, \Gamma, \circ, \leq)$ is an ordered $\Gamma$-semihypergroup and $s \in S$. Let $m, n$ be non-negative integers. Then, the intersection of all ordered (generalized) $(m,n)$-$\Gamma$-hyperideals of $S$ containing $s$, denoted by $[s]_{m,n}$, is an ordered (generalized) $(m,n)$-$\Gamma$-hyperideal of $S$ containing $s$.

**Proof.** Let $\{A_i : i \in I\}$ be the set of all ordered (generalized) $(m,n)$-$\Gamma$-hyperideals of $S$ containing $s$. Obviously, $\bigcap_{i \in I} A_i$ is a sub-$\Gamma$-semihypergroup of $S$ containing $s$. Let $j \in I$. As $\bigcap_{i \in I} A_i \subseteq A_j$, we have

$$(\bigcap_{i \in I} A_i)^m \circ \Gamma \circ S \circ \Gamma \circ (\bigcap_{i \in I} A_i)^n \subseteq A_j^m \circ \Gamma \circ S \circ \Gamma \circ A_j^n \subseteq A_j.$$

Therefore,

$$(\bigcap_{i \in I} A_i)^m \circ \Gamma \circ S \circ \Gamma \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i.$$

Let $a \in \bigcap_{i \in I} A_i$ and $b \in S$ so that $b \leq a$. Therefore, $b \in \bigcap_{i \in I} A_i$.
Hence, $\bigcap_{i \in I} A_i$ is an ordered (generalized) $(m,n)$-$\Gamma$-hyperideal of $S$ containing $s$.

**Theorem 3.2.** Suppose $(S, \Gamma, \circ, \leq)$ is an ordered $\Gamma$-semihypergroup and $s \in S$. Then, we have the following:

**(i)** $[s]_{m,n} = (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ for any positive integers $m, n$.
**(ii)** $[s]_{m,0} = (\bigcup_{i=1}^{m} s^i \cup s^m \circ \Gamma \circ S]$ for any positive integer $m$.
**(iii)** $[s]_{0,n} = (\bigcup_{i=1}^{n} s^i \cup s^n]$ for any positive integer $n$.

**Proof.** (i) $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n] \neq \emptyset$. Let $a, b \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ be such that $a \leq x$ and $b \leq y$ for some $x, y \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. If $x, y \in s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$ or $x \in \bigcup_{i=1}^{m+n} s^i$, $y \in s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$ or $x \in s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$, $y \in \bigcup_{i=1}^{m+n} s^i$, then, $x \circ \gamma \circ y \subseteq s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$, and therefore, $x \circ \gamma \circ y \subseteq \bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$ for $\gamma \in \Gamma$. It follows that $a \circ \gamma \circ b \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. Let $x, y \in \bigcup_{i=1}^{m+n} s^i$. Then, $x = s^p, y = s^q$ for some $1 \leq p, q \leq m+n$. Now, two cases arise: If $1 \leq p+q \leq m+n$, then, $x \circ \gamma \circ y \subseteq \bigcup_{i=1}^{m+n} s^i$. If $m+n < p+q$, then, $x \circ \gamma \circ y \subseteq s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$. So, $x \circ \gamma \circ y \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. This implies that $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ is a sub-$\Gamma$-semihypergroup of $S$. Moreover, we have

$$\begin{aligned}
(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^m \circ \Gamma \circ S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S] \circ \Gamma \circ S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \circ \Gamma \circ S \cup s^m \circ \Gamma \circ S \circ \Gamma \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-2} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S] \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-2} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-2} \circ \Gamma \circ (s^2 \circ \Gamma \circ S] \\
&\;\;\vdots \\
&\subseteq (s^m \circ \Gamma \circ S].
\end{aligned}$$

In a similar fashion, $S \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^n \subseteq (S \circ \Gamma \circ s^n]$. Therefore, $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^m \circ \Gamma \circ S \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^n \subseteq (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n] \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. So, $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ is an $(m,n)$-$\Gamma$-hyperideal of $S$ containing $s$; hence, $[s]_{m,n} \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. For the reverse inclusion, suppose $a \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ is such that $a \leq t$ for some $t \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. If $t = s^j$ for some $1 \leq j \leq m+n$, then, $t \in [s]_{m,n}$, therefore, $a \in [s]_{m,n}$. If $t \in s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$, by

$$s^m \circ \Gamma \circ S \circ \Gamma \circ s^n \subseteq ([s]_{m,n})^m \circ \Gamma \circ S \circ \Gamma \circ ([s]_{m,n})^n \subseteq [s]_{m,n},$$

then, $t \in [s]_{m,n}$; hence, $a \in [s]_{m,n}$. This implies that $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n] \subseteq [s]_{m,n}$.
Hence, $[s]_{m,n} = (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$.
(ii) and (iii) can be proved in a similar fashion.

**Lemma 3.3.** Suppose $(S, \Gamma, \circ, \leq)$ is an ordered $\Gamma$-semihypergroup and $s \in S$. Suppose $m, n$ are non-negative integers. Then, we have the following:

**(i)** $([s]_{m,0})^m \circ \Gamma \circ S \subseteq (s^m \circ \Gamma \circ S]$.
**(ii)** $S \circ \Gamma \circ ([s]_{0,n})^n \subseteq (S \circ \Gamma \circ s^n]$.
**(iii)** $([s]_{m,n})^m \circ \Gamma \circ S \circ \Gamma \circ ([s]_{m,n})^n \subseteq (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$.

**Proof.** (i) Using Theorem 3.2, we have

$$\begin{aligned}
([s]_{m,0})^m \circ \Gamma \circ S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^m \circ \Gamma \circ S \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S] \circ \Gamma \circ S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \circ \Gamma \circ S \cup s^m \circ \Gamma \circ S \circ \Gamma \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&\;\;\vdots \\
&\subseteq (s^m \circ \Gamma \circ S].
\end{aligned}$$

Hence, $([s]_{m,0})^m \circ \Gamma \circ S \subseteq (s^m \circ \Gamma \circ S]$.
(ii) can be proved similarly as (i).

(iii) Applying Theorem 3.2, we have

$$
\begin{aligned}
([s]_{m,n})^m\circ\Gamma\circ S &= (\bigcup_{i=1}^{m+n} s^i\cup s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]^m\circ\Gamma\circ S\\
&= (\bigcup_{i=1}^{m+n} s^i\cup s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]^{m-1}\circ\Gamma\circ(\bigcup_{i=1}^{m+n} s^i\cup s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]\circ\Gamma\circ S\\
&\subseteq (\bigcup_{i=1}^{m+n} s^i\cup s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]^{m-1}\circ\Gamma\circ(\bigcup_{i=1}^{m+n} s^i\circ\Gamma\circ S\cup s^m\circ\Gamma\circ S\circ\Gamma\circ s^n\circ\Gamma\circ S]\\
&= (\bigcup_{i=1}^{m+n} s^i\cup s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]^{m-1}\circ\Gamma\circ(s\circ\Gamma\circ S]\\
&\cdot\\
&\cdot\\
&\cdot\\
&= (s^m\circ\Gamma\circ S].
\end{aligned}
$$

Therefore, $([s]_{m,n})^m\circ\Gamma\circ S\subseteq(s^m\circ\Gamma\circ S]$. In a similar fashion, $S\circ\Gamma\circ([s]_{m,n})^n\subseteq(S\circ\Gamma\circ s^n]$. So,

$$
\begin{aligned}
([s]_{m,n})^m\circ\Gamma\circ S\circ\Gamma\circ([s]_{m,n})^n &\subseteq (s^m\circ\Gamma\circ S]\circ\Gamma\circ([s]_{m,n})^n\\
&\subseteq (s^m\circ\Gamma\circ(S\circ\Gamma\circ([s]_{m,n})^n)]\\
&\subseteq (s^m\circ\Gamma\circ(S\circ\Gamma\circ s^n]]\\
&\subseteq (s^m\circ\Gamma\circ S\circ\Gamma\circ s^n].
\end{aligned}
$$

Hence, (iii) holds.

**Theorem 3.4.** Suppose $(S,\Gamma,\circ,\leq)$ is an ordered $\Gamma$-semihypergroup and $m,n$ are non-negative integers. Let $\mathscr{R}_{(m,0)}$ and $\mathscr{L}_{(0,n)}$ be the set of all ordered $(m,0)$-$\Gamma$-hyperideals and the set of all ordered $(0,n)$-$\Gamma$-hyperideals of $S$, respectively. Then,

**(i)** $S$ is $(m,0)$-regular if and only if for all $R\in\mathscr{R}_{(m,0)}, R=(R^m\circ\Gamma\circ S]$.
**(ii)** $S$ is $(0,n)$-regular if and only if for all $L\in\mathscr{R}_{(0,n)}, L=(S\circ\Gamma\circ L^n]$.

**Proof.**(i) Suppose $S$ is $(m,0)$-regular. Then,

$$\text{for all } s\in S, s\in(s^m\circ\Gamma\circ S]. \tag{3.1}$$

Suppose $R\in\mathscr{R}_{(m,0)}$. As $R^m\circ\Gamma\circ S\subseteq R$, and $R=(R]$, we have $(R^m\circ\Gamma\circ S]\subseteq R$. If $s\in R$, by Equation (3.1), we obtain $s\in(s^m\circ\Gamma\circ S]\subseteq(R^m\circ\Gamma\circ S]$, therefore, $R\subseteq(R^m\circ\Gamma\circ S]$. So, $(R^m\circ\Gamma\circ S]=R$.
Conversely, suppose

$$\text{for all } R\in\mathscr{R}_{(m,0)}, R=(R^m\circ\Gamma\circ S]. \tag{3.2}$$

Suppose $s\in S$. Therefore, $[s]_{m,0}\in\mathscr{R}_{(m,0)}$. By Equation (3.2), we obtain

$$[s]_{m,o}=(([s]_{m,0})^m\circ\Gamma\circ S].$$

Applying Lemma 3.3, we obtain

$$[s]_{m,o}\subseteq(s^m\circ\Gamma\circ S].$$

Therefore, $s\in(s^m\circ\Gamma\circ S]$.
Hence, $S$ is $(m,0)$-regular.
(ii) It can be proved analogously.

**Theorem 3.5.** Suppose $(S,\Gamma,\circ,\leq)$ is an ordered $\Gamma$-semihypergroup and $m,n$ are non-negative integers. Suppose $\mathscr{A}_{(m,n)}$ is the set of all ordered $(m,n)$-$\Gamma$-hyperideals of $S$. Then,

$$S \textit{ is } (m,n)-\textit{regular} \iff \text{for all } A\in\mathscr{A}_{(m,n)}, A=(A^m\circ\Gamma\circ S\circ\Gamma\circ A^n]. \tag{3.3}$$

**Proof.** Consider the following four conditions:
Case (i): $m=0$ and $n=0$. Then, Equation (3.3) implies
$S$ is (0, 0)-regular $\iff$ for all $A\in\mathscr{A}_{(0,0)}, A=S$ because $\mathscr{A}_{(0,0)}=\{S\}$ and $S$ is (0, 0)-regular.
Case (ii): $m=0$ and $n\neq 0$. Therefore, Equation (3.3) implies
$S$ is (0, n)-regular$\iff$ for all $A\in\mathscr{A}_{(0,n)}, A=(S\circ\Gamma\circ A^n]$. This follows by Theorem 3.4(ii).
Case (iii): $m\neq 0$ and $n=0$. This can be proved applying Theorem 3.4(i).
Case (iv): $m\neq 0$ and $n\neq 0$. Suppose $S$ is (m, n)-regular. Therefore,

$$\text{for all } s\in S, s\in(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]. \tag{3.4}$$

Let $A\in\mathscr{A}_{(m,n)}$. As $A^m\circ\Gamma\circ S\circ\Gamma\circ A^n\subseteq A$ and $A=(A]$, we obtain $(A^m\circ\Gamma\circ S\circ\Gamma\circ A^n]\subseteq A$. Suppose $s\in A$. Applying Equation (3.4), $s\in(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]\subseteq(A^m\circ\Gamma\circ S\circ\Gamma\circ A^n]$. Therefore, $A\subseteq(A^m\circ\Gamma\circ S\circ\Gamma\circ A^n]$. Hence, $A=(A^m\circ\Gamma\circ S\circ\Gamma\circ A^n]$.
Conversely, suppose $A=(A^m\circ\Gamma\circ S\circ\Gamma\circ A^n]$ for all $A\in\mathscr{A}_{(m,n)}$. Suppose $s\in S$. As $[s]_{m,n}\in\mathscr{A}_{(m,n)}$, we have

$$[s]_{m,n}=(([s]_{m,n})^m\circ\Gamma\circ S\circ\Gamma\circ([s]_{m,n})^n].$$

Applying Lemma 3.3(iii), we obtain $[s]_{m,n}\subseteq(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]$, therefore, $s\in(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]$.
Hence, $S$ is $(m,n)$-regular.

**Theorem 3.6.** Suppose $(S,\Gamma,\circ,\leq)$ is an ordered $\Gamma$-semihypergroup and $m,n$ are non-negative integers. Suppose $\mathscr{R}_{(m,0)}$ and $\mathscr{L}_{(0,n)}$ is the set of all $(m,0)$-$\Gamma$-hyperideals and $(0,n)$-$\Gamma$-hyperideals of $S$, respectively. Then,

$$S \textit{ is } (m,n)\text{–regular ordered } \Gamma-\text{semihypergroup} \iff \text{for all } \mathrm{R}\in\mathscr{R}_{(\mathrm{m},0)},\ \text{for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \quad R\cap L=(R^m\circ\Gamma\circ L\cap R\circ\Gamma\circ L^n]. \tag{3.5}$$

**Proof.** Consider the following four cases:
Case (i): $m=0$ and $n=0$. Therefore, Equation (3.5) implies
$S$ is (0, 0)-regular $\iff$ for all $\mathrm{R}\in\mathscr{R}_{(0,0)}$ for all $\mathrm{L}\in\mathscr{L}_{(0,0)}, \mathrm{R}\cap\mathrm{L}=(\mathrm{L}\cap\mathrm{R}]$ because $\mathscr{R}_{(0,0)}=\mathscr{L}_{(0,0)}=\{S\}$ and $S$ is $(0,0)$-regular.
Case (ii): $m=0$ and $n=0$. Therefore, Equation (3.5) implies
$S$ is $(0,n)$-regular $\iff$ for all $\mathrm{R}\in\mathscr{R}_{(0,\mathrm{n})}$ for all $\mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}=(\mathrm{L}\cap\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]$. Suppose $S$ is $(0,n)$-regular. Suppose $R\in\mathscr{R}_{(0,0)}$ and $L\in\mathscr{L}_{(0,n)}$. By Theorem 3.4(ii), $L=(S\circ\Gamma\circ L^n]$. As $R\in\mathscr{R}_{(0,0)}$, we have $R=S$, therefore, $R\cap L=L$. Therefore,

$$(L\cap R\circ\Gamma\circ L^n]=(L\cap S\circ\Gamma\circ L^n]=((S\circ\Gamma\circ L^n]\cap S\circ\Gamma\circ L^n]=(S\circ\Gamma\circ L^n]=L=R\cap L.$$

Conversely, suppose

$$\text{for all } \mathrm{R}\in\mathscr{R}_{(0,0)},\ \text{for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}=(\mathrm{L}\cap\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]. \tag{3.6}$$

If $R\in\mathscr{R}_{(0,0)}$, then $R=S$. If $L\in\mathscr{L}_{(0,n)}$, $S\circ\Gamma\circ L^n\subseteq L$ and $L=(L]$. Therefore, Equation (3.6) implies

$$\text{for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{L}=(\mathrm{S}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}].$$

Applying Theorem 3.4(ii), $S$ is $(0,n)$-regular.
Case (iii): $m\neq 0$ and $n=0$. This can be proved as before.
Case (iv): $m\neq 0$ and $n\neq 0$. Suppose that $S$ is $(m,n)$-regular. Suppose $R\in\mathscr{R}_{(m,0)}$ and $L\in\mathscr{L}_{(0,n)}$. To prove that $R\cap L\subseteq(R^m\circ\Gamma\circ L]\cap(R\circ\Gamma\circ L^n]$, suppose $s\in R\cap L$. We have

$$s\in(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]\subseteq(s^m\circ\Gamma\circ L]\subseteq(R^m\circ\Gamma\circ L] \text{ and } \mathrm{s}\in(\mathrm{s}^{\mathrm{m}}\circ\Gamma\circ\mathrm{S}\circ\Gamma\circ\mathrm{s}^{\mathrm{n}}]\subseteq(\mathrm{R}\circ\Gamma\circ\mathrm{s}^{\mathrm{n}}]\subseteq(\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}].$$

Hence, $R\cap L\subseteq(R^m\circ\Gamma\circ L]\cap(R\circ\Gamma\circ L^n]$. As

$$(R^m\circ\Gamma\circ L]\subseteq(R^m\circ\Gamma\circ S]\subseteq(R]=R \text{ and } (\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]\subseteq(\mathrm{S}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]\subseteq(\mathrm{L}]=\mathrm{L}.$$

This implies that $(R^m\circ\Gamma\circ L]\cap(R\circ\Gamma\circ L^n]\subseteq R\cap L$, therefore, $R\cap L=(R^m\circ\Gamma\circ L]\cap(R\circ\Gamma\circ L^n]$.
Conversely, suppose

$$\text{for all } \mathrm{R}\in\mathscr{R}_{(\mathrm{m},0)},\ \text{for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}=(\mathrm{R}^{\mathrm{m}}\circ\Gamma\circ\mathrm{L}\cap\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]. \tag{3.7}$$

Suppose $R=[s]_{m,0}$ and $L=S$. Applying Equation (3.7), we obtain $[s]_{m,0}\subseteq(([s]_{m,0})^m\circ\Gamma\circ S]$. Applying Lemma 3.3, we obtain

$$[s]_{m,0}\subseteq(s^m\circ\Gamma\circ S]. \tag{3.8}$$

In a similar fashion, we obtain

$$[s]_{0,n}\subseteq(S\circ\Gamma\circ s^n]. \tag{3.9}$$

As $R^m\subseteq R$ and $L^n\subseteq L$, by Equation (3.7), we have

$$\text{for all } \mathrm{R}\in\mathscr{R}_{(\mathrm{m},0)},\ \text{for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}\subseteq(\mathrm{R}\circ\Gamma\circ\mathrm{L}].$$

As $(s^m\circ\Gamma\circ S]\in\mathscr{R}_{(m,0)}$ and $(S\circ\Gamma\circ s^n]\in\mathscr{L}_{(0,n)}$, we obtain

$$(s^m\circ\Gamma\circ S]\cap(S\circ\Gamma\circ s^n]\subseteq((s^m\circ\Gamma\circ S]\circ\Gamma\circ(S\circ\Gamma\circ s^n]]\subseteq(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n].$$

Applying Equations (3.8) and (3.9), we obtain

$$[s]_{m,0}\cap[s]_{0,n}\subseteq(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n].$$

Hence, $S$ is $(m,n)$-regular.

## 4. Conclusion

In this paper, we introduced the concepts of $(m,n)$-$\Gamma$-hyperideal (resp. generalized $(m,n)$-$\Gamma$-hyperideal) and $(m,n)$-regular ordered $\Gamma$-semihypergroup, where $m,n$ are non-negative integers and studied some properties of $(m,n)$-$\Gamma$-hyperideals in ordered $\Gamma$-semihypergroups. In particular, we studied $(m,n)$-regular ordered $\Gamma$-semihypergroups. We proved that if $(S,\Gamma,\circ,\leq)$ is an ordered $\Gamma$-semihypergroup, where $m,n$ are non-negative integers and if $\mathscr{A}_{(m,n)}$ is the set of all ordered $(m,n)$-$\Gamma$-hyperideals of $S$. Then, $S \textit{ is } (m,n)-\textit{regular} \iff \text{for all } \mathrm{A}\in\mathscr{A}_{(\mathrm{m},\mathrm{n})}, \mathrm{A}=(\mathrm{A}^{\mathrm{m}}\circ\Gamma\circ\mathrm{S}\circ\Gamma\circ\mathrm{A}^{\mathrm{n}}]$. We also proved that if $(S,\Gamma,\circ,\leq)$ is an ordered $\Gamma$-semihypergroup, where $m,n$ are non-negative integers; and if $\mathscr{R}_{(m,0)}$, $\mathscr{L}_{(0,n)}$ is the set of all $(m,0)$-$\Gamma$-hyperideals and $(0,n)$-$\Gamma$-hyperideals of $S$, respectively. Then, $S \textit{ is } (m,n)\text{–regular ordered } \Gamma-\text{semihypergroup} \iff \text{for all } \mathrm{R}\in\mathscr{R}_{(\mathrm{m},0)},\ \text{for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}=(\mathrm{R}^{\mathrm{m}}\circ\Gamma\circ\mathrm{L}\cap\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]$. The results of this article can also be applied on semihypergroups and on ordered semihypergroups by some suitable modifications. We hope that this work will provide the basis for further study on ordered $\Gamma$-semihypergroups.

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# The Third Isomorphism Theorem on UP-Bialgebras

**Daniel A. Romano**

*International Mathematical Virtual Institute, Banja Luka, Bosnia and Herzegovina*

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### Abstract

The concept of UP-bialgebras was introduced and analyzed by Mosrijai and Iampan at the beginning of 2019. Theorem that we can look at as the First theorem on UP-biisomorphism between the UP-bialgebras is given in our forthcoming text [9]. In this article we construct a form of the third theorem on UP-biisomorphism between UP-bialgebras.

## 1. Introduction

The concept of UP-algebras developed by Iampan in [1]. Examining the substructures in this algebra are done for example in articles [2, 3]. This author took part in analyzing the properties of UP-algebras and their substructures, also [4]-[6]. Some forms of the isomorphism theorem between UP-algebras can be found in [2, 3, 5, 6].

The concept of bi-algebraic structures was studied by Vasantha Kandasamy in 2003 [7]. The concept of UP-bialgebras with the associated substructures and their mutual connections can be found in [8]. In the forthcoming article [9], this author offered one form the first theorem of the isomorphism between the UP-bialgebras.

In this article we expose a form of the second isomorphism theorem between UP-bialgebras.

## 2. Preliminaries

In this section, we will present the necessary previous concepts of UP-algebras, their substructures and UP-homomorphisms taken from texts [1, 2, 3, 8]. We will also expose their mutual relationships in the form of proclaims necessary for our intention.

### 2.1. UP-algebras

In this subsection we will describe some elements of UP-algebras and their substructures necessary for our intentions in this text.

**Definition 2.1** ([1]). *An algebra $L = (L, \cdot, 0)$ of type $(2,0)$ is called a UP-algebra where $L$ is a nonempty set, $'\cdot'$ is a binary operation on $L$, and $0$ is a fixed element of $L$ (i.e. a nullary operation) if it satisfies the following axioms:*
*(UP-1)* $(\forall x, y \in L)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
*(UP-2)* $(\forall x \in L)(0 \cdot x = x)$,
*(UP-3)* $(\forall x \in L)(x \cdot 0 = 0)$, *and*
*(UP-4)* $(\forall x, y \in L)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

**Definition 2.2** ([1]). *A nonempty subset $J$ of a UP-algebra $(L, \cdot, 0)$ is called*
(1) *a UP-subalgebra of $L$ if $(\forall x, y \in J)(x \cdot y \in J)$.*
(2) *a UP-ideal of $L$ if*
*(i)* $0 \in J$; *and*
*(ii)* $(\forall x, y, z \in L)((x \cdot (y \cdot z) \in J \wedge y \in J) \Longrightarrow x \cdot z \in J)$.

***Email addresses and ORCID numbers***: bato49@hotmail.com, https://orcid.org/0000-0003-1148-3258 (D. A. Romano)

The set $\{0\}$ is a trivial UP-subalgebra (trivial UP-ideal) of $L$.
In the article [6], Theorem 3.3, it has been shown that the conditions (i) and (ii) in the preceding definition are equivalent to the following conditions
(iii) $(\forall x, y \in L)((x \cdot y \in J \wedge x \in J) \Longrightarrow y \in J)$,
(iv) $(\forall x, y \in L)(y \in J \Longrightarrow x \cdot y \in J)$.

**Definition 2.3** ([1]). *Let $(L, \cdot, 0_L)$ and $(M, \cdot^{'}, 0_M)$ be two UP-algebras. A mapping $f : L \longrightarrow M$ is called a UP-homomorphism if*

$$(\forall x, y \in L)(f(x \cdot y) = f(x) \cdot^{'} f(y)).$$

*A UP-homomorphism $f : L \longrightarrow M$ is called*
(3) *a UP-epimorphism if $f$ is surjective,*
(4) *a UP-monomorphism if $f$ is injective, and*
(5) *a UP-isomorphism if $f$ is bijective.*

Let $f$ be a mapping form UP-algebra $L$ to UP-algebra $M$, and let $A$ and $B$ be nonempty subsets of $L$ and of $M$, respectively. The set $f(A) = \{f(x) | x \in A\}$ is called the image of $A$ under $f$. In particular, $f(L)$ which denoted by $Im(f)$ is called the image of $f$. The dually set $f^{-1}(B) = \{x \in L | f(x) \in B\}$ is called the inverse image of $B$ under $f$. Especially, the set $Ker(f) = f^{-1}(\{0_M\}) = \{x \in L : f(x) = 0_M\}$ is called the kernel of $f$.

A relation of congruence on UP-algebras is introduced in [1] by Definition 3.1 and Proposition 3.5 on this way: If $J$ is a UP-ideal of a UP-algebra $L$, then the relation $\sim_J$ defined by

$$(\forall x, y \in L)(x \sim_J y \Longleftrightarrow (x \cdot y \in J \wedge y \cdot x \in J))$$

is a UP-congruence on $L$. Further on, any relation of congruence on UP-algebras has this form according to the claim (1) of Theorem 3.6 and the claim (1) of Theorem 3.7 in [1]. In particular, if $f : L \longrightarrow M$ is a UP-homomorphism between UP-algebras, then the relation $\sim_f$ determined by $Ker(f)$ is a UP-congruence in $L$. The factor-set $L/\sim_J = \{[x]_{\sim_J} : x \in L\}$ is a UP-algebra according to the claim (4) of Theorem 3.7 in [1]. We also use the following notion $L/J = \{[x]_J : x \in L\}$ to denote this factor algebra.

### 2.2. UP-bialgebras

The concept of UP-bialgebras and some their substructures were introduced and analyzed by Mosrijai and Iampan in the recently published work [8]. In this subsection, taking into account their determinations, we describe the concept of UP-bialgebras and some notions connected with them. So, in this subsection, we will repeat the concept of UP-bialgebras and the notions of UP-bisubalgebras and UP-biideals of UP-bialgebras, and will expose some results related to substructures of such algebras.

**Definition 2.4** ([8], Definition 3.1). *An algebra $L = (L, \cdot, *, 0)$ of type $(2,2,0)$ is called a UP-bialgebra where $L$ is a nonempty set, $\cdot$ and $*$ two are binary internal operations on $L$, and $0$ is a fixed element of $L$ if there exist two distinct proper subsets $L_1$ and $L_2$ of $L$ with respect to $\cdot$ and $*$, respectively, such that*
*(UPB-1)* $L = L_1 \cup L_2$;
*(UPB-2)* $(L_1, \cdot, 0)$ *is a UP-algebra, and*
*(UPB-3)* $(L_2, *, 0)$ *is a UP-algebra.*
*We will denote the UP-bialgebra by $L = L_1 \uplus L_2$. In case of $L_1 \cap L_2 = \{0\}$, we call $L$ zero disjoint.*

**Definition 2.5** ([8], Definition 3.7). *A nonempty subset $J$ of a UP-bialgebra $L = L_1 \uplus L_2$ is called a UP-biideal (UP-bisubalgebra) of $L$ if there exist subsets $J_1$ of $L_1$ and $J_2$ of $L_2$ with respect to $\cdot$ and $*$, respectively, such that*
(6) $J_1 \neq J_2$ *and* $J = J_1 \cup J_2$;
(7) $(J_1, \cdot, 0)$ *is a UP-ideal (UP-subalebra) of* $(L_1, \cdot, 0)$, *and*
(8) $(J_2, *, 0)$ *is a UP-ideal (UP-subalgebra) of* $(L_2, *, 0)$.
*In case of $J_1 \cap L_2 = \{0\} = L_1 \cap J_2$, we call $S$ zero disjoint.*

The important relationship between these notions is the following:

**Proposition 2.6** ([9]). *If $J \supset \{0\}$ is a UP-subalgebra (resp., UP-ideal) of UP-algebra $L_1$ (of UP-algebra $L_2$, respectively), such that $\{0\} \neq J$, then on $J$ can be seen as a zero disjoint UP-bisubgebra (resp., UP-biideal) of UP-bialgebra $L = L_1 \uplus L_2$.*

### 2.3. UP-bihomomorphisms

Let $f : L \longrightarrow M$ be a function from a set $L$ to a set $M$ and $C \subseteq L$. Then the restriction of $f$ to $C$ is the function $f_{[C]} : C \longrightarrow M$.

**Definition 2.7** ([8], Definition 4.1). *Let $L = L_1 \uplus L_2$ be a UP-bialgebra with two binary operations $\cdot$ and $*$, and let $M = M_1 \uplus M_2$ be a UP-bialgebra with two binary operations $\cdot^{'}$ and $*^{'}$. A mapping $f$ form $L = L_1 \uplus L_2$ to $M = M_1 \uplus M_2$ is called a UP-bihomomorphism if it satisfies the following properties:*
(9) $f_{[L_1]} : L_1 \longrightarrow M_1$ *is a UP-homomorphism, and*
(10) $f_{[L_2]} : L_2 \longrightarrow M_2$ *is a UP-homomorphism.*
*We say that these restrictions are natural restrictions. A UP-bihomomorphism $f : L \longrightarrow M$ is called*
*- a UP-biepimorphism if the natural restriction are UP-epimorphisms,*
*- a UP-bimonomorphism if the natural restriction are UP-monomorphisms, and*
*- a UP-biisomorphism if the natural restriction are UP-isomorphisms.*

**Proposition 2.8** ([8]). *let $f : L_1 \uplus L_2 \longrightarrow M_1 \uplus M_2$ be a UP-bihomomorphism. Then the following statements hold:*
(11) *$f(0_L) = 0_M$, and*
(12) *$Ker(f) = \{0_L\}$ if and only if $f$ is an injective mapping;*
(13) *if $J$ is a UP-bisubalgebra of $L$, then the image $f(J)$ is a UP-bisubalgebra of $B$;*
(14) *if $J = J_1 \cup J_2$ is a UP-biideal of $L$, and $J_1$ and $J_2$ are subsets of $L_1$ and of $L_2$, respectively, with $Ker(f) \subseteq J_1 \cap J_2$, then the image $f(J)$ is a UP-biideal of $M$;*
(15) *if $D$ is a UP-bisubalgebra of $M$, then the inverse image $f^{-1}(D)$ is a a UP-bisubalgebra of $L$; and*
(16) *if $D$ is a UP-biideal of $M$, then the inverse image $f^{-1}(D)$ is a UP-biideal of $L$.*

## 3. The main results

In our forthcoming article [9], we formulated and proved a form of the first isomorphism theorem between UP-bialgebras. To this direction, we used the following lemma.

**Lemma 3.1** ([9]). *Let $L = L_1 \uplus L_2$ and $M = M_1 \uplus M_2$ be two UP-bialgebras and let $f : L \longrightarrow M$ be a UP-bihomomorphism. Then the set $Ker(f_{[A_1]}) \cup Ker(f_{[A_2]})$ is a UP-biideal of $L$ and $Ker(f) = Ker(f_{[L_1]}) \uplus Ker(f_{[L_2]})$ holds.*

Let $L = L_1 \uplus L_2$ be a UP-bialgebra with two binary operations $\cdot$ and $*$, and let $M = M_1 \uplus M_2$ be a UP-bialgebra with two binary operations $\cdot'$ and $*'$ and let $f : L \longrightarrow M$ be a UP-bihomomorphism. Let $\sim_1$ is the congruence on $L_1$ generated by the UP-ideal $Ker(f_{[L_1]})$

$$\forall x, y \in L_1)(x \sim_1 y \Longleftrightarrow (x \cdot y \in Ker(f_{[L_1]}) \wedge y \cdot x \in Ker(f_{[L_1]})))$$

and let $\sim_2$ be the congruence on $L_2$ generated by the UP-ideal $Ker(f_{[L_2]})$

$$(\forall x, y \in L_2)(x \sim_2 y \Longleftrightarrow (x * y \in Ker(f_{[L_2]}) \wedge y * x \in Ker(f_{[L_2]}))).$$

Then we can construct the factor-UP-algebra $L_1/\sim_1$ and the factor-UP-algebra $L_2/\sim_2$. So, $L_1/\sim_1 \uplus L_2/\sim_2$ is a UP-bialgebra with two binary operation $'\odot'$ and $'\circledast'$ defined by

$$(\forall [x]_{\sim_1}, [y]_{\sim_1} \in L_1/\sim_1))([x]_{\sim_1} \odot [y]_{\sim_1} = [x \cdot y]_{\sim_1})$$

and

$$(\forall [x]_{\sim_2}, [y]_{\sim_2} \in L_2/\sim_2))([x]_{\sim_2} \circledast [y]_{\sim_2} = [x * y]_{\sim_2}).$$

Previous analysis enables us to introduce the following determination: Let $L = L_1 \uplus L_2$ be a UP-bialgebra. For a pair $(\sim_1, \sim_2)$ the relation of congruence $\sim_1$ on $L_1$ and $\sim_2$ on $L_2$ we write $L_1 \uplus L_2/(\sim_1, \sim_2)$ instead of $L_1/\sim_1 \uplus L_2/\sim_2$. If $\pi_1 : L_1 \longrightarrow L_1/\sim_1$ and $\pi_2 : L_2 \longrightarrow L_2/\sim_2$ are canonical UP-epimorphisms, then there is a unique canonical UP-epimorphism $\pi : L_1 \uplus L_2 \longrightarrow L_1 \uplus L_2/(\sim_1, \sim_2)$ such that $\pi_{[L_1]} = \pi_1$ and $\pi_{[L_2]} = \pi_2$ . Particulary, there is a unique UP-epimorphism $\pi : L_1 \uplus L_2 \longrightarrow (L_1 \uplus L_2)/(Ker(f_{[L_1]}), Ker(f_{[L_2]}))$. The first theorem of isomorphism between UP-bialgebras has the form in which for simplicity we write $A/Ker(f)$ instead of $A/(Ker(f_{[A_1]}), Ker(f_{[A_2]}))$.

**Theorem 3.2** ([9]). *Let $f : L \longrightarrow M$ be a UP-bihomomorphism. Then there exists the unique UP-bihomomorphism $g : L/Ker(f) \longrightarrow M$ such that $f = g \circ \pi$. In addition, for the UPB-subalgebra $f(L)$ of $M$ holds $L/Ker(f) \cong f(L)$.*

Let us analyze now the following situation:

Let $J$ and $K$ be UP-biideals of a UP-bialgebra $L$ such that $J \subseteq K$. Then there exist UP-ideals $J_1$ and $K_1$ of the UP-algebra $L_1$ and there exist UP-ideals $J_2$ and $K_2$ of the UP-algebra $L_2$ such that $J_1 \neq J_2$ and $J = J_1 \cup J_2$, and $K_1 \neq K_2$ and $K = K_1 \cup K_2$, by Definition 2.5. If $J_1 \subseteq K_1$ and $J_2 \subseteq K_2$ hold, then $K_1/J_1$ is a UP-ideal of UP-algebra $L_1/J_1$ and $K_2/J_2$ is a UP-ideal of UP-algebra $L_2/J_2$. From here follows $L_1/K_1 \cong (L_1/J_1)/(K_1/J_1)$ according to Theorem 3.10 in [6]. We also have it $L_2/K_2 \cong (L_2/J_2)/(K_2/J_2)$ according to same theorem. So, the set $K_1/J_1 \uplus K_2/J_2$ is a UP-biideal of the UP-bialgebra $L_1/J_1 \uplus L_2/J_2$. Thus, the mapping $g_1 : L_1/J_1 \longrightarrow L_1/K_1$ has $Ker(g_1) = K_1/J_1$. Analogously, the mapping $g_2 : L_2/J_2 \longrightarrow L_2/K_2$ has $Ker(g_2) = K_2/J_2$ as core. Therefore, the homomorphism $g : L/(J_1, J_2) \longrightarrow L/(K_1, K_2)$, determined by $g_{[L_1/J_1]} = g_1$ and $g_{[L_2/J_2]} = g_2$ has the core exactly $K_1/J_1 \uplus K_2/J_2$.

The previous analysis is a motivation for the following theorem can be seen as the Third isomorphism theorem between UP-bialgebras.

**Theorem 3.3.** *Let $L = L_1 \uplus L_2$ be a UP-bialgebra and let $J = J_1 \uplus J_2$ and $K = K_1 \uplus K_2$ be UP-biideals such that $J_1 \subseteq K_1$ and $J_2 \subseteq K_2$. Then*

$$L/(K_1, K_2) \cong (L/(J_1, J_2))/(K_1/J_1, K_2/J_2)$$

*holds.*

### Final Observation

The concept of UP-algebras introduced and first results on them given by Iampan 2017 [1]. This author took part in analyzing the properties of UP-algebras and their substructures, also [4, 5, 6]. Algebraic bi-strukture was analyzed by Vasantha Kandasamy in 2003 [7]. The concept of UP-bialgebras introduced and the first results ware given by Mosrijai and Iampan at the beginning of 2019 [8]. Using by the concept of UP-bihomorphisms, introduced in [8], in this article we formulated and proved the theorem (Theorem 3.3) , which can be viewed as the Third isomorphism theorem between the UP-bialgebras.

Of course, there remains an open possibility of formulating and trying to prove other forms of these two isomorphism theorems between the UP-bialgebra.

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# The Nyström Method and Convergence Analysis for System of Fredholm Integral Equations

**Huimin Zhou[1] and Qisheng Wang[1*]**

[1]*School of Mathematics and Computational Science, Wuyi University, Jiangmen, Guangdong, P.R. China*
[*]*Corresponding author E-mail: 282228006@qq.com*

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**Abstract**

In this paper, the efficient numerical solutions of a class of system of Fredholm integral equations are solved by the Nyström method, which discretizes the system of integral equations into solving a linear system. The existence and uniqueness of the exact solutions are proved by the Banach fixed point theorem. The format of the Nyström solutions is given, especially with the composite Trapezoidal and Simpson rules. The results of error estimation and convergence analysis are obtained in the infinite norm sense. The validity and reliability of the theoretical analysis are verified by numerical experiments.

## 1. Introduction

In this paper, we consider a class of system of Fredholm integral equations of the form

$$\begin{cases} u(x) = f(x) + \int_a^b [k_{11}(x,y)u(y) + k_{12}(x,y)v(y)]dy \\ v(x) = g(x) + \int_a^b [k_{21}(x,y)u(y) + k_{22}(x,y)v(y)]dy, \end{cases} \tag{1.1}$$

where the known functions $f(x), g(x) \in C[a,b]$, $k_{pq}(x,y) \in C([a,b]\times[a,b])$, $p,q = 1,2$, $u(x)$, $v(x) \in C[a,b]$ are the unknown functions. The integral equation problem has been two hundred years old, and the integral equation is widely used in the study of physics, especially in mechanics, magnetism, architecture and etc. Since the exact solution of the integral equation problem is difficult to find, its high-precision numerical solutions are often studied. Many numerical methods are used for numerical solution of Fredholm integral equation, for instance, Taylor collocation method [1], Galerkin projection and Least squares approximation method [2], variational iteration and fixed point iterative method [3], Nyström method and mechanical quadrature method [4]-[7], meshless methods [8] and multiscale methods [9], and so on. However, there is not much paper about solving the system of integral equations. This paper will study the Nyström method of the system of Fredholm integral equations.

## 2. A sufficient condition for the existence and uniqueness of exact solutions

According to Banach fixed point theorem, a sufficient condition for the existence and uniqueness of exact solution of system of Fredholm integral equations (1.1) is proposed. First, for (1.1), we structure a function vector space

$$V^2[a,b] = \{s(x) = [s_1(x), s_2(x)]^T, s_i(x) \in C[a,b], \ \ i = 1,2\}$$

and a functional matrix space

$$V^{2\times 2}([a,b]\times[a,b]) = \{(s_{ij}(x,y))_{2\times 2}, s_{ij}(x,y) \in C([a,b]\times[a,b]), \ \ i,j = 1,2\}.$$

For

$$K(x,y) = (k_{pq}(x,y))_{2\times 2} \in V^{2\times 2}([a,b]\times[a,b]), \ \ p,q = 1,2$$

***Email addresses and ORCID numbers*: 1303943677@qq.com, 0000-0003-3078-7652 (H. Zhou), 282228006@qq.com, 0000-0001-6254-2938 (Q. Wang)**

and

$$U(x) = [u(x), v(x)]^T \in V^2[a,b],$$

we write the numerical integral operator $\mathscr{K}$ defined as

$$(\mathscr{K}U)(x) = \int_a^b K(x,y)U(y)dy.$$

The norm of the numerical integral operator $\mathscr{K}$ discussed in this paper is defined as

$$\|\mathscr{K}\|_\infty = \max_{1\le p\le 2}[\sum_{q=1}^{2} \max_{a\le x\le b} \int_a^b |k_{pq}(x,y)|dy].$$

**Theorem 2.1.** *If $\|\mathscr{K}\|_\infty < 1$ holds, then the exact solutions of the system of Fredholm integral equations (1.1) is existential and unique.*

*Proof.* For all $U_i(x) \in V^2[a,b], \ \ i = 1,2$, one has

$$TU_i = F(x) + \int_a^b K(x,y)U(y)dy, \ \ i = 1,2.$$

Then we have

$$\begin{aligned} \|TU_1 - TU_2\|_\infty &= \| \textstyle\int_a^b K(x,y)U_1(y)dy - \int_a^b K(x,y)U_2(y)dy\|_\infty \\ &= \| \textstyle\int_a^b K(x,y)[U_1(y) - U_2(y)]dy\|_\infty \\ &\le \| \textstyle\int_a^b K(x,y)dy\|_\infty \cdot \|U_1 - U_2\|_\infty \\ &\le \|\mathscr{K}\|_\infty \cdot \|U_1 - U_2\|_\infty. \end{aligned}$$

Since $\|\mathscr{K}\|_\infty < 1$, $T$ is a contraction mapping. Consider that Banach fixed point theorem, then (1.1) exists a unique solution $U^* \in V^2[a,b]$ such that $TU^* = U^*$ holds. □

## 3. The Nyström method

In this section, we use the numerical quadrature scheme to obtain a general algorithm for the Nyström method of the system of Fredholm integral equations.

Applying numerical quadrature scheme to integral terms of (1.1), we can have

$$\begin{aligned} &\int_a^b [k_{p1}(x,y)u(y) + k_{p2}(x,y)v(y)]dy \\ &= \sum_{i=0}^{n} \omega_i [k_{p1}(x,x_i)u(x_i) + k_{p2}(x,x_i)v(x_i)] + R_p^{(n)}, \ \ p = 1,2, \end{aligned}$$

where $\omega_i (i = 0,1,...,n)$ are coefficients of quadrature and $x_i (i = 0,1,...,n)$ are the quadrature node points and $R_1^{(n)}, R_2^{(n)}$ are remainder terms, such that (1.1) can be rewritten as

$$\begin{cases} u(x) = f(x) + \sum_{i=0}^{n} \omega_i [k_{11}(x,x_i)u(x_i) + k_{12}(x,x_i)v(x_i)] + R_1^{(n)} \\ v(x) = g(x) + \sum_{i=0}^{n} \omega_i [k_{21}(x,x_i)u(x_i) + k_{22}(x,x_i)v(x_i)] + R_2^{(n)}. \end{cases} \tag{3.1}$$

We take the collocation points $x = x_i$, and let $f(x_i) = f_i$, $g(x_i) = g_i$, $k_{pq}(x_i,x_j) = k_{pq}^{ij}$, $u(x_i) = u_i$, $v(x_i) = v_i$, $i = 0,1,...,n; \ \ p,q = 1,2.$
Then we ignore the remainder terms and obtain the approximating linear system with respect to $u_0$, $v_0$, $u_1$, $v_1$, ... , $u_n$, $v_n$ as

$$\begin{cases} u_i = f_i + \sum_{j=0}^{n} \omega_j (k_{11}^{ij} u_j + k_{12}^{ij} v_j) \\ v_i = g_i + \sum_{j=0}^{n} \omega_j (k_{21}^{ij} u_j + k_{22}^{ij} v_j). \end{cases} \tag{3.2}$$

Remove the terms of (3.2), then we obtain

$$\begin{cases} -\sum_{j\ne i} \omega_j k_{11}^{ij} u_j + (1 - \omega_i k_{11}^{ii})u_i - \sum_{j=0}^{n} \omega_j k_{12}^{ij} v_j = f_i \\ -\sum_{j=0}^{n} \omega_j k_{21}^{ij} u_j - \sum_{j\ne i} \omega_j k_{22}^{ij} v_j + (1 - \omega_i k_{22}^{ii})v_i = g_i. \end{cases} \tag{3.3}$$

Solve linear system (3.3), we can get $u_i = u_i^*$, $v_i = v_i^*$, $i = 0,1,...,n$.
Take $u_i^*$, $v_i^*$ into (3.1) omitting the remainder terms, we have

$$\begin{cases} u_n(x) = f(x) + \sum_{i=0}^{n} \omega_i [k_{11}(x,x_i)u_i^* + k_{12}(x,x_i)v_i^*] \\ v_n(x) = g(x) + \sum_{i=0}^{n} \omega_i [k_{21}(x,x_i)u_i^* + k_{22}(x,x_i)v_i^*]. \end{cases} \tag{3.4}$$

Thus $u_n(x)$, $v_n(x)$ can be called the Nyström solutions with numerical quadrature scheme (3.1).
Meanwhile, it can be noted that

$$\begin{cases} u_n(x_i) = u_i^* \\ v_n(x_i) = v_i^*, \end{cases} \tag{3.5}$$

so $u_n(x)$, $v_n(x)$ are also the Nyström interpolation functions under the interpolation condition (3.2).

## 4. Error estimation

To carry out an error analysis for the Nyström method, we first give the following useful Lemma.

**Lemma 4.1.** *Let $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, and $x$, $y$ are positive real numbers. Assume*

$$\begin{cases} x \le a_1 + b_1 x + c_1 y \\ y \le a_2 + b_2 x + c_2 y. \end{cases}$$

*For $b_1 + c_2 < 1$ and $(1-b_1)(1-c_2) > b_2 c_1$, then*

$$x + y \le \frac{(1+b_2-c_2)a_1 + (1+c_1-b_1)a_2}{(1-b_1)(1-c_2) - b_2 c_1}.$$

The proof of this Lemma can be given directly and we omit it. The result of error estimation is given below.

**Theorem 4.2.** *Let $u(x)$, $v(x)$ are the exact solutions and let $u_n(x)$, $v_n(x)$ are the Nyström solutions of system of Fredholm integral equations (1.1). Assume $M_{11} + M_{22} < 1$, and $(1-M_{11})(1-M_{22}) > M_{12}M_{21}$, then*

$$\|u-u_n\|_\infty + \|v-v_n\|_\infty \le \frac{(1-M_{22}+M_{21})\|R_1^{(n)}\|_\infty + (1-M_{11}+M_{12})\|R_2^{(n)}\|_\infty}{(1-M_{11})(1-M_{22}) - M_{21}M_{12}},$$

*where $M_{pq} = (b-a)\|k_{pq}(x,y)\|_\infty$, $p,q = 1,2$.*

*Proof.* Consider (3.5) and subtract (3.4) from (3.1) to get

$$\begin{aligned} u - u_n &= \textstyle\sum_{i=0}^n \omega_i [k_{11}(x,x_i)(u(x_i) - u_i^*) + k_{12}(x,x_i)(v(x_i) - v_i^*)] + R_1^{(n)} \\ &= \textstyle\sum_{i=0}^n \omega_i [k_{11}(x,x_i)(u(x_i) - u_n(x_i)) + k_{12}(x,x_i)(v(x_i) - v_n(x_i))] + R_1^{(n)}, \end{aligned}$$

then

$$\|u-u_n\|_\infty \le \|R_1^{(n)}\|_\infty + \|\sum_{i=0}^n |\omega_i k_{11}(x,x_i)|\|_\infty \cdot \|u-u_n\|_\infty + \|\sum_{i=0}^n |\omega_i k_{12}(x,x_i)|\|_\infty \cdot \|v-v_n\|_\infty.$$

Similarly, we have

$$\|v-v_n\|_\infty \le \|R_2^{(n)}\|_\infty + \|\sum_{i=0}^n |\omega_i k_{21}(x,x_i)|\|_\infty \cdot \|u-u_n\|_\infty + \|\sum_{i=0}^n |\omega_i k_{22}(x,x_i)|\|_\infty \cdot \|v-v_n\|_\infty.$$

From the intermediate value theorem of continuous function, we can get

$$\begin{aligned} \|\textstyle\sum_{i=0}^n |\omega_i k_{pq}(x,x_i)|\|_\infty &= (b-a)\|k_{pq}(x,\eta_{pq})|\|_\infty \\ &\le (b-a)\|k_{pq}(x,y)|\|_\infty \\ &= M_{pq}, \end{aligned}$$

where $\eta_{pq} \in [a,b]$, $p,q = 1,2$.
It follows that a system of inequalities

$$\begin{cases} \|u-u_n\|_\infty \le \|R_1^{(n)}\|_\infty + M_{11}\|u-u_n\|_\infty + M_{12}\|v-v_n\|_\infty \\ \|v-v_n\|_\infty \le \|R_2^{(n)}\|_\infty + M_{21}\|u-u_n\|_\infty + M_{22}\|v-v_n\|_\infty. \end{cases}$$

From Lemma 4.1, we can obtain

$$\begin{cases} \|u-u_n\|_\infty \le \frac{(1-M_{22})\|R_1^{(n)}\|_\infty + M_{12}\|R_2^{(n)}\|_\infty}{(1-M_{11})(1-M_{22})-M_{21}M_{12}} \\ \|v-v_n\|_\infty \le \frac{M_{21}\|R_1^{(n)}\|_\infty + (1-M_{11})\|R_2^{(n)}\|_\infty}{(1-M_{11})(1-M_{22})-M_{21}M_{12}}, \end{cases}$$

when $M_{11} + M_{22} < 1$ and $(1-M_{11})(1-M_{22}) > M_{21}M_{12}$. □

In particular, for the composite trapezoidal rules, we have

$$\begin{cases} h = \frac{b-a}{n} \\ \omega_0 = \omega_n = \frac{h}{2} \\ \omega_1 = \omega_2 = \ldots = \omega_{n-1} = h \\ x_i = x_0 + nh, \quad i = 0,1,\ldots,n, \end{cases}$$

so

$$\begin{aligned} \textstyle\sum_{i=0}^n |\omega_i k_{pq}(x,x_i)| &= \tfrac{b-a}{n}[\tfrac{1}{2}(|k_{pq}(x,x_0)| + |k_{pq}(x,x_n)|) + \textstyle\sum_{i=1}^{n-1} |k_{pq}(x,x_i)|] \\ &= (b-a)|k_{pq}(x,\eta_{pq})|, \quad \eta_{pq} \in [a,b]. \end{aligned}$$

Conseqently, $\|\sum_{i=0}^n |\omega_i k_{pq}(x,x_i)|\|_\infty = M_{pq}$, $p,q = 1,2$.
Let

$$\begin{cases} k_1(x,\xi) = \frac{\partial^2}{\partial y^2}[k_{11}(x,y)u(y) + k_{12}(x,y)v(y)]_{y=\xi}, a < \xi < b \\ k_2(x,\eta) = \frac{\partial^2}{\partial y^2}[k_{21}(x,y)u(y) + k_{22}(x,y)v(y)]_{y=\eta}, a < \eta < b. \end{cases}$$

To sum up, we can draw the following corollary.

**Corollary 4.3.** *If $M_{11}+M_{22}<1$ and $(1-M_{11})(1-M_{22})>M_{12}M_{21}$, then the Nyström solutions with the composite trapezoidal quadrature formula have the following error estimation:*

$$\|u-u_n\|_\infty+\|v-v_n\|_\infty \le \frac{b-a}{12}\frac{(1-M_{22}+M_{21})\|k_1(x,y)\|_\infty+(1-M_{11}+M_{12})\|k_2(x,y)\|_\infty}{(1-M_{11})(1-M_{22})-M_{21}M_{12}}h^2.$$

By a similar argument, for the composite Simpson rules, we have

$$\begin{cases} h=\frac{b-a}{n} \\ \omega_0=\omega_{2n}=\frac{h}{6} \\ \omega_1=\omega_3=\ldots=\omega_{2n-1}=\frac{2}{3}h \\ \omega_2=\omega_4=\ldots=\omega_{2n-2}=\frac{1}{3}h \\ x_i=x_0+i\frac{h}{2},\quad i=0,1,\ldots,2n, \end{cases}$$

hence

$$\begin{aligned} \textstyle\sum_{i=0}^{2n}|\omega_i k_{pq}(x,x_i)| &= \tfrac{b-a}{n}[\tfrac{1}{6}(|k_{pq}(x,x_0)+k_{pq}(x,x_{2n})|)+\tfrac{1}{3}\textstyle\sum_{i=1}^{n-1}|k_{pq}(x,x_{2i})|+\tfrac{2}{3}\textstyle\sum_{i=1}^{n}|k_{pq}(x,x_{2i-1})|] \\ &=(b-a)k_{pq}(x,\eta_{pq}),\quad \eta_{pq}\in[a,b]. \end{aligned}$$

Then $\|\sum_{i=0}^{2n}|\omega_i k_{pq}(x,x_i)|\|_\infty=M_{pq},\quad p,q=1,2.$
Let

$$\begin{cases} \bar{k}_1(x,\xi)=\frac{\partial^4}{\partial y^4}[k_{11}(x,y)u(y)+k_{12}(x,y)v(y)]_{y=\xi}, a<\xi<b \\ \bar{k}_2(x,\eta)=\frac{\partial^4}{\partial y^4}[k_{21}(x,y)u(y)+k_{22}(x,y)v(y)]_{y=\eta}, a<\eta<b. \end{cases}$$

Again, we can draw the following corollary.

**Corollary 4.4.** *If $M_{11}+M_{22}<1$ and $(1-M_{11})(1-M_{22})>M_{12}M_{21}$, then the Nyström solutions with the composite Simpson quadrature formula have the following error estimation:*

$$\|u-u_n\|_\infty+\|v-v_n\|_\infty \le \frac{b-a}{180}\frac{(1-M_{22}+M_{21})\|\bar{k}_1(x,y)\|_\infty+(1-M_{11}+M_{12})\|\bar{k}_2(x,y)\|_\infty}{(1-M_{11})(1-M_{22})-M_{21}M_{12}}(\frac{h}{2})^4.$$

## 5. Numerical examples

In order to verify the validity of the proposed numerical method, two numerical examples are given and the exact solutions are compared with the approximate solutions by using Matlab.R2015a. The convergence rate is defined by

$$Ratio=\frac{\|u-u_n\|_\infty+\|v-v_n\|_\infty}{\|u-u_{2n}\|_\infty+\|v-v_{2n}\|_\infty}.$$

**Example 5.1.** *Consider the following system of Fredholm integral equations*

$$\begin{cases} u(x)=x^2+\frac{5}{24}x-\frac{7}{24}+\int_0^1[(-\frac{1}{2}x+\frac{1}{2}y)u(y)+(\frac{1}{12}x-y)v(y)]dy \\ v(x)=\frac{7}{9}x-\frac{19}{18}+\int_0^1[xyu(y)+\frac{1}{6}(xy-2y)v(y)]dy, \end{cases}$$

*with $0\le x\le 1$ and the exact solutions $u(x)=x^2$, $v(x)=x-1$.*

We choose $n=4,8,16,32$ along with $h=\frac{1}{n}$ and get $x_i=ih,\quad i=0,1,\ldots,n$. The curve graph of the exact solutions $u(x)=x^2$, $v(x)=x-1$ and the approximations $u_n(x)$, $v_n(x)$ obtained using the Nyström method are given in Figure 5.1(a), and then the maximum error $\|u-u_n\|_\infty+\|v-v_n\|_\infty$ listed in Table 1.

| | Composite trapezoidal | | Composite Simpson | |
|---|---|---|---|---|
| $n$ | $\|u-u_n\|_\infty+\|v-v_n\|_\infty$ | Ratio | $\|u-u_n\|_\infty+\|v-v_n\|_\infty$ | Ratio |
| 4 | 1.7800e-02 | | 0 | |
| 8 | 4.5000e-03 | 3.9556 | 0 | 0 |
| 16 | 1.1000e-03 | 4.0909 | 5.7465e-19 | 0 |
| 32 | 2.7778e-04 | 3.9600 | 0 | 0 |

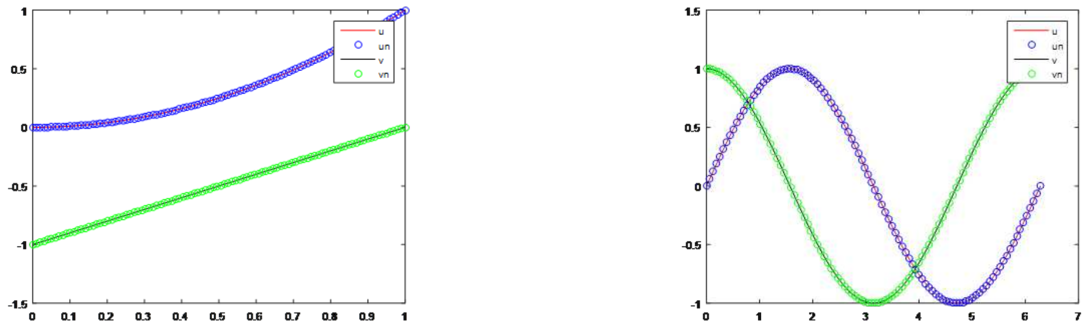
**Table 1:** Error calculation result of Example 5.1.

**Example 5.2.** *Consider the following system of Fredholm integral equations*

$$\begin{cases} u(x)=\sin x+\int_0^{2\pi}[(\frac{1}{20}\sin x-\cos y)u(y)+(\frac{1}{40}y\sin x)v(y)]dy \\ v(x)=\cos x-\frac{11\pi}{20}+\int_0^{2\pi}[(\frac{1}{50}\sin x-\frac{1}{40}y)u(y)+(\frac{1}{18}\sin x+\frac{1}{2}\cos y)v(y)]dy, \end{cases}$$

*with $0\le x\le 2\pi$ and the exact solutions $u(x)=\sin x$, $v(x)=\cos x$.*

We also choose $n=4,8,16,32$ along with $h=\frac{1}{n}$ and get $x_i=ih,\quad i=0,1,\ldots,n$. The curve graph of the exact solutions $u(x)=\sin x$, $v(x)=\cos x$ and the approximations $u_n(x)$, $v_n(x)$ obtained using the Nyström method are given in Figure 5.1(b), and then the maximum errors $\|u-u_n\|_\infty+\|v-v_n\|_\infty$ listed in Table 2.




(a) Example 5.1 (b) Example 5.2

**Figure 5.1:** The exact solutions and the Nyström solutions of Example 5.1 and Example 5.2 when n=16

| | Composite trapezoidal | | Composite Simpson | |
|---|---|---|---|---|
| $n$ | $\|u-u_n\|_\infty + \|v-v_n\|_\infty$ | Ratio | $\|u-u_n\|_\infty + \|v-v_n\|_\infty$ | Ratio |
| 4 | 6.4300e-02 | | 6.8765e-04 | |
| 8 | 1.5700e-02 | 4.0955 | 4.0590e-05 | 16.9414 |
| 16 | 3.9000e-03 | 4.0256 | 2.5019e-06 | 16.2237 |
| 32 | 9.6947e-04 | 4.0228 | 1.5583e-07 | 16.0553 |

**Table 2:** Error calculation result of Example 5.2.

## 6. Conclusion

In this paper, The Nyström method is proposed to handle approximate solutions of system of Fredholm integral equations and two numerical examples are provided to illustrate the validity and feasibility of the present method. For the simple system of integral equations such as polynomial integral equations, we appear to get the exact solutions directly by the Nyström method with the composite Simpson rule.
In the future, the Nyström method can be extended to solve Hammerstein integral equations. A two-grid iteration method for the Nyström method for system of Fredholm integral equations will also be further studied.

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# On Quasi-Sasakian Manifolds

**Ahmet Sazak[1] and Ahmet Yıldız[2*]**

[1]*Department of Mathematics, Faculty of Science and Arts, Muş Alparslan University, Mus, Turkey*
[2]*Department of Mathematics, Education Faculty, Inonu University, Malatya, Turkey*
[*]*Corresponding author E-mail: a.yildiz@inonu.edu.tr*

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**Abstract**

In this paper we study three-dimensional quasi-Sasakian manifolds admitting the Schouten-van Kampen connection. Also, we study D-homothetic deformations on three-dimensional quasi-Sasakian manifolds admitting Schouten-van connection and projectively flat three-dimensional quasi-Sasakian manifolds admitting scv connection.

## 1. Introduction

An important class of almost contact metric (shortly a.c.m.) manifolds is the class consisting of those which are normal. However, the curvature nature of such manifolds is not known in general, except for Sasakian or cosymplectic manifolds. If the almost contact structure (shortly a.c.s.) is normal and the fundamental 2-form is closed then the manifold $M$ is called a quasi-Sasakian manifold (shortly q.S.).
First examples of q.S. manifolds were given by D. E. Blair [1]. Also, some remarks on q.S. structures given by S. Tanno [2]. Then, on a three-dimensional q.S. manifold the structure function $\gamma$ was introduced by Z. Olszak [3].
The Schouten-van Kampen connection (shortly S.K.con.) has been introduced of non-holomorphic manifolds. Then the S.K.con. was applied to a.c.m. structure by Z. Olszak and he characterized some classes of a.c.m manifolds [4]. Also, A. Yildiz studied three-dimensional $f$-Kenmotsu manifolds according to this connection [5].
In the present paper, we study three-dimensional q.S. manifolds with a $\mathscr{D}_\alpha$-homothetic deformation admitting the S.K.con..

## 2. Preliminaries

Let $\varphi$ is $(1,1)$-type tensor field, $\xi$ is a locally defined vector field tangent to $M$ and $\eta$ is a 1-form on $M$. Then $M(\varphi,\xi,\eta,g)$ is called an a.c.m. manifold whose elementary properties are [6]-[8]

$$\varphi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1,$$

$$g(\varphi U,\varphi V) = g(U,V) - \eta(U)\eta(V).$$

$$\varphi\xi = 0, \quad \eta\circ\varphi = 0, \quad \eta(U) = g(U,\xi).$$

The fundamental 2-form $\theta$ is defined by

$$\theta(U,V) = g(U,\varphi V).$$

Thus $\theta(U,\xi) = 0$, for $U\in TM$. If the a.c.s. $(\varphi,\xi,\eta)$ is normal, i.e., $[\varphi,\varphi](U,U) + d\eta(U,V)\xi = 0$ and the fundamental 2-form $\theta$ is closed, i.e. $d\theta = 0$, then $M$ is called a q.S. manifold. An a.c.m. manifold $M$ is a three-dimensional q.S. manifold if and only if [9]

$$\nabla_U\xi = -\gamma\varphi U, \tag{2.1}$$

***Email addresses and ORCID:*** **a.sazak@alparslan.edu.tr, https://orcid.org/0000-0002-5620-6441 (Ahmet Sazak), a.yildiz@inonu.edu.tr, https://orcid.org/0000-0002-9799-1781 (Ahmet Yıldız)**

for a function $\gamma$ on $M$ satisfying $\xi\gamma = 0$. Also if $\gamma = 0$ then a q.S. manifold is a cosymplectic manifold [10], the converse is true. From (2.1), we have [9]

$$(\nabla_U \varphi)V = \gamma\{g(U,V)\xi - \eta(U)V\}. \tag{2.2}$$

Again from (2.1) and (2.2), we get

$$R(U,V)\xi = -U[\gamma]\varphi V + V[\gamma]\varphi U + \gamma^2\{\eta(V)U - \eta(U)V\}.$$

Using (2.1) and (2.2), we obtain

$$R(U,\xi)\xi = \gamma^2\{U - \eta(U)\xi\},$$

and

$$R(U,\xi)V = -U[\gamma]\varphi V - \gamma^2\{g(U,V)\xi - \eta(V)U\}.$$

In a three-dimensional Riemannian manifold, the curvature tensor is written

$$R(U,V)W \quad = \quad g(V,W)QU - g(U,W)QV + S(V,W)U - S(U,W)V - \frac{r}{2}\{g(V,W)U - g(U,W)V\}.$$

Let $M$ be a three-dimensional q.S. manifold. The Ricci tensor $S$ of $M$ is

$$S(V,W) \quad = \quad (\frac{r}{2} - \gamma^2)g(V,W) + (3\gamma^2 - \frac{r}{2})\eta(V)\eta(W) - \eta(V)d\gamma(\varphi W) - \eta(W)d\gamma(\varphi V). \tag{2.3}$$

From (2.3), we get

$$QV \quad = \quad (\frac{r}{2} - \gamma^2)V + (3\gamma^2 - \frac{r}{2})\eta(V)\xi + \eta(V)(\varphi \operatorname{grad}\gamma) - d\gamma(\varphi V)\xi,$$

where $d\gamma(V) = g(\operatorname{grad}\gamma, V)$. Again from (2.3), we obtain

$$S(U,\xi) = 2\gamma^2\eta(U) - d\gamma(\varphi U).$$

As a consequence of (2.1), we have

$$(\nabla_U \eta)W = g(\nabla_U \xi, W) = -\gamma g(\varphi U, V). \tag{2.4}$$

## 3. Three-dimensional q.S. manifolds admitting S.K.con.

For an a.c.m. manifold $M$, the S.K.con. $\widetilde{\nabla}$ is given by [4]

$$\widetilde{\nabla}_U V = \nabla_U V - \eta(V)\nabla_U \xi + (\nabla_U \eta)(V)\xi. \tag{3.1}$$

Let $M^3$ be a q.S. manifold. Then using (3.1), we have

$$\widetilde{\nabla}_U V = \nabla_U V + \gamma\eta(V)\varphi U + \gamma g(U,\varphi V)\xi. \tag{3.2}$$

Now we put equation (3.1) in the definition of the Riemannian curvature tensor, we can write

$$\widetilde{R}(U,V)W = \widetilde{\nabla}_U\widetilde{\nabla}_V W - \widetilde{\nabla}_V\widetilde{\nabla}_U W - \widetilde{\nabla}_{[U,V]}W. \tag{3.3}$$

Using (3.2) in (3.3), we obtain

$$\begin{aligned}\widetilde{R}(U,V)W \quad = \quad & \widetilde{\nabla}_U(\nabla_V W + \gamma\eta(W)\varphi V + \gamma g(V,\varphi W)\xi) \\ & -\widetilde{\nabla}_V(\nabla_U W + \gamma\eta(W)\varphi U + \gamma g(U,\varphi W)\xi) \\ & -(\nabla_{[U,V]}W + \gamma\eta(W)\varphi[U,V] + \gamma g([U,V],\varphi W)\xi.\end{aligned} \tag{3.4}$$

Again using (2.2) and (2.4) in (3.4), we have

$$\begin{aligned}\widetilde{R}(U,V)W \quad = \quad & R(U,V)W + U[\gamma]\{g(V,\varphi W)\xi + \eta(W)\varphi V\} \\ & -V[\gamma]\{g(U,\varphi W)\xi + \eta(W)\varphi U\} \\ & +\gamma^2\{g(U,W)\eta(V)\xi - g(V,W)\eta(U)\xi + \eta(U)\eta(W)V \\ & -\eta(V)\eta(W)U + g(U,\varphi W)\varphi V - g(V,\varphi W)\varphi U\},\end{aligned} \tag{3.5}$$

which gives

$$\begin{aligned}g(\widetilde{R}(U,V)W,Z) \quad = \quad & g(R(U,V)W,Z) \\ & +U[\gamma]\{g(V,\varphi W)\eta(Z) + g(\varphi V,Z)\eta(W)\} \\ & -Y[\gamma]\{g(U,\varphi W)\eta(Z) + g(\varphi U,Z)\eta(W)\} \\ & +\gamma^2\{g(U,W)\eta(V)\eta(Z) - g(V,W)\eta(U)\eta(Z) \\ & +g(V,Z)\eta(U)\eta(W) - g(U,Z)\eta(V)\eta(W) \\ & +g(U,\varphi W)g(\varphi V,Z) - g(V,\varphi W)g(\varphi U,Z)\}.\end{aligned} \tag{3.6}$$

Putting $U = Z = e_i$ ,$\{i = 1,2,3\}$, in (3.6), we get

$$\widetilde{S}(V,W) = S(V,W) + (\varphi V)[\gamma]\eta(W) - 2\gamma^2\eta(V)\eta(W). \tag{3.7}$$

From (3.7), we have

$$\widetilde{Q}V = QV + (\varphi V)[\gamma]\xi - 2\gamma^2\eta(V)\xi.$$

Again putting $V = W = e_i$ in (3.7), then we obtain

$$\widetilde{r} = r - 2\gamma^2,$$

From (3.5) and (3.6), we have

$$\widetilde{R}(U,V)W + \widetilde{R}(V,U)W = 0,$$

$$g(\widetilde{R}(U,V)W,Z) + g(\widetilde{R}(U,V)Z,W) = 0.$$

and

$$\begin{aligned}\widetilde{R}(U,V)W + \widetilde{R}(V,W)U + \widetilde{R}(W,U)V \quad = \quad & U[\gamma]\{2g(V,\varphi W)\xi + \eta(W)\varphi V - \eta(V)\varphi W\} \\ & + V[\gamma]\{2g(W,\varphi U)\xi + \eta(U)\varphi W - \eta(W)\varphi U\} \\ & + W[\gamma]\{2g(U,\varphi V)\xi + \eta(V)\varphi U - \eta(U)\varphi V\}.\end{aligned}$$

If $\gamma$ is a constant, then we have

$$\widetilde{R}(U,V)W + \widetilde{R}(V,W)U + \widetilde{R}(W,U)V = 0.$$

## 4. Three-dimensional q.S. manifolds and $\mathscr{D}_\alpha$-homothetic deformations

In this section, we study a $\mathscr{D}_\alpha$-homothetic deformation on a q.S. manifold $M^3$.
For a $(2n+1)$-dimensional a.c.m. manifold $(M,\ \varphi,\xi,\eta,g)$ if $\eta = 0$, then there is an $2n$-dimensional distribution $\mathscr{D}_\alpha$ on $M$. Also an $2n$-dimensional homothetic deformation or a $\mathscr{D}_\alpha$-homothetic deformation is defined by

$$\begin{aligned}\eta^\alpha &= \alpha\eta, \quad \xi^\alpha = \frac{1}{\alpha}\xi, \quad \varphi^\alpha = \varphi, \\ g^\alpha &= \alpha g + \alpha(\alpha-1)\eta\otimes\eta,\end{aligned} \tag{4.1}$$

where $\alpha =$ constant $> 0$. If $(M,\ \varphi,\xi,\eta,g)$ is an a.c.m. structure then $(M,\ \varphi^\alpha,\xi^\alpha,\eta^\alpha,g^\alpha)$ is also an a.c.m. structure [2].
Now we have the followings:

**Lemma 4.1.** *Let $M^3$ be a q.S. manifold admitting a $\mathscr{D}_\alpha$-homothetic deformation. Then*

$$\nabla^\alpha_U V = \nabla_U V - (\alpha-1)\gamma\{\eta(U)\varphi V + \eta(V)\varphi U\}. \tag{4.2}$$

*Proof.* From Kozsul's formula, we have

$$2g^\alpha(\nabla^\alpha_U V,W) \quad = \quad Ug^\alpha(V,W) + Vg^\alpha(U,W) - Wg^\alpha(U,V) - g^\alpha(U,[V,W]) - g^\alpha(V,[U,W]) + g^\alpha(W,[U,V]),$$

for any vector fields $U,V,W$. From (4.1), we obtain

$$\begin{aligned}2\{\alpha g(\nabla^\alpha_U V,W) + \alpha(\alpha-1)\eta(\nabla^\alpha_U V)\eta(W)\} \quad = \quad & U\{\alpha g(V,W) + \alpha(\alpha-1)\eta(U)\eta(W)\} \\ & + V\{\alpha g(U,W) + \alpha(\alpha-1)\eta(U)\eta(W)\} \\ & - W\{\alpha g(U,V) + \alpha(\alpha-1)\eta(U)\eta(V)\} \\ & - \alpha\{g(U,[V,W]) + (\alpha-1)\eta(U)\eta([V,W])\} \\ & - \alpha\{g(V,[U,W]) + (\alpha-1)\eta(V)\eta([U,W])\} \\ & + \alpha\{g(W,[U,V]) + (\alpha-1)\eta(W)\eta([U,V])\}.\end{aligned}$$

After some calculations, we get

$$\begin{aligned}2\{\alpha g(\nabla^\alpha_U V,W) + \alpha(\alpha-1)\eta(\nabla^\alpha_U V)\eta(W)\} \quad = \quad & \alpha\{g(\nabla_U V,W) + g(V,\nabla_U W)\} \\ & + \alpha(\alpha-1)\{U(\eta(V)\eta(W)) + V(\eta(U)\eta(W)) \\ & - W(\eta(U)\eta(V)) - \eta(U)\eta(\nabla_V W) + \eta(U)\eta(\nabla_W V) \\ & - \eta(V)\eta(\nabla_U W) + \eta(V)\eta(\nabla_W U) \\ & + \eta(W)\eta(\nabla_U V) - \eta(W)\eta(\nabla_V U)\}.\end{aligned}$$

Thus we have

$$\begin{aligned}2\{\alpha g(\nabla^\alpha_U V,W) + \alpha(\alpha-1)\eta(\nabla^\alpha_U V)\eta(W)\} \quad = \quad & 2\alpha g(\nabla_U V,W) + 2a(a-1)\eta(\nabla_U V)\eta(W) \\ & + a(a-1)\{g(V,\nabla_U\xi)\eta(W) + g(W,\nabla_U\xi)\eta(V) \\ & + g(U,\nabla_V\xi)\eta(W) + g(W,\nabla_V\xi)\eta(U) \\ & - g(U,\nabla_W\xi)\eta(V) - g(V,\nabla_W\xi)\eta(U)\}.\end{aligned} \tag{4.3}$$

Using (2.1) in (4.3), we get

$$\begin{aligned}2\{\alpha g(\nabla^{\alpha}_{U}V,W)+\alpha(\alpha-1)\eta(\nabla^{\alpha}_{U}V)\eta(W)\} &= 2\alpha g(\nabla_{U}V,W)+2a(a-1)\eta(\nabla_{U}V)\eta(W)\\ &\quad -\alpha(\alpha-1)\gamma\{g(V,\varphi U)\eta(W)+g(W,\varphi U)\eta(V)\\ &\quad +g(\varphi V,U)\eta(W)+g(W,\varphi V)\eta(U)\\ &\quad -g(U,\varphi W)\eta(V)-g(V,\varphi W)\eta(U)\}.\end{aligned}$$

After some calculations, we obtain

$$g(\nabla^{\alpha}_{U}V,W)+(\alpha-1)\eta(\nabla^{\alpha}_{U}V)\eta(W) = g(\nabla_{U}V,W)+(\alpha-1)\eta(\nabla_{U}V)\eta(W)-(\alpha-1)\gamma\{g(W,\varphi U)\eta(V)+g(W,\varphi V)\eta(U)\},$$

which implies (4.2). □

**Proposition 4.2.** *Let $M^3$ be a q.S. manifold with a $\mathscr{D}_a$-homothetic deformation. Then*

$$\begin{aligned}R^{\alpha}(U,V)W &= R(U,V)W\\ &\quad -(\alpha-1)U[\gamma]\{\eta(V)\varphi W+\eta(W)\varphi V\}\\ &\quad +(\alpha-1)V[\gamma]\{\eta(U)\varphi W+\eta(W)\varphi U\}\\ &\quad -(\alpha-1)\gamma^2\{\eta(U)\eta(W)V-\eta(V)\eta(W)U\}\\ &\quad +(\alpha-1)\gamma^2\{g(V,W)\eta(U)\xi-g(U,W)\eta(V)\xi\\ &\quad -2\eta(U)\eta(W)V+2\eta(V)\eta(W)U-2g(U,\varphi V)\varphi W\\ &\quad -g(U,\varphi W)\varphi V+g(V,\varphi W)\varphi U\}.\end{aligned} \tag{4.4}$$

*Proof.* The definition of the Riemannian curvature tensor, we can write

$$R^{\alpha}(U,V)W=\nabla^{\alpha}_{U}\nabla^{\alpha}_{V}W-\nabla^{\alpha}_{V}\nabla^{\alpha}_{U}W-\nabla^{\alpha}_{[U,V]}W. \tag{4.5}$$

Using (4.2) in (4.5) and after long calculations, we have

$$\begin{aligned}R^{\alpha}(U,V)W &= R(U,V)W\\ &\quad -(\alpha-1)\gamma[\eta(U)\varphi\nabla_{V}W-\eta(U)\nabla_{V}\varphi W-\eta(V)\varphi\nabla_{U}W\\ &\quad +\eta(V)\nabla_{U}\varphi W-\eta(W)\varphi\nabla_{U}V+\eta(W)\nabla_{U}\varphi V\\ &\quad +\eta(W)\varphi\nabla_{V}U-\eta(W)\nabla_{V}\varphi U\\ &\quad +a\gamma\{2g(U,\varphi V)\varphi W+g(U,\varphi W)\varphi V-g(V,\varphi W)\varphi U\}\\ &\quad +(\alpha-1)\gamma\{\eta(U)\eta(W)V-\eta(V)\eta(W)U\}]\\ &\quad -(\alpha-1)\{(U(\gamma))(\eta(V)\varphi W+\eta(W)\varphi V)\\ &\quad -(V(\gamma))(\eta(U)\varphi W+\eta(W)\varphi U)\}.\end{aligned} \tag{4.6}$$

Using (2.2) in (4.6), we obtain (4.4). □

From (4.4), we have

$$R^{\alpha}(U,V)W+R^{\alpha}(V,U)W=0,$$

and

$$R^{\alpha}(U,V)W+R^{\alpha}(V,W)U+R^{\alpha}(W,U)V=0.$$

## 5. $\mathscr{D}_{\alpha}$-homothetic deformations on three-dimensional q.S. manifolds admitting the S.K.con.

In this section, we study how a $\mathscr{D}_{\alpha}$-homothetic deformation affects a three-dimensional q.S. manifold $M$ admitting the S.K.con..

**Lemma 5.1.** *Let $M^3$ be a q.S. manifold with a $\mathscr{D}_{\alpha}$-homothetic deformation admitting the S.K.con.. Then*

$$\widetilde{\nabla}^{\alpha}_{U}V=\nabla_{U}V-(\alpha-1)\gamma\eta(U)\varphi V+\gamma\eta(V)\varphi U+\gamma g(U,\varphi V)\xi. \tag{5.1}$$

*Proof.* Using (3.1) and (4.2), we obtain

$$\begin{aligned}\widetilde{\nabla}^{\alpha}_{U}V &= \nabla_{U}V-\gamma\eta^{\alpha}(V)\varphi^{\alpha}U+\gamma g^{\alpha}(U,\varphi V)\xi^{\alpha}\\ &= \nabla_{U}V-(\alpha-1)\gamma\{\eta(U)\varphi V+\eta(V)\varphi U\}-\gamma\alpha\eta(V)\varphi U\\ &\quad +\gamma[g(U,\varphi V)+(\alpha-1)\eta(U)\eta(\varphi V)]\xi\\ &= \nabla_{U}V-(\alpha-1)\gamma\eta(U)\varphi V+\gamma\eta(V)\varphi U+\gamma g(U,\varphi V)\xi.\end{aligned}$$

□

**Proposition 5.2.** *Let $M^3$ be a q.S. manifold with a $\mathscr{D}_\alpha$-homothetic deformation admitting the S.K.con.. Then*

$$\begin{aligned}\widetilde{R}^\alpha(U,V)W &= R(U,V)W \\ &+U[\gamma]\{g(V,\varphi W)\xi+\eta(W)\varphi V-(\alpha-1)\eta(V)\varphi W\} \\ &-V[\gamma]\{g(U,\varphi W)\xi+\eta(W)\varphi U-(\alpha-1)\eta(U)\varphi W\} \\ &+\gamma^2\{g(U,W)\eta(V)\xi-g(V,W)\eta(U)\xi+g(U,\varphi W)\varphi V \\ &-g(V,\varphi W)\varphi U+\eta(U)\eta(W)V-\eta(V)\eta(W)U\} \\ &-2(\alpha-1)\gamma^2 g(U,\varphi V)\varphi W.\end{aligned} \tag{5.2}$$

*Proof.* Using (4.1), (3.5) and (4.4), we have

$$\begin{aligned}\widetilde{R}^\alpha(U,V)W &= R^\alpha(U,V)W+U[\gamma]\{g^\alpha(V,\varphi^\alpha W)\xi^\alpha+\eta^\alpha(W)\varphi^\alpha V\} \\ &-V[\gamma]\{g^\alpha(U,\varphi^\alpha W)\xi^\alpha+\eta^\alpha(W)\varphi^\alpha U\} \\ &+\gamma^2\{g^\alpha(U,W)\eta^\alpha(V)\xi^\alpha-g^\alpha(V,W)\eta^\alpha(U)\xi^\alpha \\ &+\eta^\alpha(U)\eta^\alpha(W)V-\eta^\alpha(V)\eta^\alpha(W)U \\ &+g^\alpha(U,\varphi^\alpha W)\varphi^\alpha V-g^\alpha(V,\varphi^\alpha W)\varphi^\alpha U\}.\end{aligned} \tag{5.3}$$

Using (4.2) in (5.3), we obtain (5.2). □

Now taking the inner product with $Z$ and putting $V=W=e_i$, $\{i=1,2,3\}$, in (5.2), we get

$$\widetilde{S}^\alpha(U,Z) = S(U,Z)+(\varphi U)[\gamma]\eta(Z)+(\alpha-1)(\varphi Z)[\gamma]\eta(U)-2\gamma^2\eta(U)\eta(Z). \tag{5.4}$$

If we use (2.3) in (5.4), we have

$$\begin{aligned}\widetilde{S}^\alpha(U,Z) &= (\frac{r}{2}-\gamma^2)g(U,Z)+(3\gamma^2-\frac{r}{2})\eta(U)\eta(Z) \\ &-(\varphi U)[\gamma]\eta(Z)-(\varphi Z)[\gamma]\eta(U) \\ &+(\varphi U)[\gamma]\eta(Z)+(\alpha-1)(\varphi Z)[\gamma]\eta(U) \\ &-2\gamma^2\eta(U)\eta(Z),\end{aligned}$$

i.e.,

$$\widetilde{S}^\alpha(U,Z) = (\frac{r}{2}-\gamma^2)\{g(U,Z)-\eta(U)\eta(Z)\}+(\alpha-2)(\varphi Z)[\gamma]\eta(U).$$

Also we take $U=Z=e_i$ in (5.4), we get

$$\widetilde{r}^\alpha=r-2\gamma^2.$$

## 6. Main result

In this section, we study a projectively flat q.S. manifold $M^3$ with a $\mathscr{D}_\alpha$-homothetic deformation admitting the S.K.con..
In a q.S. manifold $M^3$ with a $\mathscr{D}_\alpha$-homothetic deformation admitting the S.K.con. $\widetilde{\nabla}^a$, the projective curvature tensor $\widetilde{P}^a$ is given by

$$\widetilde{P}^\alpha(U,V)W=\widetilde{R}^\alpha(U,V)W-\frac{1}{2}\{\widetilde{S}^\alpha(V,W)U-\widetilde{S}^\alpha(U,W)V\}.$$

Now let $M^3$ be a projectively flat q.S. manifold with a $\mathscr{D}_\alpha$-homothetic deformation admitting the S.K.con. $\widetilde{\nabla}^a$ (i.e. $\widetilde{P}^a=0$). Then we have

$$\widetilde{R}^\alpha(U,V)W=\frac{1}{2}\{\widetilde{S}^\alpha(V,W)U-\widetilde{S}^\alpha(U,W)V\}. \tag{6.1}$$

Using (5.2) and (5.4) in (6.1), we get

$$\begin{aligned}&R(U,V)W \\ &+U[\gamma]\{g(V,\varphi W)\xi+\eta(W)\varphi V-(\alpha-1)\eta(V)\varphi W\} \\ &-V[\gamma]\{g(U,\varphi W)\xi+\eta(W)\varphi U-(\alpha-1)\eta(U)\varphi W\} \\ &+\gamma^2\{g(U,W)\eta(V)\xi-g(V,W)\eta(U)\xi+g(U,\varphi W)\varphi V \\ &-g(V,\varphi W)\varphi U+\eta(U)\eta(W)V-\eta(V)\eta(W)U\} \\ &+2(\alpha-1)\gamma^2 g(U,\varphi V)\varphi W \\ =\ &\frac{1}{2}[S(V,W)U+(\varphi V)[\gamma]\eta(W)U+(\alpha-1)(\varphi W)[\gamma]\eta(V)U \\ &-2\gamma^2\eta(V)\eta(W)U-S(U,W)V-(\varphi U)[\gamma]\eta(W)V \\ &-(\alpha-1)(\varphi W)[\gamma]\eta(U)V+2\gamma^2\eta(U)\eta(W)W],\end{aligned}$$

which gives

$$\begin{aligned}
& g(R(U,V)W,Z) \\
& +U[\gamma]\{g(V,\varphi W)\eta(Z)+\eta(W)g(\varphi V,Z)-(\alpha-1)\eta(V)g(\varphi W,Z)\} \\
& -V[\gamma]\{g(U,\varphi W)\eta(Z)+\eta(W)g(\varphi U,Z)-(\alpha-1)\eta(U)g(\varphi W,Z)\} \\
& +\gamma^2\{g(U,W)\eta(V)\eta(Z)-g(V,W)\eta(U)\eta(Z)+g(U,\varphi W)g(\varphi V,Z) \\
& -g(V,\varphi W)g(\varphi U,Z)+\eta(U)\eta(W)g(V,Z)-\eta(V)\eta(W)g(U,Z)\} \\
& +2(\alpha-1)\gamma^2 g(U,\varphi V)g(\varphi W,Z) \\
= & \frac{1}{2}[S(V,W)g(U,Z)+(\varphi V)[\gamma]\eta(W)g(U,Z) \\
& +(\alpha-1)(\varphi W)[\gamma]\eta(V)g(U,Z)-2\gamma^2\eta(V)\eta(W)g(U,Z) \\
& -S(U,W)g(V,Z)-(\varphi U)[\gamma]\eta(W)g(V,Z) \\
& -(\alpha-1)(\varphi W)[\gamma]\eta(U)g(V,Z)+2\gamma^2\eta(U)\eta(W)g(V,Z)].
\end{aligned} \tag{6.2}$$

Putting $U=Z=\xi$ in (6.2), we have

$$S(V,W)=2\gamma^2\eta(V)\eta(W)-(\varphi W)[\gamma]\eta(V)-(\varphi V)[\gamma]\eta(W). \tag{6.3}$$

If we put (6.3) in (5.4), we get

$$\widetilde{S}^{\alpha}(V,W)=(\alpha-2)(\varphi W)[\gamma]\eta(V). \tag{6.4}$$

Clearly, if $\gamma$ is a constant or $\alpha=2$, then from (6.4), we have $\widetilde{S}^{\alpha}=0$. If $\widetilde{S}^{\alpha}=0$, then from (6.1), we get $\widetilde{R}^{\alpha}=0$. Conversely if $\widetilde{R}^{\alpha}=0$ then we have $\widetilde{S}^{\alpha}=0$ and from (6.4) we obtain $\gamma$ is a constant or $\alpha=2$.
Thus the above discussion leads us to state the following:

**Theorem 6.1.** *Let $M^3$ be a projectively flat q.S. manifold with a $\mathscr{D}_{\alpha}$-homothetic deformation admitting the S.K.con.. Then the followings hold:* $(i)$ *$\gamma$ is a constant or $\alpha=2$.* $(ii)$ *The manifold $M$ is a Ricci-flat manifold,* $(iii)$ *The manifold $M$ is a flat manifold.*

## 7. Example

In this section, we give an example of three-dimensional q.S. manifolds with a $\mathscr{D}_{\alpha}$-homothetic deformation admitting the S.K.con..
Let $M=\{(x,y,z)\in\Re^3 : x\neq 0\}$ be a three-dimensional manifold, where $(x,y,z)$ are standard coordinates in $\Re^3$and $\{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3\}$ be linearly independent global frame on $M$ is given by

$$\tilde{e}_1=2\frac{\partial}{\partial y},\quad \tilde{e}_2=2\frac{\partial}{\partial x}-4\frac{\partial}{\partial y}+y\frac{\partial}{\partial z},\quad \tilde{e}_3=\frac{\partial}{\partial z}.$$

Let $g$ be the Riemannian metric, $\eta$ be the 1-form and $\varphi$ be the $(1,1)$-type tensor field given by

$$\begin{aligned}
g(\tilde{e}_1,\tilde{e}_3) &= g(\tilde{e}_1,\tilde{e}_2)=g(\tilde{e}_2,\tilde{e}_3)=0, \\
g(\tilde{e}_1,e_1) &= g(\tilde{e}_2,\tilde{e}_2)=g(\tilde{e}_3,\tilde{e}_3)=1,
\end{aligned} \tag{7.1}$$

$$\eta(W)=g(W,\tilde{e}_3), \tag{7.2}$$

$$\varphi\tilde{e}_1=\tilde{e}_2,\quad \varphi\tilde{e}_2=-\tilde{e}_1,\quad \varphi\tilde{e}_3=0, \tag{7.3}$$

respectively. Using the linearity of $\varphi$ and $g$, we have

$$\eta(\tilde{e}_3)=1,\quad \varphi^2 W=-W+\eta(W)\tilde{e}_3,$$

and

$$g(\varphi W,\varphi Z)=g(W,Z)-\eta(W)\eta(Z).$$

Thus for $\tilde{e}_3=\xi$, $(\varphi,\xi,\eta,g)$ defines a c.m.s. on $M^3$. Thus we have

$$[\tilde{e}_1,\tilde{e}_2]=2\tilde{e}_3,\quad [\tilde{e}_1,\tilde{e}_3]=0,\quad [\tilde{e}_2,\tilde{e}_3]=0.$$

Recall Koszul's formula

$$2g(\nabla_U V,W) = Ug(V,W)+Vg(U,W)-Wg(U,V)-g(U,[V,W])-g(V,[U,W])+g(W,[U,V]),$$

Taking $\tilde{e}_3=\xi$ and using the above formula for Riemannian metric $g$, we get

$$\begin{aligned}
\nabla_{\tilde{e}_1}\tilde{e}_3 &= -\tilde{e}_2, & \nabla_{\tilde{e}_2}\tilde{e}_3&=\tilde{e}_1, & \nabla_{\tilde{e}_3}\tilde{e}_3&=0, \\
\nabla_{\tilde{e}_3}\tilde{e}_1 &= -\tilde{e}_2, & \nabla_{\tilde{e}_1}\tilde{e}_2&=\tilde{e}_3, & \nabla_{\tilde{e}_2}\tilde{e}_1&=-\tilde{e}_3, \\
\nabla_{\tilde{e}_2}\tilde{e}_2 &= 0, & \nabla_{\tilde{e}_3}\tilde{e}_2&=\tilde{e}_1, & \nabla_{\tilde{e}_1}\tilde{e}_1&=0.
\end{aligned} \tag{7.4}$$

Hence from (2.1), the manifold $M^3$ is a q.S. manifold with $\gamma = 1$. Using (7.1), (7.2), (7.3) and (7.4) in (5.1), we have $D_a$-homothetic deformation of the manifold $M^3$ admitting the S.K.con. given by

$$\begin{aligned}
\widetilde{\nabla}^{\alpha}_{\tilde{e}_1}\tilde{e}_3 &= 0, \qquad \widetilde{\nabla}^{\alpha}_{\tilde{e}_2}\tilde{e}_3 = 0, \qquad \widetilde{\nabla}^{\alpha}_{e_3}\tilde{e}_3 = 0,\\
\widetilde{\nabla}^{\alpha}_{\tilde{e}_3}\tilde{e}_1 &= -\alpha\tilde{e}_2, \quad \widetilde{\nabla}^{\alpha}_{\tilde{e}_1}\tilde{e}_2 = 0, \qquad \widetilde{\nabla}^{\alpha}_{\tilde{e}_2}\tilde{e}_1 = 0,\\
\widetilde{\nabla}^{\alpha}_{\tilde{e}_2}\tilde{e}_2 &= 0, \qquad \widetilde{\nabla}^{\alpha}_{\tilde{e}_3}\tilde{e}_2 = \alpha e_1, \quad \widetilde{\nabla}^{\alpha}_{\tilde{e}_1}\tilde{e}_1 = 0.
\end{aligned} \tag{7.5}$$

Using (7.5), we obtain

$$\begin{aligned}
\widetilde{R}^{\alpha}(\tilde{e}_1,\tilde{e}_2)\tilde{e}_1 &= 2\alpha\tilde{e}_2, \quad \widetilde{R}^{\alpha}(\tilde{e}_1,\tilde{e}_2)\tilde{e}_2 = -2\alpha\tilde{e}_2, \quad \widetilde{R}^{\alpha}(\tilde{e}_1,\tilde{e}_2)\tilde{e}_3 = 0,\\
\widetilde{R}^{\alpha}(\tilde{e}_1,\tilde{e}_3)\tilde{e}_1 &= 0, \qquad \widetilde{R}^{\alpha}(\tilde{e}_1,\tilde{e}_3)\tilde{e}_2 = 0, \qquad \widetilde{R}^{\alpha}(\tilde{e}_1,\tilde{e}_3)\tilde{e}_3 = 0,\\
\widetilde{R}^{\alpha}(\tilde{e}_2,\tilde{e}_3)\tilde{e}_1 &= 0, \qquad \widetilde{R}^{\alpha}(\tilde{e}_2,\tilde{e}_3)\tilde{e}_2 = 0, \qquad \widetilde{R}^{\alpha}(\tilde{e}_2,\tilde{e}_3)\tilde{e}_3 = 0.
\end{aligned} \tag{7.6}$$

Thus from (7.6), the manifold $M^3$ is a flat manifold. Since a flat manifold is a Ricci flat manifold, from the Theorem 6.1 the manifold $M^3$ is a projectively flat manifold.

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# Neimark-Sacker Bifurcation of a Third Order Difference Equation

**Asmaa Shareef[1] and Marwan Aloqeili[1*]**

[1]*Department of Mathematics, Faculty of Science, Birzeit University, Birzeit, Palestine*
[*]*Corresponding author E-mail: maloqeili@birzeit.edu*

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**Abstract**

In this paper, we investigate the bifurcation of a third order rational difference equation. Firstly, we show that the equation undergoes a Neimark-Sacker bifurcation when the parameter reaches a critical value. Then, we consider the direction of the Neimark-Sacker bifurcation. Finally, we give some numerical simulations of our results.

## 1. Introduction

Bifurcation is an important dynamic behavior of some dynamical systems. Some difference equations exhibits different kinds of bifurcation including period-doubling bifurcation, saddle-node bifurcation and Neimark-Sacker bifurcation. In this paper, we show that a third order rational difference equation exhibits Neimark-Sacker bifurcation. This type of bifurcation exits when the Jacobian matrix of a system of difference equations has complex eigenvalues of modulus one. In [1], the author studied the dynamics of the third order difference equation

$$x_{n+1}=\frac{\beta x_n+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}} \tag{1.1}$$

Using appropriate change of variables, equation (1.1) becomes

$$x_{n+1}=\frac{\beta x_n+x_{n-2}}{A+Bx_n+x_{n-1}}$$

where $A\geq 0,\beta,B>0$. The author gives dynamic properties of solutions of this equation. In [2], the authors considered the difference equation

$$x_{n+1}=\frac{\beta x_n+\alpha x_{n-2}}{1+x_{n-1}}$$

They show that this equation undergoes a Neimark-Sacker bifurcation and give the direction of the bifurcation. In this paper, we consider the third order rational difference equation

$$x_{n+1}=\frac{\beta x_n+x_{n-2}}{A+x_{n-1}} \tag{1.2}$$

where $A\in(0,1)$, $\beta>0$ and nonnegative initial conditions $x_{-2},x_{-1}$ and $x_0$. Firstly, we show that the unique positive equilibrium $X^*=\beta-A+1$ is locally asymptotically stable if $\beta>(1-A)/(1+A)$. Then, we show that equation (1.2) undergoes a Neimark-Sacker bifurcation by converting this equation to a first order system and showing that the Jacobian matrix of the linearized system has a pair of complex

***Email addresses and ORCID numbers*: asma_aqel@yahoo.com, https://orcid.org/0000-0001-8109-5811 (A. Shareef), https://orcid.org/0000-0002-9030-1717 (M. Aloqeili),maloqeili@birzeit.edu**

conjugated eigenvalues of modulus one and a real eigenvalue in the interval $(0,1)$. Equation (1.2) is a special case of the following one which was considered in [3]

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+x_{n-2}}{A+x_{n-1}} \tag{1.3}$$

setting $\alpha=\gamma=0$, we get equation (1.2). The authors in [3] proved that every solution of equation (1.3) is bounded.
The rest of the article is organized as follows: in section 2, we give condition for local asymptotic stability. Then, we show in section 3 that equation (1.2) undergoes a Neimark-Sacker bifurcation. In section 4, the direction of bifurcation is considered. Finally, some numerical simulations are given.

## 2. Local stability

In this section, we study local stability of the unique positive equilibrium of equation (1.2). We apply Jury's test to the characteristic polynomial of the linearized equation. **Jury's conditions** provide an algebraic test that determines whether the roots of a polynomial lie within the unit circle. Jury's conditions consist of a test for necessary conditions and a test for sufficient conditions. For a polynomial of the form:

$$f(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$$

The necessary conditions for stability are: $f(1)>0$ and $(-1)^nf(-1)>0$, while the sufficient conditions for stability are given by:

$$|a_0|<a_n,\quad |b_0|>|b_{n-1}|,\quad |c_0|>|c_{n-2}|,\cdots$$

where $b_k=\begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$, $c_k=\begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}$
We need the following theorem

**Theorem 2.1.** *(**Viète Theorem [4]**) Consider the following polynomial of degree n*

$$f(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$$

*Then, the n roots of f (counting multiplicities) $z_1,z_2,\cdots,z_n$ satisfy the following relations*

$$z_1+z_2+\cdots+z_{n-1}+z_n=\frac{-a_{n-1}}{a_n}$$

$$(z_1z_2+z_1z_3+\cdots+z_1z_n)+(z_2z_3+z_2z_4+\cdots+z_2z_n)+\cdots+z_{n-1}z_n=\frac{a_{n-2}}{a_n}$$

$$\vdots$$

$$z_1z_2\cdots z_n=(-1)^n\frac{a_0}{a_n}$$

Firstly, we convert the third order equation (1.2) to the first order system

$$\begin{aligned} x_{n+1} &= \frac{\beta x_n+z_n}{A+y_n} \\ y_{n+1} &= x_n \\ z_{n+1} &= y_n \end{aligned}$$

The system has two fixed points, the first one is the zero fixed point $(0,0,0)$ and a positive fixed point

$$X^*=(\beta-A+1,\beta-A+1,\beta-A+1),\quad \beta+1>A$$

Viète's theorem will be used to show that the Jacobian matrix of the above system has a pair of complex eigenvalues of modulus one.
The following theorem gives a condition for local stability of $X^*$. Let

$$\beta^*=\frac{1-A}{1+A}$$

**Theorem 2.2.** *The positive fixed point is stable if $\beta>\beta^*$ and unstable if $\beta<\beta^*$*

*Proof.* The Jacobian matrix of the system is

$$J=\begin{pmatrix} \frac{\beta}{A+y_n} & \frac{-(\beta x_n+z_n)}{(A+y_n)^2} & \frac{1}{A+y_n} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

At the positive fixed point

$$J=\begin{pmatrix} \frac{\beta}{\beta+1} & \frac{-(\beta-A+1)}{\beta+1} & \frac{1}{\beta+1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix $J$ is

$$p(\lambda)=|\lambda I-J|=\lambda^3-\frac{\beta}{\beta+1}\lambda^2+\frac{\beta-A+1}{\beta+1}\lambda-\frac{1}{\beta+1} \tag{2.1}$$

To study the stability of $X^*$ we use Jury's conditions

$$p(1)=\frac{\beta-A+1}{\beta+1}>0$$

$$(-1)^3p(-1)=2+\frac{\beta-A+1}{\beta+1}>0$$

The sufficient conditions are, $|a_0|<a_3$ and $|b_0|>|b_2|$ where

$$a_0=\frac{-1}{\beta+1},\quad a_1=\frac{\beta-A+1}{\beta+1},\quad a_2=\frac{-\beta}{\beta+1},\quad a_3=1$$

and

$$b_0=\begin{vmatrix} a_0 & a_3 \\ a_3 & a_0 \end{vmatrix},\quad b_2=\begin{vmatrix} a_0 & a_1 \\ a_3 & a_2 \end{vmatrix}$$

The condition $|a_0|<a_3$ is trivially satisfied. Now,

$$b_0=\frac{1}{(\beta+1)^2}-1,\quad \text{thus}\quad |b_0|=1-\frac{1}{(\beta+1)^2}=\frac{\beta^2+2\beta}{(\beta+1)^2}$$

and

$$b_2=\begin{vmatrix} \frac{-1}{\beta+1} & \frac{\beta-A+1}{\beta+1} \\ 1 & \frac{-\beta}{\beta+1} \end{vmatrix}=\frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2}$$

We consider two cases. The first case is

$$\frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2}>0$$

then $|b_2|=\frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2}$. The condition $|b_0|>|b_2|$ is satisfied if and only if

$$\frac{\beta^2+2\beta}{(\beta+1)^2}>\frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2}$$

which is equivalent to

$$\frac{2\beta+1-A}{\beta+1}>0$$

the last inequality is satisfied since $\beta-A+1>0$. The second case is when

$$\frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2}<0$$

So

$$|b_2|=\frac{\beta^2+\beta+1-A\beta-A}{(\beta+1)^2}$$

Now, $|b_0|>|b_2|$ if

$$\frac{\beta^2+2\beta}{(\beta+1)^2}>\frac{\beta^2+\beta+1-A\beta-A}{(\beta+1)^2}$$

which is satisfied if and only if

$$\beta>\frac{1-A}{1+A}$$

The proof is complete. ■

## 3. Existence of Neimark-Sacker bifurcation

In this section, we show that equation (1.2) undergoes a Neimark-Sacker bifurcation by proving the existence of a pair of complex conjugate eigenvalues of modulus one.

**Theorem 3.1.** *When $\beta = \beta^* = \frac{1-A}{1+A}$, polynomial (2.1) has two complex conjugate roots of modulus one and another real root that lies inside the unit circle. Moreover for $A \in (0,1)$ the Neimark Sacker bifurcation conditions are satisfied.*

The theorem will be proved through the following lemmas

**Lemma 3.1.** *The characteristic polynomial (2.1) has two complex roots, $\lambda_1, \lambda_2 = \bar{\lambda}_1$ and a real root $\lambda_3$ in the interval $(0,1)$.*

*Proof.* The derivative of $p(\lambda)$ is given by

$$p'(\lambda) = 3\lambda^2 - \frac{2\beta}{\beta+1}\lambda + \frac{\beta - A + 1}{\beta+1}$$

If the discriminant of $p'(\lambda)$ is negative then $p(\lambda)$ has complex roots,

$$\Delta p'(\lambda) = \frac{-8\beta^2 + 12(\beta A + A - 1) - 24\beta}{(\beta+1)^2}$$

Using the condition $\beta(A+1) + A - 1 = 0$, we find that

$$\Delta p'(\lambda) = \frac{-8\beta^2 - 36\beta}{(\beta+1)^2} < 0$$

So $p'(\lambda)$ has complex roots. Hence, $p(\lambda)$ has complex roots as well. Since $p(0) = \frac{-1}{\beta+1} < 0$ and $p(1) > 0$, then there exists $\lambda_3 \in (0,1)$ such that $p(\lambda_3) = 0$, this is the unique real root inside the unit circle. ■

**Lemma 3.2.** *The complex roots of polynomial (2.1) have modulus one when $\beta = \beta^*$. Moreover, the real root $\lambda_3 = \frac{1}{1+\beta}$.*

*Proof.* Suppose that $\lambda_1, \lambda_2, \lambda_3$ are the roots of $p(\lambda)$ where $\lambda_2 = \bar{\lambda}_1$ and $\lambda_3 = r_0$. We apply *Viète* theorem to the polynomial $p(\lambda)$. If $|\lambda_1| = |\lambda_2| = 1$ and $\lambda_3 = r_0$ then

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{-a_2}{a_3} = \frac{\beta}{\beta+1} \tag{3.1}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{a_1}{a_3} = \frac{\beta - A + 1}{\beta+1} \tag{3.2}$$

$$\lambda_1\lambda_2\lambda_3 = \frac{-a_0}{a_3} = \frac{1}{\beta+1}$$

It follows that

$$\lambda_1\lambda_2\lambda_3 = \lambda_3 = \frac{1}{\beta+1}$$

Plugging this value of $\lambda_3$ into (3.2) and using the fact that $\lambda_1\lambda_2 = 1$, we find

$$\lambda_1 + \lambda_2 = -A$$

Then substitute for $\lambda_3$ in (3.1) to get

$$\lambda_1 + \lambda_2 = \frac{\beta}{\beta+1} - \frac{1}{\beta+1} = \frac{\beta-1}{\beta+1}$$

Therefore,

$$\lambda_1 + \lambda_2 = \frac{\beta-1}{\beta+1} = -A$$

which implies that

$$\beta = \frac{1-A}{1+A}$$

It follows, from the above argument, that there exist a conjugate pair of complex roots on the unit circle. ■

The roots of the characteristic polynomial depend on the parameters $A$ and $\beta$. Hence, at $\beta^* = (1-A)/(1+A)$, these roots are functions of $A$, and will be denoted by $\lambda_1^*(A) = \lambda_1(A, \beta^*(A)), \lambda_2^*(A) = \lambda_2(A, \beta^*(A))$ and $\lambda_3^* = \lambda_3(A, \beta^*(A))$

**Lemma 3.3.** *The complex roots of polynomial (2.1) are* $\lambda^*_{1,2}(A) = \exp \pm i\theta^*$ *where*

$$\theta^* = \arccos\left(\frac{-A}{2}\right)$$

*Proof.* Let $e^{i\theta}, e^{-i\theta}$ be the roots of $p(\lambda)$, then

$$e^{3i\theta} - \frac{\beta}{\beta+1}e^{2i\theta} + \frac{\beta-A+1}{\beta+1}e^{i\theta} - \frac{1}{\beta+1} = 0$$

$$\cos 3\theta + i\sin 3\theta - \frac{\beta}{\beta+1}(\cos 2\theta + i\sin 2\theta) + \frac{\beta-A+1}{\beta+1}(\cos\theta + i\sin\theta) - \frac{1}{\beta+1} = 0$$

Separation of real and imaginary parts gives

$$\cos 3\theta - \frac{\beta}{\beta+1}\cos 2\theta = -\frac{\beta-A+1}{\beta+1}\cos\theta + \frac{1}{\beta+1}$$

and

$$\sin 3\theta - \frac{\beta}{\beta+1}\sin 2\theta = -\frac{\beta-A+1}{\beta+1}\sin\theta$$

Square both sides of previous equations and add them up, we find that

$$\cos^2 3\theta + \sin^2 3\theta + \left(\frac{\beta}{\beta+1}\right)^2(\cos^2 2\theta + \sin^2 2\theta) - \frac{2\beta}{\beta+1}(\cos 2\theta\cos 3\theta + \sin 2\theta \sin 3\theta)$$

$$= \left(\frac{1}{\beta+1}\right)^2 + \left(\frac{\beta-A+1}{\beta+1}\right)^2(\cos^2\theta + \sin^2\theta) - \frac{2(\beta-A+1)}{(\beta+1)^2}\cos\theta$$

It follows that

$$1 + \left(\frac{\beta}{\beta+1}\right)^2 - \left(\frac{1}{\beta+1}\right)^2 - \left(\frac{\beta-A+1}{\beta+1}\right)^2 = \left(\frac{2\beta}{\beta+1} - \frac{2(\beta-A+1)}{(\beta+1)^2}\right)\cos\theta$$

Simplifying we get

$$\cos\theta = \frac{(\beta+A-1)}{2\beta}$$

Then, evaluating at $\beta = \beta^* = \frac{1-A}{1+A}$

$$\cos\theta = \frac{-A}{2}$$

Hence for $A \in (0,1)$, $\frac{-1}{2} < \cos\theta < 0$. Therefore, there exists $\theta^* \in (\frac{\pi}{2}, \frac{2\pi}{3})$ such that

$$\theta^* = \arccos\left(\frac{-A}{2}\right)$$

Moreover, $\theta^* \neq 0, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pm\pi$. Consequently, $e^{ik\theta^*} \neq 1$ for $k \in \{1,2,3,4\}$. ∎

**Lemma 3.4.** *The condition* $\frac{d|\lambda|^2}{d\beta}\mid_{\beta=\beta^*} \neq 0$ *is fulfilled at* $\beta = \beta^*$.

*Proof.* Note that

$$p(\lambda) = \lambda^3 - \frac{\beta}{\beta+1}\lambda^2 + \frac{\beta-A+1}{\beta+1}\lambda - \frac{1}{\beta+1}$$

$$\frac{d|\lambda|^2}{d\beta}\mid_{\beta=\beta^*} = \frac{d(\lambda\bar{\lambda})}{d\beta} = \lambda\frac{\partial\bar{\lambda}}{\partial\beta} + \bar{\lambda}\frac{\partial\lambda}{\partial\beta}$$

$$\begin{aligned}
\frac{d|\lambda|^2}{d\beta} &= \lambda\left(\frac{\partial p(\bar{\lambda})}{\partial\beta}\frac{\partial\bar{\lambda}}{\partial p(\bar{\lambda})}\right) + \bar{\lambda}\left(\frac{\partial p(\lambda)}{\partial\beta}\frac{\partial\lambda}{\partial p(\lambda)}\right) \\
&= \lambda\left(\frac{-\bar{\lambda}^2 + A\bar{\lambda} + 1}{(\beta+1)^2(3\bar{\lambda}^2 - \frac{2\beta}{\beta+1}\bar{\lambda} + \frac{\beta-A+1}{\beta+1}}\right) + \bar{\lambda}\left(\frac{-\lambda^2 + A\lambda + 1}{(\beta+1)^2(3\lambda^2 - \frac{2\beta}{\beta+1}\lambda + \frac{\beta-A+1}{\beta+1}}\right)
\end{aligned}$$

After some calculations, the right hand side of the last equation can be written as

$$\frac{3A(\beta+1)(\bar{\lambda}^2+\lambda^2) - 2\beta A(\lambda+\bar{\lambda}) + 6i(\beta+1)\sin\theta(\lambda^2-\bar{\lambda}^2) + 4i\beta\sin\theta(\bar{\lambda}-\lambda) + 2AX}{(\beta+1)(3(\beta+1)\bar{\lambda}^2 - 2\beta\bar{\lambda} + X)(3(\beta+1)\lambda^2 - 2\beta\lambda + X)}$$

where $X = \beta - A + 1$. But

$$\lambda + \bar{\lambda} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta$$

$$\lambda^2 + \bar{\lambda}^2 = (\cos\theta + i\sin\theta)^2 + (\cos\theta - i\sin\theta)^2 = 2(\cos^2\theta - \sin^2\theta) = 2\cos(2\theta)$$

$$\lambda^2 - \bar{\lambda}^2 = (\cos\theta + i\sin\theta)^2 - (\cos\theta - i\sin\theta)^2 = 4i\cos\theta\sin\theta = i2\sin(2\theta)$$

Consequently, we have

$$\frac{d|\lambda|^2}{d\beta} = \frac{6A(\beta+1)\cos(2\theta) - 4\beta A\cos\theta + (-24(\beta+1)\cos\theta + 8\beta)\sin^2\theta + 2AX}{(\beta+1)(3(\beta+1)\bar{\lambda}^2 - 2\beta\bar{\lambda} + X)(3(\beta+1)\lambda^2 - 2\beta\lambda + X)}$$

Now, at $\theta = \theta^*$, $\beta = \beta^*$, the last expression becomes

$$\frac{d|\lambda|^2}{d\beta}\,|_{\beta=\beta^*} = \frac{8A\frac{2}{1+A} - 2A^2 + 8\frac{1-A}{1+A}}{(\beta^*+1)(3(\beta^*+1)\bar{\lambda}^2 - 2\beta\bar{\lambda} + X^*)(3(\beta^*+1)\lambda^2 - 2\beta^*\lambda + X^*)}$$

$$= \frac{-2(A-2)(A+2)(A+1)}{(1+A)(\beta^*+1)(3(\beta^*+1)\bar{\lambda}^2 - 2\beta^*\bar{\lambda} + X^*)(3(\beta^*+1)\lambda^2 - 2\beta^*\lambda + X^*)}$$

where $X^* = \beta^* - A + 1$. It follows that

$$\frac{d|\lambda|^2}{d\beta}\,|_{\beta=\beta^*} = -\frac{(A-2)(A+2)(A+1)}{\alpha_1^2 + \alpha_2^2}$$

where

$$\begin{aligned}
\alpha_1 &= 3(\beta^*+1)\cos(2\theta^*) - 2\beta^*\cos\theta^* + (\beta^* - A + 1) \\
\alpha_2 &= 3(\beta^*+1)\sin(2\theta^*) - 2\beta^*\sin\theta^*
\end{aligned}$$

We conclude that $\frac{d|\lambda|^2}{d\beta}\,|_{\beta=\beta^*} \neq 0$ for $A \in (0,1)$ which is the required result. ■

This completes also the proof of theorem (3.1).

## 4. Direction of Neimark-Sacker bifurcation

We have shown that system (2.1) undergoes a Neimark-Sacker bifurcation. In this section, we determine the direction of stability of the invariant closed curve bifurcating from the positive fixed point. We follow the the normal form theory of Neimark-Sacker bifurcation as in Kuznetsove, [5], see also [2].

Now, we shift the fixed point to the origin by taking $u_n = x_n - x^*$, $v_n = y_n - y^*$, $w_n = z_n - z^*$. System (2.1) takes the form

$$\begin{aligned}
u_{n+1} &= \frac{B(u_n + X^*) + w_n + X^*}{A + v_n + X^*} - X^* \\
v_{n+1} &= u_n \\
w_{n+1} &= v_n
\end{aligned}$$

Which can be written as

$$Y_{n+1} = JY_n + G(Y_n) + O(\|Y\|^4) \tag{4.1}$$

where

$$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y), \quad \text{and} \quad Y_n = (u_n, v_n, w_n)^T \in \mathbb{R}^3$$

and

$$B(Y,Y) = (B_1(Y,Y), 0, 0)^T \quad \text{and} \quad C(Y,Y,Y) = (C_1(Y,Y,Y), 0, 0)^T$$

where

$$B_i(\xi,\zeta) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 Y_i(\phi)}{\partial\phi_j\partial\phi_k}\right|_{\phi=0} \xi_j\zeta_k$$

and

$$C_i(\xi,\zeta,\eta) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 Y_i(\phi)}{\partial\phi_j\partial\phi_k\partial\phi_l}\right|_{\phi=0} \xi_j\zeta_k\eta_l$$

$$B_1(\xi,\zeta)=\frac{-\beta}{(\beta+1)^2}(\xi_2\zeta_1+\xi_1\zeta_2)+\frac{2(\beta-A+1)}{(\beta+1)^2}\xi_2\zeta_2-\frac{1}{(\beta+1)^2}(\xi_3\zeta_2+\xi_2\zeta_3)$$

$$C_1(\xi,\zeta,\eta)=\frac{2}{(\beta+1)^3}(\xi_1\zeta_2\eta_2+\xi_2\eta_1\zeta_2+\xi_2\zeta_2\eta_1)$$

$$+\frac{2\beta}{(\beta+1)^3}(\xi_2\zeta_2\eta_3+\xi_2\zeta_3\eta_2+\xi_3\zeta_2\eta_2)-\frac{6(\beta-A+1)}{(\beta+1)^3}\xi_2\zeta_2\eta_2$$

Let $q^*\in\mathbb{C}^3$ be an eigenvector of $J$ corresponding to the eigenvalue $e^{i\theta^*}$ and $p^*\in\mathbb{C}^3$ be an eigenvector of $J^T$ corresponding to the eigenvalue $e^{-i\theta^*}$; that is,

$$Jq^*=e^{i\theta^*}q^*, J^Tp^*=e^{-i\theta^*}p^*$$

Solving $(J-\lambda I)q^*=(J-e^{i\theta}I)q^*=0$, we get $q^*\sim(1,e^{-i\theta^*},e^{-2i\theta^*})^T$ and solving $(J-\lambda I)^Tp^*=(J-e^{-i\theta^*}I)^Tp^*=0$ we get $p^*\sim(1,e^{-i\theta^*}-\frac{\beta}{\beta+1},\frac{e^{i\theta^*}}{\beta+1})^T$. Now, we want to normalize $p^*$ and $q^*$ so that $\langle q^*,p^*\rangle=1$, where $\langle .,.\rangle$ is the standard scalar product in $\mathbb{C}^3$. Note that

$$\langle q^*,p^*\rangle=\sum_{i=1}^{3}\overline{q_i}p_i=2-\frac{\beta e^{i\theta^*}}{\beta+1}+\frac{e^{3i\theta^*}}{\beta+1}$$

So let $q=\varphi q^*$ where $\varphi=(2-\frac{\beta e^{i\theta^*}}{\beta+1}+\frac{e^{3i\theta^*}}{\beta+1})^{-1}$ and $p=p^*$. The real eigenspace $T^c$ corresponding to $\lambda_{1,2}$ is two-dimensional and is spanned by $\{\mathrm{Re}(q),\ \mathrm{Im}(q)\}$. The real eigenspace $T^s$ corresponding to the real eigenvalue of $J$ is one-dimensional. Any vector $x\in\mathbb{R}^3$ may be decomposed as

$$x=zq+\bar{z}\bar{q}+y$$

where $z\in\mathbb{C}^1$, and $\bar{z}\bar{q}\in T^c$, $y\in T^{su}$. The complex variable $z$ is a coordinate on $T^c$. We have

$$\begin{aligned} z &= \langle p,x\rangle \\ y &= x-\langle p,x\rangle q-\langle\bar{p},x\rangle\bar{q} \end{aligned}$$

In these coordinates, the map (4.1) takes the form

$$\begin{aligned} z &\mapsto e^{i\theta^*}z+\langle p,G(zq+\bar{z}\bar{q}+y)\rangle \\ y &\mapsto Jy+G(zq+\bar{z}\bar{q}+y)-\langle p,G(zq+\bar{z}\bar{q}+y)\rangle q-\langle\bar{p},G(zq+\bar{z}\bar{q}+y)\rangle\bar{q} \end{aligned}$$

Using Taylor expansions, the previous system can be written in the form:

$$\begin{aligned} z &\mapsto e^{i\theta^*}z+\frac{1}{2}G_{20}z^2+G_{11}z\bar{z}+\frac{1}{2}G_{02}\bar{z}^2+\frac{1}{2}G_{21}z^2\bar{z}+\cdots \\ y &\mapsto Jy+\frac{1}{2}H_{20}z^2+H_{11}z\bar{z}+\frac{1}{2}H_{02}\bar{z}^2+\cdots \end{aligned}$$

Where

$$G_{20}=\langle p,B(q,q)\rangle, G_{11}=\langle p,B(q,\bar{q})\rangle, G_{02}=\langle p,B(\bar{q},\bar{q})\rangle \tag{4.2}$$

and

$$G_{21}=\langle p,C(q,q,\bar{q})\rangle \tag{4.3}$$

$$H_{20} = B(q,q)-\langle p,B(q,q)\rangle q-\langle\bar{p},B(q,q)\rangle\bar{q} \tag{4.4}$$
$$H_{11} = B(q,\bar{q})-\langle p,B(q,\bar{q})\rangle q-\langle\bar{p},B(q,\bar{q})\rangle\bar{q} \tag{4.5}$$

and the scalar product in $\mathbb{C}^3$ is used. From the center manifold theorem, there exists a center manifold $W^c$ which can be approximated as

$$Y=V(z,\bar{z})=\frac{1}{2}w_{20}z^2+w_{11}z\bar{z}+\frac{1}{2}w_{02}\bar{z}^2+O(|z|^3)$$

where $<p,w_{ij}>=0$. The vectors $w_{ij}\in\mathbb{C}^3$ can be found from the linear equations

$$\begin{aligned} w_{20} &= (e^{2i\theta^*}I_3-J)^{-1}H_{20} \\ w_{11} &= (I_3-J)^{-1}H_{11} \\ w_{02} &= (e^{-2i\theta^*}I_3-J)^{-1}H_{02} \end{aligned}$$

So $z$ can be expressed as

$$z\mapsto e^{i\theta^*}z+\frac{1}{2}G_{20}z^2+G_{11}z\bar{z}+\frac{1}{2}G_{02}\bar{z}^2+\frac{1}{2}(G_{21}+2\langle p,B(q,(I-J)^{-1}H_{11})\rangle$$

$$+\langle p,B(\bar{q},(e^{2i\theta^*}I-J)^{-1}H_{20}))\rangle z^2\bar{z} \tag{4.6}$$

Substituting equations (4.2)-(4.5) into (4.6) and taking into account the identities

$$(I-J)^{-1}q=\frac{1}{1-e^{i\theta^*}}q, \quad (e^{2i\theta^*}I-J)^{-1}q=\frac{e^{-i\theta^*}}{e^{i\theta^*}-1}q$$

and

$$(I-J)^{-1}\bar{q}=\frac{1}{1-e^{-i\theta^*}}\bar{q}, \quad (e^{2i\theta^*}I-J)^{-1}\bar{q}=\frac{e^{i\theta^*}}{e^{3i\theta^*}-1}\bar{q}$$

We can express $z$ using the map

$$z\mapsto e^{i\theta^*}z+\sum_{k+l\geq 2}\frac{1}{k!j!}g_{kj}z^k\bar{z}^j$$

Finally, the restricted map can be written as

$$z\mapsto e^{i\theta^*}z(1+d(\beta^*))|z|^2+O(|z|^4)$$

where the real number $A(\beta^*)=\mathrm{Re}(d(\beta^*))$ determines the direction of bifurcation of the closed invariant curve and can be computed using the formula

$$A(\beta^*)=\mathrm{Re}\left(\frac{e^{-i\theta^*}g_{21}}{2}\right)-\mathrm{Re}\left(\frac{(1-2e^{i\theta^*})e^{-2i\theta^*}}{2(1-e^{i\theta^*})}g_{20}g_{11}\right)-\frac{1}{2}|g_{11}|^2-\frac{1}{4}|g_{02}|^2$$

The coefficients $g_{20},g_{11},g_{02}$ and $g_{21}$ can be readily calculated using simple, but tedious, calculations. Firstly, we have

$$B(q,q)=\begin{pmatrix}\frac{2(\beta-A+1)e^{-2i\theta^*}-2\beta e^{-i\theta^*}-2e^{-3i\theta^*}}{(\beta+1)^2}\\0\\0\end{pmatrix}$$

It follows that

$$g_{20}=\langle p,B(q,q)\rangle=\frac{2(\beta-A+1)e^{-2i\theta^*}-2\beta e^{-i\theta^*}-2e^{-3i\theta^*}}{(\beta+1)\left(2(\beta+1)-\beta e^{i\theta^*}+e^{3i\theta^*}\right)}$$

whereas

$$B(q,\bar{q})=\begin{pmatrix}\frac{2(\beta-A+1)-2(\beta+1)\cos\theta^*}{(\beta+1)^2}\\0\\0\end{pmatrix}$$

Hence,

$$g_{11}=\langle p,B(q,\bar{q})\rangle=\frac{2(\beta-A+1)-2(\beta+1)\cos\theta^*}{(\beta+1)\left(2(\beta+1)-\beta e^{i\theta^*}+e^{3i\theta^*}\right)}$$

and

$$B(\bar{q},\bar{q})=\begin{pmatrix}\frac{2(\beta-A+1)e^{2i\theta^*}-2\beta e^{i\theta^*}-2e^{3i\theta^*}}{(\beta+1)^2}\\0\\0\end{pmatrix}$$

Then

$$g_{02}=\langle p,B(\bar{q},\bar{q})\rangle=\frac{2(\beta-A+1)e^{2i\theta^*}-2\beta e^{i\theta^*}-2e^{3i\theta^*}}{(\beta+1)\left(2(\beta+1)-\beta e^{i\theta^*}+e^{3i\theta^*}\right)}$$

Finally, to find $g_{21}$ we use the formula

$$g_{21}=\langle p,C(q,q,\bar{q})\rangle+2\langle p,B(q,(I-J)^{-1}B(q,\bar{q}))\rangle+$$

$$\langle p,B(\bar{q},(e^{2i\theta^*}I-J)^{-1}B(q,q))\rangle+\frac{e^{-i\theta^*}(1-2e^{i\theta^*})}{1-e^{i\theta^*}}\langle p,B(q,q)\rangle\langle p,B(q,\bar{q})\rangle$$

$$-\frac{2}{1-e^{-i\theta^*}}|\langle p,B(q,\bar{q})\rangle|^2-\frac{e^{i\theta^*}}{e^{3i\theta^*}-1}|\langle p,B(\bar{q},\bar{q})\rangle|^2$$

where

$$C(q,q,\bar{q})=\begin{pmatrix}\frac{-6(B-A+1)e^{-i\theta^*}+2B(1+2e^{-2i\theta^*})+2(2+e^{-2i\theta^*})}{(B+1)^3}\\0\\0\end{pmatrix}$$

So

$$\langle p, C(q,q,\bar{q})\rangle = \frac{-6(\beta-A+1)e^{-i\theta^*}+2\beta(1+2e^{-2i\theta^*})+2(2+e^{-2i\theta^*})}{(\beta+1)^2\left(2(\beta+1)-\beta e^{i\theta^*}+e^{3i\theta^*}\right)}$$

and

$$\langle p, B(\bar{q},(e^{2i\theta^*}I-J)^{-1}B(q,q)\rangle = \frac{L\left(2(\beta-A+1)e^{3i\theta^*}-\beta(e^{2i\theta^*}+e^{5i\theta^*})-(e^{i\theta^*}+e^{4i\theta^*})\right)}{K(2(\beta+1)-\beta e^{i\theta^*}+e^{3i\theta^*})}$$

where

$$K=(\beta+1)e^{6i\theta_0}-\beta e^{4i\theta^*}+(\beta-A+1)e^{2i\theta^*}-1,\ L=\frac{2(\beta-A+1)-2(\beta+1)\cos\theta^*}{(\beta+1)^2}$$

Depending on the above calculation, we find that $A(\beta^*)=-0.91<0$ when $A=0.5, \beta=\beta^*=1/3$, so the closed invariant curve is supercritical (stable) according to the following theorem.

**Theorem 4.1.** *If $A(\beta^*)<0$ (respectively, $>0$), then the Neimark-Sacker bifurcation at $\beta=\beta^*$ is supercritical (respectively, subcritical) and there exists a unique invariant closed curve bifurcates from the fixed point which is asymptotically stable (respectively, unstable).*

## 5. Computer simulation

In this section, we present some numerical simulations of equation (1.2) that supports our analytical results. The first figure is a bifurcation diagram for equation (1.2) when $A=0.5$, $x_{-2}=x_{-1}=x_0=0.2$. In this case, the positive equilibrium point is stable if $\beta>\frac{1}{3}$ and unstable if $\beta<\frac{1}{3}$. In figures 2 and 3, we plot phase portraits in the $(x(n),x(n-2))$ plane. In Figure 2, $A=0.5, \beta=\beta^*$, and $x_{-2}=x_{-1}=x_0=0.2$. Notice the existence of a closed invariant curve at the bifurcation value. In figure 3, $A=0.5, \beta=0.4$, and $x_{-2}=x_{-1}=x_0=0.5$.

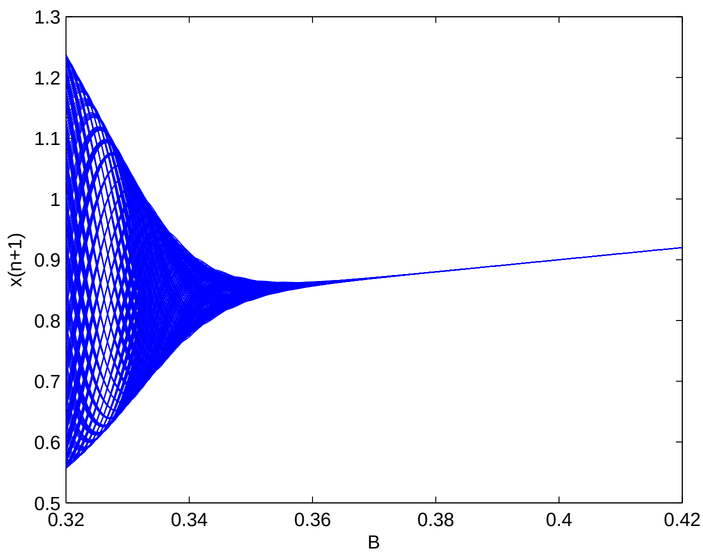


**Figure 1:** Bifurcation diagram of Eq.(1.2) in $(\beta,X)$ plane for $A=0.5$

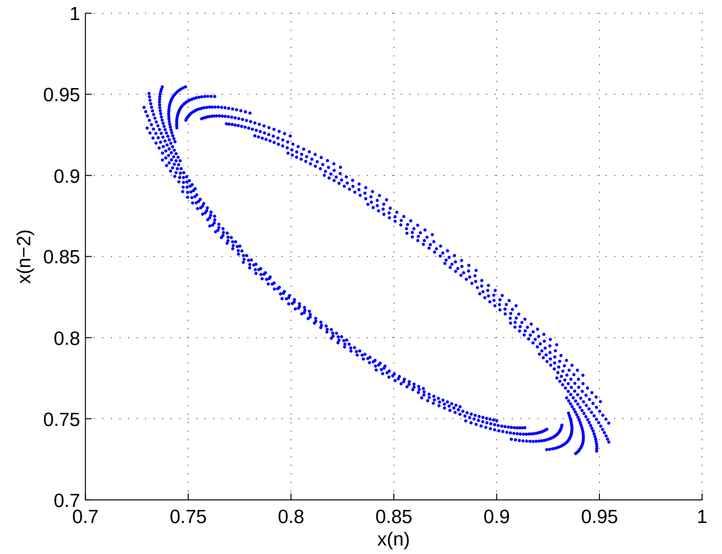


**Figure 2:** Phase portrait of Eq.(1.2) in $(x(n), x(n-2))$ plane for $A = 0.5$, $\beta = 1/3$

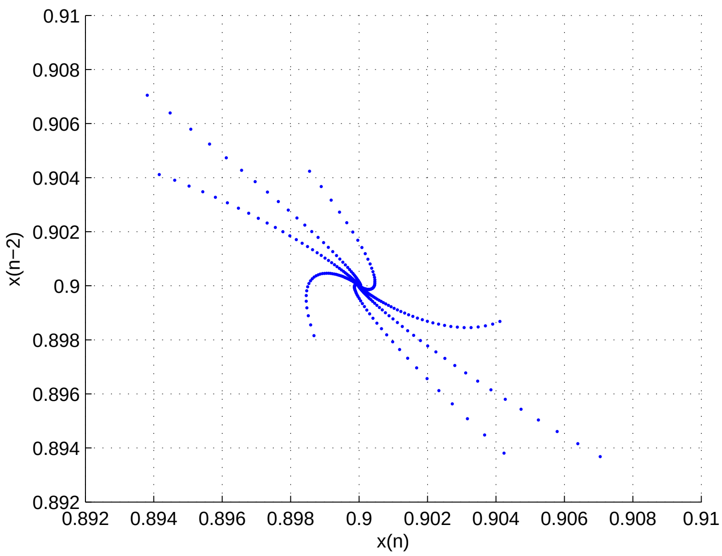


**Figure 3:** Phase portrait of Eq.(1.2) in $(x(n), x(n-2))$ plane for $A = 0.5$, $\beta = 0.4$

## 6. Conclusion

In this paper, we have used normal form theory to show that a third order difference equation undergoes a Neimar-Sacker bifurcation. All conditions for the existence of A Neimark-Sacker bifurcation have been checked. In the last section, we gave some numerical simulations that support our analytical results. Notice the stability of the invariant curve and the fixed point in figure 2 and figure 3, respectively, as predicted by Theorem 4.1.

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# On Simultaneously and Approximately Simultaneously Diagonalizable $m$-tuples of Matrices

**Maria Isabel García-Planas**

*Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain*

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**Abstract**

In this paper, the problem of simultaneous diagonalization of $m$-tuples of $n$-order square complex matrices, is analyzed and some necessary and some necessary and sufficient conditions for this property to be fulfilled are presented. This study has an interest in its applications in different areas as for example in engineering and physical sciences. For example, they appear founding when we must give the instanton solution of Yang-Mills field presented in an octonion form, and it can be represented by triples of traceless matrices.
In the case where the $m$-tuple does not simultaneously diagonalize, the possibility of to find near of the given $m$-tuple, an m-tuple that diagonalize simultaneously is studied.

## 1. Introduction

Let $\mathfrak{M}$ be the manifold of $m$-tuples of $n$-order square complex matrices $T = (X_1, \dots, X_m)$ representing polynomial matrices $P_T(x) = X_1 + xX_2 + \dots + x^{m-1}X_m$ that appear in a natural way modeling tools in several research areas of applied mathematics, sciences and engineering, and in a special manner in systems theory ([1]-[3]). Studying control problems by means the polynomial matrix approach, the solution of these problems are reformulated in terms of polynomial matrix equations, where solutions are based on structural properties of the involved matrices, where the simultaneous diagonalization of each and every one of the matrices is a great advantage for solving the problem. The simultaneously diagonalization is related to sets of commuting matrices and it can be found some results (see [4], [5], for example).
Among families of $m$-tuples of matrices, have some interest the families of traceless triples because the Lie algebra is related to gauge fields because they appear in the Lagrangian describing the dynamics of the field, then they are associated to 1-forms that take values on a certain Lie algebra. It is also of interest to note that triples of traceless matrices have some relevance for supergravity theories ([6]). Another application is found when we must give the instanton solution of Yang-Mills field can be presented in an octonion form, and it can be represented by triples of traceless matrices ([7]).
In the space of $n$-square complex matrices, it is well known that the subset of diagonalizable matrices is generic in the sense that this subset is an open and dense set, then any no diagonalizable matrix can be diagonalized by a small perturbation of its entries. This property cannot be generalized to the case of simultaneous diagonalization of an $m$-tuple of $n$-order complex square matrices. We are interested in analyzing the collection of $m$-tuples of matrices that simultaneously diagonalize and the collection that simultaneously diagonalize under small perturbations, some properties in this sense appear in [8].
The simultaneous diagonalization of two real symmetric matrices has long been of interest and largely studied [9]. In this paper, we generalize to the problem of deciding whether the elements of $\mathfrak{M}$ can be simultaneously diagonalized, and in the case where the $m$-tuple does not simultaneously diagonalize, we study the possibility of to find near of the given $m$-tuple, an $m$-tuple that diagonalize simultaneously.

## 2. Simultaneous similarity of $m$-tuples of $n$-order matrices

**Definition 2.1.** *Let $T = (X_1, \dots, X_m), T' = (Y_1, \dots, Y_m) \in \mathfrak{M}$ be two m-tuples of matrices. Then, $T$ is simultaneous similar to $T'$ if and only if there exists $P \in Gl(n;\mathbb{R})$ such that*

$$(Y_1, \dots, Y_m) = (PX_1P^{-1}, \dots, PX_mP^{-1}). \tag{2.1}$$

***Email addresses and ORCID numbers***: **maria.isabel.garcia@upc.edu, https://orcid.org/0000-0001-7418-7208, (M. I. García-Planas)**

For simplicity, we will write $PTP' = T'$.
We are interested on the simultaneous diagonalizable $m$-tuples.

**Definition 2.2.** *The m-tuples of matrices $T = (X_1, \dots, X_m) \in \mathfrak{M}$ is simultaneously diagonalizable if and only if there exist an equivalent m-tuple formed by diagonal matrices.*

From definition we have

**Corollary 2.3.** *Let $T = (X_1, \dots, X_m)$ be an m-tuple of square matrices. The m-tuple is simultaneous diagonalizable if and only if there exist diagonal matrices $D_i$, $i = 1, \dots, m$ and a invertible matrix $P$ (the same matrix $P$ for all $i$) such that*

$$(X_i^t \otimes I_n - I_n \otimes D_i)\operatorname{vec} P = 0, \ \forall 1 \le i \le m.$$

**Remark 2.4.** *Let $A = (a_{ij})$ and $B$, the Kronecker product is defined as $A \otimes B = (a_{ij}B)$.*

*Proof.* From $D_i = PX_iP^{-1}$ for all $i = 1, \dots, m$ we have $PX_i - D_iP = 0$, for all $i = 1, \dots, m$
Then, computing the Kronecker product and applying the vectorizing operator we deduce the result. □

Clearly, necessary conditions for simultaneous diagonalizable $m$-tuples are the following

**Proposition 2.5.** *Let $T = (X_1, \dots, X_m)$ be a simultaneous diagonalizable m-tuple. Then all matrices $X_i$ must be diagonalizable.*

Obviously, the reciprocal is false

**Example 2.6.** *Clearly, matrices $X_1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ are diagonalizable, but none of the matrices $P_i = Q_i^{-1}$ with $Q_1 = \begin{pmatrix} a & b \\ 0 & b \end{pmatrix}$ and $Q_2 = \begin{pmatrix} b & a \\ b & 0 \end{pmatrix}$ with $ab \neq 0$, diagonalizing $X_1$ can diagonalize $X_2$.*

**Proposition 2.7.** *Let $T = (X_1, \dots, X_m)$ be a simultaneous diagonalizable m-tuple. Then $X_iX_j = X_jX_i$.*

*Proof.* Let $T = (X_1, \dots, X_m)$ be a simultaneously diagonalizable $m$-tuple, then there exist $P \in Gl(n;\mathbb{C})$ such that $PX_iP^{-1} = D_i$ for $i = 1, \dots, m$. So, taking into account that $D_iD_j = D_jD_i$, for all $i, j = 1, \dots, m$, we have $P^{-1}D_iPP^{-1}D_jP = P^{-1}D_jPP^{-1}D_iP$, that is to say $X_iX_j = X_jX_i$, for all $i, j = 1, \dots, m$.
□

**Theorem 2.8.** *Let $T = (X_1, \dots, X_m)$ be a m-tuple of commuting n-order square matrices and suppose that the matrix $X_j$ for some $j$ is diagonalizable with simple eigenvalues ($\lambda_k \neq \lambda_\ell$ for all $k \neq \ell$, $k, \ell = 1, \dots n$). Then $T$ is a m-tuple of simultaneously diagonalizable matrices*

*Proof.* For simplicity we consider $X_1$ the diagonalizable matrix.
Let $\{v_1, \dots, v_n\}$ be a basis of eigenvectors corresponding to eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of $X_j$.
Let us consider $X_iX_1v_j$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

$$\begin{aligned} X_iX_1v_j &= X_i\lambda_jv_j = \lambda_jX_iv_j \\ X_jX_iv_j &= \lambda_jX_iv_j \end{aligned}$$

So, if $X_iv_j \neq 0$ it is an eigenvector of $X_1$ of eigenvalue $\lambda_j$, but condition $\lambda_k \neq \lambda_\ell$ implies that $\dim \operatorname{Ker}(X_1 - \lambda_jI) = 1$, then, $X_iv_j = \mu_iv_j$, that is to say $v_j$ is an eigenvector for $X_i$ of eigenvalue $\mu_i$. If $X_iv_j = 0$ $v_j$ is an eigenvector of $X_i$ of eigenvalue equal zero. That is to say $\{v_1, \dots, v_n\}$ is a basis of eigenvectors for each $X_i$, $i = 1, \dots, m$ and $T$ is $m$-tuple of simultaneous diagonalizable matrices with $P = \begin{pmatrix} v_1^t & \dots & v_n^t \end{pmatrix}^{-1}$. □

**Remark 2.9.** *The other matrices not necessary have simple eigenvalues.*

**Theorem 2.10.** *Let $T = (X_1, \dots, X_m)$ be an m-tuple of commuting and diagonalizable n-order square matrices. Then, they diagonalize simultaneously.*

*Proof.* Let $P_1$ be an invertible matrix such that $D_1 = P_1X_1P^{-1} = \begin{pmatrix} D_1^1 & & \\ & \ddots & \\ & & D_{r_1}^1 \end{pmatrix}$ with $D_i^1 = \lambda_i^1 I \in M_{n_i}(\mathbb{C})$, $1 \le i \le r$ and $n_1 + \ldots + n_r = n$.
Let us consider $v_{1_1}, \dots, v_{n_1}, \dots, v_{1_r}, \dots, v_{n_r}$ the vector columns of $P^{-1}$, then

$$\begin{aligned} X_jX_\ell v_{i_\ell} &= X_j\lambda_\ell v_{i_\ell} = \lambda_\ell X_jv_{i_\ell} \\ X_jX_\ell v_{i_\ell} &= X_\ell X_jv_{i_\ell} \end{aligned}$$

Consequently $X_jv_{i_\ell}$ is an eigenvector of $X_\ell$ of eigenvalue $\lambda_\ell$ or $X_jv_{i_\ell} = 0$, in any case we have that $X_jv_{i_\ell} \in [v_{1_\ell}, \dots, v_{n_\ell}] = F_\ell$, consequently, the subspace $F_\ell$ is $X_j$ invariant for all $1 \le \ell \le r$ and $1 \le j \le m$.

So, $P_1X_jP_1^{-1} = \begin{pmatrix} Y_1^j & & \\ & \ddots & \\ & & Y_r^j \end{pmatrix}$, for $2 \le j \le m$.

If all matrices $Y_k^j$ are diagonal the proof is concluded, otherwise and taking into account that all matrices $X_i$ diagonalize all submatrices $Y_k^j$ diagonalize.

Consider $P_2 = \begin{pmatrix} P_2^1 & & \\ & \ddots & \\ & & P_1^r \end{pmatrix}$ where $P_2^j$ diagonalizes $Y_j^2$ for $1 \le j \le r$.

Obviously $P_2$ diagonalizes $D_1$:

$$\begin{pmatrix} P_2^1 & & \\ & \ddots & \\ & & P_2^r \end{pmatrix}\begin{pmatrix} D_1^1 & & \\ & \ddots & \\ & & D_r^1 \end{pmatrix}\begin{pmatrix} P_2^1 & & \\ & \ddots & \\ & & P_2^r \end{pmatrix}^{-1} = \begin{pmatrix} P_2^1 D_1^1 (P_2^1)^{-1} & & \\ & \ddots & \\ & & P_2^r D_r^1 (P_2^r)^{-1} \end{pmatrix} =$$
$$\begin{pmatrix} P_2^1 \lambda_1^1 I (P_2^1)^{-1} & & \\ & \ddots & \\ & & P_2^r \lambda_r^1 I (P_2^r)^{-1} \end{pmatrix} = \begin{pmatrix} D_1^1 & & \\ & \ddots & \\ & & D_r^1 \end{pmatrix}$$

Then $P_2P_1$ diagonalizes $X_1$ and $X_2$, now partitioning the matrices $P_2^jY_j^2(P_2^j)^{-1}$ into blocks corresponding to the same eigenvalue (it is possible that different blocks $Y_j^2$ have common eigenvalues but we partition according to each block).
Now we consider $P_2P_1X_j(P_2P_1)^{-1}$, if all matrices are diagonal the proof is concluded, otherwise we repeat the processus with $P_2P_1X_3(P_2P_1)^{-1}$ taking into account the new partition in scalar matrices. The process ends at most when reaches to the last matrix. □

After these results it is easy to obtain the following geometrical result.

**Theorem 2.11.** *Let $T = (X_1,\dots,X_m)$ be an m-tuple of n-order square matrices and suppose that all matrices $X_i$ are diagonalizable, then a necessary and sufficient condition for simultaneous diagonalization is there exist a basis $\{v_1,\dots,v_n\}$ of $v \in \mathbb{C}^n$ such that*

$$v_j \in \cap_{i=1}^m \operatorname{Ker}(X_i - \lambda_i^j)I, \text{ where } \lambda_j^i \in Spec X_i = \{\lambda_1^i,\dots,\lambda_n^i\}$$

**Corollary 2.12.**

$$P = \begin{pmatrix} v_1^t & \dots & v_n^t \end{pmatrix}^{-1}$$

*verifies that $PX_iP^{-1} = D_i$*

**Example 2.13.** *Let $T = (X_1,X_2,X_3)$ be a triple with*

$$X_1 = \begin{pmatrix} 5.5 & 2 & -3.5 \\ 3 & 3 & -3 \\ 4.5 & 2 & -2.5 \end{pmatrix}, X_2 = \begin{pmatrix} 3 & -2 & 2 \\ 1.5 & 4 & -1.5 \\ 0 & -2 & 5 \end{pmatrix}, X_3 = \begin{pmatrix} 15.5 & 10 & -13.5 \\ 3 & 7 & -3 \\ 10.5 & 10 & -8.5 \end{pmatrix}$$

$$SpecX_1 = \{1,2,3\},\ SpecX_2 = \{3,4,5\},\ SpecX_3 = \{2,5,7\}$$

$v_1 = (0.6667, 0.3333, 0.6667) \in Ker(X_1 - 3I) \cap Ker(X_2 - 4I) \cap Ker(X_3 - 7I)$
$v_2 = (-0.2294, -0.6882, -0.6882) \in Ker(X_1 - I) \cap Ker(X_2 - 3I) \cap Ker(X_3 - 5I)$
$v_3 = (0.7071, 0, 0.7071) \in Ker(X_1 - 2I) \cap Ker(X_2 - 5I) \cap Ker(X_3 - 2I)$
*Then there exist*

$$P = \begin{pmatrix} 4.5005 & 3.0003 & -4.5005 \\ 2.1796 & 0.0000 & -2.1796 \\ -2.1220 & -2.8289 & 3.5362 \end{pmatrix} = \begin{pmatrix} 0.6667 & -0.2294 & 0.7071 \\ 0.3333 & -0.6882 & 0.0000 \\ 0.6667 & -0.6882 & 0.7071 \end{pmatrix}^{-1}$$

*such that*

$$PX_1P^{-1} = \begin{pmatrix} 3 & & \\ & 1 & \\ & & 2 \end{pmatrix}, PX_2P^{-1} = \begin{pmatrix} 4 & & \\ & 3 & \\ & & 5 \end{pmatrix}, PX_3P^{-1} = \begin{pmatrix} 7 & & \\ & 5 & \\ & & 2 \end{pmatrix}$$

*(Calculations made with MatlabR2012b).*

In this case, all possible matrices $P$ diagonalizing $X_i$ for some $i = 1,\dots,m$, (that they are such that $P = Q^{-1}$ where $Q$ is a matrix whose columns are the eigenvectors corresponding to each of the eigenvalues of $X_i$ for some $i = 1,\dots n$), are matrices that diagonalizes all matrices simultaneously obtaining $D_i$ or permutations of this matrices. In fact, we have the following proposition.

**Proposition 2.14.** *If the set of matrices $\{X_1,\dots,X_m\}$ are simultaneously diagonalizable and for some i, $X_i$ has simple eigenvalues, all matrices P diagonalizing $X_i$ diagonalize $X_j$ for all $j = 1 = \dots,m$.*

**Remark 2.15.** *If no matrix has simple eigenvalues then the result fails*

**Example 2.16.** *Let $T = (X_1,X_2,X_3)$ be a triple with*

$$X_1 = \begin{pmatrix} 3 & 1 & -2 & -1 & 0 \\ 5 & -1 & -6 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 \\ 5 & -3 & -6 & 3 & 2 \\ 1 & -1 & -2 & 0 & 3 \end{pmatrix}, X_2 = \begin{pmatrix} 7 & 3 & -4 & -3 & 0 \\ 13 & -3 & -16 & 3 & 6 \\ 0 & 3 & 3 & -3 & 0 \\ 13 & -9 & -16 & 9 & 6 \\ 5 & -3 & -8 & 0 & 9 \end{pmatrix}, X_3 = \begin{pmatrix} 6 & 4 & -4 & -4 & 0 \\ 16 & -6 & -20 & 4 & 8 \\ 0 & 4 & 2 & -4 & 0 \\ 16 & -12 & -20 & 10 & 8 \\ 8 & -4 & -12 & 0 & 10 \end{pmatrix}$$

*This triple diagonalize simultaneously, because there exists $Q \in Gl(n;\mathbb{C})$ with* $Q = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 & -1 \end{pmatrix}$,

*such that* $P = Q^{-1} = \begin{pmatrix} -2 & 1 & 3 & 0 & -1 \\ 2 & -2 & -2 & 1 & 1 \\ 3 & -2 & -4 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}$

$$D_i = PX_iP^{-1}$$

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, D_2 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}, D_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

*But, in this case not all matrices diagonalizing one of this matrices diagonalize all set of matrices of m-tuple of smultaneously diagonalizable matrices, because taking* $P = \begin{pmatrix} -2 & 1 & 3 & 0 & -1 \\ 2 & -2 & -2 & 1 & 1 \\ 3 & -2 & -4 & 1 & 2 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}$

*Then* $P^{-1}X_1P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$, $P^{-1}X_2P = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & -1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$, $P^{-1}X_3P = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$

*We observe that the matrix P only diagonalise* $X_3$

## 3. Approximately simultaneously diagonalizable $m$-tuples of matrices

It is well known that near of a squre matrix there is a diagonalizable matrix having simple eigenvalues. We ask if this result can be extended to the case of $m$-tuples of square matrices. We will try to obtain an answer using geometrical tools.

### 3.1. Group Lie action

The equivalence relation defined in (2.1) can be seen as the action over $\mathfrak{M}$ in the following manner
Let us consider the following map

$$\begin{aligned} \alpha : Gl(n;\mathbb{C}) \times \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (P,T) &\longrightarrow PTP^{-1} = (PX_1P^{-1}, \dots, PX_mP^{-1}) \end{aligned}$$

that verifies

i) If $I \in Gl(n;\mathbb{C})$ is the identity element, then $\alpha(I,T) = T$ for all $T \in \mathfrak{M}$.
ii) If $P_1$ and $P_2$ are in $Gl(n;\mathbb{C})$ , then $\alpha(P_1, \alpha(P_2,T)) = \alpha(P_1P_2,T)$ for all $T \in \mathfrak{M}$.
$\alpha(P_1, \alpha(P_2,T)) = \alpha(P_1, P_2TP_2^{-1}) = P_1P_2TP_2^{-1}P_1^{-1} = (P_1P_2)T(P_1P_2)^{-1} = \alpha(P_1P_2,T)$

So, the map $\alpha$ defines an action of $Gl(n;\mathbb{C})$ over $\mathfrak{M}$.
Fixing $T \in \mathfrak{M}$ we can consider the map

$$\begin{aligned} \alpha_T : Gl(n;\mathbb{C}) &\longrightarrow \mathfrak{M} \\ P &\longrightarrow \alpha_T(P) = \alpha(P,T) \end{aligned}$$

We consider the following sets

$$\begin{aligned} &\operatorname{Im}\alpha_T = \mathscr{O}(T) = \{(Y_1, \dots, Y_m) = (PX_1P^{-1}, \dots, PX_mP^{-1}), \forall P \in Gl(n;\mathbb{C})\} \\ &Stab(T) = \{P \in Gl(n;\mathbb{C}) \mid \alpha_T(P) = T\} \end{aligned}$$

Fixing $P \in Gl(n;\mathbb{C})$ we can consider the map

$$\begin{aligned} \alpha_P : \mathfrak{M} &\longrightarrow \mathfrak{M} \\ T &\longrightarrow \alpha_P(T) = \alpha(P,T) \end{aligned}$$

Notice that $\alpha_P$ is a bijection: if $\alpha(P,T_1) = \alpha(P,T_2)$ then $PT_1P^{-1} = PT_2P^{-1}$ and $T_1 = T_2$, so it is injective; for all $T \in \mathfrak{M}$, there exists $\bar{T} = P^{-1}TP$ such that $\alpha(P,\bar{T}) = T$, then it is surjective.

### 3.2. Approximately simultaneously diagonalizability

It is well known that close to any matrix there is a nearby that diagonalizes. Then the question is: given an $m$-tuple of square matrices, it is possible to found an $m$-tuple diagonalizing simultaneously?
In the case where that it is possible we say that the $m$-tuple is approximately simultaneously diagonalizable (abbreviated ASD), more concretely

**Definition 3.1.** *[8] The m-tuple* $T = (X_1, \dots, X_n)$ *is approximately simultaneously diagonalizable if and only if for any* $\varepsilon > 0$, *there exist a m-tuple of matrices* $(Y_1, \dots, Y_m)$ *which are simultaneously diagonalizable and satisfy* $\|Y_i - X_i\| < \varepsilon$ *for* $i = 1, \dots, m$.

O'Meara and Vinsonhaler in [8], analyze approximately simultaneously diagonalizable matrices for the case where the matrices of the $m$-tuple commute.

**Proposition 3.2.** *Let* $T = (X_1, \dots, X_n)$ *be an m-tuple simultaneously diagonalizable. Then, each* $T' \in \mathscr{O}(T)$ *is an m-tuple simultaneously diagonalizable.*

*Proof.* Taking into account that $T = (X_1,\ldots,X_n)$ is an $m$-tuple simultaneously diagonalizable there exist $P \in Gl(n;\mathbb{C})$ such that $PTP^{-1} = (PX_1P^{-1},\ldots,PX_mP^{-1}) = (D_1,\ldots,D_m) = D$ with $D_i$ diagonal matrices for all $i = 1,\ldots,m$.
Let $T' \in \mathscr{O}(T)$, then, there exist $P' \in Gl(n;\mathbb{C})$ such that $T' = P'T(P')^{-1} = (P'X_1(P')^{-1},\ldots,P'X_m(P')^{-1})$.
So, $T' = P'P^{-1}DP(P')^{-1} = (P'P^{-1}D_1P(P')^{-1},\ldots,P'P^{-1}D_mP(P')^{-1}) = (P''D_1(P'')^{-1},\ldots,P''D_m(P'')^{-1})$, with $P'' = P'P^{-1} \in Gl(n;\mathbb{C})$. □

Consequently, and taking into account that if $T' \in \mathscr{O}(T)$ is $\mathscr{O}(T) = \mathscr{O}(T')$, we can use miniversal deformations to study approximately simultaneously diagonalizability.

### 3.3. Miniversal deformations

**Definition 3.3.** *A deformation of an element $X_0 \in \mathfrak{M}$ is a family of elements of $\mathfrak{M}$ indexed by $\lambda \in \Lambda$ $\varphi : \Lambda \longrightarrow \mathfrak{M}$ where $\Lambda \subset \mathbb{F}^\ell$ is a neighborhood of 0, and where $\varphi(0) = X_0$ and $\varphi$ depends holomorphically (smoothly) on the parameters.*

**Definition 3.4.** *A deformation $\varphi(\lambda) = \varphi(\lambda_1,\ldots,\lambda_\ell)$ of $X_0$ is versal if and only if for any deformation $\varphi'(\mu_1,\ldots,\varphi_k) \in \mathfrak{M}$ of $X_0$, $\varphi'(\mu)$ is induced by $\varphi(\lambda)$, i.e., there exists a neighborhood $V$ of 0 in $\mathbb{F}^k$, a map $\psi : V \longrightarrow \mathbb{F}^\ell$ with $\psi(0) = 0$, and a map $g : V \longrightarrow G$ with $g(0) = I$ such that $\forall \mu \in V$, $\varphi'(\mu) = g(\mu)\varphi(\psi(\mu))g^{-1}(\mu)$ with $\psi$ and $g$ holomorphic (smooth).*

It is obvious that if we have a versal deformation of an element automatically we have a versal deformation of any element that is equivalent to it, since if $X = \alpha(g,X_0)$ is an equivalent element of $X_0$ and $\varphi(\lambda)$ is a versal deformation of $X'$ then $\alpha(g^{-1},X(\lambda))$ is a versal deformation of $X_0$.
A versal deformation having minimal number of parameters is called *miniversal*.
The following result was proved by Arnold [10], in the case where $\mathrm{Gl}(n;\mathbb{C})$ acts on $M_n(\mathbb{C})$, and was generalized by Tannenbaum [11], in the case where a Lie group acts on a complex manifold. It provides the relationship between a versal deformation of $X_0$ and the local structure of the orbit.

**Theorem 3.5** ([11])**.**
1. *A deformation $\varphi(\lambda)$ of $(X_0)$ is versal if and only if it is transversal to the orbit $\mathscr{O}(X_0)$ at $(X_0)$.*
2. *Minimal number of parameters of a versal deformation is equal to the codimension of the orbit of $X_0$ in $\mathfrak{M}$, $\ell = \operatorname{codim}\mathscr{O}(X_0)$.*

**Corollary 3.6.** *Then $\varphi(\lambda) = X_0 + (T_{X_0}\mathscr{O}(X_0))^\perp$ for some scalar product is a miniversal deformation.*

Let $d\alpha_{X_0} : T_I\mathscr{G} \longrightarrow \mathfrak{M}$ be the differential of $\alpha_{X_0}$ at the unit element $I$. It is easy to compute $d\alpha_{X_0}(P)$:

$$d\alpha_T(P) = ([X_1,P],\ldots,[X_m,P]) \in \mathfrak{M}, \quad P \in T_I\mathscr{G}.$$

If we define scalar products in $\mathfrak{M}$ and $T_I\mathscr{G}$, we can consider the adjoint application of $d\alpha_{X_0}$. The Euclidean scalar products considered in this paper are defined as follows:
For all $T_i = (X_1^i,\ldots,X_m^i) \in \mathfrak{M}$ and for all $P_i \in T_I\mathscr{G}$

$$\langle T_1,T_2\rangle_1 = \operatorname{trace}(X_1^1X_1^{2*}) + \ldots + \operatorname{trace}(X_1^mX_1^{m*}),$$
$$\langle P_1,P_2\rangle_2 = \operatorname{trace}(P_1P_2^*),$$

where $X^*$ denotes the conjugate transpose of a matrix $X$.
The adjoint linear mapping $d\alpha_T^* : \mathfrak{M} \longrightarrow T_I\mathscr{G}$ is defined by the relation

$$\langle d\alpha_{X_0}(P),Z\rangle_1 = \langle P,d\alpha_{x_0}^*(Z)\rangle_2, \quad P \in T_I\mathscr{G},\ Z \in \mathfrak{M}.$$

It is straightforward to find

$$d\alpha_{X_0}^*(W) = ([X^*,A_0] + [Y^*,B_0] + [Z^*,C_0]) \in T_I\mathscr{G}, \quad W = (X,Y,Z) \in \mathfrak{M}.$$

The mappings $d\alpha_{X_0}$ and $d\alpha_{X_0}^*$ provide a simple description of the tangent spaces $T_{X_0}\mathscr{O}(X_0)$, $T_I\mathscr{S}tab(X_0)$ and their normal complements $(T_{X_0}\mathscr{O}(X_0))^\perp$, $(T_I\mathscr{S}(X_0))^\perp$.

**Theorem 3.7.** *The tangent spaces to the orbit of the m-tuple of matrices $T$ and the corresponding normal complementary subspace can be found in the following form*

1. $T_T\mathscr{O}(X_0) = \operatorname{Im} d\alpha_T \subset \mathfrak{M}$.
2. $(T_T\mathscr{O}(X_0))^\perp = \operatorname{Ker} d\alpha_T^* \subset \mathfrak{M}$,

After this theorem, it is easy to compute these spaces.

**Corollary 3.8.**
1. $T_{X_0}\mathscr{O}(X_0) = \{([P,A_0],[P,B_0],[P,C_0]) \mid P \in T_I\mathscr{G}\}$
2. $(T_{X_0}(\mathscr{O}(X_0))^\perp = \{(X,Y,Z) \in \mathfrak{M} \mid [X^*,A_0] + [Y^*,B_0] + [Z^*,C_0] = 0\}$

**Remark 3.9.** *Let $X_0 = (X_1^0,\ldots,X_m^0)$ be an n-tuple of matrices and we consider $\mathfrak{X}_i = (0,\ldots,0,X_i,0,\ldots,0)$ an m-tuple of matrices such that $X_i^0 + X_i$ is a miniversal deformation of $X_i^0$. Then $\mathfrak{X}_i \in (T_{X_0}(\mathscr{O}(X_0))^\perp$ and consequently $\mathfrak{X} = \sum \mathfrak{X}_i \in T_{X_0}(\mathscr{O}(X_0))^\perp$.*

Consequently, we have the following proposition.

**Proposition 3.10.** *Let $X_0 = (X_1^0,\ldots,X_m^0)$ be an n-tuple of matrices. Then, for all $\varepsilon > 0$ there exist $\mathfrak{X} = (X_1,\ldots,X_m)$ such that $X_i^0 + X_i$ is diagonalizable, for all $i = 1,\ldots,m$.*

**Remark 3.11.** *Given any n-tuple of matrices, we can find in a neighborhood, an n-tuple of matrices in which all matrices are diagonalizable but not necessarily all matrices in the n-tuple diagonalize simultaneously.*

**Example 3.12.** *Consider the following pair of matrices* $\left(\begin{pmatrix}2&0\\1&2\end{pmatrix},\begin{pmatrix}3&1,\\0&3\end{pmatrix}\right)$ *that they are no diagonalizable, and the following family of perturbations of the pair:* $\left(\begin{pmatrix}2+\varepsilon_1&0\\1&2+\varepsilon_2\end{pmatrix},\begin{pmatrix}3+\varepsilon_3&1,\\0&3+\varepsilon_4\end{pmatrix}\right)$ *for all* $\varepsilon_i$ *with* $\varepsilon_1\neq\varepsilon_2$ *and* $\varepsilon_3\neq\varepsilon_4$.
*Clearly, both matrices are diagonalizable.*
*For simultaneously diagonalization it is necessary that both matrices commute, but*

$$\begin{pmatrix}2+\varepsilon_1&0\\1&2+\varepsilon_2\end{pmatrix}\begin{pmatrix}3+\varepsilon_3&1,\\0&3+\varepsilon_4\end{pmatrix}\neq\begin{pmatrix}3+\varepsilon_3&1,\\0&3+\varepsilon_4\end{pmatrix}\begin{pmatrix}2+\varepsilon_1&0\\1&2+\varepsilon_2\end{pmatrix}$$

*for all* $\varepsilon_1$, $\varepsilon_2$*, so both matrices diagonalize but not diagonalize simultaneously.*
*Now, we consider the following perturbation* $\left(\begin{pmatrix}2&\varepsilon_1\\1&2\end{pmatrix},\begin{pmatrix}3&1,\\\varepsilon_2&3\end{pmatrix}\right)$ *for all* $\varepsilon_i$ *with* $\varepsilon_1\cdot\varepsilon_2\neq 0$*. Clearly, both matrices are diagonalizable. Analyzing commutativity*

$$\begin{pmatrix}2&\varepsilon_1\\1&2\end{pmatrix}\begin{pmatrix}3&1,\\\varepsilon_2&3\end{pmatrix}=\begin{pmatrix}3&1,\\\varepsilon_2&3\end{pmatrix}\begin{pmatrix}2&\varepsilon_1\\1&2\end{pmatrix}$$

*equivalently* $\varepsilon_1\varepsilon_2=1$
*So, taking* $\varepsilon_1=\varepsilon_2=1$*, both matrices diagonalice simultanoeulsy, (it suffices to consider* $P^{-1}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$*)*
*The near pair of matrices in this family diagonalizing simultaneously is with* $(\varepsilon_1,\varepsilon_2)$ *minimizing distance of the variety* $V=\{(\varepsilon_1,\varepsilon_2)\mid\varepsilon_1\varepsilon_2=1\}$

In general, a lower bound at the distance of the a $n$-tuple of matrices to a one $n$-tuple diagonalizing simultaneously is given tn the following proposition

**Proposition 3.13.** *Let* $T=(X_1,\dots,X_m)$ *be a n-tuple of matrices and* $T(\lambda)=(X_1(\lambda),\dots,X_m(\lambda))$ *with* $\lambda\in\mathbb{C}^\ell$ *a family of n-tuples such that in a neigborhood of* $0\in\mathbb{C}^\ell$ *is a miniversal deformation of the given n-tuple. A lower bound at the distance of the a n-tuple of matrices to a one n-tuple diagonalizing simultaneously is*

$$\inf\{\mathrm{dist}(0,\lambda),0,\lambda\in\mathbb{C}^\ell\mid X_i(\lambda)X_j(\lambda)=X_j(\lambda)X_i(\lambda)\forall 1\leq i,j\leq m\}$$

**Example 3.14.** *Let* $T=\left(\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\right)$ *be a triple of matrices no diagonalizing. Let us consider the family of triples* $T(\varepsilon)=\left(\left(\begin{smallmatrix}\varepsilon_1&\varepsilon_3\\1+\varepsilon_2&\varepsilon_4\end{smallmatrix}\right),\left(\begin{smallmatrix}\varepsilon_5&1\\\varepsilon_6&\varepsilon_7\end{smallmatrix}\right),\left(\begin{smallmatrix}\varepsilon_8&0\\0&1+\varepsilon_9\end{smallmatrix}\right)\right)$ *in such a way that for some* $\varepsilon$ *with* $\|\varepsilon\|>0$ *is a miniversal (no orthogonal) deformation.*
*The subset of the commuting triples in the family is*

$$V=\left\{T\in\{T\in T(\varepsilon)\mid\begin{array}{rl}\varphi_1(\varepsilon)=\varepsilon_3\cdot\varepsilon_6-\varepsilon_2-1&=0,\\\varphi_2(\varepsilon)=\varepsilon_1-\varepsilon_4-\varepsilon_3\cdot\varepsilon_5+\varepsilon_3\cdot\varepsilon_7&=0,\\\varphi_3(\varepsilon)=\varepsilon_4\cdot\varepsilon_6-\varepsilon_1\cdot\varepsilon_6+\varepsilon_5\cdot(\varepsilon_2+1)-\varepsilon_7\cdot(\varepsilon_2+1)&=0,\\\varphi_4(\varepsilon)=\varepsilon_9-\varepsilon_8+1&=0.\end{array}\right\}$$

*We can compute the minimal distance by means the Lagrange's undetermined multipliers method, from the function:*

$$f(\varepsilon,\lambda)=\sum_{i=1}^{9}\varepsilon_i^2+\sum_{i=1}^{4}\lambda_i\varphi_i(\varepsilon)$$

*The minimal distance is* $\sqrt{3/2}$*, a triple minimizing this distance is a triple of commuting matrices with* $\varepsilon_2=-1$, $\varepsilon_8=1/2=-\varepsilon_9$ *and* $\varepsilon_i=0$ *for* $i=1,3,4,5,6,7$ *but no diagonalize simultaneously.*
*Taking the solution* $\varepsilon_3=\varepsilon_6=\sqrt{2}$, $\varepsilon_2=1$, $\varepsilon_8=\dfrac{1}{2}=-\varepsilon_9$*, and* $\varepsilon_i=0$ *for* $i=1,4,5,7$ *with distance* $\sqrt{11/2}$ *we have a triple of commuting matrices and they diagonalize simultaneously with* $P^{-1}=\begin{pmatrix}1&1\\2^{1/4}&-2^{1/4}\end{pmatrix}$.

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# Difference Sequence Spaces Derived by using Pascal Transform

**Saadettin Aydın[1] and Harun Polat[2*]**

[1]*Department of Mathematics, Faculty of Science and Arts, Kilis 7 Aralık University, Kilis, Turkey*
[2]*Department of Mathematics, Faculty of Science and Arts, Muş Alparslan University, Muş, Turkey*
[*]*Corresponding author E-mail: h.polat@alparslan.edu.tr*

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**Abstract**

The essential goal of this manuscript is to investigate some novel sequence spaces of $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ which are comprised by all sequence spaces whose differences are in Pascal sequence spaces $p_\infty$, $p_c$ and $p_0$, respectively. Furthermore, we determine both $\gamma$-, $\beta$-, $\alpha$- duals of newly defined difference sequence spaces of $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$. We also obtain bases of the newly defined difference sequence spaces of $p_c(\Delta)$ and $p_0(\Delta)$. Finally, necessary and sufficient conditions on an infinite matrix belonging to the classes $(p_c(\Delta) : l_\infty)$ and $(p_c(\Delta) : c)$ are characterized.

## 1. Introduction

Real or complex valued sequences spaces are represented by $w$ along with the manuscript. Each sub-classes of real or complex valued sequences spaces is known as a sequence space. A sequence space of null, convergent, and bounded sequences are respectively demonstrated by $c_0$ , $c$, and $l_\infty$. Moreover $cs$, $l_1$, $bs$ depict convergent, absolutely convergent, and bounded series respectively.
$K$ space is defined by any sequence space $\lambda$ with a linear topology satisfying following transformation for a continuous term of $p_s(m) = m_s$ $s \in N$ such that $p_s : \lambda a \to C$ , where $N = \{0,1,2,\dots\}$ and $C$ represents the set of complex number. If $\lambda$ is a complete linear metric space then $K$-space is named by $FK$- space. $BK$-space is defined as normable topological space of $FK$-space [1].
Infinite matrix of complex or real numbers $A = (a_{nk})$ is defined for $n, k \in N$. Let $X$ and $Y$ be any two sequence spaces. Then, $A$ is defined as a transformation between $X$ to $Y$ such that following equality holds.

$$(Ax)_n = \sum_k a_{nk}x_k \tag{1.1}$$

for each $n \in N$. $(X : Y)$, shows the family of matrices where $A : X \to Y$. Hence series given by the (1.1) converges for every $x \in X$ and each $n \in N$ iff $A \in (X : Y)$. One also has $Ax = \{(Ax)_n\} \in Y$. Here collection of entire finite subsets on $K$ and $N$ is denoted by $F$, where $N \subset F$.
Studies on the sequence space have been mainly focused on some elementary concepts which are inclusions of sequence spaces, matrix mapping, determination of topologies, [2]. Let $X$ be a sequence space and $A$ be an infinite matrix in $X$ then the domain of matrix is determined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}$$

In general limitation matrix $A$ produces novel sequence space $X_A$ and it is either contraction or the expansion of the original space. Indeed, it is obviously clear that inclusion relations of $X \subset X_\Delta$ and $X_S \subset X$ are decidedly satisfied for $X \in \{c, l_\infty, c_0\}$ [3]. In particular, the the difference operator and sequence spaces which are fundamental samples for the matrix $A$ and they have been investigated comprehensively through the mentioned methods.
Let $P$ represeents the means of Pascal which is described by the matrix of Pascal [4] then it is defined by

$$P = [p_{nk}] = \begin{cases} \binom{n}{n-k}, & (0 \le k \le n) \\ 0, & (k > n) \end{cases} \quad , (n, k \in N)$$

***Email addresses and ORCID numbers***: **saadettinaydin@gmail.com, https://orcid.org/0000-0002-9559-0730 (S. Aydın), h.polat@alparslan.edu.tr, https://orcid.org/0000-0003-3955-9197 (H. Polat)**

and the inverse of matrix of Pascal $P_n = (p_{nk})$ is defined by

$$P^{-1} = [p_{nk}]^{-1} = \begin{cases} (-1)^{n-k}\binom{n}{n-k}, & (0 \le k \le n) \\ 0 & , (k > n) \end{cases}, (n,k \in N).$$

Pascal matrix contains some fascinating features. For instance; we can form three types of matrices: symmetric, lower triangular, and upper triangular, for any integer $n > 0$. The n-th order symmetric Pascal matrix $n$ is given by

$$S_n = (s_{ij}) = \binom{i+j-2}{j-1}, \text{ for } i, j = 1,2,....,n, \tag{1.2}$$

$n-$th order lower triangular Pascal matrix is presented by

$$L_n = (l_{ij}) = \begin{cases} \binom{i-1}{j-1}, & (0 \le j \le i) \\ 0, & (j > i) \end{cases}, \tag{1.3}$$

and the n-th order upper triangular Pascal matrix of order is presented by

$$U_n = (u_{ij}) = \begin{cases} \binom{j-1}{i-1}, & (0 \le i \le j) \\ 0, & (j > i) \end{cases}. \tag{1.4}$$

We notice that $U_n = (L_n)^T$, n is any natural number.
**i.** Let $S_n$ be the n-th order symmetric Pascal matrix given by (1.2), $L_n$ be the n-th order lower triangular Pascal matrix given by (1.3), and $U_n$ be the n-th order upper triangular Pascal matrix given by (1.4), then $S_n = L_n U_n$ and $det(S_n) = 1$ [5].
**ii.** Let $S_n$ be the n-th order symmetric Pascal matrix given by (1.2), then $S_n$ is similar to its inverse $S_n^{-1}$ [5].
**iii.** Let $A$ and $B$ be $n \times n$ matrices. It is already known obviously that $A$ is similar to $B$ if one can define $n \times n$ invertible matrix $P$i which satisfies following
$P^{-1}AP = B$ [6].
**iv.** Let $L_n$ be the n-th order Pascal matrix. It is also assumed that it is a lower triangular matrix which is given by (1.3), then $L_n^{-1} = ((-1)^{i-j} l_{ij})$ [7].
Recently, Pascal sequence spaces was investigated by Polat [8] $p_\infty$, $p_c$ and $p_0$ like as follows:

$$p_\infty = \left\{ x = (x_k) \in w : \sup_n \left| \sum_{k=0}^{n} \binom{n}{n-k} x_k \right| < \infty \right\},$$

$$p_c = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} x_k \text{ exists} \right\},$$

and

$$p_0 = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} x_k = 0 \right\}.$$

$l_\infty(\Delta) = \{x \in w : (x_k - x_{k+1}) \in l_\infty\}$, $c(\Delta) = \{x \in w : (x_k - x_{k+1}) \in c\}$ and $c_0(\Delta) = \{x \in w : (x_k - x_{k+1}) \in c_0\}$ are known as difference sequence space and they are firstly defined by Kızmaz [9]. Further, various authors have defined and studied the difference sequence spaces, which can be seen in the following papers [10]-[15].
In this manuscript, Pascal difference sequence spaces of $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ are defined. They contain entire sequences whose differences are in Pascal sequence spaces $p_\infty$, $p_c$ and $p_0$, respectively. What is more, we determine the bases of the novel difference sequence spaces $p_c(\Delta)$ and $p_0(\Delta)$, and the $\alpha$-, $\beta$- of the difference sequence spaces $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$. Finally, we give the characterization of the necessary and sufficient conditions on an infinite matrix belonging to families of $(p_c(\Delta) : l_\infty)$ and $(p_c(\Delta) : c)$.

## 2. Inverse formula of the Pascal matrix and Pascal sequence spaces

We define the operators $\Delta : w \to w$ here and after it may be written for the sequence $(x_k - x_{k-1})$ that $(\Delta x)_k = \Delta x$. The well known difference matrix and the inverse of the difference matrix are defined as follows:

$$\left(\Delta^{(1)}\right)_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \le k \le n) \\ 0, & (0 \le k < n-1 \text{ or } k > n) \end{cases}, (n,k \in N)$$

and

$$\left(\left(\Delta^{(1)}\right)^{-1}\right)_{nk} = \begin{cases} 1, & (0 \le k \le n) \\ 0, & (k > n) \end{cases}, (n,k \in N).$$

Pascal difference sequence spaces are defined by $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ by

$$p_\infty(\Delta) = \{x = (x_k) \in w : (x_k - x_{k-1}) \in p_\infty\},$$

$$p_c(\Delta) = \{x = (x_k) \in w : \ (x_k - x_{k-1}) \in p_c\},$$

and

$$p_0(\Delta) = \{x = (x_k) \in w : \ (x_k - x_{k-1}) \in p_0\}.$$

Let be a sequence $y = \{y_n\}$, which is generally utilized as $H$- mapping or $H$- transformation of a sequence $x = (x_k)$ and $H = P\Delta^{(1)}$ i.e.,

$$y_n = (Hx)_n = \sum_{k=0}^{n} \binom{n}{n-k} (x_k - x_{k-1}) \tag{2.1}$$

$$= \sum_{k=0}^{n} \left[ \binom{n}{k} - \binom{n}{k+1} \right] x_k$$

for each $n \in N$. It can be easily shown that $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ are linear and normed spaces by the following norm:

$$\|x\|_\Delta = \|y\|_\infty = \sup_n |y_n|. \tag{2.2}$$

**Theorem 2.1.** *$p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ sequence spaces are Banach spaces provided with the norm function given by (2.2).*

*Proof.* In the space of $p_\infty(\Delta)$, let we define following sequence and suppose that it is a Cauchy sequence $\{x^i\}$ such that $\{x^i\} = \{x_k^i\} = \left\{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots\right\} \in p_\infty(\Delta)$ for every $i \in N$. For a given $\varepsilon > 0$ it may be found a positive integer $N_0(\varepsilon)$ such that $\left\|x_i^k - x_i^n\right\|_\Delta < \varepsilon$ for all $k$, $n > N_0(\varepsilon)$. Hence

$$\left|H(x_i^k - x_i^n)\right| < \varepsilon$$

for all $k, n > N_0(\varepsilon)$ and for each $i \in N$. Therefore, following sequence is a reeal Cauchy sequence $\{(Hx^k)_i\} = \{(Hx^0)_i, (Hx^1)_i, (Hx^2)_i, \ldots\}$ for every fixed $i \in N$. Since real number of set $R$ is complete, it converges, say

$$\lim_{i\to\infty} (Hx^i)_k \to (Hx)_k$$

for each $k \in N$. So, we have

$$\lim_{n\to\infty} \left|H(x_i^k - x_i^n)\right| = \left|H(x_i^k - x_i)\right| \le \varepsilon$$

for each $k \ge N_0(\varepsilon)$. This implies that $\left\|x^k - x\right\|_\Delta < \varepsilon$ for $k \ge N_0(\varepsilon)$, that is, $x^i \to x$ as $i \to \infty$.
Now, we must show that $x \in p_\infty(\Delta)$. We have

$$\|x\|_\Delta = \|Hx\|_\infty = \sup_n \left| \sum_{k=0}^{n} \binom{n}{n-k} \Delta x_k \right|$$

$$= \sup_n \left| \sum_{k=0}^{n} \binom{n}{n-k} (x_k - x_{k-1}) \right|$$

$$= \sup_n \left| \sum_{k=0}^{n} \left[ \binom{n}{k} - \binom{n}{k+1} \right] x_k \right|$$

$$\le \sup_n \left|H(x_k^i - x_k)\right| + \sup_n \left|Hx_k^i\right|$$

$$\le \left\|x^i - x\right\|_\Delta + \left|Hx_k^i\right| < \infty$$

for all $i \in N$. This implies that $x = (x_i) \in p_\infty(\Delta)$. Therefore $p_\infty(\Delta)$ is a Banach space.
It can be shown that $p_c(\Delta)$ and $p_0(\Delta)$ are closed subspaces of $p_\infty(\Delta)$ which implies that $p_c(\Delta)$ and $p_0(\Delta)$ are also Banach spaces. Moreover, $p_\infty(\Delta)$ is a $BK$- space due to the fact that it is a Banach space with continuous coordinates □

## 3. The Bases of the sequence spaces $p_c(\Delta)$ and $p_0(\Delta)$

In this part, it is firstly gien the Schauder basis for the spaces $p_0(\Delta)$ and$p_c(\Delta)$. In normed sequence space $X$, Schauder basis (or briefly bases) is a sequnce of $\left\{b^{(k)}\right\}_{k\in\mathbb{N}}$ such that $x \in \lambda$ and $(\lambda_k)$ of scalars such that

$$\lim_{n\to\infty} \|x-(\lambda_0 x_0+\lambda_1 x_1+\ldots+\lambda_n x_n)\| = 0.$$

**Theorem 3.1.** *Let $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathbb{N}}$ be the sequence of elements of the space $p_0(\Delta)$ for each $k\in\mathbb{N}$ by*

$$b_n^{(k)} = \begin{cases} 0, & (0\le n<k) \\ \sum\limits_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}, & (n\ge k \end{cases}$$

*Then the following assertions are true:*
***i.** The sequence $\left\{b^{(k)}\right\}_{k\in\mathbb{N}}$ is a basis for the space $p_0(\Delta)$, and for any $x\in p_0(\Delta)$ there exists a unique representation of the given form*

$$x=\sum_k \lambda_k(\Delta)\, b^{(k)}.$$

***ii.** The set $\left\{t, b^{(1)}, b^{(2)}, b^{(3)}, \ldots\right\}$ is a basis for the space $p_c(\Delta)$, and for any $x\in p_0(\Delta)$ there exists a unique representation of the given form*

$$x = lt+\sum_k(\lambda_k(\Delta)-l)\, b^{(k)},$$

*where $t=\{t_n\}$ with $t_n=\sum\limits_{k=0}^{n}\left[\sum\limits_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}\right]$, $\lambda_k(\Delta)=(Hx)_k$, $k\in\mathbb{N}$ and $l=\lim\limits_{k\to\infty}(Hx)_k$.*

**Theorem 3.2.** *The sequence spaces $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ are linearly isomorphic to given spaces $l_\infty$, $c$ and $c_0$ respectively, i.e., $p_\infty(\Delta)\cong l_\infty$, $p_c(\Delta)\cong c$ and $p_0(\Delta)\cong c_0$.*

*Proof.* To begin the proof of $p_0(\Delta)\cong c_0$, it is firstly needed to indicate the presence of a linear bijection among spaces $p_0(\Delta)$ and $c_0$. Let we also take the map $T$ described by the (2.1), from $p_0(\Delta)$ to $c_0$ by $x\to y=Tx$ . $T$ is trivially linear. It is also evident that $x=0$ since $Tx=0$ and thus $T$ is an injective.
Let $y\in c_0$ and define the sequence $x=\{x_n\}$ by

$$x_n=\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}\right]y_k.$$

Then,

$$\lim_{n\to\infty}(Hx)_k=\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{n-k}\Delta x_k=\sum_{k=0}^{n}\binom{n}{n-k}(x_k-x_{k-1})$$

$$=\sum_{k=0}^{n}\left[\binom{n}{k}-\binom{n}{k+1}\right]x_k=\lim_{n\to\infty}y_n=0.$$

Thus, we have $x\in p_0(\Delta)$. Finally, T is is norm preserving and surjective. Thus, T is a linearly bijective. Therfore $p_0(\Delta)$ and $c_0$ spaces are linearly isomorphic. Similarly, it might be demonstated that $p_\infty(\Delta)$ and $p_c(\Delta)$ are respectively linearly isomorphic to $l_\infty$ and $c$. $\square$

## 4. The $\alpha$-, $\beta$- and $\gamma$- duals of the sequence spaces $p_\infty(\Delta)$ , $p_c(\Delta)$ and $p_0(\Delta)$

Here we present some facts together with their proofs to determine $\alpha$-, $\beta$- and $\gamma$- duals of Pascal difference sequence spaces $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$. Let $\lambda$ and $\mu$ be two sequence space and let we determine the set $S(\lambda,\mu)$ where

$$S(\lambda,\mu)=\{z=(z_k)\in w: xz=(x_k z_k)\in\mu \text{ for all } x\in\lambda\}. \tag{4.1}$$

From the (4.1), duals of $\alpha$-, $\beta$- and $\gamma$- of the sequence space $\lambda$ that are denoted severally by $\lambda^\alpha$, $\lambda^\beta$ and $\lambda^\gamma$ formed by Garling [17] as the following manner,

$$\lambda^\alpha=S(\lambda,l_1),\ \lambda^\beta=S(\lambda,cs) \text{ and } \lambda^\gamma=S(\lambda,bs).$$

Following facts presented by Tietz and Stieglitz [18] are useful to prove following theorems.

**Lemma 4.1.** $A\in(c_0:l_1)$ *if and only if*

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K}a_{nk}\right|<\infty.$$

**Lemma 4.2.** *$A \in (c_0 : c)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty,$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0.$$

**Lemma 4.3.** *$A \in (c_0 : l_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty.$$

**Theorem 4.4.** *Let $a = (a_k) \in w$ and the matrix $B = (b_{nk})$ by*

$$b_{nk} = \left[\sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_n\right].$$

*Then the $\alpha$- dual of the spaces $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ is the set*

$$b = \left\{ a = (a_n) \in w : \sup_{K\in F} \sum_n \left| \sum_{k\in K} \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_n \right| < \infty \right\}.$$

*Proof.* Let us assume to have $a = (a_n) \in w$ and specially defined matrix $B$ such that rows of the given matrix are the products of the rows of the given matrix $\left(\Delta^{(1)}\right)^{-1} P^{-1}$. From the (2.1), it is derived immediately that

$$a_n x_n = \sum_{k=0}^{n} \left[\sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_n\right] y_k = \sum_{k=0}^{n} b_{nk} y_k = (By)_n \tag{4.2}$$

$i, n \in \mathbb{N}$. We therefore see from the (4.2) that $ax = (a_n x_n) \in l_1$ when $x \in p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ iff $By \in l_1$ whenever $y \in l_\infty$, $c$ and $c_0$. Consequently, it is obtained from the first lemma that

$$\sup_{K\in F} \sum_n \left| \sum_{k\in K} \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{i-k} a_n \right| < \infty$$

which yields the consequence that $[p_\infty(\Delta)]^\alpha = [p_c(\Delta)]^\alpha = [p_0(\Delta)]^\alpha = b$.

□

**Theorem 4.5.** *Let $a = (a_k) \in w$ and the matrix $C = (c_{nk})$ by*

$$c_{nk} = \begin{cases} \sum\limits_{i=k}^{n} \sum\limits_{j=k}^{i} (-1)^{j-k} \binom{j}{j-k} a_i \text{ if } 0 \le k \le n, \\ 0 \text{ if } k > n, \end{cases}$$

*and define sets $c_1$, $c_2$, $c_3$ and $c_4$ by*

$$c_1 = \left\{ a = (a_k) \in w : \sup_n \sum_k |c_{nk}| < \infty \right\},$$

$$c_2 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} c_{nk} \text{ exists for each } k \in N \right\},$$

$$c_3 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k |c_{nk}| = \sum_k \left| \lim_{n\to\infty} c_{nk} \right| \right\},$$

*and*

$$c_4 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k c_{nk} \text{ exists} \right\}.$$

*Then $[p_0(\Delta)]^\beta$, $[p_c(\Delta)]^\beta$ and $[p_\infty(\Delta)]^\beta$ is $c_1 \cap c_2$, $c_1 \cap c_2 \cap c_4$ and $c_2 \cap c_3$, respectively.*

*Proof.* We solely present the proof for $p_0(\Delta)$ space. Since the rest of proof is accomplished by using the similar argument for $p_c(\Delta)$ and $p_\infty(\Delta)$. Let we take the following equation

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[ \sum_{i=k}^{n} \sum_{j=k}^{i} (-1)^{j-k} \binom{j}{j-k} y_j \right] a_k$$

$$= \sum_{k=0}^{n} \left[ \sum_{i=k}^{n} \sum_{j=k}^{i} (-1)^{j-k} \binom{j}{j-k} a_i \right] y_k$$

$$= (Cy)_n .$$

Hence, it is deduced by the second lemma and aforementioned equality that $ax = (a_n x_n) \in cs$ when $x \in p_0(\Delta)$ iff $Cy \in c$ whenever $y \in c_0$. Consequently, it may be shown due to the second lemma that $\{p_0(\Delta)\}^\beta = c_1 \cap c_2$. □

**Theorem 4.6.** *The $\gamma$-dual of the spaces $p_\infty(\Delta)$, $p_c(\Delta)$ and $p_0(\Delta)$ is the set $c_1$*

*Proof.* Proof is accomplished by utilizing the similar method as in the above case. □

## 5. Some matrix transformations on the sequence spaces $p_c(\Delta)$

We shall for brevity that

$$\tilde{a}_{nk} = \sum_{i=k}^{\infty} \sum_{j=k}^{i} (-1)^{j-k} \binom{j}{j-k} a_{ni}$$

and

$$\hat{g}_{nk} = \sum_{i=k}^{n} \sum_{j=k}^{i} (-1)^{j-k} \binom{j}{j-k} a_{ni}$$

In this part, some classes $(p_c(\Delta) : l_\infty)$ and $(p_c(\Delta) : c)$ are characterized. Following proofs of theorems is finalized by considering familiar approaches. Detais left to the reader.

**Theorem 5.1.** *$A \in (p_c(\Delta) : l_\infty)$ if and only if*

$$\sup_n \sum_k |\hat{g}_{nk}| < \infty, \tag{5.1}$$

$$\lim_{n\to\infty} \sum_k \hat{g}_{nk} \text{ exists for all } m \in N, \tag{5.2}$$

$$\sup_{n\in N} \sum_k |\tilde{a}_{nk}| < \infty, \ (n \in N) \tag{5.3}$$

*and*

$$\lim_{n\to\infty} \tilde{a}_{nk} \text{ exists for all } n \in N. \tag{5.4}$$

**Theorem 5.2.** *$A \in (p_c(\Delta) : c)$ iff (5.1)-(5.4) hold, and*

$$\lim_{n\to\infty} \sum_k \tilde{a}_{nk} = \alpha,$$

$$\lim_{n\to\infty} (\tilde{a}_{nk}) = \alpha_k , \ (k \in N) .$$

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# Spectral Theory, Jacobi Matrices, Continued Fractions and Difference Operators

**Ahmed Lesfari**

*Chouaïb Doukkali University, Faculty of Sciences, Department of Mathematics, B.P. 20, 24000 El Jadida, Morocco.*

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**Abstract**

The aim of this paper is to describe some connections between spectral theory in infinite dimensional Lie algebras, deformation theory and linearization of nonlinear dynamical systems. We explain how results from isospectral deformations, cohomology groups and algebraic geometry can be used to obtain insight into integrable systems. Another part will be dedicated to the study of infinite continued fractions and isospectral deformation of periodic Jacobi matrices and general difference operators from an algebraic geometrical point of view. Also, the notion of algebraically completely integrable systems is explained and techniques to solve such systems are presented. Several nonlinear problems in mathematical physics illustrate these results.

## 1. Introduction

The discovery towards the end of the 19th century by Poincaré [1] that complete integrability is an exceptional a phenomenon for Hamiltonian dynamical systems marked the end of a long and fruitful interaction between Hamiltonian mechanics and algebraic geometry and the interest in integrable systems disappeared almost completely; it has been a dormant subject for more than half a century. In fact many algebraic geometrical results such that elliptic and hyperelliptic curves, Abelian integrals, Riemann surfaces, etc., have their origin in problems of mechanics. Fortunately the discovery, 50 years ago that the Korteweg-de Vries (KdV) equation [2] could be integrated by spectral methods have generated an enormous number of new ideas in the area of Hamiltonian completely integrable dynamical systems. The resolution of this problem has led to unexpected connections between mechanics, spectral theory, Lie groups, algebraic geometry and even differential geometry, which has provided new insights into the old mechanical problems of the last centuries and many new ones as well. With respect to this, some questions arise: how do you decide about the complete integrability of a Hamiltonian system? Once you have found necessary conditions of complete integrability on the parameters involved in a Hamiltonian system, how do you prove that the system is effectively completely integrable and how to determine its solutions explicitly? It is well known that solving explicitly a nonlinear Hamiltonian system by quadrature (i.e., by a finite number of algebraic operations including the inverting of functions), was a central theme in mechanics during the 19-th century but the methods of resolution were something very unsystematic and required a great deal of luck and ingenuity. Jacobi [3] himself was very much aware of this difficulty in his famous "*Vorlesungen über Dynamik*", in the context of geodesic flow on the ellipsoid (before introducing the elliptic coordinates). Difficulties come from the fact that in most problems the quadratures were obtained in terms of elliptic or hyperelliptic integrals and where it was often necessary to find remarkable coordinates algebraically related to the originally given ones, in which the Hamilton-Jacobi equation could be solved by separation of variables. In recent years, important results have been obtained following studies on the Korteweg-de Vries (K-dV) and Kadomtsev-Petviashvili (KP) hierarchies. The use of tau functions related to infinite dimensional Grassmannians, Fay identities, vertex operators and the Hirota's bilinear formalism led to obtaining important results concerning these algebras of infinite order differential operators. In addition, many problems related to algebraic geometry, combinatorics, probabilities and quantum gauge theory,..., have been solved explicitly by methods inspired by techniques from the study of dynamical integrable systems. An account of these results will appear elsewhere. This circle of ideas are far from being completely understood, but it is a gold mine of research problems.

***Email addresses and ORCID numbers***: **lesfariahmed@yahoo.fr, https://orcid.org/0000-0001-6213-4301, (A. Lesfari)**

The purpose of this paper is to describe some connections between spectral theory, Jacobi matrices, continued fractions and difference operators and it is organized as follows: Section 2 concerns nonlinear integrable dynamical systems which can be written as Lax equations with a spectral parameter. Such equations have no a priori Hamiltonian content. However, through the Adler-Kostant-Symes construction, we can produce Hamiltonian dynamical systems on coadjoint orbits in the dual space to a Lie algebra whose equations of motion take the Lax form. We outline an algebraic-geometric interpretation of the flows of these systems, which are shown to describe linear motion on a complex torus via the van Moerbeke-Mumford linearization method. We also present Griffith's method of studying these problems without reference to Kac-Moody's algebras. These results are exemplified by several problems of dynamical integrable systems: Euler-Arnold equations for the geodesic flow on the special orthogonal group (the rotation group), Jacobi geodesic flow on the ellipsoid, Neumann problem on the sphere, Lagrange top, periodic infinite band matrix, $n$-dimensional rigid body and Toda lattice. Section 3 is devoted to the study of some connections between continued fractions, isospectral deformation of Jacobi matrices, difference operators, Cauchy-Stieltjes transform and Abelian integrals from an algebraic geometrical point of view. In Section 4 the notion of algebraically completely integrable Hamiltonian systems are explained and techniques to solve such systems are presented. Some important problems will be studied such that: the periodic 5-particle Kac-van Moerbeke lattice, generalized periodic Toda systems, Ramani-Dorizzi-Grammaticos (RDG) series of integrable potentials and a generalized Hénon-Heiles system.

## 2. Coadjoint orbits in Kac-Moody Lie algebras, isospectral deformations and linearization

Assume a Hamiltonian system having the Lax form (with a rational indeterminate $h$) :

$$\dot{A} \equiv \frac{dA}{dt} = [A,B] \text{ or } [B,A], \qquad A = \sum_{j=l}^{n} A_j h^j, \qquad B = \sum_{j=l}^{n} B_j h^j, \tag{2.1}$$

where $A_j$ and $B_j$ are matrices.

**Theorem 2.1.** *For every $h \in \mathbb{C}$, the flow (2.1) preserves the spectrum of $A$. For almost all $(z,h) \in \mathbb{C}^2$, the spectral curve defined by*

$$C = \{(z,h) \in \mathbb{C}^2 : P(z,h) \equiv \det(A - zI) = 0\}, \tag{2.2}$$

*is time independent and its coefficients $tr(A^n)$ are first integrals.*

The matrix $A - zI$, has a one-dimensional null-space, defining a holomorphic line bundle on the curve $C$. Whenever the entries of the $A_j$ are moving in time, the curve $C$ does not move, inducing a motion on the set of line bundles. The set of holomorphic line bundles on an algebraic curve form a group for the operation of tensoring $\otimes$ and the full set with a given topological type is parametrized by the points of a $g$-dimensional complex algebraic torus, where $g$ is the genus of the curve. This torus that we note, $Jac(C)$, is the Jacobian or Picard variety of the curve. When $C$ is an elliptic curve, $Jac(C)$ is isomorphic to $C$. Since the flow (2.1) induces deformations of line bundles, their topological type remains unchanged and therefore it induces a motion on the Jacobian variety; under some checkable condition on $A$ and $B$, du to Griffiths [4] (see further for details).
We state the Adler-Kostant-Symes theorem [5]-[7] valid for any Lie algebra :

**Theorem 2.2.** *Let $\mathscr{G}$ be a Lie algebra with a non-degenerate, ad-invariant metric $\langle , \rangle$. Assume that $\mathscr{G} = \mathscr{L} \oplus \mathscr{K}$ as a vector space decomposition, where $\mathscr{L}$ is an ideal and $\mathscr{K}$ is a Lie sub-algebra.*
*a) Then we have the split $\mathscr{G} = \mathscr{G}^* = \mathscr{L}^\perp + \mathscr{K}^\perp$, $\mathscr{L}^\perp \simeq \mathscr{K}^*$ coupled with $\mathscr{K}$ via an induced form $\langle\langle , \rangle\rangle$ inherits the Kostant-Kirillov coadjoint symplectic structure. The Poisson bracket of the latter, between functions $F$ and $G$ on $\mathscr{K}^*$, is given by*

$$\{F,G\}(A) = \langle\langle A, [\nabla_{\mathscr{K}^*} F, \nabla_{\mathscr{K}^*} G] \rangle\rangle, \quad A \in \mathscr{K}^*.$$

*b) Let $M \subset \mathscr{K}^*$ be an invariant manifold under the above coadjoint action of $\mathscr{K}$ on $\mathscr{K}^*$. Then the functions $H$ defined on a neighborhood of $M$ invariants under the coadjoint action of $\mathscr{G}$, lead to commuting vector fields of the Lax isospectral flows*

$$\dot{A} \equiv \frac{dA}{dt} = [A, pr_{\mathscr{L}}(\nabla H)],$$

*where $pr_{\mathscr{L}}$ is the projection on $\mathscr{L}$.*

The reader interested in the most general form of this theorem can consult with profit the recent paper [8]. This is a general theorem for constructing fully dynamical Hamiltonian integrable systems on the coadjoint orbits of a Lie algebra. We will see explicitly how to apply, with some precautions, this theorem to certain Lie algebras of infinite dimension.
Any finite dimensional semi-simple Lie algebra $\mathscr{G}$ leads to an infinite dimensional Lie algebras, the so-called Kac-Moody extensions (that we also note $\mathscr{G}$) :

$$\mathscr{G} = \left\{ \sum_{-\infty}^{n} A_j h^j | n \in \mathbb{Z} \text{ free} \right\},$$

with bracket

$$\left[\sum A_i h^i, \sum B_j h^j\right] = \sum_k \sum_{i+j=k} [A_i, B_j]\, h^k$$

and ad-invariant, symmetric forms

$$\left\langle \sum A_i h^i, \sum B_j h^j \right\rangle_k = \sum_{i+j=-k} \langle A_i, B_j \rangle,$$

depending on $k \in \mathbb{Z}$. Obviously if the form $\langle , \rangle$ is non degenerate, then the form $\langle , \rangle_k$ is also.
Let $\mathscr{G}_r^s$ $(r \leq s)$ be the vector subspace of $\mathscr{G}$, corresponding to powers of $h$ between $r$ and $s$. A first interesting class of problems is obtained by taking $\mathscr{G} = \mathscr{G}l(n,\mathbb{R})$ and by putting the form $\langle , \rangle_1$ on the Kac-Moody extension. Then we have the decomposition into Lie sub-algebras

$$\mathscr{G} = \mathscr{G}_0^{\infty} + \mathscr{G}_{-\infty}^{-1} \equiv \mathscr{L} + \mathscr{K},$$

with $\mathscr{L} = \mathscr{L}^{\perp}$, $\mathscr{K} = \mathscr{K}^{\perp}$ and $\mathscr{L} = \mathscr{K}^*$.
Another class is obtained by choosing any semi-simple Lie algebra $\mathscr{G}$. Then the Kac-Moody extension $\mathscr{G}$ equipped with the form $\langle , \rangle = \langle , \rangle_0$ has the natural level decomposition

$$\mathscr{G} = \sum_{i\in\mathbb{Z}} G_i, \left[G_i, G_j\right] \subset G_{i+j}, \qquad [G_0, G_0] = 0, \qquad G_i^* = G_{-i}.$$

Let $A_+ = \sum_{i\geq 0} G_i$ and $A_- = \sum_{i<0} G_i$. Then the product Lie algebra $\mathscr{G} \times \mathscr{G}$ has the following bracket and pairing

$$[(a_1,a_2),((b_1,b_2)] = ([a_1,b_1], -[a_2,b_2]), \qquad \langle (a_1,a_2),(b_1,b_2)\rangle = \langle a_1,b_1\rangle - \langle a_2,b_2\rangle.$$

It admits the decomposition into $\mathscr{L} + \mathscr{K}$ with

$$\mathscr{L} = \{(a,-a) : a \in \mathscr{G}\}, \quad \mathscr{L}^{\perp} = \{(a,a) : l \in \mathscr{G}\},$$

$$\mathscr{K} = \{(c_1,c_2) : c_1 \in A_-, c_2 \in A_+, Pr_0(c_1) = Pr_0(c_2)\}, \qquad \mathscr{K}^{\perp} = \{(c_1,c_2) : c_1 \in A_-, c_2 \in A_+, Pr_0(c_2+c_1) = 0\},$$

where $Pr_0$ denotes projection onto $G_0$. Then from the last theorem, the orbits in $\mathscr{K}^* = \mathscr{L}^{\perp}$ possesses a lot of commuting Hamiltonian vector fields of Lax form.
We consider the invariant manifold $M_n$, $n \geq 1$, in $\mathscr{L} = \mathscr{K}^*$ defined by the set of

$$A = \sum_{j=1}^{n-1} A_j h^j + \mu h^j, \quad \mu \equiv diag(\mu_1, ..., \mu_n) \text{ fixed}, \qquad diag\,(A_{n-1}) = 0,$$

as well as the $\mathscr{K}$-invariant manifolds $M_{-j}^k$ defined by

$$M_{-j}^k = \sum_{i=-j}^{k} L_i \subseteq \mathscr{G} \simeq \mathscr{L}^{\perp}.$$

We state the following theorem [9]-[11] :

**Theorem 2.3.** *a) Let $H = \langle f(Ah^{-j}), h^k \rangle_1$, be functions defined on the manifold $M_n$ where $f$ are differentiable functions. Then, the equations*

$$\dot{A} = \left[A, Pr_{\mathscr{K}}(f'(Ah^{-j})h^{k-j})\right], \qquad A = \sum_{i=0}^{n-1} A_i h^i + \mu h^n,$$

*determine integrable Hamiltonian systems whose linearization is carried out on the Jacobian of the curve $C$ of genus $(m-1)(mn-2)/2$ defined by (2.2). Moreover, especially for $j = n$, $k = n+1$, the flow*

$$\dot{A} = \left[A \ , \ ad_{\nu}\, ad_{\mu}^{-1} A_{n-1} + \nu h\right], \tag{2.3}$$

*depends on $f$ by the relation $\nu_i = f'(\mu_i)$ only.*
*b) Let $H(a_1,a_2) = f(a_1)$, be functions defined on the manifold $M_{-j}^k$ where $f$ are differentiable functions. We have*

$$\dot{a} = \left[a, (Pr^+ - \frac{1}{2} pr_0)\nabla H(a)\right],$$

*where $Pr^+$ is the projection onto $A_+$ and these Lax equations are linearized on the Jacobian of a curve whose affine equation is given by the characteristic polynomial of elements in $M_{-j}^k$, considered as functions of h.*

Using the van Moerbeke-Mumford approach [11], one can construct an algebraic map from the complex invariant manifolds of linearizable dynamical systems to the Jacobi variety $Jac(C)$ or one of its sub-manifolds such as Prym varieties, associated with an algebraic curve determined by the spectral curve $C$ (2.1). The equations that linearize the dynamic system are given by

$$\sum_{j=1}^{g} \int_{s_j(0)}^{s_j(t)} \omega_k = c_k t, \quad 1 \leq k \leq g,$$

where $(\omega_1, ..., \omega_g)$ is a basis in the space of holomorphic differentials on the curve $C$ of genus $g$.

**1)** As a first example for $M_1$ in the above theorem, we consider $A = X + \mu h$, with $X \in so(n)$. It is deduced that the Hamiltonian flow (2.3), where $\mu_i$ and $\nu_i$ can be taken arbitrarily, is the $0^{th}$-order in $h$

$$\dot{X} = [X, \Lambda(X)], \quad \Lambda(X)_{ij} = \lambda_{ij} X_{ij}, \quad \lambda_{ij} = \lambda_{ji}, \quad \lambda_{ij} = \frac{\nu_i - \nu_j}{\mu_i - \mu_j},$$

and an identity to first order in $h$. This flow expresses the Euler-Arnold equations [12] for the geodesic flow on the group $SO(n)$, for a left invariant diagonal metric $\Lambda$. The algebraic curve

$$C=\{(z,h)\in\mathbb{C}^2:\det(X+\mu h-zI)=0\},$$

has an involution

$$\sigma:C\longrightarrow C,\quad (z,h)\longmapsto(-z,-h),$$

exchanging sheets of $C$ over $C_0=C/\sigma$. In such a situation, this involution extends by linearity to a map (which will again be denoted by $\sigma$), $\sigma:Jac(C)\longrightarrow Jac(C)$ and up some points of order two, the Jacobi variety $Jac(C)$ splits into an even part, i.e., $Jac(C)$ and an odd part (Prym variety) denoted $Prym(C/C_0)$ and defined by

$$Prym(C/C_0)=\left(H^0\left(C,\Omega^1\right)^-\right)^*/H_1\left(C,\mathbb{Z}\right)^-,$$

where $\Omega^1$ is the sheaf of holomorphic 1-forms on the curve $C$ and $-$ means the $-1$ eigenspace for a vector space on which the involution $\sigma$ acts. We have

$$Jac(C)=Jac(C_0)\oplus Prym(C/C_0).$$

The phase space for this problem is an orbit defined in the group $SO(n)$ by $[\frac{n}{2}]$ orbit invariants. By Theorem 2.3, the problem linearizes on $Prym(C/C_0)$ of dimension $\frac{\frac{n(n-1)}{2}-[\frac{n}{2}]}{2}$.

**2)** For another example, consider the case of $M_2$ in the above theorem with

$$A=\mu h^2-y\otimes y-hx\wedge y,\quad (x,y\in\mathbb{R}^n),$$

where $(x,y)\in\mathbb{R}^{2n}$. In this case, equation (2.3) is reduced to the study of

$$\dot{A}=\left[A,\nu h+ad_\nu ad_\mu^{-1}(y\wedge x)\right],$$

where $\nu_i=f'(\mu_i)$. Explicitly, we can rewrite this equation in the form of a nonlinear dynamic system :

$$\begin{aligned}\dot{x}&=-\frac{\partial H_\nu}{\partial y}=-\nu y-(ad_\nu ad_\mu^{-1}(y\wedge x))x,\\ \dot{y}&=\frac{\partial H_\nu}{\partial x}=-(ad_\nu ad_\mu^{-1}(y\wedge x))y,\end{aligned}$$

where

$$H_\nu=\frac{1}{2}\sum_i\nu_i\left(y_i^2+\sum_{j\neq i}\frac{(x_iy_j-x_jy_i)^2}{\mu_i-\mu_j}\right).$$

Note that in the particular case where

$$f(z)=\log z,\qquad \nu_i=\frac{1}{\mu_i},$$

then this problem is reduced to the study of the well known Jacobi geodesic flow on the ellipsoid :

$$\frac{x_1^2}{\nu_1^2}+\cdots+\frac{x_n^2}{\nu_n^2}=1.$$

Another special case is where $f(z)=\frac{1}{2}z^2$, $\nu_i=\mu_i$. Here the problem is reduced to the study of the Neumann movement (under the influence of the force $-\mu x$) of a point on the sphere [13] :

$$S^{n-1}:x_1^2+\cdots+x_n^2=1.$$

According to theorem 2.3, the linearization of the problem related to these two cases is carried out on the Jacobi variety $Jac(\mathscr{H})$, where $\mathscr{H}$ is a hyperelliptic curve of genus $n-1$. For an interesting geometric interpretation of these motions and their relationship with confocal quadrics, theorem of Chasles, geodesic, intersection of two quadrics, K-dV equation, etc., see for example [13]-[17].

**3)** Another example of $M_2$, $n=3$, in the above theorem, concerns the Lagrange spinning top [18]. It expresses a particular case of the rotational motion of a solid body around a fixed point. Here, we have

$$A=mh+\gamma+lh^2,$$

where $m\in so(3)$ (angular momentum), $\gamma\in so(3)$ (unit vector in the direction of gravity), $l=(\alpha+\beta)\varepsilon$ with $\varepsilon\in so(3)$ (coordinates of the center of mass) and where $(\alpha+\beta,\alpha+\beta,2\alpha)$ (inertia tensor in diagonalized form). Here, the linearization of the problem takes place on the Jacobi variety of an elliptic curve, i.e., on the curve itself (see [19] and for higher-dimensional generalizations [20]).

**4)** As an example of $M_{-j}^{k}$ in theorem 2.3, b) (see [9]-[11]), we consider the periodic infinite band matrix $M$ of period $n$ having $j+h+1$ diagonals; the spectrum of $M$ is defined by the points $(z,h)\in\mathbb{C}^2$ such that

$$Mv(h)=zv(h),\quad v(h)=(...,h^{-1}v,v,hv,...),\quad v\in\mathbb{C}^n.$$

Let $M_h$ be the square matrix obtained from $M$ and let $C$ be the curve whose affine equation is $\det(M_h-zI)=0$. Then the set of infinite band matrices with $j+k+1$ diagonals, in higher dimensions many partial results seem to lead to rigidity. In fact, it was shown that a discrete 2-dimensional Laplacian cannot be deformed, given its periodic spectrum; the proof can be summarized by the observation that the Picard variety of most algebraic surfaces are trivial; the proof that the specific spectral surface defined by the 2-dimensional Laplacian has trivial Picard variety is based on the technique of toroidal embedding, which reduces cohomological computations to combinatorial questions. Finally, inspired by the dynamical systems, Mumford [21] has given a beautiful description of hyperelliptic Jacobians of dimension $g$.
Griffiths [4] has given a necessary and sufficient condition on $B$ (easily checkable), for an equation of the type (2.1) to be linearizable on the Jacobi variety $\mathrm{Jac}(C)$ of its spectral curve defined by (2.2) (although, without reference to Kac-Moody Lie algebras). Indeed, suppose that for every $p(z,h)$ belonging to the curve $C$ of affine equation (2.2), with $\dim\ker(A-zI)=1$ (i.e., the corresponding eigenspace of $A$ is one-dimensional) and generated by a vector $v(t,p)\in\mathbb{C}^n$. So, we can find a family of holomorphic mappings which send $(z,h)\in C$ to $\ker(A-zI)$ :

$$\phi_t:C\longrightarrow\mathbb{P}^n(\mathbb{C}),\quad p\longmapsto\mathbb{C}v(t,p),$$

called the eigenvector map associated to the equation (2.1). Let

$$\Psi(t)=\phi_t^*\left(\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}(1)\right)\in\mathrm{Pic}^s(C),\quad s=\deg\,\phi_t(C),$$

where $\mathrm{Pic}^s(C)\cong\mathrm{Jac}(C)$ is the Picard variety of the curve $C$ and $\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}(1)$ is the hyperplane line bundle in $\mathbb{P}^n(\mathbb{C})$. Obviously the degree of $\Psi(t)$ does not vary with $t$ Let $H$ be the hyperplane class in $\mathbb{P}^n(\mathbb{C})$. The Poincaré dual of the class $[C]$ of $C$ coincides with the degree of $C$,

$$\deg\,\Psi(t)=\int_C\phi_t^*H=\int_{\phi_t(C)}H=\deg(C).$$

Since $\Psi(t)$ moves in $\mathrm{Pic}^s(C)$ when $t$ varies, then by fixing a line bundle $\Psi(0)\in\mathrm{Pic}^s(C)$, the line bundle $\Psi(0)^{-1}\otimes\Psi(t)$ moves in the Jacobian variety $\mathrm{Jac}(C)$. We will determine a necessary and sufficient condition of a cohomological nature on $B$ so that the flow

$$t\longmapsto\Psi(t), \tag{2.4}$$

is linearizing on $\mathrm{Jac}(C)$. By applying cohomological techniques of the theory of deformation, we can find necessary and sufficient conditions to linearize the flow (2.4). Indeed, this is because the tangent space for any deformation is in a proper cohomology group, and according to the theory of duality on algebraic curves, the higher cohomology can always be eliminated. Let's see that with a little more detail. Let $X$ be a complex manifold and

$$\phi:C\longrightarrow X, \tag{2.5}$$

a non-constant holomorphic map. Let $\theta_C$, $\theta_X$ be the respective tangent sheaves and $\phi_*$ the differential of $\phi$. The normal sheaf of $C$ in $X$ is defined by the following exact sequence

$$0\longrightarrow\theta_C\xrightarrow{\phi_*}\phi^*\theta_X\longrightarrow N_\phi\longrightarrow 0 \tag{2.6}$$

and let $H^0(C,N_\phi)$ (that we also note $H^0(N_\phi)$) be the Kodaira-Spencer tangent space [22] to the moduli space of (2.5). Let $\phi_t:C\longrightarrow X$, $\phi_0=\phi$, be a deformation of (2.5). In local product coordinates $(z,t)$ on $\bigcup_t C_t$, $w=(w^1,w^2,...,w^n)\in X$, we show that $\phi_t$ is given by $(t,\xi)\longmapsto w(t,\xi)$, i.e., the section $\dot{\phi}\in H^0(N_\phi)$ and is locally given by

$$\left.\frac{\partial w(t,\xi)}{\partial t}\right|_{t=0}\text{ modulo }\frac{\partial w(0,\xi)}{\partial z}.$$

The corresponding cohomological sequence of (2.6) is

$$H^0(\theta_C)\longrightarrow H^0(\phi^*\theta_X)\longrightarrow H^0(N_\phi)\xrightarrow{\overline{\partial}}H^1(\theta_C).$$

Consider the tangent space $H^1(\theta_C)$ to the moduli space of the curve $C$ as well as the tangent $\dot{C}\equiv\overline{\partial}(\dot{\phi})\in H^1(\theta_C)$ to the family of curves $\{C_t\}$. Hence, $H^0(\phi^*\theta_X)/H^0(\theta_C)\subset H^0(N_\phi)$ is the tangent space to deformations of (2.5) where, according to theorem 2.1, the curve $C$ is independent of $t$.
Consider now the Euler exact sequence of vector bundles

$$0\longrightarrow\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}\xrightarrow{i}\mathbb{C}^n\otimes\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}(1)\xrightarrow{p}\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}\longrightarrow 0$$

Therefore, the following sequences $(\Psi(0)=\phi^*\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}(1))$ :

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathscr{O}_C & & & & \\
 & & & & \downarrow v & & & & \\
 & & & & \mathbb{C}^n\otimes\Psi(0) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \theta_C & \xrightarrow{\phi_*} & \phi^*\Theta_{\mathbb{P}^n(\mathbb{C})} & \longrightarrow & N_\phi & \longrightarrow & 0\\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

are exact and the cohomology diagram corresponding to these sequences contains the following part :

$$\begin{array}{ccccccc} & & H^0(C,\mathbb{C}^n\otimes\Psi(0)) & & & & \\ & & \downarrow\tau & & & & \\ H^0(C,\theta_C) & \longrightarrow & H^0(C,\phi^*\theta_{\mathbb{P}^n(\mathbb{C})}) & \stackrel{j}{\longrightarrow} & H^0(C,N_\phi) & \stackrel{\overline{\delta}}{\longrightarrow} & H^1(C,\theta_C) \\ & & \downarrow\delta & & & & \\ & & H^1(C,\mathscr{O}_C) & & & & \end{array}$$

Let $(t,\xi)\longmapsto v(t,\xi)\in\mathbb{C}^n\backslash\{0\}$, be a position vector mapping with $\xi$ a local coordinate on $C$. In other words, a local lift $v_t$ (which is a time-dependent map $C\longrightarrow\mathbb{C}^n\backslash\{0\}$) of the family of holomorphic maps $\phi_t : C\longrightarrow\mathbb{P}^n(\mathbb{C})$, to $\mathbb{C}^n\backslash\{0\}$, such that

$$\phi_t(\xi)=\mathbb{C}.v(t,\xi)\subset\mathbb{C}^n.$$

As this lift exists only locally, it will have to find an independent object of this lift but which exists globally. The solution is to use this lift to determine such an object that we note $\dot{v}$. Notice that the space $\mathbb{C}v_t(p)$ and the fibre of $\phi^*\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}(-1)$ at a point $p\in C$ identify and define the maps $\phi^*\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}(-1)\mathbb{C}^n\otimes\mathscr{O}_C$ and

$$v_t:\mathscr{O}_C\longrightarrow\mathbb{C}^n\otimes\Psi(t),\quad \phi\longmapsto\phi v_t,$$

(here, $v_0$ coincides with the application $v$ mentioned in the previous diagram). In the case where $\eta$ will be another lift such that :

$$\eta(t,\xi)=\kappa(t,\xi)v(t,\xi),\quad \kappa\neq 0,$$

then we will have $\dot{\eta}=\kappa\dot{v}+\dot{\kappa}v$. The inclusion $\mathscr{O}_C\stackrel{v}{\hookrightarrow}\mathbb{C}^n\otimes L$, $L=f^*\mathscr{O}_{\mathbb{P}^n(\mathbb{C})}(1)$, is locally given by $\mathscr{O}_{\mathscr{C}}\ni\phi\longmapsto\phi.v$, and then (modulo $v(t,\xi)$), the expression

$$\dot{v}(\xi)=\left.\frac{\partial v(t,\xi)}{\partial t}\right|_{t=0}\in H^0(C,\mathbb{C}^n\otimes L/\mathscr{O}_C)=H^0(C,f^*\theta_{\mathbb{P}^n(\mathbb{C})})$$

is well-defined independently of the choice of the lift, and we have $\sigma(\dot{v})=\phi$. We are interested in the tangent vector

$$\dot{\Psi}(0)\equiv\left.\frac{d\Psi(t)}{dt}\right|_{t=0}\in H^1(C,\mathscr{O}_C).$$

**Theorem 2.4.** *If $\dot{v}$ is an infinitesimal variation of $\phi_t : C\longrightarrow\mathbb{P}^n(\mathbb{C})$, then*

$$\dot{\Psi}(0)\equiv\left.\frac{d\Psi(t)}{dt}\right|_{t=0}=\delta(\dot{v})\in H^1(C,\mathscr{O}_C).$$

*In addition if $\tau$ is the map mentioned in the diagram above, then there is an equivalence between the fact that $\dot{\Psi}(0)=0$ and $\dot{v}=\tau(w)$ for some $w\in H^0(\mathbb{C}^n\otimes\Psi(0))$.*

Let $h_0$, $h_1$ be homogeneous coordinates and consider $h$ as an affine coordinate on $\mathbb{P}^1(\mathbb{C})$ which is the base of the covering $\pi : C\longrightarrow\mathbb{P}^1(\mathbb{C})$. Note that $B(t,h)$ can be written in the form

$$B(t,h)=\sum_{k=0}^{N}B_k(t)h^k=\sum_{k=0}^{N}B_k(t)h^N\left(\frac{h_1}{h_0}\right)^k\in H^0(C,\mathrm{Hom}(\mathscr{F},\mathscr{F}(N))),$$

where $\mathscr{F}$ is the sheaf of sections of the trivial bundle $C\times\mathscr{F}$. We have $\mathscr{F}(D)=\mathscr{F}\otimes\mathscr{O}_C(D)$ and $B(t,h)$ can be seen as a holomorphic section of the bundle $\mathrm{Hom}(\mathscr{F},\mathscr{F})\otimes\mathscr{O}_C(N)$, $\mathscr{O}_C(N)=\pi^*\mathscr{O}_{\mathbb{P}^1}(N)$. In other words, we visualize $h=[h_0:h_1]$ as a homogeneous coordinate on $\mathbb{P}^1(\mathbb{C})$ pulled up to $C$. We have $\frac{B}{h_0^K}\in H^0(C,\mathrm{Hom}(\mathscr{F},\mathscr{F}(D)))$, $v\in H^0(V\otimes L)$ where $D=(h_0^N)$, is the divisor $N.\pi^{-1}(\infty)$ on the curve $C$ and $\mathscr{F}(D)\cong\mathscr{F}(N)$ are the sections of $\mathscr{F}\otimes\mathscr{O}_C(D)$. It should be noted that here $\frac{B}{h_0^N}$ is a matrix in $\mathrm{Hom}(\mathscr{F},\mathscr{F})$ with meromorphic functions in $H^0(C,\mathscr{O}_C(D))$ as entries, i.e., $\frac{h_1}{h_0}$ is seen as a function on $H^O(C,\mathscr{O}_C(D))$. We deduce that $\left(\frac{B}{h_0^N}\right).v\in H^0(C,\mathscr{F}\otimes\Psi(0)(D))$ and the Lax equation can be interpreted in cohomological form as follows:

**Theorem 2.5.** *The following conditions are equivalent :*
*(i)*

$$\dot{v}=\tau\left(\frac{B}{h_0^N}.v\right)\qquad\dot{\Psi}(0)=0,$$

*(ii) There is a meromorphic function $\varphi\in H^O(C,\mathscr{O}_C(D))$ such that $\frac{B}{h_0^N}.v+\varphi v\in H^0(C,\mathscr{F}\otimes L(D))$ is holomorphic.*

By differentiating the eigenvalue problem $Av(t,p) = zv(t,p)$ (in the neighborhood of the point $p = (h,z) \in C$) with respect to $t$, and taking into account the Lax equation in the form $\dot{A} = [B,A]$, we immediately obtain the expression $A(\dot{v} - Bv) = z(\dot{v} - Bv)$. Since eigenvalues have (generically) a multiplicity of 1, then for a some $\lambda$, we have

$$Bv = \dot{v} + \lambda v. \tag{2.7}$$

This equation can be written in the form $Bv = \dot{v} + \lambda_j v$, where $\lambda_j$ is the main part of Laurent series expansion of $\lambda$ in the neighborhood of $p$. Then given the curve $C$ defined by (2.2) and $p \in C$, Griffiths defines

$$L_p \equiv [\text{Laurent tail}(B)]_p \quad = \quad \{\text{main part of Laurent series expansion of } \lambda \text{ in the neighborhood of } p\},$$

and shows that the linearization of the Lax flow takes place on the Jacobi variety $\mathrm{Jac}(C)$ if and only if $p \in (h)_\infty$ (divisor of the poles of $h$), we have for any meromorphic function $f$ on $C$ such that : $(f) \geq n(h)_\infty$,

$$\frac{dL_p}{dt} \in \text{ linear combination } \{L_p;\text{ Laurent tail at } p \text{ of } f\}.$$

Let $P(h,g) \in \mathbb{C}[h,g]$ and note that if we replace $B$ by $B + P(h,A)$ in equation (2.1), we see that this equals invariant which shows that $B$ is not unique and that its natural place is somewhere in a cohomology group.
Consider a positive divisor

$$D = \left(\frac{1}{h}\right)(\infty) = \sum_j n_j p_j, \quad n_j \geq 0,$$

on $C$, where $h$ is seen as a meromorphic function. The polynomial $B(t,h) = \sum_{k=0}^{n} B_k h^k$ of degree $n$ should be interpreted as an element of $H^0(C, \mathrm{Hom}(V, V(D))$ where $V$ is the sheaf of sections of the trivial bundle $C \times V$ and $V(D) = V \otimes \mathscr{O}_C(D)$. A section of $\mathscr{O}_D(D)$ is written

$$\varphi = \sum \varphi_j, \qquad \varphi_j = \sum_{k=-n_j}^{-1} a_k z_j^k,$$

where $z_j$ is a local coordinate around $p_j$. This is a principal part (Laurent tail) centered on $p_j$. A question arises : Given a main part $\varphi_j$, determine conditions for a function $\varphi \in H^0(C, \mathscr{O}_C(D))$ such that $\varphi - \varphi_j$ is holomorphic in the neighborhood of $p_j$. The answer to this question (known as the Mittag-Leffler problem) is provided by

**Theorem 2.6.** *Let $\{\varphi_j\}$ be a Laurent tail and let $D = \sum_j a_j p_j$. The following conditions are equivalent :*
*(i) There exist $\varphi \in H^0(C, \mathscr{O}_C(D))$ such that $\varphi - \varphi_j$ is holomorphic near $p_j$.*
*(ii) For every holomorphic differential $\omega$ on $C$, we have*

$$\sum_j Res_{p_j}(\varphi_j.\omega) = 0.$$

The residue of $B$, denoted by $\varsigma(B) \in H^0(C, \mathscr{O}_D(\mathscr{D})$, is the collection of Laurent tails $\{\lambda_j\}$ given above, where $\lambda_j$ is the main part of the Laurent series expansion of $\lambda$ around $p$.
We will say that the flow $\Psi(t)$ (2.4) is linearized if there is a complex number $c$ such that

$$\frac{d^2\Psi(t)}{dt^2} = c\frac{d\Psi(t)}{dt}.$$

The Griffiths theorem is as follows :

**Theorem 2.7.** *1) We have*

$$\dot{\Psi}(0) = \left.\frac{d\Psi(t)}{dt}\right|_{t=0} = \delta_1(\varsigma(B)).$$

*2) The following conditions are equivalent :*
*(i) The flow $\Psi(t)$ (2.4) is linearized in $Pic^s(C)$.*
*(ii) We have*

$$\varsigma(\dot{B}) = 0 \; mod.(\varsigma(B), Im\ res),$$

*where $Im\ res \subset H^0(C, \mathscr{O}_D(D)$ is the Laurent tails of meromorphic functions in $H^0(C, \mathscr{O}_D(D))$.*
*(iii) We have*

$$\sum_j Res_{p_j}(\dot{\varsigma}_j(B))\omega) = t\sum_j Res_{p_j}(\varsigma_j(B))\omega), \quad \omega \in H^0(C, \Omega_C)$$

It follows from the above theorem that the linearized flow on $\text{Jac}(C)$ is provided by the bilinear map

$$(t,\omega) \longmapsto t\sum_j \text{Res}_{p_j}(\varsigma_j(B))\omega) = t\sum_j \text{Res}_{p_j}(\lambda_j\omega). \tag{2.8}$$

As an example, consider Euler's problem of a free rigid body in $\mathbb{R}^n$. This one is described by the following equations :

$$\dot{M} = [M,\Omega], \qquad M = \Omega J + J\Omega \in so(n), \qquad \Omega(t) \in so(n),$$

where $J = \text{diag}(\lambda_1,...,\lambda_n)$, $\lambda_j > 0$. These equations form a Hamiltonian system on each adjoint orbit of $so(n)$ and whose Hamiltonian is explicitly described by $H(M) = \frac{1}{2}(M,\Omega) = -\frac{1}{4}Tr(M\Omega)$. Manakov [23] observed that these equations admit a Lax equation with an indeterminate parameter $h$,

$$\dot{\overbrace{(M+J^2h)}} = [M+J^2h, \Omega + Jh].$$

Hence $D = \left(\frac{1}{h}\right)(\infty) = \sum_j p_j$, is the divisor with $n$ distinct points $p_j$ located on $h=\infty$. We deduce from equation (2.7) with $B = \Omega + Jh$, the following relation : $\varsigma(B) = \sum_j \frac{\lambda_j}{z_j}$, where $z_j = \frac{1}{h}$ is a local coordinate on $C$ around $p_j$. We have $\varsigma(\dot{B}) = 0$ since $\lambda_i$ are constant, and consequently, the flow is linearized on $Jac(C)$. Taking into account that $A = M + J^2h$, $M + M^\top = 0$, $J^2 - J^{2^\top} = 0$, we obtain $P(h,z) = (-1)^n P(-h,-z)$. The curve $C$ has an involution $\sigma : C \longrightarrow C$, $\quad (h,z) \longmapsto (-h,-z)$. Here the linearization of the problem necessitates the knowledge of $\frac{1}{2}\dim\mathscr{O}$ independent first integrals and in involution (this is because $\Omega$ moves on an adjoint orbit $\mathscr{O} \subset so(n)$). In general, we have

$$\dim\mathscr{O} = \frac{n(n-1)}{2} - \left[\frac{n}{2}\right], \tag{2.9}$$

Let $g(C) = \frac{(n-1)(n-2)}{2}$ be the genus of the algebraic curve $C$ and $g(C_0)$ the genus of the quotient $C_0 = C/\sigma$ of $C$ by the involution $\sigma$. Using the Riemann-Hurwitz formula, we get

$$g(C) - g(C_0) = \frac{1}{2}\left(\frac{n(n-1)}{2} - \left[\frac{n}{2}\right]\right). \tag{2.10}$$

Note that $\sigma(\varsigma(B)) = -\varsigma(B)$ and the linearization of the problem in question is carried out on the Prym variety $Prym(C/C_0)$ of the curve $C$ for the involution $\sigma$, interchaging the sheets of the double covering $C \longrightarrow C_0$. From (2.10) it follows that

$$\dim Prym(C/C_0) = \begin{cases} \frac{n(n-2)}{4} & n \equiv 0 \text{ mod.}2 \\ \frac{(n-1)^2}{4} & n \equiv 1 \text{ mod.}2 \end{cases} \tag{2.11}$$

and taking into account (2.9), we finally get $\dim Prym(C/C_0) = \frac{1}{2}\dim\mathscr{O}$. The linearization of the Euler equations is carried out on the Prym variety $Prym(C/C_0)$ of exactly the correct dimension.

## 3. Infinite continued fraction and spectral theory for periodic Jacobi operators

A Jacobi matrix is a doubly infinite matrix $(a_{ij})$ with entries $i,j$ such that : $a_{ij} = 0$ for $|i-j|$ large enough. The set of these matrices is an associative algebra and consequently a Lie algebra by anti-symmetrization. Consider the Jacobi matrix

$$\Gamma = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & & \vdots \\ 0 & a_2 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & 0 & & \end{pmatrix},$$

where all the $b_j$ are real and all the $a_j$ are positive, and let

$$\varphi(z) = \cfrac{a_0^2}{z - b_1 - \cfrac{a_1^2}{z - b_2 - \cfrac{a_2^2}{z - b_3 - \ddots}}} \tag{3.1}$$

be the associated continued $\Gamma$-fraction, where $a_0$ is a positive real number. By cutting off the $\Gamma$-fraction $\varphi(z)$ at the $k$-th term, we obtain the $k$-th Padé approximant $\frac{A_k(z)}{B_k(z)}$ of $\varphi(z)$, i.e.,

$$\varphi(z) = \lim_{k\to\infty}\frac{A_k(z)}{B_k(z)}. \tag{3.2}$$

We show that $\varphi(z)$ admits formal series expansion arount the point $z=0$ (pole),

$$\varphi(z)=\frac{c_0}{z}+\frac{c_1}{z^2}+\frac{c_2}{z^3}+\cdots=\sum_{k=0}^{\infty}\frac{c_k}{z^{k+1}}.$$

Note that the characteristic polynomial

$$B_k(z)=\det\begin{pmatrix} b_1-z & a_1 & 0 & \cdots & 0\\ a_1 & b_2-z & a_2 & & \vdots\\ 0 & a_2 & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & a_{k-1}\\ 0 & \cdots & 0 & a_{k-1} & b_k-z \end{pmatrix},$$

of $\Gamma$, is the last term of the second order recursion

$$B_k(z)=(b_k-z)B_{k-1}(z)-a_{k-1}^2B_{k-2}(z).$$

The polynomials $A_k(z)$, $B_k(z)$ form a pair of solutions of a finite difference equation of the second order (the eigenvectors of the Jacobi matrix from which we remove the first row and the first column) :

$$a_{k+1}y_{k+2}+b_{k+1}y_{k+1}+a_ky_k=zy_{k+1},\quad k\in\mathbb{N}$$

with the boundary conditions : $y_0\neq 0$, $y_1=0$, $y_{N+1}=0$. In addition, these solutions are linearly independent and we have also the following relation :

$$a_{k-1}\left(A_{k-1}(z)B_k(z)-A_k(z)B_{k-1}(z)\right)=1,\quad k\in\mathbb{N}^*$$

The polynomials $B_k$ form an orthogonal system with respect to the Stieltjes measure $d\sigma(x)$ on $\mathbb{R}$,

$$\int_{-\infty}^{\infty}B_k(x)B_l(x)d\sigma(x)=\delta_{kl}.$$

Conversely, if a family of polynomials $P_n(x)$ is orthogonal for $d\sigma(x)$, then $P_n(x)$ satisfies the following recurrence relation :

$$P_k(x)-(\lambda_kx-\mu_j)P_{k-1}(x)+\gamma_{k-1}P_{k-2}(x)=0,$$

where $\lambda_k>0$, $\mu$ and $\gamma_k>0$ are constants. Moreover, if we consider the continued fraction

$$\psi(z)=\cfrac{\gamma_0}{\lambda_1z-\mu_1-\cfrac{\gamma_1}{\lambda_2z-\mu_2-\cfrac{\gamma_2}{\lambda_3z-\mu_3-\ddots}}}$$

and realize an equivalent transformation

$$\psi(z)=\cfrac{\gamma_0}{z-\frac{\mu_1}{\lambda_1}-\cfrac{\frac{\gamma_1}{\lambda_1\lambda_2}}{z-\frac{\mu_2}{\lambda_2}-\cfrac{\frac{\gamma_2}{\lambda_2\lambda_3}}{z-\frac{\mu_3}{\lambda_3}-\ddots}}}$$

we reconstruct the $\Gamma$-fraction corresponding to $d\sigma(x)$ (where we can put $\dfrac{\gamma_k}{\lambda_k\lambda_{k+1}}=a_k^2$ and $\dfrac{\mu_k}{\lambda_k}=b_k$). As a result, there is a one-to-one correspondence between the set of orthogonal polynomial systems on $\mathbb{R}$ and that of Jacobi matrices. In fact, if the orthogonal polynomials

$$P_n=\frac{\gamma_0}{\prod_{k=1}^{n-1}a_k}B_{n-1}(x),\quad 1\leq n<\infty$$

form a basis of the vector space consisting of all the polynomials, then the Jacobi matrix represents the multiplication by $x$.

As an example of $V_{-j,k}$ (theorem 2.3, b)), consider the infinite matrix :

$$A=\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & b_0 & a_0 & 0 & \cdots & 0 & \\ & a_0 & b_1 & a_1 & & \vdots & \\ & 0 & a_1 & \ddots & \ddots & 0 & \\ & \vdots & & \ddots & \ddots & a_{N-1} & \\ & 0 & \cdots & 0 & a_{N-1} & b_N & \ddots\\ & & & & & \ddots & \ddots \end{pmatrix},\quad (a_i,b_i\in\mathbb{C})\tag{3.3}$$

The matrix $A$ is $N$-periodic when

$$a_{i+N}=a_i,\qquad b_{i+N}=b_i,$$

for all $i\in\mathbb{Z}$. We denote by $f=(...,f_{-1},f_0,f_1,...)$ the (infinite) column vector and by $D$ (shift operator) the operator passage of degree +1, $Df_i=f_{i+1}$. Since the matrix $A$ is $N$-periodic, we have $AD^N=D^NA$. Reciprocally, this relation of commutation means that $N$ is the period of $A$. Consider the finite Jacobi matrix (symmetric tridiagonal and $N$-periodic) :

$$A(h)=\begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_Nh^{-1}\\ a_1 & b_2 & a_2 & & \vdots\\ 0 & a_2 & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & a_{N-1}\\ a_Nh & \cdots & 0 & a_{N-1} & b_N \end{pmatrix},$$

where $h\in\mathbb{C}^*$. The determinant of the matrix

$$A(h)-zI=\begin{pmatrix} b_1-z & a_1 & 0 & \cdots & a_Nh^{-1}\\ a_1 & b_2-z & a_2 & & \vdots\\ 0 & a_2 & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & a_{N-1}\\ a_Nh & \cdots & 0 & a_{N-1} & b_N-z \end{pmatrix}, \tag{3.4}$$

is

$$\det(A(h)-zI)=(-1)^{N+1}\left(\alpha(h+h^{-1})-P_N(z)\right)\equiv F(h,h^{-1},z), \tag{3.5}$$

where $(z,h)\in\mathbb{C}\times\mathbb{C}^*$, $\alpha=a_1a_2\cdots a_N$, and $P(z)$ is given by the following polynomial of degree $N$ (with real coefficients) :

$$P(z)=\det\begin{pmatrix} b_1-z & a_1 & 0 & \cdots & 0\\ a_1 & b_2-z & a_2 & & \vdots\\ 0 & a_2 & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & a_{N-1}\\ 0 & \cdots & 0 & a_{N-1} & b_N-z \end{pmatrix}-a_0^2\det\begin{pmatrix} b_2-z & a_2 & 0 & \cdots & 0\\ a_2 & b_3-z & a_3 & & \vdots\\ 0 & a_3 & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & a_{N-2}\\ 0 & \cdots & 0 & a_{N-2} & b_{N-1}-z \end{pmatrix}=z^N+\cdots$$

Let $C$ be the Riemann surface defined by the set of $(z,h)\in\mathbb{C}\times\mathbb{C}^*$ such that : $Af=zf$ and $D^Nf=hf$. In other words, we have

$$C=\left\{(z,h)\in\mathbb{C}\times\mathbb{C}^*:F(h,h^{-1},z)=0\right\}. \tag{3.6}$$

Assuming that $\alpha\neq 0$, we derive from (3.5) and (3.6) the following relation :

$$h=\frac{P(z)\pm\sqrt{P^2(z)-4\alpha^2}}{2\alpha}.$$

Note that $C$ is a hyperelliptic curve with $2N$ branch points over the roots of the equation : $P(z)=\pm 2\alpha$ and two points at infinity $\mathscr{P}$ and $\mathscr{Q}$; the point $\mathscr{P}$ covering the case $z=\infty$, $h=\infty$ while the point $\mathscr{Q}$ is relative to the case $z=\infty$, $h=0$. The hyperelliptic involution on the curve $C$ maps $(z,h)$ into $(z,h^{-1})$ and $C$ can be singular. Using Riemann-Hurwitz formula, we find $g=N-1$ (= genus of $C$). The meromorphic function $h$ has neither zero nor poles except in the neighborhood of $z=\infty$. When $z\nearrow\infty$, we have

$$h\simeq\frac{P(z)}{\alpha}=\frac{z^N}{\alpha}+\cdots,$$

on the sheet +, which shows that $h$ has a pole of order $N$. Similarly, when $z\nearrow\infty$, we have

$$h=\frac{P(z)-\sqrt{P^2(z)-4\alpha^2}}{2\alpha}=\frac{2\alpha}{P(z)+\sqrt{P^2(z)-4\alpha^2}}\simeq\alpha z^{-N}+\cdots,$$

on the sheet -, and therefore $h$ has a zero of order $N$. Let $\mathscr{P}$ be the point covering $\infty$ on the sheets + and $\mathscr{Q}$ the two point covering $\infty$ on the sheets -. Therefore the divisor $(h)$ of the function $h$ on the curve $C$ is

$$(h)=-N\mathscr{P}+N\mathscr{Q}.$$

The curve $C$ has an antiholomorphic involution

$$\sim: C\longrightarrow C,\quad (z,h)\longmapsto(\bar{z},1/\bar{h}),$$

i.e., the map $\sim: p\longmapsto\widetilde{p}$ is such that : $\widetilde{\mathscr{P}}=\mathscr{Q}$. Since the finite matrix $A(h)$ for $|h|=1$ is self-adjoint, then it admits a real spectrum. Therefore, the fixed points of this involution form a set that we write $C^{\sim}$. The latter is determined by the set of $p\in C$ such that : $\widetilde{p}=p$, or it is the set of $(z,h)$ such that : $h=1/\overline{h}$ and $\bar{z}=z$, or what amounts to the same, is the set of $(z,h)$ such that : $|h|=1$. Let $C_+$ (repectively $C_-$) the set of $p\in C$ such that : $|h|>1$ (repectively $|h|<1$). Note that $C_+$ contains the point $\mathscr{P}$ and $C_-$ contains the point $\mathscr{Q}$. We have $C\backslash C^\sim = C_+\cup C_-$, which shows that the set $C^\sim$ divides $C$ into two distinct regions $C_+$ and $C_-$, and so

$$C=C_+\cup C^\sim \cup C_-.$$

In fact, $C^\sim$ is homologous to zero because $\mathscr{C}^\sim$ can be thought of as the boundary between $C_+$ and $C_-$. Moreover, the involution $\sim$ extends to an involution $*$ on the field of meromorphic functions as follows: $\varphi^*(p)=\overline{\varphi(\widetilde{p})}$, and on the differential space as follows : $(\varphi d\psi)^*=\varphi^* d\psi^*$, which shows that : $h^*=\dfrac{1}{h}$ and $z^*=z$. The matrices $A$ and $D^N$ have an eigenvector $f=(...,f_{-1},f_0,f_1,...)$ in common. Such a condition is parameterized by the Riemann surface $C$ (3.6). In the following, appropriate standardization is used by selecting $f_0\equiv 1$, from where $F_N=h$. Let us therefore $\overline{f}=(f_1,f_2,...,f_{N-1})^\top$. Since $\overline{f}$ satisfies $(A(h)-zI)\overline{f}=0$, then we have

$$f_k=\frac{C_{1,k}}{C_{1,l}}f_l=\frac{C_{2,k}}{C_{2,l}}f_l=\cdots=\frac{C_{N,k}}{C_{N,l}}f_l, \quad 1\le k,l\le N,$$

where $C_{k,l}$ is the $(k,l)$-cofactor of $(A(h)-zI)$, that is to say,

$$C_{k,l}=(-1)^{k+l}M_{k,l}. \tag{3.7}$$

where $M_{kl}$ is the $(k,l)$ minor of the matrix $(A(h)-zI)$ , i.e., the determinant of the $N-1$ submatrix obtained by removing the $k^{th}$-line and the $l^{th}$-column of the matrix $(A(h)-zI)$). In particular, we have

$$f_k=\frac{C_{N,k}}{C_{N,N}}h=\frac{C_{k,k}}{C_{k,N}}h.$$

According to matrix (3.4), we note that

$$\begin{aligned} C_{N,1} &= a_1a_2\cdots a_{N-1}+(-1)^N\frac{a_N}{h}P_{N-1},\\ C_{1,N} &= a_1a_2\cdots a_{N-1}+(-1)^N a_N h P_{N-1}, \end{aligned}$$

where $P_{N-1}\equiv(-z)^{N-2}+\cdots$, and similarly, $C_{N,N}=(-z)^{N-1}+\cdots$. To determine the divisor structure of $f_k$, one proceeds as follows : for $f_1$, we have

$$\begin{aligned} (f_1)_\infty &= (C_{N,1})_\infty+(h)-(C_{N,N})_\infty,\\ &= -(2N-2)\mathscr{Q}-N\mathscr{P}+N\mathscr{Q}+(N-1)\mathscr{P}+(N-1)\mathscr{Q},\\ &= \mathscr{Q}-\mathscr{P}, \end{aligned}$$

and for the other $f_k$, we consider first the matrix (3.4) shifted by one, i.e.,

$$\begin{pmatrix} b_2-z & a_2 & 0 & \cdots & a_1h^{-1}\\ a_2 & b_3-z & a_3 & & \vdots\\ 0 & a_3 & \ddots & \ddots & 0\\ \vdots & & \ddots & b_N-z & a_N\\ a_1h & \cdots & 0 & a_N & b_1-z \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} b_2-z & a_2 & 0 & \cdots & a_1h^{-1}\\ a_2 & b_3-z & a_3 & & \vdots\\ 0 & a_3 & \ddots & \ddots & 0\\ \vdots & & \ddots & b_N-z & a_N\\ a_1h & \cdots & 0 & a_N & b_1-z \end{pmatrix}\begin{pmatrix} \frac{f_2}{f_1}\\ \frac{f_3}{f_1}\\ \vdots\\ \frac{f_N}{f_1}\\ h \end{pmatrix}=0,$$

and as above, we have $\left(\frac{f_2}{f_1}\right)_\infty=\mathscr{Q}-\mathscr{P}$, which implies that

$$(f_2)_\infty=\left(\frac{f_2}{f_1}\right)_\infty+(f_1)_\infty=2\mathscr{Q}-2\mathscr{P},$$

and in general, we get

$$(f_k)_\infty=k\mathscr{Q}-k\mathscr{P}.$$

Note that the degree of a minimal positive divisor $D$ on the curve $C$ such that : for all $k\in\mathbb{Z}$, $(f_k)+D\ge -k\mathscr{P}+k\mathscr{Q}$, is given by

$$\deg D=g=N-1.$$

We have

$$\dim \mathscr{L}(D+k\mathscr{P}-(k+1)\mathscr{Q})=0,$$

for all $k\in\mathbb{Z}$, i.e., the divisor $D$ is regular. To be convinced of this, it suffices to show that the divisor $D$ is general. It means that $(\omega_1(p_1),...,\omega_g(p_g))\neq(0,...,0)$ where $(\omega_1,...,\omega_g)$ is a normalized base of differential forms on $C$ and $p_1,,...,p_g\in C$, or what is equivalent if $\dim\mathscr{L}(D)=1$ where $\mathscr{L}(D)$ denotes the set of meromorphic functions $f$ such that : $(f)+\mathscr{D}\geq 0$, or what amounts to the same if $\dim\Omega(-D)=0$ where $\Omega(D)$ denotes the set of meromorphic differential forms $\omega$ such that the divisor $(\omega)+D\geq 0$. From the regularity of the divisor $D$ and the Riemann-Roch theorem, we deduce

$$\begin{aligned}\dim\mathscr{L}(D+k\mathscr{P}) &= \dim\Omega(-D-k\mathscr{P})+g+k-g+1,\\ &= \dim\Omega(-D-k\mathscr{P})+k+1,\end{aligned}$$

for an integer $k>g-2$. Therefore, we have $\dim\mathscr{L}(D+k\mathscr{P})=k+1$, because $\dim\Omega(-D-k\mathscr{P})=0$. Moreover, $\mathscr{L}(D+j\mathscr{P})$ is strictly larger than $\mathscr{L}(D+(j-1)\mathscr{P})$, and therefore by lowering the index $j$ down to 0, it follows that $\dim\mathscr{L}(D)=1$, which shows that the divisor $D$ is general. Let's show now that $\mathscr{D}$ is regular. It suffices to proceed by induction. We have just shown that $\dim\mathscr{L}(D)=1$. Since $f_0=1\notin\mathscr{L}(D-\mathscr{Q})$, it means that $\mathscr{L}(D-\mathscr{Q})\subsetneqq\mathscr{L}(D)$ and that the function $f_0=1$ does not belong to the first space but belongs to the second and then, $\dim\mathscr{L}(D-\mathscr{Q})=0$. Assuming that $\dim\mathscr{L}(D+k\mathscr{P}(k+1)\mathscr{Q})=0$, we obtain (taking into account the Riemann-Roch theorem) immediately

$$\dim\mathscr{L}(D+(k+1)\mathscr{P}-(k+2)\mathscr{Q})\leq\dim\mathscr{L}(D+k\mathscr{P}-(k+1)\mathscr{Q})+1=1,$$

which implies equality because $f_{k+1}$ belongs to the first space. In addition, we have

$$\dim\mathscr{L}(D+(k+1)\mathscr{P}-(k+2)\mathscr{Q})=0,$$

because $f_{k+1}$ does not belong to the space $\mathscr{L}(D+(k+1)\mathscr{P}-(k+2)\mathscr{Q})$, but belongs to the space $\mathscr{L}(D+(k+1)\mathscr{P}-(k+1)\mathscr{Q})$.

Consider now, the differential of $F$ (3.5) while taking into account that $z$ appears only on the diagonal of the matrix $A(h)-zI$. Therefore, we have

$$-\sum_{i=1}^{N}C_{ii}dz+h\frac{\partial F}{\partial h}\frac{dh}{h}=0,$$

and either

$$\omega=\frac{-iC_{NN}dz}{h\frac{\partial F}{\partial h}}.$$

We have

$$\begin{aligned}\omega &= \frac{-i\frac{dh}{h}}{\sum_{i=1}^{N}\frac{C_{ii}}{C_{NN}}},\\ &= \frac{-i\frac{dh}{h}}{\sum_{i=1}^{N}\frac{C_{ii}}{C_{iN}}\cdot\frac{C_{iN}}{C_{NN}}},\\ &= \frac{-i\frac{dh}{h}}{\sum_{i=1}^{N}\frac{C_{Ni}}{C_{NN}}\cdot\frac{C_{iN}}{C_{NN}}}.\end{aligned}$$

Taking into account that $C_{iN}=C_{Ni}^*$, $1\leq i\leq N$, we obtain

$$\begin{aligned}\omega &= \frac{-i\frac{dh}{h}}{\sum_{i=1}^{N}\frac{C_{Ni}}{C_{NN}}\cdot\left(\frac{C_{iN}}{C_{NN}}\right)^*},\\ &= \frac{-i\frac{dh}{h}}{\sum_{i=1}^{N}f_if_i^*},\\ &= \pm\frac{C_{NN}dz}{\sqrt{P^2(z)-4\alpha^2}}.\end{aligned}$$

From this we deduce that $\omega^*=\omega$ and in addition, we have $\omega\geq 0$ on $C^{\sim}$. We also have a relation which shows that the scalar product between $f_k$ and $f_l$ is

$$\langle f_k,f_l\rangle=\int_{\mathscr{C}^{\sim}}f_k.f_l^*\omega=\begin{cases}0 & \text{si } k\neq l\\ >0 & \text{si } k=l\end{cases}$$

That is, the functions $f_k$, $k\in\mathbb{Z}$, are orthogonal to $\mathscr{C}^{\sim}$ with respect to $\omega$. We deduce from these properties that the divisor of $\omega$ is $(\omega)=D+\widetilde{D}-\mathscr{P}-\mathscr{Q}$, for the involution $\sim$ introduced previously. Given a matrix of the form $A$ (3.3), we have obtained a series of data $\{C,z,h,D,\omega\}$. What is remarkable is that the reverse is also true (for further information, see [11]) :

**Theorem 3.1.** *Consider the following two sets of data :*

*1) Let $a_i, b_i \in \mathbb{C}$, $a_i \neq 0$, where $a_{i+N} = a_i$, $b_{i+N} = b_i$, $-\infty < i < +\infty$. An infinite N-periodic matrix*

$$\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & b_0 & a_0 & 0 & \cdots & 0 & \\ & \overline{a}_0 & b_1 & a_1 & & \vdots & \\ & 0 & \overline{a}_1 & \ddots & \ddots & 0 & \\ & \vdots & & \ddots & \ddots & a_{N-1} & \\ & 0 & \cdots & 0 & \overline{a}_{N-1} & b_N & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix},$$

*modulo conjugation by N-periodic diagonal matrices with real entries.*
*2) Let $\mathscr{P}$ and $\mathscr{Q}$ two points on a curve of genus $N-1$ and $D$ be a divisor of degree $N-1$ on $C$ such that:*

$$(h) = -N\mathscr{P} + N\mathscr{Q}, \qquad (z) = -\mathscr{P} - \mathscr{Q} + S,$$

*where $h$, $z$ are two meromorphic functions on $C$ and $S$ is a positive divisor not containing the points $\mathscr{P}$ and $\mathscr{Q}$. The curve $C$ is equipped with an antiholomorphic involution $\sim$: $(z,h) \longmapsto (\overline{z}, \frac{1}{h})$, for which $C = \mathscr{C}_+ \cup C^{\sim} \cup C_-$, where $\mathscr{C}^{\sim}$ is the set of $p \in C$ such that : $\widetilde{p} = p$, i.e., the set of $(z,h)$ such that : $|h| = 1$, and $C_+$ (repectively $C_-$) is the set of $p \in C$ such that : $|h| > 1$ (repectively $|h| < 1$) containing $\mathscr{P}$ (repectively $\mathscr{Q}$). By introducing an involution $\star$ acting on the space of all meromorphic functions on $C$ and on the differential space in a way $\varphi^\star(p) = \widetilde{\varphi(\widetilde{p})}$ and $(\varphi d\psi)^\star = \varphi^\star d\psi^\star$, then $h^\star = h^{-1}$ and $z^\star = z$ and the divisor of a differential form $\omega$ on $C$ is*

$$(\omega) = D + \widetilde{D} - \mathscr{P} - \mathscr{Q}.$$

*Then, there is a one-to-one correspondence and equivalence of these sets of data.*

For any difference operator $X$, we define

$$(X_+)_{ij} = \begin{cases} X_{ij} & \text{si } i < j, \\ \frac{1}{2}X_{ij} & \text{si } i = j, \\ 0 & \text{si } i > j, \end{cases}, \qquad X_- = X - X^{[+]}.$$

Let $\mathscr{M}$ be the vector space of infinite $N$-periodic matrices $A$ such that for some $k$, $a_{ij} = 0$ if $|i-j| > K$. On $\mathscr{M}$, we introduce the following scalar product :

$$\langle A, B\rangle = Tr(AB^\top) = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij}b_{ij}.$$

We call a functional $F$ differentiable if there exists a matrix $\dfrac{\partial F}{\partial A}$ in $\mathscr{M}$ such that :

$$\lim_{\varepsilon\to 0} \frac{F(A+\varepsilon B) - F(A)}{\varepsilon} = \left\langle \frac{\partial F}{\partial A}, B\right\rangle,$$

for every $B$. The following bracket

$$\{F,G\} = \left\langle \left[\left(\frac{\partial F}{\partial X}\right)_+, \left(\frac{\partial G}{\partial X}\right)_+\right] - \left[\left(\frac{\partial F}{\partial X}\right)_-, \left(\frac{\partial G}{\partial X}\right)_-\right], X\right\rangle, \qquad (F, G \in \mathscr{M}),$$

satisfies the Jacobi identity. Let $P(A,S,S^{-1})$ be a polynomial in $S+S^{-1}$ and $A$ with real coefficients. Consider the following Lax equation:

$$\dot{A} = \left[P(A,S,S^{-1})_+ - P(A,S,S^{-1})_-, A\right]. \tag{3.8}$$

When the matrix $A(t)$ deforms with $t$, then only the divisor $D$ varies while $\{C, z, h, \mathscr{P}, \mathscr{Q}\}$ remain fixed. As we have already shown, the coefficients of $z^i h^j$ in equation (3.5) are invariants of this motion. The divisor $D(t)$ evolves linearly on the Jacobi variety Jac$(C)$. Any linear flow over Jac$(C)$ is equivalent to equation (3.8) and can be written in the form of a Hamitonian vector field with respect to the above bracket. For example,, the flow

$$\dot{A} = \left[A, (S^{-k}A^l]_+\right],$$

is written as follows :

$$\dot{a}_{ij} = \{F, a_{ij}\}, \qquad F = \frac{1}{l+1} Tr\left(S^{-k}A^{l+1}\right).$$

The (Poisson) bracket of two functional of the form $Tr\left(S^{-k}A^{l+1}\right)$ is zero, which means that we have a set of integrals in involution. Let $(\omega_1,...,\omega_g)$ be a holomorphic differential basis on the hyperelliptic curve $C$. We have

$$\omega_k=\frac{z^{k-1}}{\sqrt{P^2(z)-4Q^2}},\quad k=1,2,...,g$$

and let $c_k=\operatorname{Res}_p(\omega_k z^j)$, $1\le j\le g$. Since the order of the zeros of $\omega_k$ at the points at infinity $\mathscr{P}$, $\mathscr{Q}$ is equal to $g-k$, then $c_k=0$ for $k=1,2,...,g-j+1$ and $c_k\neq 0$ for $k=g-j+1$. Therefore, the flow which leaves invariant the spectrum of $A$ and $X$ is given by a polynomial $P(z)$ of degree at most equal to $g$ :

$$\dot{A}=\frac{1}{2}\left[A,P(A)_+-P(A)_-\right],$$

where $P(A)_+$ is the upper triangular part of $P(A)$ and $-P(A)_-$ is the lower triangular part of $P(A)$, including the diagonal of $P(A)$. The (Poisson) bracket between two functional $F$ and $G$ can still be written in the form

$$\{F,G\}=\left\langle\begin{pmatrix}\frac{\partial F}{\partial a} & \frac{\partial F}{\partial b}\end{pmatrix},J\begin{pmatrix}\frac{\partial G}{\partial a}\\ \frac{\partial G}{\partial b}\end{pmatrix}\right\rangle,$$

where $\frac{\partial F}{\partial a}=\left(\frac{\partial F}{\partial a_i}\right)$ and $\frac{\partial F}{\partial b}=\left(\frac{\partial F}{\partial b}_i\right)$ are the column vectors, while $J$ is the following $2n$-order antisymmetric matrix :

$$J=\begin{pmatrix}O & \mathscr{A}\\ -\mathscr{A}^\top & O\end{pmatrix},\qquad \mathscr{A}=2\begin{pmatrix}a_1 & 0 & 0 & \dots & -a_N\\ -a_1 & a_2 & 0 & & \vdots\\ 0 & -a_2 & a_3 & & \vdots\\ \vdots & & & & \vdots\\ 0 & \dots & \dots & -a_{N-1} & a_N\end{pmatrix}.$$

The symplectic structure [24] is given by

$$\omega=\sum_{j=2}^{N}db_j\wedge\sum_{j\le i\le N}\frac{da_i}{a_i}. \tag{3.9}$$

Flaschka variables [25] :

$$a_j=\frac{1}{2}e^{x_j-x_{j+1}},\qquad b_j=-\frac{1}{2}y_j,$$

applied to the form (3.9) with $x_{N+1}=0$, leads to the symplectic structure

$$\omega=\frac{1}{2}\sum_{j=2}^{N}dx_j\wedge dy_j,$$

used by Moser [26, 27] in the study of a dynamic system describing the motion of $N-1$ particles on a line, interacting under an exponential potential. See also the example below concerning the study of Toda lattice. We have

$$\det(A_h-zI)|_{h=i}=(-1)^N z^N+\beta_1 z^{N-1}+\beta_2 z^{N-2}+\cdots+\beta_N,$$

where $\beta_2,...,\beta_N$ are the $g$ invariant, functionally independent and in involution. These are given by the branch points on the hyperelliptic curve $\mathscr{C}$ or by the quantities $TrA^k$ for $k=2,3,...,N$, i.e., by the $g=N-1$ points chosen from the spectrum of $A_1$ and $A_{-1}$. With Jacobi's matrix, we can associate an operator $T$ on a separable Hilbert space $E$ as follows,

$$Te_0=b_0e_0+a_0e_1,\qquad Te_i=b_ie_j+a_{i-1}e_{i-1}+a_ie_{i+1},\quad i=1,2,...$$

where $(e_1,e_2,...)$ is an orthonormal basis in $E$. The operator $T$ is symmetric. Indeed, we have $\langle Tu_1,u_2\rangle=\langle u_1,Tu_2\rangle$ for any two finite vectors, according to the symmetry of the Jacobi matrix. Moreover, if the Carleman's condition :

$$\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_n}+\cdots=+\infty$$

is satisfied, then the spectrum of the self-adjoint operator $T$ (with $e_0$ a generating element) is simple. In this case, the information about the spectrum of $T$ is contained in the following function,

$$\varphi(z)=\left\langle(T-zI)^{-1}e_0,e_0\right\rangle=\int_{-\infty}^{\infty}\frac{d\sigma(x)}{z-x}, \tag{3.10}$$

defined at $z\notin\sigma(T)$ where $\sigma(x)=\langle I_xe_0,e_0\rangle$ and $I_x$ is the resolution of the identity operator $T$. Recall that the infinite continued fraction converges if the limit (3.2) exists. If the operator $T$ is self-adjoint, then the continued fraction $\varphi(z)$ converges uniformly in any closed bounded domain of $z$ without common points with real axis, to the analytic function defined by (3.10). If the support of $d\sigma(x)$ is bounded, then the sequence $\left(\dfrac{A_k(z)}{B_k(z)}\right)$ converges uniformly to a holomorphic function near $z=\infty$. Moreover, if a Jacobi matrix is bounded, i.e., if there exists $\rho>0$ such that, for all $j$, $|a_j|\leq\dfrac{\rho}{3}$, $|b_j|\leq\dfrac{\rho}{3}$, then the associated $\Gamma$-fraction converges uniformly on the domain $\{z:|z|\geq\rho\}$ and the support of $d\sigma(x)$ is included in $[-\rho,\rho]$. The fraction $\Gamma$ associated with a periodic Jacobi matrix (this case is obviously bounded ) converges near $z=\infty$. In addition, the function $\varphi(z)$ is written in the form (3.10) (Cauchy-Stieltjes transform of $d\sigma(x)$), which shows that $\varphi(z)$ has a first-order zero at $z=\infty$ and for any point $z$ belonging the upper-half plane, the imaginary part of $\varphi(z)$ is non positive.

We will now extend the Jacobi matrix $\Gamma$ to the infinite symmetric, tridiagonal and $N$-periodic Jacobi matrix $A$ (3.3) and use the results obtained previously. We consider $\varphi(z)$ (3.1) as being the associated $N$-periodic $\Gamma$-fraction. The latter converges near the infinite point $z=\infty$. An analytic extension of the function $\varphi(z)$ allows us to see that this coincides with the meromorphic function $a_0f_1$ on the genus $(N-1)$-hyperelliptic curve $C$ (3.6). This curve is branched at the $2N$ real zeroes $\xi_1$, $\xi_2$,...,$\xi_{2N}$ of the polynomial $P^2(z)-4\alpha^2$. We define the stable band as being the interval $[\xi_{2j-1},\xi_{2j}]$, $1\leq j\leq N$, and the unstable band the interval $[\xi_{2j},\xi_{2j+1}]$, $1\leq j\leq N-1$.

**Theorem 3.2.** *Each zero $\sigma_1<\sigma_2<\cdots<\sigma_{N-1}$ of $C_{k,l}$ (3.7), belongs to the $j$-th finite unstable band $[\lambda_{2j},\lambda_{2j+1}]$, $1\leq j\leq N-1$.*

We will see below (theorem 3.3) how to express the function $\varphi(z)$ in terms of Abelian integrals on the hyperelliptic curve $C$ (3.6). Note that for $N=1$, $B_k(x)$ is the well-known Chebyshev polynomial of the second kind. In addition, Kato [28, 29] discovered, for $N>1$, new results related to discrete measurements. We have seen that

$$\varphi(z)=a_0f_1=a_0\frac{C_{N,1}}{C_{N,N}}h,$$

belonging to $\mathscr{L}(D+\mathscr{P}-\mathscr{Q})$. Then, we have

**Theorem 3.3.** *We have*

$$\varphi(z)=\frac{Res_{\sigma_1^-}\varphi(z)}{z-\sigma_1}+\cdots+\frac{Res_{\sigma_{N-1}^-}\varphi(z)}{z-\sigma_{N-1}}+\frac{(-1)^{N+1}}{2\pi i}\left(\int_{\xi_1}^{\xi_2}\frac{\sqrt{P^2(x)-4\alpha^2}}{(z-x)C_{N,N}(x)}dx+\cdots+\int_{\xi_{2N-1}}^{\xi_{2N}}\frac{\sqrt{P^2(x)-4\alpha^2}}{(z-x)C_{N,N}(x)}dx\right), \tag{3.11}$$

*where,*

$$Res_{\sigma_j^-}\varphi(z)\equiv\frac{\alpha h(\sigma_j^-)+(-1)^N a_0^2.\Delta}{\prod_{l\neq j}(\sigma_j-\sigma_l)},\quad j=1,2,...,N-1$$

*and*

$$\Delta\equiv\det\begin{pmatrix} b_2-\sigma_j & a_2 & 0 & \cdots & 0\\ a_2 & b_3-\sigma_j & a_3 & & \vdots\\ 0 & a_3 & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & a_{N-2}\\ 0 & \cdots & 0 & a_{N-2} & b_{N-1}-\sigma_j \end{pmatrix}.$$

The differentials obtained in the previous section,

$$a\frac{C_{N,N}(x)}{\sqrt{P^2(x)-4\alpha^2}}dx,\qquad b\frac{\sqrt{P^2(x)-4\alpha^2}}{C_{N,N}(x)}dx,$$

($a$ and $b$ are constants) are positive mesures on each stable band $[\xi_{2j-1},\xi_{2j}]$. Therefore, the expression (3.11) means that $\varphi(z)$ can be obtained by the Cauchy-Stieltjes transform of

$$d\sigma=\sum_{j=1}^{N-1}\text{Res}_{\sigma_j^-}\varphi(z),\sigma_j^-).C(x-\sigma_j)dx+\frac{(-1)^{N+1}}{2\pi i}.\frac{\sqrt{P^2(x)-4\alpha^2}}{C_{N,N}(x)}dx=\text{discrete mesure }+\text{ continuous mesure},$$

as follows,

$$\varphi(z)=\int_{-\infty}^{\infty}\frac{d\sigma}{z-x}.$$

The function $\varphi(z)$ belongs to $\mathscr{L}(D'+\mathscr{P}-\mathscr{Q})$ where $D'=\sigma_1^++\cdots+\sigma_{N-1}^+$ is contained in $C_+=\{p\in C:|h|>1\}$ (see previous section). From expression (3.11), we have

$$D=\sigma_{j_1}^-+\cdots+\sigma_{j_l}^-+\sigma_{j_{l+1}}^++\cdots+\sigma_{j_{N-1}}^+,$$

where $j_1<j_2<...<j_l$ denote the numbers for which $\text{Res}_{\sigma_j^-}\varphi(z)>0$ and $j_{l+1}<j_{l+2}<...<j_{N-1}$ the numbers for which $\text{Res}_{\sigma_j^-}\varphi(z)=0$. Hence,

$$\text{Res}_{\sigma_j^-}\varphi(z)=0\text{ or }-\frac{\sqrt{P^2(\sigma_j^-)-4\alpha^2}}{\prod_{l\neq j}(\sigma_j-\sigma_l)}.$$

The Toda lattice equations [30] describe the motion of $n$ masses with exponential restoring forces :

$$H=\frac{1}{2}\sum_{j=1}^{N}y_j^2+\sum_{j=1}^{N}e^{x_j-x_{j+1}},\ \text{(Hamiltonian)}.$$

We noted above that Flaschka variables [25] : $a_j=\frac{1}{2}e^{x_j-x_{j+1}}$, $b_j=-\frac{1}{2}y_j$, can be used to express the symplectic structure $\omega$ (3.9) in terms of $x_j$ and $y_j$,

$$da_j=a_j(dx_j-dx_{j+1}),\qquad 2db_j=-dy_j,$$

then

$$\omega=\frac{1}{2}\sum_{j=2}^{N}dx_j^*\wedge dy_j^*,\qquad (x_j^*\equiv x_j-x_1,\quad y_j^*\equiv y_j).$$

We will study the integrability of this problem with the Griffiths approach. There are two cases :
$(i)$ The non-periodic case, i.e., $x_0=-\infty$, $x_{N+1}=+\infty$, where the masses are arranged on a line. In term of the Flaschka variables above, Toda's equations take the following form

$$\begin{array}{rcl}\dot{a}_j&=&a_j\left(b_{j+1}-b_j\right),\\ \dot{b}_j&=&2(a_j^2-a_{j+1}^2),\end{array}$$

with $a_{N+1}=a_1$ and $b_{N+1}=b_1$. To show that this system is completely integrable, one should find $N$ independent first integrals in involution. From the second equation, we have

$$\overbrace{(b_1+b_2+\cdots+b_N)}^{\cdot}=\dot{b}_1+\dot{b}_2+\cdots+\dot{b}_N=0,$$

and we normalize $b_i$'s by requiring that $b_1+b_2+\cdots+b_N=0$. Applying this fact to (3.9), leads to the following symplectic form :

$$\omega=\frac{1}{2}\sum_{j=2}^{N}dx_j\wedge dy_j.$$

We have obtained a first integral of the system and it will be necessary to determine $N-1$ other integrals that are functionally independent and in involution. We further define $N\times N$ matrices $A$ and $B$ with

$$A=\begin{pmatrix}b_1&a_1&0&\cdots&a_N\\ a_1&b_2&\vdots&&\vdots\\ 0&\ddots&\ddots&\ddots&0\\ \vdots&&\ddots&b_{N-1}&a_{N-1}\\ a_N&\cdots&0&a_{N-1}&b_N\end{pmatrix},\qquad B=\begin{pmatrix}0&a_1&\cdots&\cdots&-a_N\\ -a_1&0&\vdots&&\vdots\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ \vdots&&\ddots&\ddots&a_{N-1}\\ a_N&\cdots&\cdots&-a_{N-1}&0\end{pmatrix}.$$

The system in question is written in the form $\dot{A}=[B,A]$. Since $I_k=\frac{1}{k}trA^k$, $k=1,2,...,N$, are first integrals (see theorem 2.1), then

$$\dot{I}_k=tr(\dot{A}.A^{k-1})=tr([B,A].A^{k-1})=tr(BA^k-ABA^{k-1})=0.$$

Notice that $I_1$ is the first integral already know. These $N$ first integrals are functionally independent and in involution, the system in question is thus completely integrable.
$(ii)$ The periodic case, i.e., $y_{j+N}=y_j$, $x_{j+N}=x_j$, the connected masses will be arranged on a circle. We show that in this case, the spectrum of the periodic matrix

$$A(h)=\begin{pmatrix}b_1&a_1&0&\cdots&a_Nh^{-1}\\ a_1&b_2&\vdots&&\vdots\\ 0&\ddots&\ddots&\ddots&0\\ \vdots&&\ddots&b_{N-1}&a_{N-1}\\ a_Nh&\cdots&0&a_{N-1}&b_N\end{pmatrix},$$

remains invariant in time. The matrix $B(h)$ depending on the spectral parameter $h$, has the form

$$B(h)=\begin{pmatrix}0&a_1&\cdots&\cdots&-a_Nh^{-1}\\ -a_1&0&\vdots&&\vdots\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ \vdots&&\ddots&\ddots&a_{N-1}\\ a_Nh&\cdots&\cdots&-a_{N-1}&0\end{pmatrix},$$

and the rest follows from the general theory. Note that if $a_j(0) \neq 0$, then $a_j(t) \neq 0$ for all $t$. Since $A^\top(h) = A(h^{-1})$, the spectral curve $C$ is given by

$$0 = \det(A(h) - zI) = P\left(\frac{1}{h}, z\right) \equiv P(h,z).$$

By the antisymmetry of $A$, the curve $C$ has an involution

$$\tau : C \longrightarrow C, \quad (h,z) \longmapsto \left(\frac{1}{h}, z\right). \tag{3.12}$$

We choose

$$A(h) = \begin{pmatrix} 0 & \dots & a_N \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} h^{-1} + \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & & & \\ & & \ddots & & \\ & & & b_{N-1} & a_{N-1} \\ & & & b_N & a_N \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ a_N & \dots & 0 \end{pmatrix} h.$$

Note that here the matrix $A$ is meromorphic (whereas previously we considered it to be a polynomial in $h$) but we will see that we can adopt the theory explained in this section, to this situation too . We have

$$P(h,z) = -a_1.a_2...a_{N-1}.\left(h + \frac{1}{h}\right) + z^N + c_1 z^{N-1} + \cdots + c_N.$$

Let us assume that $a_1.a_2...a_{N-1} \neq 0$ and pose

$$Q(h,z) \equiv \frac{P(h,z)}{\prod_{j=1}^{N-1} a_j} = h + \frac{1}{h} + \frac{z^N + c_1 z^{N-1} + \cdots + c_N}{\prod_{j=1}^{N-1} a_j} = h + \frac{1}{h} + d_0 z^N + d_1 z^{N-1} + \cdots + d_N.$$

In $\mathbb{P}^2(\mathbb{C})$, the affine algebraic curve of equation $Q(h,z) = 0$, is singular at infinity and to determine the genus $g$ of its normalization, we proceed as follows : note that the curve $C$ appears as a double sheeted covering of $\mathbb{P}^1(\mathbb{C})$ branched into $2N$ points coinciding with the fixed points of involution $\sigma$ (3.12), that is, points where $h = \pm 1$. Using the Riemann-Hurwitz formula, we obtain

$$g = 2\left(g(\mathbb{P}^1(\mathbb{C})) - 1\right) + 1 + \frac{2N}{2} = N - 1.$$

Consider the covering $C \longrightarrow \mathbb{P}^1(\mathbb{C})$ below and $\mathscr{P}$, $\mathscr{Q}$ located on two separate sheets. By putting $\frac{1}{z}(\infty) = \mathscr{P} + \mathscr{Q}$, we see from the equation $Q(h,z) = 0$, that the divisor of $h$ is $(h) = N\mathscr{P} - N\mathscr{Q}$ and in that case, the divisor $D$ is written $D = N\mathscr{P} + N\mathscr{Q}$, which implies that $B \in H^0(D, \mathrm{Hom}(V, V(D))$. The residue $\varsigma(B) \in H^0(D, \mathscr{O}_D(D)$ satisfies the conditions of theorem 2.7, and consequently the linear flow is given by the application (2.8). To compute the residue $\varsigma(B)$ of $B$, we will determine a set of holomorphic eigenvectors, using the van Moerbeke-Mumford method described above. Let us calculate the residue in $\mathscr{Q}$ and the result will be similarly deduced in $\mathscr{P}$. Consider a general divisor $E$ of degree $g$, of the form $E = \sum_{j=1}^{g} r_j$ such that : $\dim \mathscr{L}(E + (k-1)\mathscr{P} - k\mathscr{Q}) = 0$, for all $k$. We deduce from Riemann-Roch's theorem that $\dim \mathscr{L}(E + k\mathscr{P} - k\mathscr{Q}) \geq 1$, and therefore $\dim \mathscr{L}(E + k\mathscr{P} - k\mathscr{Q}) = 1$, for all $k$. Let

$$(f_k) \in \mathscr{L}(E + k\mathscr{P} - k\mathscr{Q}) = H^0(C, \mathscr{O}_C(E + k\mathscr{P} - k\mathscr{Q})), \quad 1 \leq k \leq N$$

be a basis with $f_N = h$. We can choose a vector $v$ of the following form $v = (f_1, ..., f_N)^\top$, such that $v$ is an eigenvector of $A$, i.e., $Av = zv$, $(h,z) \in C$. Hence, $V = h^{-1} v$ is a holomorphic eigenvector. Without restricting generality, we take $N = 3$. The system $Av = zv$, is written explicitly

$$\begin{aligned} b_1 f_1 + a_2 f_2 + a_3 &= z f_1, \\ a_1 f_1 + b_2 f_2 + a_2 h &= z f_2, \\ a_3 h f_1 + a_2 f_2 + b_3 h &= z h. \end{aligned}$$

By multiplying each equation of this system by $\frac{1}{h}$, everything becomes holomorphic except the last equation, i.e., $a_3 f_1 = z + \text{Taylor}$. Recall that the section of $\mathscr{O}_D(D)$ induced by $\lambda$ in the equation (2.7) : $Bv = \dot{v} + \lambda v$, is the residue $\varsigma(B)$ of $B$. In other words,

$$Bv = \varsigma(B) v + \text{Taylor},$$

and therefore

$$\begin{pmatrix} \frac{a_1 f_2}{h} - \frac{a_3}{h} \\ -\frac{a_1 f_1}{h} + a_2 \\ a_3 f_1 - \frac{a_2 f_2}{h} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} + \text{Taylor}.$$

We deduce that $\varsigma(B) = \frac{z}{h}$, and $\varsigma(\dot{B}) = 0$. The same conclusion holds for the residue in $\mathscr{P}$. Consequently, the flow in question linearizes on the Jacobian variety of $C$.

## 4. Algebraically integrable systems

Consider the nonlinear system of differential equations :

$$\begin{aligned} \frac{dz_1}{dt} &= f_1(t, z_1, ..., z_n), \\ &\vdots \\ \frac{dz_n}{dt} &= f_n(t, z_1, ..., z_n), \end{aligned} \tag{4.1}$$

where $f_1, ..., f_n$ are functions of $n+1$ complex variables $t, z_1, ..., z_n$ and which apply a domain of $\mathbb{C}^{n+1}$ into $\mathbb{C}$. The Cauchy problem is the search for a solution $(z_1(t), ..., z_n(t))$ in a neighborhood of a point $t_0$, satisfying the initial conditions : $z_1(t_0) = z_1^0, ..., z_n(t_0) = z_n^0$. The system (4.1) can be written in vector form in $\mathbb{C}^n$,

$$\frac{dz}{dt} = f(t, z(t)), \quad z = (z_1, ..., z_n), \quad f = (f_1, ..., f_n).$$

In this case, the Cauchy problem will be to determine the solution $z(t)$ such that $z(t_0) = z_0 = (z_1^0, ..., z_n^0)$. When the functions $f_1, ..., f_n$ are holomorphic in the neighborhood of $(t_0, z_1^0, ..., z_n^0)$, then the Cauchy problem admits a holomorphic solution and only one. A question arises : can the Cauchy problem admits some non-holomorphic solution around $(t_0, z_1^0, ..., z_n^0)$? When $f_1, ..., f_n$ are holomorphic, the answer is negative. Other circumstances may arise for the Cauchy problem concerning the system of differential equations (4.1), when the holomorphic hypothesis relative to the functions $f_1, ..., f_n$ is no longer satisfied in the neighborhood of a point. In such a case, it can be seen that the behavior of the solutions can take on the most diverse aspects. In general, the singularities of the solutions are of two types : mobile or fixed, depending on whether or not they depend on the initial conditions. Important results have been obtained by Painlevé [31]. Suppose that the system (4.1) is written in the form

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{P_1(t, z_1, ..., z_n)}{Q_1(t, z_1, ..., z_n)}, \\ &\vdots \\ \frac{dz_n}{dt} &= \frac{P_n(t, z_1, ..., z_n)}{Q_n(t, z_1, ..., z_n)}, \end{aligned}$$

where

$$P_k(t, z_1, ..., z_n) = \sum_{0 \le i_1, ..., i_n \le p} A^{(k)}_{i_1, ..., i_n}(t) z_1^{i_1} ... z_n^{i_n}, \ 1 \le k \le n,$$

$$Q_k(t, z_1, ..., z_n) = \sum_{0 \le j_1, ..., j_n \le q} B^{(k)}_{j_1, ..., j_n}(t) z_1^{j_1} ... z_n^{j_n}, \ 1 \le k \le n,$$

polynomials with several indeterminate $z_1, ..., z_n$ and algebraic coefficients in $t$. There are two cases: **(i)** the fixed singularities are constituted by four sets of points. The first set contains the singular points of the coefficients $A^{(k)}_{i_1, ..., i_n}(t)$, $B^{(k)}_{j_1, ..., j_n}(t)$ intervening in the polynomials $P_k(t, z_1, ..., z_n)$ and $Q_k(t, z_1, ..., z_n)$. In general this set contains $t = \infty$. The second set consists of the points $\alpha_j$ such that : $Q_k(t, z_1, ..., z_n) = 0$, which occurs if all the coefficients $B^{(k)}_{j_1, ..., j_n}(t)$ vanish for $t = \alpha_j$. The third is the set of points $\beta_l$ such that for some values $(z_{1'}, ..., z_{n'})$ of $(z_1, ..., z_n)$, we have $P_k(\beta_l, z_{1'}, ..., z_{n'}) = Q_k(\beta_l, z_{1'}, ..., z_{n'}) = 0$. Then the second members of the above system are presented in the indeterminate form $\frac{0}{0}$ at the points $(\beta_l, z_{1'}, ..., z_{n'})$. Finally, the set of points $\gamma_n$ such that there exist $u_1, ..., u_n$, for which $R_k(\gamma_n, u_1, ..., u_n) = S_k(\gamma_n, u_1, ..., u_n) = 0$, where $R_k$ and $S_k$ are polynomials in $u_1, ..., u_n$ obtained from $P_k$ and $Q_k$ by setting $z_1 = \frac{1}{u_1}, \dots, z_n = \frac{1}{u_n}$. Each of these sets contains only a finite number of elements. The system in question has a finite number of fixed singularities. **(ii)** the mobile singularities of solutions of this system are algebraic : poles and (or) algebraic critical points. There are no essential singular points for the solution $(z_1, ..., z_n)$.

We will use the method of indeterminate coefficients to find sufficient conditions for the existence and uniqueness of the meromorphic solution of the Cauchy problem concerning the system (4.1). The solution will be expressed in the form of Laurent expansions in $t$ and such a solution is formal because we obtain it by performing on various series, which we assume a priori convergent, various operations whose validity remains to be justified. The problem of convergence will therefore arise. The result will therefore be established as soon as we have verified that these series are convergent. This will be done using the majorant method [32]-[34]. Without restricting the generality, we consider the Cauchy problem relative to the normal system (4.1) where $f_1, ..., f_n$ do not depend explicitly on $t$, i.e.,

$$\begin{aligned} \frac{dz_1}{dt} &= f_1(z_1, ..., z_n), \\ &\vdots \\ \frac{dz_n}{dt} &= f_n(z_1, ..., z_n). \end{aligned} \tag{4.2}$$

We suppose that $f_1, ..., f_n$ are rational functions in $z_1, ..., z_n$ and that the system (4.2) is weight-homogeneous, i.e., there exist positive integers $l_1, ..., l_n$ such that :

$$f_i(\alpha^{l_1} z_1, ..., \alpha^{l_n} z_n) = \alpha^{l_i+1} f_i(z_1, ..., z_n), \quad 1 \le i \le n,$$

for each non-zero constant $\alpha$. Note that if the determinant $\det\left(z_j\frac{\partial f_i}{\partial z_j}-\delta_{ij}f_i\right)_{1\leq i,j\leq n}\not\equiv 0$, then the numbers $s_1,\ldots,s_n$ are unique. In order to facilitate the notations, we will assume (without loss of generalities) that $t_0=z_0=0$.

**Theorem 4.1.** *Suppose that*

$$z_i=\frac{1}{t^{k_i}}\sum_{k=0}^{\infty}z_i^{(k)}t^k,\quad 1\leq i\leq n,\quad z^{(0)}\neq 0 \tag{4.3}$$

*($k_i\in\mathbb{Z}$, some $k_i>0$) is the formal solution (Laurent series), obtained by the method of undetermined coefficients of the weight-homogeneous system (4.2). Then the coefficients $z_i^{(0)}$ satisfy the nonlinear equation*

$$k_iz_i^{(0)}+f_i(z_1^{(0)},\ldots,z_n^{(0)})=0,\quad 1\leq i\leq n,$$

*while $z_i^{(1)},z_i^{(2)},\ldots$ are solution of the following system of linear equations :*

$$(L-k\mathscr{I})z^{(k)}=\textit{some polynomial in the } z^{(j)},\quad 0\leq j\leq k,$$

*where $z^{(k)}=(z_1^{(k)},\ldots,z_n^{(k)})^\top$ and $L\equiv\left(\frac{\partial f_i}{\partial z_j}(z^{(0)})+\delta_{ij}k_i\right)_{1\leq i,j\leq n}$, is the Jacobian matrix. Moreover, the formal series (4.3) are convergent.*

The coefficients $z_i^{(k)}$ are determined unequivocally with the adopted method of calculation which explains why the series (4.3) is the only meromorphic solution. Moreover, the result of the previous theorem applies to the following quasi-homogeneous differential equation of order $n$ :

$$\frac{d^nz}{dt^n}=f\left(z,\frac{dz}{dt},\ldots,\frac{d^{n-1}z}{dt^{n-1}}\right).$$

$f$ being a rational function in $z,\frac{dz}{dt},\ldots,\frac{d^{n-1}z}{dt^{n-1}}$ and $z(0)=z_1^0$, $\frac{dz}{dt}(0)=z_2^0,\ldots,\frac{d^{n-1}z}{dt^{n-1}}(0)=z_n^0$. Indeed, the differential equation above reduces to the following system :

$$z(t)=z_1(t),\quad \frac{dz}{dt}(t)=z_2(t),\ldots,\frac{d^{n-1}z}{dt^{n-1}}(t)=z_n(t).$$

We thus obtain

$$\frac{dz_1}{dt}=z_2,\frac{dz_2}{dt}=z_3,\ldots,\frac{dz_{n-1}}{dt}=z_n,\frac{dz_n}{dt}=f(z_1,z_2,\ldots,z_n).$$

Such a system constitutes a particular case of the normal system (4.2).

Let $X_H$ be a Hamiltonian vector field defined by

$$\dot z\equiv\frac{dz}{dt}=J\frac{\partial H}{\partial z}\equiv f(z),\quad z\in\mathbb{R}^m \tag{4.4}$$

where $J=J(z)$ is a skew-symmetric matrix polynomial in $z$ of rank $2n$, such that the Poisson bracket $\{H,F\}=\left\langle\frac{\partial H}{\partial z},J\frac{\partial F}{\partial z}\right\rangle$ satisfies the Jacobi identity. The system (4.4) is algebraic complete integrable (in abbreviated form : a.c.i.) when $J$ has polynomial entries and when the following conditions hold :

**i)** The system is completely integrable with polynomial invariants $H_1,\ldots,H_{n+k}$. It means that besides the $k$ invariants $H_1,\ldots,H_k$ (Casimir functions), i.e., such that $J\frac{\partial H_i}{\partial z}(z)=0$, $1\leq i\leq k$, the system admits $n=\frac{m-k}{2}$ invariants $H_{k+1}=H,\ldots,H_{k+n}$ in involution, i.e., such that $\{H_i,H_j\}=0$. These give rise to $n$ commuting vector fields. For generic $c_i$, the invariant manifolds (level surfaces)

$$\bigcap_{i=1}^{n+k}\{z\in\mathbb{R}^m:H_i=c_i\},$$

are compact, connected and therefore real tori according to the Arnold-Liouville theorem [12].

**ii)** The invariant manifolds (level surfaces) thought of as lying in $\mathbb{C}^m$,

$$\bigcap_{i=1}^{n+k}\{z\in\mathbb{C}^m:H_i=c_i\},$$

are related, for generic $c_i$, to Abelian varieties $T^n=\mathbb{C}^n/Lattice$ (complex algebraic tori) as follows :

$$\bigcap_{i=1}^{n+k}\{z\in\mathbb{C}^m:H_i=c_i\}=T^n\backslash D,$$

where $D$ is a divisor (codimension one subvarieties) in $T^n$. The coordinates $z_i$ are meromorphic on $T^n$ and $D$ is the minimal divisor on $T^n$ where the variables $z_i$ blow up. The flows (4.4) run with complex time are straight-line motions on $T^n$.

As the reader has surely noted, we have insisted in the above definition that invariants must be polynomials. But it must be understood that the existence of a sufficient number of polynomial invariants does not necessarily imply the algebraic complete integrability of the system in question. To convince ourselves of this, it is enough to consider the following Hamiltonian system whose Hamiltonian is

$$H((x,y)=\frac{x^2}{2}+P_n(y),$$

where $P_n(y)$ is a polynomial in $y$ of degree $n$. We show that such a system is algebraically completely integrable if and only if $n=3$ or 4, and the explicit resolution of the system is done using elliptic functions. So a natural question arises : given a completely integrable system with polynomial invariants, what makes it algebraically completely integrable? Mumford gives in his book [21] a definition of the algebraic complete integrability including also non compact and explains (although it has nothing to do with the system above) this extra feature as follows : the vector fields $X_{H_1},\dots,X_{H_n}$ define on the real torus $M_c=\bigcap_{i=1}^{n+k}\{H_i=c_i\}\subset\mathbb{R}^{2n}$ an addition law

$$\oplus: M_c\times M_c\longrightarrow M_c,\ (x,y)\longmapsto x\oplus y=g_{t+s}(p),\quad p\in M_c,$$

with $x=g_t(p)$, $y=g_s(p)$, $g_t(p)=g_{t_1}^{X_1}\dots g_{t_n}^{X_n}(p)$, where $g_{t_i}^{X_i}(p)$ denote the flow of $X_{H_i}$. From the polynomial nature of the vector fields X, this addition law will always be real analytic. The algebraic complete integrability of the system in question means that this law of addition is rational. In other words, we have $(x\oplus y)_j=R_j(x_i,y_i,c)$, where $R_j(x_i,y_i,c)$ is a rational function of the coordinates $x_i$, $y_i$ for all $i=1,2,\dots,n$. Putting $x=p$, $y=g_t^{X_i}(p)$, in the above formula, we notice that on the real torus $M_c$, the flows $g_t^{X_i}(p)$ depend rationally on the initial condition $p$. Moreover, a Weierstrass theorem on the functions admitting a law of addition, affirms that the coordinates $x_i$ restricted to the real torus :

$$\mathbb{R}^n/lattice\longrightarrow M_c,\quad (t_1,\dots,t_n)\longmapsto z_i(t_1,\dots,t_n),$$

are Abelian functions. Geometrically, this means that the real torus $M_c\approx\mathbb{R}^n/lattice$ is the affine part of an algebraic complex torus (Abelian variety) $T^n\simeq\mathbb{C}^n/lattice$ and the real functions $z_i(t_1,\dots,t_n)$, $(t_i\in\mathbb{R})$, are the restrictions to this real torus of meromorphic functions $z_i(t_1,\dots,t_n)$, $(t_i\in\mathbb{C})$ of $n$ complex variables, with $2n$ real periods (of which $n$ real periods and $n$ imaginary periods). It must be said that Mumford's explanation of the algebraic complete integrability of a completely integrable Hamiltonian system with polynomial invariants, is of purely theoretical interest. Indeed, how do you recognise from the differential equations that, on a given level manifold, the commuting vector fields define a rational addition law ? Painlevé [31] provides the following provocative example, among many others not necessarily in the context of Hamiltonian mechanics. Consider on $\mathbb{C}^2$ the two polynomial commuting vector fields :

$$\begin{array}{lcll} X_1 & : & \dot{x}=x, & \dot{y}=xy,\\ X_2 & : & \dot{x}=0, & \dot{y}=y. \end{array}$$

The flow

$$g_t^{X_1}(x_0,y_0)=\left(x_0e^t,y_0^{x_0(e^t-1)}\right),$$

doest not depend rationally on the initial condition $(x_0,y_0)$. Therefore, simply looking at the face of the equations does not answer the question of whether the problem is algebraically completely integrable. The only method was to solve the problem explicitly in terms of Abelian integrals.

Now if the system (4.4) is algebraically completely integrable, it means that the variables $z_i$ restricted to a generic complex invariant manifold of the flows, are meromorphic functions on a complex torus $\mathbb{C}^n/lattice$; in fact these are Abelian functions. By compactness, these functions must blow up along a divisor (a codimension one subvariety) $D\subset\mathbb{C}^n/lattice$. Expanding the solutions of the system (4.4) near this divisor and allowing the constants of the motion to vary, one gets meromorphic solutions depending on $\dim D+\sharp H_i=m-1$ parameters, because $\dim D=n-1$ and $\sharp H_i=n+k$ is the number of constants of the motion. The fact that algebraic complete integrable systems possess $(m-1)$-dimensional families of Laurent solutions, was implicitly used by Kowalewski [35] in her classification of integrable rigid body motions. The following necessary condition was developed and used by Adler-van Moerbeke [36] :

**Theorem 4.2.** *Suppose that the Hamiltonian system (4.4) is algebraically completely integrable with Abelian functions $z_i$ and for generic $c$, the invariant tori related to this system do not contain elliptic curves. Then this system must admit enough meromorphic Laurent expansion solutions in $t\in\mathbb{C}$ such that : each $z_i$ blows up at least once and Laurent expansion of $z_i$, depend on $m-1$, free parameters. In addition, the system in question has families of Laurent solutions depending on $m-2$, $m-3$, ..., $m-n$, parameters and the coefficients of each of these solutions are rational functions on affine algebraic varieties of dimensions $m-1$, $m-2$, $m-3$,...,$m-n$.*

The question is whether this criterion is sufficient and how it can be used to detect algebraically completely integrable systems. The idea of the direct proof given by Adler-van Moerbeke[37, 38] is closely related to the geometric spirit of the real Arnold-Liouville theorem [12]. Namely, a compact complex $n$-dimensional variety on which there exist $n$ holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a $n$-dimensional complex torus $\mathbb{C}^n/Lattice$ and the complex flows generated by the vector fields are straight lines on this complex torus. Now a complex affine algebraic variety is never compact, unless it is 0-dimensional. So the main problem will be to complete the affine variety $M_c=\bigcap_{i=1}^{n+k}\{z\in\mathbb{C}^m,H_i=c_i\}$, into a non-singular compact complex algebraic variety $\overline{M}_c=M_c\cup D$ in such a way that the vector fields extend holomorphically along $D$ and remain independent there. If this is possible, $\overline{M}_c$ is an Abelian variety (an algebraic complex torus) and the coordinates $z_i$ restricted to $M_c$ are Abelian functions. To compactifize $M_c$ into an algebraic complex torus, a naive guess would be to take the natural compactification

$$\overline{M}_c=\bigcap_{i=1}^{n+k}\left\{Z\in\mathbb{P}^m(\mathbb{C}),H_i(Z)=c_iZ_0^{\deg H_i}\right\},$$

of $M_c$ by projectivizing the equations. Indeed, this can never work for a general reason: an Abelian variety $\widetilde{M}_c$ of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space $\mathbb{P}^n(\mathbb{C})$ by $n$-dim $\widetilde{M}_c$ global polynomial homogeneous equations. In other words, if $M_c$ is to be the affine part of an Abelian variety, $\overline{M}_c$ must have a singularity somewhere along the locus at infinity, i.e., along all or part of the hyperplane section $\{Z_0 = 0\}$ at infinity. The trajectories of the vector fields (4.4) hit every point of the singular locus at infinity and ignore the smooth locus at infinity. In fact, the existence of meromorphic solutions to the differential equations (4.4) depending on some free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor. A partial converse to theorem 4.2, can be formulated as follows [36] :

**Theorem 4.3.** *We assume that condition **i)** in the above definition of the algebraic complete integrability is satisfied. In addition, suppose that the system (4.4) with $k+n$ polynomial invariants have a coherent tree of Laurent solutions, i.e., it possesses families of Laurent solutions in t, depending on $n-1$, $n-2$,..., $m-n$, free parameters. Then, this system is algebraic complete integrable and moreover, there are no other Laurent solutions of $m-1$ dimension than those provided by the coherent set.*

The study of the algebraic complete integrability of Hamiltonian systems, includes several passages to prove rigorously. Here we mention the main passages. We saw that if the flow is algebraically completely integrable, the differential equations (4.4) must admits Laurent series solutions

$$z_i(t) = \frac{1}{t^{k_i}}\left(z_i^{(0)} + z_i^{(1)} + \cdots\right), \quad k_i \in \mathbb{Z}, \quad i = 1,2,...$$

where $z_i^{(0)}, z_i^{(1)}, \ldots$ are rational functions depending on $m-1$, free parameters. We must have $k_i = l_i$ and coefficients in the series must satisfy at the $0^{th}$step nonlinear equations,

$$f_i\left(z_1^{(0)}, \ldots, z_m^{(0)}\right) + g_i z_i^{(0)} = 0,\ 1 \le i \le m, \tag{4.5}$$

and at the $k^{th}$step, linear systems of equations :

$$(\mathscr{M} - kI)\, z^{(k)} = \begin{cases} 0 & \text{for } k = 1 \\ \text{polynomials in} & z^{(1)}, \ldots, z^{(k-1)} \text{ for } k > 1, \end{cases} \tag{4.6}$$

where $\mathscr{M} = \dfrac{\partial f}{\partial z} + gI\mid_{z=z^{(0)}}$ is the Jacobian matrix of the equations (4.5). If $m-1$, free parameters are to appear in the Laurent series, they must either come from the nonlinear equations (4.5) or from the eigenvalue problem (4.6), i.e., $\mathscr{M}$ must have at least $m-1$, integer eigenvalues. These are much less conditions than expected, because of the fact that the homogeneity $k$ of the constant $H$ must be an eigenvalue of $L$. The formal series solutions are convergent as a consequence of the majorant method. By substituting these series solutions into the constants of motion $H_i(z)$, $1 \le i \le n+k$, one eliminates some parameters linearly, leading to an algebraic relation between the remaining parameters, which is nothing but the equation of the divisor $D$ along which the $z_i$ blow up; if the differential equations admit $l$ families of Laurent meromorphic solutions of the form above, it means that $D$ is formed by $l$ algebraic curves. More precisely, you have to prove that the set

$$D \equiv \left\{z_i(t), 1 \le i \le m, \text{ Laurent solutions such that} : H_j\left(z_i(t)\right) = c_j + \text{Taylor part}\right\}$$

defines one or several $n-1$ dimensional algebraic varieties ("Painlevé" divisor) having the property that $\bigcap_{i=1}^{n+k} \{z \in \mathbb{C}^m : H_i = c_i\} \cup D$, is a smooth compact, connected variety with $n$ commuting vector fields independent at every point, i.e., a complex algebraic torus $\mathbb{C}^n/lattice$. Note that the system of coordinates $z_1, \ldots, z_m$ can be enlarged to a new set $z_0 = 1, z_1, \ldots, z_N$ having the property that for fixed but arbitrary $0 \le j \le N$, we have

$$\overbrace{\left(\frac{z_i}{z_j}\right)}^{\cdot} = \frac{\dot{z}_i z_j - z_i \dot{z}_j}{z_j^2} = \sum_{k,l} a_{k,l}\left(\frac{z_k}{z_j}\right)\left(\frac{z_l}{z_j}\right),$$

i.e., the ratios $\dfrac{z_i}{z_j}$ form a closed system of coordinates under differentiation. Indeed, consider a point $p \in D$, a chart $U_j$ around $p$ on the torus and a function $z_j$ in $L(D)$ having a pole of maximal order at $p$. Then the vector $\left(\dfrac{1}{z_j}, \dfrac{z_1}{z_j}, \ldots, \dfrac{z_N}{z_j}\right)$ provides a good system of coordinates in $U_j$. Then taking the derivative with regard to one of the flows $\overbrace{\left(\dfrac{z_i}{z_j}\right)}^{\cdot}$ are finite on $U_j$ as well. Therefore, since $z_j^2$ has a double pole along $D$, the numerator must also have a double pole (at worst), i.e., $\dot{z}_i z_j - z_i \dot{z}_j \in L(2D)$. Hence, when the divisor $D$ is projectively normal, i.e., whenever $L(kD) = L(D)^{\otimes k}$ which means that the space $L(kD)$ is generated by homogeneous polynomials of degree $k$ in some basis elements of $L(D)$. At the bad points, the concept of projective normality play an important role: this enables one to show that $\dfrac{z_i}{z_j}$ is a bona fide Taylor series starting from every point in a neighborhood of the point in question. Therefore, the flows $J\dfrac{\partial H_{k+i}}{\partial z}$,..., $J\dfrac{\partial H_{k+n}}{\partial z}$ are straight line motions on this torus (for concrete applications, see for example [32, 36, 39, 40, 41, 42, 43]). Let's point out that having computed the space of functions $\mathscr{L}(D)$ with simple poles at worst along with the expansions, it is often important to compute the space of functions $\mathscr{L}(kD)$ of functions having $k$-fold poles at worst along with the expansions. These functions play a crucial role in the study of the procedure for embedding the invariant tori into projective space. As mentioned previously, the idea of the Adler-van Moerbeke's proof [37, 38] consists of using arguments similar to those used in the proof of the real Arnold-Liouville theorem [12], and we can call this result the Liouville-Arnold-Adler-van Moerbeke theorem:

**Theorem 4.4.** *Let $\widetilde{M}$ be an n-dimensional complex compact manifold with n independent meromorphic functions. Assume that :*
*(i) For some divisor D, there exist n non-vanishing holomorphic vector fields $X_1,...,X_n$ on the affine variety $\widetilde{M}\backslash D = M$ which commute and are independent at every point.*
*(ii) One vector field, say $X_k$ (where $1 \le k \le n$), extends holomorphically on $\widetilde{M}$ and having the property that, for all $p \in D$,*

$$\left\{ g_{t_k}^{X_k}(p) : 0 < \mid t \mid < \varepsilon, t \in \mathbb{C} \right\} \subset M,$$

*where $g_{t_k}^{X_k}$ denote the flow of $X_k$. This condition means that the orbits of $X_k$ through D go immediately into the affine part M and in particular, the vector field $X_k$ does not vanish on any point of D.*
*Then, $\widetilde{M}$ is an Abelian variety and the vector fields $X_1,...,X_n$ extend holomorphically and remain independent on $\widetilde{M}$.*

1) As an example, consider the Kac-van Moerbeke periodic lattice [44] given by the following system :

$$\dot{x}_j = x_j(x_{j-1} - x_{j+1}), \quad j = 1,...,5$$

where $(x_1,...,x_5) \in \mathbb{C}^5$ and $x_j = x_{j+5}$. This vector field forms a Hamiltonian system for the Poisson structure

$$\{x_j, x_k\} = x_j x_k(\delta_{j,k+1} - \delta_{j+1,k}), \quad 1 \le j,k \le 5$$

and admits three independent first integrals

$$\begin{aligned} H_1 &= x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2, \\ H_2 &= x_1 + x_2 + x_3 + x_4 + x_5, \\ H_3 &= x_1x_2x_3x_4x_5. \end{aligned}$$

Note that $H_1$ and $H_2$ are involution while $H_3$ is a Casimir, and the system in question is therefore integrable. The affine manifold

$$\bigcap_{j=1}^{3} \left\{ x = (x_1,x_2,x_3,x_4,x_5) \in \mathbb{C}^5 : H_j(x) = c_j \right\}, \quad (c_1,c_2,c_3) \in \mathbb{C}^3, \quad c_3 \neq 0$$

is isomorphic to $\mathrm{Jac}(C)\backslash D$ where $C$ is a curve of genus 2 given by the equation.

$$w^2 = \left(z^3 - c_1z^2 + c_2z\right)^2 - 4z,$$

and $D$ consists of five copies of $C$ in the Jacobian variety $\mathrm{Jac}(C)$. The flows generated by $H_1$ et $H_2$ are linearized on $\mathrm{Jac}(C)$ and the system is algebraically completely integrable. The reader interested in the study of this system via various methods can find further information with more detail in [39] as well as in [45].

2) The problem we are going to study now is the generalized periodic Toda systems. We consider $l+1$ vectors $e_0,...,e_l$ in the Euclidean vector space $(\mathbb{R}^{l+1}, \langle .|.\rangle)$, $l \ge 1$, linearly dependent and such that they are $l$ to $l$ linearly independent (i.e, for all $j$, the vectors $e_0,...,\widehat{e_j},...,e_l$ are linearly independent). Suppose that the non-zero reals $\xi_0,...,\xi_l$ satisfying $\sum_{j=0}^{l} \xi_j e_j = 0$ are non-zero sum; that is, $\sum_{j=0}^{l} \xi_j \neq 0$. Let $\Omega = (a_{ij})_{0 \le i,j \le l}$ be the matrix where

$$a_{ij} = 2\frac{\langle e_i|e_j\rangle}{\langle e_j|e_j\rangle}, \quad 0 \le i,j \le l.$$

We consider the vector field $X_\Omega$ on $\mathbb{C}^{2(l+1)}$, defined by

$$\dot{x} = x.y \qquad \dot{y} = Ax, \qquad (x,y \in \mathbb{C}^{l+1}),$$

where $x.y = (x_0y_0,...,x_ly_l)$. It has been shown [32] that if $X_\Omega$ is an integrable vector field of an irreducibly algebraically completely integrable system, then $\Omega$ is the Cartan matrix of a twisted affine Lie algebra. Specific detailed results concerning this problem can be found on the technical paper [39] and also in [32, 46] and references therein, about link between Abelian varieties, Dynkin diagrams, singularities and Toda lattice. The periodic $l+1$ particle Toda lattices are associated to extended Dynkin diagrams. They are completely integrable and have as many polynomial invariants as points in the Dynkin diagram. The affine variety defined by the intersections of the constants of the motion is completed into an Abelian variety by the addition of a specific divisor $D$. The latter consists of $l+1$ irreducible components $D_j$ each associated with a root $\alpha_j$ of the extended Dynkin diagram $\Delta$. The intersection of $k$ components $D_{j_1},...,D_{j_k}$ satisfies the following relation : the intersection multiplicity of the intersection of $k$ components of the divisor equals $\frac{\mathrm{order}(W)}{\det(\Omega)}$ where $W$ and $A$ are the Weyl group and the Cartan matrix going with the sub-Dynkin diagram $\alpha_{j_1},...,\alpha_{j_k}$ associated with the $k$ components. The intersection of all the divisors except one is a discrete set of points whose number is explicitly determined, but on the other hand the intersection of all the divisors are empty. The set-theoretical number of points is given (in terms of the Dynkin diagram) by

$$\text{Number of}\left(\bigcap_{\beta\neq\alpha} D_\beta\right) = \frac{p_\alpha}{p_0}\left(\frac{\text{order(Weyl group of the Dynkin diagram } \Delta\backslash\alpha_0)}{\text{order(Weyl group of the Dynkin diagram } \Delta\backslash\alpha)}\right),$$

where the integers $p_\alpha$, are given by the null vector of the Cartan matrix (going with the extended Dynkin diagram $\Delta$). The singularities of the divisor are canonically associated to semi-simple Dynkin diagrams and those of each component occur only at the intersections with other components and their multiplicities at the intersection with other divisors are expressed in terms of how a corresponding root is located in the sub-Dynkin diagram determined by this root and those of the members of the above divisor intersection. We have

$$\mathrm{sing}(D_k) \subseteq D_k \cap \sum_{\substack{0 \le j \le l \\ j \ne k}} D_j, \quad k = 0, ..., l$$

and this inclusion is valid for the singular locus $\mathrm{sing}(D_k)$ of $D_k$. The multiplicity of the singularity of a particular component $D_k$, at its intersection with $m$ other divisors is entirely specified by the way the corresponding root $\alpha_k$ are located in the sub-Dynkin diagram $\alpha_k, \alpha_{j_1}, ..., \alpha_{j_m}$. (See [39], for proof of these results as well as other information).

There are many examples of Hamiltonian systems, called algebraic completely integrable in the generalized sense, for which all movable singularities of the general solution have only a finite number of branches and the complex invariant manifolds are coverings of Abelian varieties. These systems of differential equations possess solutions which are Laurent expansions containing $n$-th root terms of type $\sqrt[n]{t}$ ($t$ being complex time) and whose coefficients depend rationally on certain algebraic parameters. In other words, for these systems just replace in the above definition of the complete algebraic integrability of Hamiltoian systems, the condition ***ii)*** by by this one,

**iii)** the invariant manifolds $\bigcap_{i=1}^{n+k} \{z \in \mathbb{C}^m : H_i = c_i\}$ are related to an $l$-fold cover $\widetilde{T}^n$ of the torus $T^n$ ramified along a divisor $D$ in $T^n$ as follows :

$$\bigcap_{i=1}^{n+k} \{z \in \mathbb{C}^m : H_i = c_i\} = \widetilde{T}^n \backslash D.$$

Let $H_m$ be a family of Hamiltonians [47, 48] :

$$H_m(x,y) = \frac{1}{2}(p_x^2 + p_y^2) + \alpha_m V_m(x,y), \quad m = 1,2,...$$

where

$$V_m(x,y) = \sum_{k=0}^{[m/2]} \frac{(m-k)! 2^{m-2k}}{k!(2k-m)!} x^{2k} y^{m-2k}, \quad m = 1,2,...$$

It is easy to verify that the associated Hamiltonian systems have a second first integral :

$$F_m(x,y) = p_x(xp_y - yp_x) + \alpha_m x^2 V_{m-1}(x,y), \quad m = 1,2,...$$

and they are Liouville integrable. The study of the systems corresponding to the cases $m \ge 3$ is not obvious contrary to the cases $m = 1$ and $m = 2$ whose study is immediate. For $m = 3$, the study is reduced to that of the Hénon-Heiles system [49]:

$$\begin{aligned} \dot{y}_1 &= x_1, \\ \dot{y}_2 &= x_2, \\ \dot{x}_1 &= -\varepsilon y_1 - 2y_1 y_2, \\ \dot{x}_2 &= -y_1^2 - 16\varepsilon y_2 - 16y_2^2, \end{aligned} \tag{4.7}$$

corresponding to a generalized Hénon-Heiles Hamiltonian

$$H = \frac{1}{2}(x_1^2 + x_2^2) + \frac{\varepsilon}{2}(y_1^2 + 16y_2^2) + y_1^2 y_2 + \frac{16}{3} y_2^3,$$

where $y_1, y_2, x_1, x_2$ are canonical coordinates and momenta respectively and $\varepsilon$ a constant parameter. The associated Hamiltonian system has the following second constant of motion :

$$F = 3x_1^4 + 6\varepsilon x_1^2 y_1^2 + 12x_1^2 y_1^2 y_2 - 4x_1 x_2 y_1^3 - 4\varepsilon y_1^4 y_2 - 4y_1^4 y_2^2 + 3\varepsilon^2 y_1^4 - \frac{2}{3} y_1^6.$$

The functions $H$ and $F$ commute : $\{H,F\} = \sum_{k=1}^{2} \left( \frac{\partial H}{\partial x_k} \frac{\partial F}{\partial y_k} - \frac{\partial H}{\partial y_k} \frac{\partial F}{\partial x_k} \right) = 0$. The system (4.7) admits Laurent solutions in $\sqrt{t}$, depending on three free parameters : $\alpha$, $\beta$, $\gamma$ and they are explicitly given as follows

$$\begin{aligned} y_1 &= \frac{\alpha}{\sqrt{t}} + \beta t\sqrt{t} - \frac{\alpha}{18} t^2 \sqrt{t} + \frac{\alpha\varepsilon^2}{10} t^3 \sqrt{t} - \frac{\alpha^2 \beta}{18} t^4 \sqrt{t} + \cdots, \\ y_2 &= -\frac{3}{8t^2} - \frac{\varepsilon}{2} + \frac{\alpha^2}{12} t - \frac{2\varepsilon^2}{5} t^2 + \frac{\alpha\beta}{3} t^3 - \gamma t^4 + \cdots, \\ x_1 &= -\frac{1}{2} \frac{\alpha}{t\sqrt{t}} + \frac{3}{2} \beta \sqrt{t} - \frac{5}{36} \alpha t \sqrt{t} + \frac{7}{20} \alpha \varepsilon^2 t^2 \sqrt{t} - \frac{1}{4} \alpha^2 \beta t^3 \sqrt{t} + \cdots, \\ x_2 &= \frac{3}{4t^3} + \frac{1}{12} \alpha^2 - \frac{4}{5} \varepsilon^2 t + \alpha\beta t^2 - 4\gamma t^3 + \cdots \end{aligned} \tag{4.8}$$

As previously mentioned, the convergence of these series results from the majorant method. By replacing these series in the equations $H = a$, $F = b$, one eliminates one parameter linearly, leading to an algebraic relation between the two remaining parameters, which is nothing but the equation of an algebraic curve $D$ along which the $(y_1(t), y_2(t), x_1(t), x_2(t))$ blow up. To be more precise, we have

$$\begin{aligned} H &= \frac{1}{9}\alpha^2 - \frac{21}{4}\gamma + \frac{13}{288}\alpha^4 + \frac{4}{3}\varepsilon^3 = a, \\ F &= -144\alpha\beta^3 + \frac{294}{5}\alpha^3\beta\varepsilon^2 + \frac{8}{9}\alpha^6 - 33\gamma\alpha^4 = b, \end{aligned}$$

which implies that

$$144\alpha\beta^3 - \frac{294\varepsilon^2}{5}\alpha^3\beta + \frac{143}{504}\alpha^8 - \frac{4}{21}\alpha^6 + \frac{44}{21}\left(4\varepsilon^3 - 3a\right)\alpha^4 + b = 0.$$

Let

$$A = \bigcap_{k=1}^{2} \{(y_1, y_2, x_1, x_2) \in \mathbb{C}^4 : H(y_1, y_2, x_1, x_2) = a, F(y_1, y_2, x_1, x_2) = b\}, \tag{4.9}$$

be the smooth affine surface defined by putting the two invariants $H$ and $F$ equal to generic constants $a$ and $b$. The Laurent expansions above where $(y_1(t), y_2(t), x_1(t), x_2(t))$ blow up contain square root terms of the type $\sqrt{t}$ and admit three free parameters and in addition these solutions restricted to the surface $A$ are parameterized by the curve $D$. We will see that (4.7) is in fact a generalized algebraic completely integrable system but is part of a new system that is algebraically completely integrable. This latter is a system of five nonlinear differential equations with five unknowns having three first integrals, two of which are cubic and one is quartic. By inspection of the expansions (4.8), we look for polynomials in $(y_1, y_2, x_1, x_2)$ without fractional exponents, which suggests considering the following change of variables :

$$z_1 = y_1^2, \qquad z_2 = y_2, \qquad z_3 = x_2, \qquad z_4 = y_1 x_1, \qquad z_5 = 3x_1^2 + 2y_1^2 y_2. \tag{4.10}$$

Note that this change of variables determines a morphism on the affine variety $A$ (4.9). Using the two first integrals $H$, $F$ and differential equations (4.7), we obtain the following system :

$$\begin{aligned} \dot{z}_1 &= 2z_4, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= -z_1 - 16\varepsilon z_2 - 16z_2^2, \\ \dot{z}_4 &= -\varepsilon z_1 - \frac{8}{3}z_1 z_2 + \frac{1}{3}z_5, \\ \dot{z}_5 &= 2z_1 z_3 - 8z_2 z_4 - 6\varepsilon z_4, \end{aligned} \tag{4.11}$$

having two cubic and one quartic invariants (constants of motion),

$$\begin{aligned} G_1 &= \frac{1}{2}\varepsilon z_1 + \frac{1}{6}z_5 + 8\varepsilon z_2^2 + \frac{1}{2}z_3^2 + \frac{2}{3}z_1 z_2 + \frac{16}{3}z_2^3, \\ G_2 &= 9\varepsilon^2 z_1^2 + z_5^2 + 6\varepsilon z_1 z_5 - 2z_1^3 - 24\varepsilon z_1^2 z_2 - 12z_1 z_3 z_4 + 24 z_2 z_4^2 - 16 z_1^2 z_2^2, \\ G_3 &= z_1 z_5 - 3z_4^2 - 2z_1^2 z_2. \end{aligned}$$

This new system is a completely integrable Hamiltonian system where $G_1$ is the Hamiltonian whose structure is determined by the bracket

$$\{H, F\} = \left\langle \frac{\partial H}{\partial z}, J\frac{\partial F}{\partial z} \right\rangle,$$

the anti-symmetric matrix $J$ defines a Poisson structure for which the corresponding Poisson bracket satisfies the Jacobi identity. The two first integrals $G_1$ and $G_2$ are in involution while the latter $G_3$ is trivial (i.e., a Casimir function). For generic values of constants $c_1$, $c_2$ and $c_3$, the invariant variety

$$B = \bigcap_{k=1}^{3} \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : G_k(z_1, z_2, z_3, z_4, z_5) = c_k\}, \tag{4.12}$$

is a smooth affine surface. The differential equations (4.11) admit Laurent series expansions restricted to the surface $B$ (4.12); these solutions can be read off from (4.8) and the change of variable (4.10) and depend on four free parameters. We have shown that the change of variables (4.10) transforms the system (4.7) into an algebraic completely integrable system (4.11) of five differential equations in five unknowns and parallel to that, the affine variety $A$ (4.9) is transformed into the affine part $B$ (4.12) of an Abelian variety $\widetilde{B}$. The Hamiltonian system (4.7) is a generalized algebraic complete integrable system, the invariant surface $A$ (4.9) can be completed as a cyclic double cover $\overline{A}$ of an Abelian surface $\widetilde{B}$ and in addition, $\overline{A}$ is smooth except at the point lying over the singularity of type $A_3$ whose resolution $\widetilde{A}$ of $\overline{A}$ is a surface of general type. This explains (among other) why the asymptotic solutions to the differential equations (4.7) contain fractional powers. All this is summarized as follows [50] :

**Theorem 4.5.** *The system* (4.7) *admits Laurent solutions with fractional powers depending on three free parameters and is algebraic complete integrable in the generalized sense. In addition, this system is part of a new system of differential equations* (4.11) *in five unknowns having two cubic and one quartic invariants (constants of motion). This last system possesses Laurent expansions (but without fractional powers) depending on four free parameters and it is algebraically completely integrable.*

The case $m=4$ corresponds to Ramani-Dorizzi-Grammaticos (RDG) potential [47, 48], whose corresponding system is given by

$$\ddot{q}_1-q_1\left(q_1^2+3q_2^2\right)=0, \qquad \ddot{q}_2-q_2\left(3q_1^2+8q_2^2\right)=0. \tag{4.13}$$

These equations can be written in the form of an integrable Hamiltonian system whose Hamiltonian is given by

$$H_1=\frac{1}{2}(p_1^2+p_2^2)-\frac{3}{2}q_1^2q_2^2-\frac{1}{4}q_1^4-2q_2^4.$$

The second first integral being

$$H_2=p_1^4-6q_1^2q_2^2p_1^2+q_1^4q_2^4-q_1^4p_1^2+q_1^6q_2^2+4q_1^3q_2p_1p_2-q_1^4p_2^2+\frac{1}{4}q_1^8.$$

The first integrals $H_1$ and $H_2$ are obviously in involution. For generic $(b_1,b_2)\in\mathbb{C}^2$, the affine variety $B$ defined by

$$B=\bigcap_{k=1}^{2}\{z\in\mathbb{C}^4 : H_k(z)=b_k\}, \tag{4.14}$$

is a smooth surface. The solutions of the differential equations (4.13) in the form of Laurent's series depend on three free parameters $u,v,w$. and are written

$$\begin{aligned}
q_1 &= \frac{1}{\sqrt{t}}\left(u-\frac{1}{4}u^3t+vt^2-\frac{5}{128}u^7t^3+\frac{1}{8}u(\frac{3}{4}u^3v-\frac{7}{256}u^8+3\kappa w)t^4+\cdots\right),\\
q_2 &= \frac{1}{t}\left(\frac{1}{2}\kappa-\frac{1}{4}\kappa u^2t+\frac{1}{8}\kappa u^4t^2+\frac{1}{4}\kappa u(\frac{1}{32}u^5-3v)t^3+wt^4+\cdots\right),\\
p_1 &= \frac{1}{2t\sqrt{t}}\left(-u-\frac{1}{4}u^3t+3vt^2-\frac{25}{128}t^3u^7+\frac{7}{8}u\left(\frac{3}{4}u^3v-\frac{7}{256}u^8+3\kappa w\right)t^4+\cdots\right),\\
p_2 &= \frac{1}{t^2}\left(-\frac{1}{2}\kappa+\frac{1}{8}\kappa u^4t^2+\frac{1}{2}\kappa u\left(\frac{1}{32}u^5-3v\right)t^3+3wt^4+\cdots\right),
\end{aligned} \tag{4.15}$$

where $\kappa=\pm1$. The convergence of these series derives from the the majorant method. Note that these solutions contain square root terms of type $\sqrt{t}$, and we will see that these terms can be removed by introducing the variables $z_1,z_2,z_3,z_4,z_5$ (4.17) which restores the Painlevé property (that is, the only singularities are poles) of the system in question. Substituting (4.15) in the invariants $H_1=b_1$ and $H_2=b_2$, after eliminating the parameter $w$, we obtain the following equation (of a curve of genus 16 denoted $\Gamma$) connecting the parameters $u$ and $v$ :

$$a_1uv^3+a_2u^6v^2-a_3u^{11}v+a_4b_1u^3v-a_5u^{16}-a_6b_1u^8+b_2+a_7=0. \tag{4.16}$$

where $a_1=\frac{65}{4}$, $a_2=\frac{93}{64}$, $a_3=\frac{29487}{8192}$, $a_4=\frac{78336}{8192}$, $a_5=\frac{10299}{65536}$, $a_6=\frac{123}{256}$, $a_7=\frac{1536298731}{52}$. Consider on the variety $B$ (4.14), the following morphism

$$\psi:\mathscr{B}\longrightarrow\mathbb{C}^5, \quad (q_1,q_2,p_1,p_2)\longmapsto(z_1,z_2,z_3,z_4,z_5),$$

where

$$z_1=q_1^2, \quad z_2=q_2, \quad z_3=p_2, \quad z_4=q_1p_1, \quad z_5=p_1^2-q_1^2q_2^2. \tag{4.17}$$

These variables are easily obtained by simple inspection of the series (4.15). By using the variables (4.17) and differential equations (4.13), one obtains

$$\begin{aligned}
\dot{z}_1 &= 2z_4,\\
\dot{z}_2 &= z_3,\\
\dot{z}_3 &= z_2(3z_1+8z_2^2),\\
\dot{z}_4 &= z_1^2+4z_1z_2^2+z_5,\\
\dot{z}_5 &= 2z_1z_4+4z_2^2z_4-2z_1z_2z_3.
\end{aligned} \tag{4.18}$$

This new system on $\mathbb{C}^5$ admits the following three first integrals

$$\begin{aligned}
F_1 &= \frac{1}{2}z_5-z_1z_2^2+\frac{1}{2}z_3^2-\frac{1}{4}z_1^2-2z_2^4,\\
F_2 &= z_5^2-z_1^2z_5+4z_1z_2z_3z_4-z_1^2z_3^2+\frac{1}{4}z_1^4-4z_2^2z_4^2,\\
F_3 &= z_1z_5+z_1^2z_2^2-z_4^2.
\end{aligned} \tag{4.19}$$

The first integrals $F_1$ and $F_2$ are in involution , while $F_3$ is trivial (Casimir function). The invariant variety $A$ defined by

$$A=\bigcap_{k=1}^{3}\{(z_1,z_2,z_3,z_4,z_5)\in\mathbb{C}^5 : F_k(z_1,z_2,z_3,z_4,z_5)=c_k\}, \tag{4.20}$$

is a smooth affine surface for generic values of $(c_1,c_2,c_3)\in\mathbb{C}^3$. The system (4.18) is completely integrable and possesses Laurent series solutions which depend on four free parameters $\alpha,\beta,\gamma$ et $\theta$ :

$$\begin{aligned}
z_1 &= \frac{1}{t}\alpha-\frac{1}{2}\alpha^2+\beta t-\frac{1}{16}\alpha\left(\alpha^3+4\beta\right)t^2+\gamma t^3+\cdots,\\
z_2 &= \frac{1}{2t}\kappa-\frac{1}{4}\kappa\alpha+\frac{1}{8}\kappa\alpha^2 t-\frac{1}{32}\kappa\left(-\alpha^3+12\beta\right)t^2+\theta t^3+\cdots,\\
z_3 &= -\frac{1}{2t^2}\kappa+\frac{1}{8}\kappa\alpha^2-\frac{1}{16}\kappa\left(-\alpha^3+12\beta\right)t+3\theta t^2+\cdots,\\
z_4 &= -\frac{1}{2t^2}\alpha+\frac{1}{2}\beta-\frac{1}{16}\alpha\left(\alpha^3+4\beta\right)t+\frac{3}{2}\gamma t^2+\cdots,\\
z_5 &= \frac{1}{2t^2}\alpha^2-\frac{1}{4t}\left(\alpha^3+4\beta\right)+\frac{1}{4}\alpha\left(\alpha^3+2\beta\right)-\left(\alpha^2\beta-2\gamma+4\kappa\theta\alpha\right)t+\cdots,
\end{aligned}\tag{4.21}$$

where $\kappa=\pm1$. The convergence of these series is guaranteed by the majorant method. By replacing these series in the equations $F_1=c_1$, $F_2=c_2$, $F_3=c_3$ one eliminates two parameters $\gamma$ and $\theta$ linearly, leading to an algebraic relation between the two remaining parameters, which is the equation of an algebraic curve $C$ of genus 7,

$$64\beta^3-16\alpha^3\beta^2-4\left(\alpha^6-32\alpha^2c_1-16c_3\right)\beta+\alpha\left(32c_2-32\alpha^4c_1+\alpha^8-16\alpha^2c_3\right)=0.\tag{4.22}$$

The Laurent solutions restricted to the surface $A$ (4.20) are thus parameterized by two copies $C_{-1}$ and $C_1$ of the same Riemann surface $C$ (4.22) and we embed these curves in a hyperplane of $\mathbb{P}^{15}(\mathbb{C})$ using the sixteen functions :

$$\begin{gathered}
1,\quad z_1,\quad z_2,\quad 2z_5-z_1^2,\quad z_3+2\kappa z_2^2,\quad z_4+\kappa z_1z_2,\quad W(f_1,f_2),\quad f_1(f_1+2\kappa f_4),\quad f_2(f_1+2\kappa f_4),\quad z_4(f_3+2\kappa f_6),\\
z_5(f_3+2\kappa f_6),\quad f_5(f_1+2\kappa f_4),\quad f_1f_2(f_3+2\kappa f_6),\quad f_4f_5+W(f_1,f_4),\quad W(f_1,f_3)+2\kappa W(f_1,f_6),\quad f_3-2z_5+4f_4^2,
\end{gathered}$$

where $W(s_j,s_k)\equiv\dot{s}_js_k-s_j\dot{s}_k$ is the Wronskian. The curves $C_1$ and $C_{-1}$ have double points in common where they are tangent to each other and which are a singularity of type $A_3$ of $C_1+C_{-1}$. The Hamiltonian system (4.13) is algebraic complete integrable in the generalized sense and the invariant surface $B$ (4.14) is completed as a cyclic double cover $\overline{B}$ of the Abelian surface $\widetilde{A}$, ramified along the divisor $C_1+C_{-1}$. In addition, $\overline{B}$ is smooth except at the singularity above and the resolution $\widetilde{B}$ of $\overline{B}$ is a surface of general type. Let $G$ be a cyclic group of two elements $\{-1,1\}$ on $V_\varepsilon^j=U_\varepsilon^j\times\{\tau\in\mathbb{C}:0<|\tau|<\delta\}$, where $\tau=\sqrt{t}$ and $U_\varepsilon^j$ is an affine chart of $\Gamma_\varepsilon$ for which the Laurent expansions (4.21) are well defined. Since the action of $G$ is defined by $(-1)\circ(u,v,\tau)=(-u,-v,-\tau)$ and is without fixed points in $V_\varepsilon^j$, then the quotient $V_\varepsilon^j/G$ identifies itself with the image of the smooth map $h_\varepsilon^j:V_\varepsilon^j\longrightarrow B$ defined by the Laurent series (4.21). We have

$$(-1,1).(u,v,\tau)=(-u,-v,\tau),\qquad (1,-1).(u,v,\tau)=(u,v,-\tau),$$

which means that $G\times G$ acts separately on each coordinate and so,, identifying $V_\varepsilon^j/G^2$ with the image of $\psi\circ h_\varepsilon^j$ in $A$. Note that, except for a finite number of points, $B_\varepsilon^j=V_\varepsilon^j/G$ is smooth and the coherence of the $B_\varepsilon^j$ follows from the coherence of $V_\varepsilon^j$ and the action of $G$. After gluing various varieties $B_\varepsilon^j\backslash\{\text{some points}\}$ on $B$, we obtain a smooth complex manifold $\widehat{B}$ which is a double cover of the Abelian variety $\widetilde{A}$ ramified along $C_1+C_{-1}$, and therefore can be completed to an algebraic cyclic cover of $\widetilde{A}$. We would like to know information on the points that are missing. For this, we must examine the image of $\Gamma\times\{0\}$ in $\cup B_\varepsilon^j$. The quotient $\Gamma\times\{0\}/G$ is birationally equivalent to the curve $\Upsilon$ defined by the equation :

$$a_1y^3+a_2x^3y^2-a_3x^6y+a_4b_1x^2y-\left(a_5x^8+a_6b_1x^4-b_2-a_7\right)x=0,$$

and its genus is 7, where $a_1,\ldots,a_7$, have been defined above and $y=uv, x=u^2$. The curve $\Upsilon$ is birationally equivalent to $C$ and the only points of $\Upsilon$ fixed under $(u,v)\longmapsto(-u,-v)$ are the points at $\infty$. These correspond to the (double) ramification points of the map $\Gamma\times\{0\}\longrightarrow\Upsilon:(u,v)\longmapsto(x,y)$, and coincide with the points at $\infty$ of the curve $C$. The variety $\widehat{B}$ constructed above is birationally equivalent to the compactification $\overline{B}$ of $B$ and $\overline{B}$ is a cyclic double cover of the Abelian surface $\widetilde{A}$. The system (4.13) is algebraic complete integrable in the generalized sense and $\overline{B}$ is smooth except at the point lying over the singularity (of type $A_3$) of $C_1+C_{-1}$. The resolution $\widetilde{B}$ of singularities of $\overline{B}$, is a surface of general type with invariants : Euler characteristic of $\widetilde{B}$ =1 and geometric genus of $\widetilde{B}$=2. In conclusion, we have [51],

**Theorem 4.6.** *The Hamiltonian system* (4.13) *is algebraic complete integrable in the generalized sense and possess Laurent expansions depending on three free parameters : $u,v,w$, and containing square root terms of type $\sqrt{t}$. These Laurent solutions restricted to the affine manifold $B$* (4.14) *are parameterized by two copies $\Gamma_1$ and $\Gamma_{-1}$ of an algebraic curve $\Gamma$* (4.16) *of genus* 16*. This system is part of a new algebraically completely integrable system* (4.18) *in five unknowns and having three quartics invariants* (4.19)*. The complex invariant manifold $A$* (4.20) *defined by putting these polynomial invariants equal to generic constants is the affine part of an Abelian surface $\widetilde{A}$ with $\widetilde{A}\backslash A=C_1+C_{-1}$, where the divisor $C_1+C_{-1}$ is very ample and consists of two components $C_1$ and $C_{-1}$ of a genus* 7 *curve $C$* (4.22)*. In addition, the invariant manifold $B$ is completed into a cyclic double cover $\overline{B}$ of the Abelian surface $\widetilde{A}$, ramified along the divisor $C_1+C_{-1}$ in such a way that the vector fields extend holomophically alond this divisor and remain independent there. Moreover, $\overline{B}$ is smooth except at the point lying over the singularity (of type $A_3$) of $C_1+C_{-1}$ and the resolution $\widetilde{B}$ of $\overline{B}$ is a surface of general type with invariants : Euler characteristic of $\widetilde{B}$=1 and geometric genus of $\widetilde{B}$=2.*

## 5. Conclusion

At the end of this paper, it is worth to mention some similar problems as well as recent results. Abelian varieties, very heavily studied by algebraic geometers, enjoy certain algebraic properties which can then be translated into differential equations and their Laurent solutions. Among the results presented in this paper, there is an explicit calculation of invariants for Hamiltonian systems which cut out an open set in

an Abelian variety and various algebraic curves related to these systems are given explicitly. The integrable systems presented here are interesting problems, particular to experts of Abelian varieties who may want to see explicit examples of correspondence for varieties defined by different algebraic curves. The methods used are primarily analytical but heavily inspired by algebraic geometrical methods. The concept of algebraic complete integrability is quite effective in small dimensions and has the advantage to lead to global results, unlike the existing criteria for real analytic integrability, which, at this stage are perturbation results. In fact, the overwhelming majority of dynamical systems, Hamiltonian or not, are non-integrable and possess regimes of chaotic behavior in phase space. The methods used are primarily analytical but heavily inspired by algebraic geometrical methods. Abelian varieties and cyclic coverings of Abelian varieties, very heavily studied by algebraic geometers, enjoy certain algebraic properties which can be translated into differential equations and their Laurent solutions.
In recent years, other important results have been obtained following studies on the KP and KdV hierarchies. The use of tau functions related to infinite dimensional Grassmannians, Fay identities, vertex operators and the Hirota's bilinear formalism led to obtaining remarkable properties concerning these algebras of infinite order differential operators as for example the existence of an infinite family of first integrals functionally independent and in involution. The elaboration of powerful methods and the discovery of their common algebraic structures led to important developments concerning the study of nonlinear problems. The functions $\tau(t)$ are specific functions of time, constructed from sections of a determinant bundle on an infinite-dimensional Grassmannian manifold. These functions generalize the Riemann theta functions and they are solutions of the KP hierarchy, i.e, solutions of an infinite series of nonlinear partial differential equations connecting infinity of functions of infinity variables. The functions $\tau(t)$ can be Schur polynomials, falling within Fredholm's group representation theory or determinants. Recently, a new type of tau function has appeared, within the framework of quantum gauge theory with gauge group $SU(N)$ when $N$ is large. This led to the so-called matrix models (quantum gravity) for counting triangulations on certain surfaces (topology). The underlying models have remained relatively intractable except in two space-time dimensions; although being physically toy models, their structure is still very rich. The first tau function was introduced by Sato, Miwa, and Jimbo in relation to the theory of isomonodromic deformations. It has been defined as a correlation function of certain quantum fields associated with the poles of a Fuchsian system on the Riemann sphere. These functions give information on the topology of moduli spaces of Riemann surfaces and are closely related to the theory of representations of Virasoro algebras and W-algebras. The $\tau(t)$ functions play an important role in a large number of branches of mathematics and theoretical physics, such as integrable systems, string theories, quantum-gauge theories, isomonodromic deformations, matrix models (quantum gravity), the associated matrix integrals which have power series expansions (perturbative series) and whose terms count the triangulations on surfaces (Feynman graphs), the module problems and in many other domains. Many problems related to algebraic geometry, combinatorics, probabilities and quantum gauge theory,..., have been solved explicitly by methods inspired by techniques from the study of integrable systems. In particular, the study of random matrices, a domain that establishes links with several problems, for example with combinatorics, probabilities, number theory, models of growth and random tailings and questions of communication technology. The functions $\tau(t)$ are the source of inspiration for many mathematicians and physicists in search of new algebraic structures appearing in mathematics and physics. The vertex operators give a good device to the investigation of the matrix models and the spectrum of the stochastic matrices. An account of these results will appear elsewhere.

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# Effect of Inflation on Stochastic Optimal Investment Strategies for DC Pension under the Affine Interest Rate Model

**Kevin N. C. Njoku**[1] **and Bright O. Osu** [2*]

[1]*Department of Mathematics, Imo State University, P. M. B 2000, Owerri, Imo State, Nigeria*
[2]*Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria.*
[*]*Corresponding author E-mail: Osu.bright@mouau.edu.ng*

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**Abstract**

In this paper, we seek to investigate the effect of inflation on the optimal investment strategies for DC Pension. Our model permits the plan member to make a defined contribution, as provided in the Nigerian Pension Reform Act of 2004. The plan member is free to invest in risk-free asset and two risky assets. A stochastic differential equation of the pension wealth that takes into account certainly agreed proportions of the plan member's salary, paid as a contribution towards the pension fund, is presented. The Hamilton-Jacobi-Bellman (H-J-B) equation, Legendre transformation, and dual theory are used to obtain the explicit solution of the optimal investment strategies for CRRA utility function. Our investigation reveals that the inflation has significant negative effect on optimal investment strategy, particularly, the CCRA is not constant with the investment strategy since the inflation parameters and coefficient of CRRA utility function have insignificant input on the investment strategy.

## 1. Introduction

There are two major designs of pension plan, namely, the defined benefit (DB) pension, and the defined contribution (DC) pension plan. As the names implies, in that of the DB, the benefits of the plan member are defined, and the sponsor bears the financial risk. Whereas, in the DC pension plan, the contributions are defined, the retirement benefits depends on the contributions and the investment returns, and the contributors (the plan members) bears the financial risk. Recently, the DC pension has taken dominance over the DB pension plan in the pension scheme, since DC pension plan is fully funded, which makes it easier for the plan managers (Pension Fund Administrators (PFAs') and the Pension Fund Custodians (PFCs') to invest equitably in the market, and also makes it easier for the plan members to receive their retirement benefit as and when due.

Investment strategies of the contributions, which in turn is a strong determinant of the investment returns vis-a-vis the benefits of the contributors at retirement must be given optimum attention. Recent publications in economic Journals and other reputable Mathematics and Science Journals have brought to light, a variety of methods of optimizing investment strategies and returns. For instance, some researchers have made various contributions in this direction, particularly, in DC Pension Plan. [1] did work on, "stochastic life styling: optimal dynamic asset allocation for defined contribution pension plans. In their work, various properties and characteristics of the optimal asset allocation strategy, both with and without the presence of non-hedge able salary risk were discussed. The significance of alternative optimal strategy by pension providers was established.

In order to deal with optimal investment strategy, the need for maximization of the expected utility of the terminal wealth became necessary. Example, the Constant Relative Risk Aversion (CRRA) utility function, and (or) the Constant Absolute Risk Aversion (CARA) utility function were used to maximize the terminal wealth. [1]-[4], and [5] used CRRA to maximize terminal wealth. However, [6] used the CRRA and the CARA to maximize terminal wealth.

[7] applied the well-known H-J-B equation, Legend transform, and dual theory to obtain the explicit solutions of CRRA and CARA utility function, for the maximization of the terminal wealth. In 2012, Nan-wei Han et al took a different direction. The investigated optimal asset allocation for DC pension plans under inflation. In their work, the retired individuals receive an annuity that is indexed by inflation and a

***Email addresses*****: kevinnjoku60@yahoo.com, 0000-0002-4430-4159 (K. N. C. Njoku) Osu.bright@mouau.edu.ng, 0000-0003-2463-430X (B. O. Osu)**

downside protection on the amount of this annuity is considered. More so, in 2015, [1] considered an Inflationary market. In their work, the plan member made extra contribution to amortize the pension fund. The CRRA utility function was used to maximize the terminal wealth. This triggered our research. Ours is to investigate and view the extent of damage the inflation may have caused to enable us to introduce, not just an amortization fund, but an optimum amortization fund that would sufficiently dampen the effect of inflation. The approach used here is similar to that of [5]. The models we used is that of [8], though, we considered inflation of globally competing goods, and some real life assumptions are made to buttress this fact.

## 2. Preliminaries

We start with a complete and frictionless financial market that is continuously open over the fixed time interval $[0,T]$, for $T>0$, representing the retirement time of any plan member.
We assume that the market is composed of the risk-free asset (cash), the inflation-linked bond, and risky asset (the stock price subject to inflation). Let $(\Omega,\ F,P)$ be a complete probability space, where $\Omega$ is a real space and $P$ is a probability measure, $\{W_s(t),\ W_I(t)\}$ are two standard orthogonal Brownian motions, $\{F_I(t),\ F_s(t)\}$ are right continuous filtrations whose information are generated by the two standard Brownian motions $\{W_s(t),\ W_I(t)\}$, whose sources of uncertainties are respectively to the inflation rate and the stock market. We assume also that at the early stage of the inflation, before government intervention policy, $\{W_R(t),\ W_I(t)\}$, $\{W_s(t),\ W_R(t)\}$ are two standard orthogonal Brownian motions, respectively.
Let $C(t)$ denote the price of the risk free asset at time $t$ and it is modeled as follows

$$\frac{dC(t)}{C(t)}=r_R(t)dt, C(0)=1$$

$r(t)$ is the real interest rate process and is given by the stochastic differential equation (SDE)

$$dr_R(t)\ =\ (a\ -\ br_R\ (t))\ dt\ -\ \sigma_R dW_R(t),$$

$$\sigma_R=\ \sqrt{k_1 r_R(t)\ +\ k_2}\ ,\ \ t\geq 0,$$

where $r_R$ is a real interest rate, $r_R(0)$, $k_1$, and $k_2$ are positive real numbers. If $k_1$ (resp., $k_2$) is equal to zero, we have a special case, as in [9], [10] dynamics. So under these dynamics, the term structure of the real interest rates is affine, which has been studied by [7], [4], [11] and [2].
Let $S(t)$ denote the price of the risky asset subject to inflation and its dynamics is given based on a continuous time stochastic process at $t\geq 0$ and the dynamics of the price process is described as follows

$$\frac{dS(t)}{S(t)}\ =(\ r_R(t)+\lambda_1\sigma_s{}^s+\lambda_2\sigma^I{}_s\theta_I)dt+\ \sigma^s{}_s dW_s+\sigma^I{}_s dW_I,\ \ S(0)=\ 1 \tag{2.1}$$

premium associated with the positive volatility constants $\sigma_s{}^s$ *and* $\sigma^I{}_s$ , respectively, see [4]. $\theta_I$ represents the inflation price m with $\lambda_1$ *and* $\lambda_2$ represents the instantaneous market risk.
An inflation-linked bond with maturity $T$, whose price at time $t$ is denoted by $B(t,I(t))$ , $t\geq 0$ , and its evolution is given by the SDE below (see [8])

$$\frac{dB(t,\ I(t)),\ )}{B(t,\ I(t)),}\ =(r_R(t)+\ \sigma_I\theta_I)\,dt+\sigma_I dW_I(t),\ \ B(T,\ I(T))=\ 1 \tag{2.2}$$

Let us denote the stochastic wage of the plan member, at time $t$, by $P(t)$ which is described by

$$\frac{dP(t)}{P(t)}\ \ =\ \mu_P(t)\ dt+\ \sigma^s{}_p dW_s(t)+\sigma^I{}_p dW_I(t),$$

where, $\mu_p(t)$ denotes the expected instantaneous rate of the wage, while $\sigma^s{}_p$ *and* $\sigma^I{}_p$ denote the two volatility scale factors of stock and inflation, respectively. Since the wage is stochastic, we let the instantaneous mean of the wage to be $\mu_P(t,\ r\ (t))$=$r(t)+u_p$, where $m_p$ is a real constant.

## 3. Methodology

### 3.1. Hamilton-Jacobi-Bellman (HJB) equation

Suppose, we represent $u=(u_B,u_S)$ as the strategy and we define the utility attained by the contributor from a given state $y$ at time $t$ as

$$G_u(t,r_R,y)=E_u\ V(X(T))\mid r_R(t)=r_R,Y(t)=y],\tag{3.1}$$

where $t$ is the time, $r_R$ is the real interest rate and $y$ is the wealth. Our interest here is to find the optimal value function

$$G(t,r_R,y)=\sup_u G_u(t,r_R,y)$$

and the optimal strategy $u^*=(u_B{}^*,u_S{}^*)$ such that

$$G_{u^*}(t,r_R,y)=G(t,r_R,y)\,.$$

### 3.2. Legendre transformation

The nonlinear partial differential equation obtained in (3.1) above is transformed into a linear partial differential equation, using the Legendre transform method and Dual theory.

**Theorem 3.1.** *[12] Let $f:\ R^n \to R$ be a convex function for $z>0$, define the Legendre transform*

$$L(z) = \max_{y}\{f(y) - zy\}, \tag{3.2}$$

*where $L(z)$ is the Legendre dual of $f(y)$. Suppose, $f(y)$ is strictly convex, then the supremum (3.2) would be attained at one point, denoted by $y_0$ (i.e, the sup. exist). We write*

$$L(z) = sup_y\{f(y) - zy\} = f(y_0) - zy_0$$

By Theorem 3.1 and the assumption of convexity of the value function $G(t,r_R,y)$, we define the Legendre transform

$$\hat{G}(t,r_R,z) = \sup_{y>0} G(t,r_R,y) - zy \mid 0<y<\infty\}\ \ 0<t<T. \tag{3.3}$$

Where $z>0$ denotes the dual variable to $y$ and $\hat{G}$ is the dual function of $G$.
The value of $y$ where this optimum is attained is denoted by $h(t,r,z)$, so that

$$h(t,r_R,z) = \inf_{y>0} y \,\big|\, G(t,r_R,y) \geq zy + \hat{G}(t,r_R,z)\big\}\ \ 0<t<T. \tag{3.4}$$

from (3.4), we see that the function $h$ and $\widehat{G}$ are closely related, hence we write either of them as dual of $G$. To see this relationship,

$$\hat{G}(t,r_R,z) = G(t,r_R,h) - zh.$$

where

$$h(t,r_R,z) = y,\ G_y = z,\ \ \textit{and relating } \hat{G} \textit{ to } h \textit{ by } h = -\hat{G}_{z.}$$

Replicating the idea in (3.3) and (3.4), above, we define the Legendre transform of the utility function $U(y)$, at terminal time, thus

$$\hat{U}(z) = sup_{x>0}\, U(x) - zx \mid 0<x<\infty\},$$

where $z>0$ denotes the dual variable to $y$, and $\hat{U}$ is the dual of $U$.
Similarly, the value of $y$ where this optimum is attained is denoted by $G(z)$, such that

$$G(z) = sup_{x>0}\left\{w \mid U(x) \geq zx + \hat{U}(z)\right\}.$$

Consequently, we have

$$G(z) = (U')^{-1}(z)$$

where $G$ is the inverse of the marginal utility $U$.
Since $h(T,r_R,y) = U(y)$, then at the terminal time, $T$, we can define

$$h(T,r_R,z) = \inf_{y>0}\left\{y \middle| U(y) \geq zy + \hat{h}(T,r_R,z)\right\}$$

and

$$\hat{h}(T,r_R,z) = \sup_{y>0}\{U(y) - zy\}$$

so that

$$h(T,r_R,z) = (U')^{-1}(z). \tag{3.5}$$

## 4. Model formulation

Here, the contributions are continuously paid into the pension fund at the rate of $KP(t)$ where $K$ is the mandatory rate of contribution. Let $W(t)$ denote the wealth of pension fund at time $t \in [0,T]$. $u_B(t)$ and $u_S(t)$ represent the proportion of the pension fund invested in the bond and the stock respectively. This implies that the proportion of the pension fund invested in the risk-free asset $u_C(t) = 1 - u_B(t) - u_S(t)$. The dynamics of the pension wealth is given by

$$dW(t) = u_C W(t)\frac{dC(t)}{C(t)} + u_B W(t)\frac{dB(t,I(t))}{B(t,I(t))} + u_S W(t)\frac{dS(t)}{S(t)} + KP(t)\,dt \tag{4.1}$$

Substituting f(1), (2.1) and (2.2) in (4.1) we have

$$dW(t) = W(t)\left[r_R(t) + \sigma_I\theta_I u_B + \left(\lambda_1\sigma^s{}_s + \lambda_2\sigma^I{}_s\theta_I\right)u_S\right]dt + KP(t)\,dt + W(t)\left(\sigma_I u_B + \sigma^I{}_s u_S\right)dW_I(t) + W(t)\sigma^s{}_s u_S dW_S(t) \tag{4.2}$$

Let the relative wealth $(t)t$be defined as follows

$$Y(t) = \frac{W(t)}{P(t)} \tag{4.3}$$

Applying product rule and Ito's formula to (4.3) and making use of (2.3) and (4.2) we arrive at the following equation

$$\begin{aligned} dY(t) &= Y(t)\left\{r(t) - \mu_p + (\sigma^s{}_p)^2 + (\sigma^I{}_p)^2 + \left[\left(\lambda_1\sigma^s{}_s + \lambda_2\sigma^I{}_s\theta_I\right) - \frac{1}{2}\sigma^I{}_s\sigma^I{}_p - \frac{1}{2}\sigma^s{}_s\sigma^s{}_p\right]u_S + \left(\sigma_I\theta_I - \frac{1}{2}\sigma_I\sigma^I{}_p\right)u_B\right\}dt \\ &+ Kdt + Y(t)\left(\sigma_I u_B + \sigma^I{}_s u_s - \sigma^I{}_p\right)dW_I + Y(t)\left(\sigma^s{}_s u_s - \sigma^s{}_p\right)dW_s \qquad Y(0) = W(0)/P(0) \end{aligned}$$

Simplifying,

$$dY(t) = X(c_1 + c_2 u_s + c_3 u_B)dt + Kdt + Y(t)\left(\sigma_I u_B + \sigma^I{}_s u_S - \sigma^I{}_p\right)dW_I(t) + Y(t)\left(\sigma^s{}_s u_S - \sigma^s{}_p\right)dW_S(t), \tag{4.4}$$

where

$$c_1 = r_R(t) - \mu_p + (\sigma^s{}_p)^2 + (\sigma^I{}_p)^2$$

$$c_2 = \left(\lambda_1\sigma^s{}_s + \lambda_2\sigma^I{}_s\theta_I\right) - \frac{1}{2}\sigma^I{}_s\sigma^I{}_p - \frac{1}{2}\sigma^s{}_s\sigma^s{}_p$$

$$c_3 = \sigma_I\theta_I - \frac{1}{2}\sigma_I\sigma^I{}_p$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with (4.4) is

$$G_t + (a - br_R)G_r + \frac{1}{2}\sigma_{r_R}{}^2 G_{r_R r_R} + \sup_u\left\{\begin{array}{l} \{y(c_1 + u_s c_2 + u_B c_3)G_y + KG_y \\ +\frac{1}{2}y^2[((\sigma_I u_B + \sigma^I{}_s u_s - \sigma^I{}_p))^2 + (\sigma^s{}_s u_s - \sigma^s{}_p)^2]G_{yy} = 0 \end{array}\right\} \tag{4.5}$$

where $G_t$, $G_y$, $G_{r_R r_R}$ , $G_y$ and $G_{yy}$ are partial derivatives of first and second orders with respect to time, real interest rate, and relative wealth. Differentiating (4.5) with respect to $u_B$ and $u_S$, we obtain the first-order maximizing conditions for the optimal strategies $u_B{}^*$ *and* $u_S{}^*$ , thus

$$c_3 G_y + y\sigma_I\left(\sigma_I u_B{}^* + \sigma^I{}_s u_S{}^* - \sigma^I{}_p\right)G_{yy} = 0 \tag{4.6}$$

$$c_2 G_y + x\sigma^I{}_s\left(\sigma_I u_B{}^* + \sigma^I{}_s u_S{}^* - \sigma^I{}_p\right)G_{yy} + y\sigma^s{}_s\left(\sigma^s{}_s u_S{}^* - \sigma^s{}_p\right)G_{yy} = 0 \tag{4.7}$$

Solving (4.6) and (4.7) simultaneously we have

$$u_S{}^* = \frac{\sigma^I{}_s c_3 - c_2\sigma_I}{(\sigma^s{}_s)^2\sigma_I y}\frac{G_y}{G_{yy}} + \left(\frac{\sigma^s{}_p\sigma^I{}_s + \sigma^s{}_p\sigma^s{}_s - \sigma^I{}_p\sigma^I{}_s}{(\sigma^s{}_s)^2}\right) \tag{4.8}$$

$$u_B{}^* = \frac{\sigma^I{}_p}{\sigma_I} - \frac{\sigma^I{}_s(\sigma^s{}_p\sigma^I{}_s + \sigma^s{}_p\sigma^s{}_s - \sigma^I{}_p\sigma^I{}_s)}{(\sigma^s{}_s)^2\sigma_I} - \frac{\sigma^I{}_s(\sigma^I{}_s c_3 - c_2\sigma_I)}{(\sigma^s{}_s)^2 y}\frac{G_y}{G_{yy}} - \frac{c_3}{\sigma_I{}^2 y}\frac{G_y}{G_{yy}} \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.5), and assuming independent and identically distributed volatility scale of salary for stock and inflation (i.e., $\sigma^I{}_p = \sigma^s{}_p$), we have

$$\begin{aligned} &G_t + (a - br_R)G_{r_R} + \tfrac{1}{2}\sigma_{r_R}{}^2 G_{r_R r_R} + \left(K + y(\tfrac{1}{2}\rho_5 + \rho_1)\right)G_y \\ &+ \left(2\theta_I{}^2 + \tfrac{1}{2}(\sigma^I{}_p)^2 - \theta_I\sigma^I{}_p + \rho_2 + \rho_4\right)\frac{G_y{}^2}{G_{yy}} + \tfrac{1}{2}y^2\rho_3 = 0, \end{aligned} \tag{4.10}$$

$$\begin{aligned}\rho_1 = &\tfrac{3}{2}({\sigma^I}_p)^2+\lambda_1{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I+\lambda_2({\sigma^I}_s)^2{\theta_I}^2\sigma_I-\tfrac{1}{2}\lambda_1{\sigma^s}_s{\sigma^I}_s\sigma_I{\sigma^I}_p-\tfrac{1}{2}\lambda_2({\sigma^I}_s)^2\theta_I\sigma_I{\sigma^I}_p\\ &-\tfrac{1}{2}({\sigma^I}_s)^2\theta_I\sigma_I{\sigma^I}_p-\tfrac{1}{2}{\sigma^s}_s{\sigma^I}_p{\sigma^I}_s\sigma_I\theta_I+\tfrac{1}{4}({\sigma^I}_s)^2({\sigma^I}_p)^2\sigma_I+\tfrac{1}{4}{\sigma^s}_s({\sigma^I}_s)^2{\sigma^I}_s\sigma_I\\ &+\lambda_1{\sigma^s}_p+\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}+\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}-\tfrac{\theta_I\lambda_2({\sigma^I}_s)^2({\sigma^I}_p)^2}{({\sigma^s}_s)^2}-\tfrac{({\sigma^I}_p)^2{\sigma^I}_s}{2{\sigma^s}_s}-u_p\end{aligned}$$

$$\begin{aligned}\rho_2 = &\tfrac{\lambda_1{\sigma^I}_s{\sigma^I}_p}{2{\sigma^s}_s}-\tfrac{2\lambda_1\lambda_2{\sigma^I}_s\theta_I}{{\sigma^s}_s}-\tfrac{{\lambda_2}^2({\sigma^I}_s)^2{\theta_I}^2}{({\sigma^s}_s)^2}+\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}+\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}-\tfrac{3({\sigma^I}_s)^2({\sigma^I}_p)^2}{4{{\sigma^s}_s}^2}\\ &-\tfrac{3}{2}\tfrac{\theta_I({\sigma^I}_s)^2{\sigma^I}_p}{({\sigma^s}_s)^2}-\tfrac{\theta_1{\sigma^I}_p{\sigma^I}_s}{2{\sigma^s}_s}-\tfrac{{\theta_I}^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}+\tfrac{\theta_I{\sigma^I}_s\lambda_1}{{\sigma^s}_s}+\tfrac{({\sigma^I}_p)^2({\sigma^I}_s)^2}{2({\sigma^s}_s)^2}+\tfrac{({\sigma^s}_p)^2{\sigma^I}_s}{4{\sigma^s}_s}\end{aligned}$$

$$\begin{aligned}\rho_3 = &({\sigma^I}_s)^4\sigma_I{\theta_I}^2-({\sigma^I}_s)^4{\sigma_I}^2\theta_I{\sigma^I}_p+\tfrac{({\sigma^I}_s)^4{\sigma_I}^2({\sigma^I}_p)^2}{4}+({\sigma^s}_s)^2({\sigma^I}_s)^2{\sigma_I}^2{\theta_I}^2-({\sigma^s}_s)^2{\sigma^I}_s{\sigma_I}^2\theta_I{\sigma^I}_p\\ &+2{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p-2{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p+\tfrac{({\sigma^s}_s)^2({\sigma^I}_s)^2{\sigma_I}^2({\sigma^I}_p)^2}{4}-\tfrac{2({\sigma^I}_p)^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}\end{aligned}$$

$$\begin{aligned}\rho_4 = &\tfrac{({\sigma^I}_s)^4{\theta_I}^2}{({\sigma^s}_s)^4}+\tfrac{({\sigma^I}_s)^3\theta_I{\sigma^I}_p}{({\sigma^s}_s)^4}+\tfrac{({\sigma^I}_s)^2({\sigma^I}_p)^2}{4({\sigma^s}_s)^4}-\tfrac{2{\theta_I}^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_s{\sigma^I}_p\theta_I}{({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_p({\sigma^I}_s)^2\theta_I}{({\sigma^s}_s)^2}-\tfrac{{\sigma^I}_s({\sigma^I}_p)^2}{2({\sigma^s}_s)^2}\\ &+\tfrac{2\lambda_1\lambda_2{\sigma^I}_s\theta_I}{{\sigma^s}_s}-\tfrac{\lambda_1{\sigma^I}_s{\sigma^I}_p}{{\sigma^s}_s}+\tfrac{{\lambda_1}^2({\sigma^I}_s)^2{\theta_I}^2}{({\sigma^s}_s)^2}-\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}-\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}+\tfrac{({\sigma^I}_s)^2({\sigma^I}_p)^2}{4({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_s({\sigma^I}_p)^2}{2{\sigma^s}_s}\end{aligned}$$

$$\begin{aligned}\rho_5 &= \frac{({\sigma^I}_s)^3\sigma_I\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}-\frac{2({\sigma^I}_s)^4\sigma_I{\theta_I}^2}{({\sigma^s}_s)^2}+\frac{({\sigma^I}_s)^4\sigma_I{\sigma^I}_p\theta_I}{({\sigma^s}_s)^2}-\frac{({\sigma^I}_s)^3\sigma_I({\sigma^I}_p)^2}{2({\sigma^s}_s)^2}+2{\theta_I}^2\sigma_I({\sigma^I}_s)^2-\theta_I({\sigma^I}_s)^2\sigma_I{\sigma^I}_p\\ &+\frac{({\sigma^I}_p)^2({\sigma^I}_s)^2\sigma_I}{2}-2{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I\lambda_1-2({\sigma^I}_s)^2\sigma_I{\theta_I}^2\lambda_2+{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p+{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p\lambda_1+({\sigma^I}_s)^2\sigma_I{\sigma^I}_p\lambda_2\theta_I\\ &-\frac{({\sigma^I}_s)^2\sigma_I({\sigma^I}_p)^2}{2}-\frac{{\sigma^s}_s{\sigma^I}_s\sigma_I({\sigma^I}_p)^2}{2}-2{\sigma^s}_s\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p+\frac{2\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}\end{aligned}$$

with $G(T, r_R y,) = U(y)$.
Applying Legendre transform to (4.10), we have

$$\hat{G}_t+(a-br_R(t))\,\hat{G}_{r_R}+\frac{1}{2}{\sigma_{r_R}}^2\hat{G}_{r_Rr_R}+[K+y(\frac{1}{2}\rho_5+\rho_1)]Z-\left(2{\theta_I}^2\frac{1}{2}\left({\sigma^I}_p\right)^2-\theta_I{\sigma^I}_p+\rho_2+\rho_4\right)Z^2\hat{G}_{zz}-\frac{1}{2}y^2\rho_3\frac{1}{\hat{G}_{zz}}=0,$$

$$\begin{aligned}\rho_1 = &\tfrac{3}{2}({\sigma^I}_p)^2+\lambda_1{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I+\lambda_2({\sigma^I}_s)^2{\theta_I}^2\sigma_I-\tfrac{1}{2}\lambda_1{\sigma^s}_s{\sigma^I}_s\sigma_I{\sigma^I}_p-\tfrac{1}{2}\lambda_2({\sigma^I}_s)^2\theta_I\sigma_I{\sigma^I}_p\\ &-\tfrac{1}{2}({\sigma^I}_s)^2\theta_I\sigma_I{\sigma^I}_p-\tfrac{1}{2}{\sigma^s}_s{\sigma^I}_p{\sigma^I}_s\sigma_I\theta_I+\tfrac{1}{4}({\sigma^I}_s)^2({\sigma^I}_p)^2\sigma_I+\tfrac{1}{4}{\sigma^s}_s({\sigma^I}_s)^2{\sigma^I}_s\sigma_I+\lambda_1{\sigma^s}_p\\ &+\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}+\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}-\tfrac{\theta_I\lambda_2({\sigma^I}_s)^2({\sigma^I}_p)^2}{({\sigma^s}_s)^2}-\tfrac{({\sigma^I}_p)^2{\sigma^I}_s}{2{\sigma^s}_s}-u_p\end{aligned}$$

$$\begin{aligned}\rho_2 = &\tfrac{\lambda_1{\sigma^I}_s{\sigma^I}_p}{2{\sigma^s}_s}-\tfrac{2\lambda_1\lambda_2{\sigma^I}_s\theta_I}{{\sigma^s}_s}-\tfrac{{\lambda_2}^2({\sigma^I}_s)^2{\theta_I}^2}{({\sigma^s}_s)^2}+\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}+\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}-\tfrac{3({\sigma^I}_s)^2({\sigma^I}_p)^2}{4{{\sigma^s}_s}^2}\\ &-\tfrac{3}{2}\tfrac{\theta_I({\sigma^I}_s)^2{\sigma^I}_p}{({\sigma^s}_s)^2}-\tfrac{\theta_1{\sigma^I}_p{\sigma^I}_s}{2{\sigma^s}_s}-\tfrac{{e_I}^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}+\tfrac{\theta_I{\sigma^I}_s\lambda_1}{{\sigma^s}_s}+\tfrac{({\sigma^I}_p)^2({\sigma^I}_s)^2}{2({\sigma^s}_s)^2}+\tfrac{({\sigma^s}_p)^2{\sigma^I}_s}{4{\sigma^s}_s}\end{aligned}$$

$$\begin{aligned}\rho_3 = &({\sigma^I}_s)^4\sigma_I{\theta_I}^2-({\sigma^I}_s)^4{\sigma_I}^2\theta_I{\sigma^I}_p+\tfrac{({\sigma^I}_s)^4{\sigma_I}^2({\sigma^I}_p)^2}{4}+({\sigma^s}_s)^2({\sigma^I}_s)^2{\sigma_I}^2{\theta_I}^2-({\sigma^s}_s)^2{\sigma^I}_s{\sigma_I}^2\theta_I{\sigma^I}_p\\ &+2{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p-2{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p+\tfrac{({\sigma^s}_s)^2({\sigma^I}_s)^2{\sigma_I}^2({\sigma^I}_p)^2}{4}-\tfrac{2({\sigma^I}_p)^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}\end{aligned}$$

$$\begin{aligned}\rho_4 = &\tfrac{({\sigma^I}_s)^4{\theta_I}^2}{({\sigma^s}_s)^4}+\tfrac{({\sigma^I}_s)^3\theta_I{\sigma^I}_p}{({\sigma^s}_s)^4}+\tfrac{({\sigma^I}_s)^2({\sigma^I}_p)^2}{4({\sigma^s}_s)^4}-\tfrac{2{\theta_I}^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_s{\sigma^I}_p\theta_I}{({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_p({\sigma^I}_s)^2\theta_I}{({\sigma^s}_s)^2}-\tfrac{{\sigma^I}_s({\sigma^I}_p)^2}{2({\sigma^s}_s)^2}\\ &+\tfrac{2\lambda_1\lambda_2{\sigma^I}_s\theta_I}{{\sigma^s}_s}-\tfrac{\lambda_1{\sigma^I}_s{\sigma^I}_p}{{\sigma^s}_s}+\tfrac{{\lambda_1}^2({\sigma^I}_s)^2{\theta_I}^2}{({\sigma^s}_s)^2}-\tfrac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}-\tfrac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}+\tfrac{({\sigma^I}_s)^2({\sigma^I}_p)^2}{4({\sigma^s}_s)^2}+\tfrac{{\sigma^I}_s({\sigma^I}_p)^2}{2{\sigma^s}_s}\end{aligned}$$

$$\begin{aligned}\rho_5 &= \frac{({\sigma^I}_s)^3\sigma_I\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}-\frac{2({\sigma^I}_s)^4\sigma_I{\theta_I}^2}{({\sigma^s}_s)^2}+\frac{({\sigma^I}_s)^4\sigma_I{\sigma^I}_p\theta_I}{({\sigma^s}_s)^2}-\frac{({\sigma^I}_s)^3\sigma_I({\sigma^I}_p)^2}{2({\sigma^s}_s)^2}+2{\theta_I}^2\sigma_I({\sigma^I}_s)^2-\theta_I({\sigma^I}_s)^2\sigma_I{\sigma^I}_p\\ &+\frac{({\sigma^I}_p)^2({\sigma^I}_s)^2\sigma_I}{2}-2{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I\lambda_1-2({\sigma^I}_s)^2\sigma_I{\theta_I}^2\lambda_2+{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p+{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p e_1+({\sigma^I}_s)^2\sigma_I{\sigma^I}_p\lambda_2\theta_I\\ &-\frac{({\sigma^I}_s)^2\sigma_I({\sigma^I}_p)^2}{2}-\frac{{\sigma^s}_s{\sigma^I}_s\sigma_I({\sigma^I}_p)^2}{2}-2{\sigma^s}_s\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p+\frac{2\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}\end{aligned}$$

Differentiating equation (4.9) for $\hat{G}$ with respect to $z$ we obtain a linear PDE in terms of $h$ and its derivatives and using $y = h = -\hat{G}_z$, we have

$$
\begin{aligned}
&h_t+(a-br)h_{r_R}+\frac{1}{2}{\sigma_{r_R}}^2 h_{r_R r_R}-Zh_2\left(\frac{1}{2}\rho_5+\rho_1\right)-\left[k+h\left(\frac{1}{2}\rho_5+\rho_1\right)\right]- \\
&\left(2{\theta_I}^2\frac{1}{2}\left({\sigma^I}_p\right)^2-\theta_I{\sigma^I}_p+\rho_2+\rho_4\right)\left(h_z 2z+z^2h_{zz}\right)+\frac{1}{2}\rho_3\left(2h+\frac{h^{\,2}hh_{zz}}{{h_z}^2}\right)=0
\end{aligned}
\tag{4.11}
$$

where

$$
\begin{aligned}
\rho_1 =\ & \tfrac{3}{2}({\sigma^I}_p)^2+\lambda_1{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I+\lambda_2({\sigma^I}_s)^2{\theta_I}^2\sigma_I-\tfrac{1}{2}\lambda_1{\sigma^s}_s{\sigma^I}_s\sigma_I{\sigma^I}_p-\tfrac{1}{2}\lambda_2({\sigma^I}_s)^2\theta_I\sigma_I{\sigma^I}_p \\
&-\tfrac{1}{2}({\sigma^I}_s)^2\theta_I\sigma_I{\sigma^I}_p-\tfrac{1}{2}{\sigma^s}_s{\sigma^I}_p{\sigma^I}_s\sigma_I\theta_I+\tfrac{1}{4}({\sigma^I}_s)^2({\sigma^I}_p)^2\sigma_I+\tfrac{1}{4}{\sigma^s}_s({\sigma^I}_s)^2{\sigma^I}_s\sigma_I \\
&+\lambda_1{\sigma^s}_p+\frac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}+\frac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}-\frac{\theta_I\lambda_2({\sigma^I}_s)^2({\sigma^I}_p)^2}{({\sigma^s}_s)^2}-\frac{({\sigma^I}_p)^2{\sigma^I}_s}{2{\sigma^s}_s}-u_p
\end{aligned}
$$

$$
\begin{aligned}
\rho_2 =\ & \frac{\lambda_1{\sigma^I}_s{\sigma^I}_p}{2{\sigma^s}_s}-\frac{2\lambda_1\lambda_2{\sigma^I}_s\theta_I}{{\sigma^s}_s}-\frac{{\lambda_2}^2({\sigma^I}_s)^2{\theta_I}^2}{({\sigma^s}_s)^2}+\frac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}+\frac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}-\frac{3({\sigma^I}_s)^2({\sigma^I}_p)^2}{4{{\sigma^s}_s}^2} \\
&-\frac{3}{2}\frac{\theta_I({\sigma^I}_s)^2{\sigma^I}_p}{({\sigma^s}_s)^2}-\frac{\theta_1{\sigma^I}_p{\sigma^I}_s}{2{\sigma^s}_s}-\frac{{\theta_I}^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}+\frac{\theta_I{\sigma^I}_s\lambda_1}{{\sigma^s}_s}+\frac{({\sigma^I}_p)^2({\sigma^I}_s)^2}{2({\sigma^s}_s)^2}+\frac{({\sigma^s}_p)^2{\sigma^I}_s}{4{\sigma^s}_s}
\end{aligned}
$$

$$
\begin{aligned}
\rho_3 =\ & ({\sigma^I}_s)^4\sigma_I{\theta_I}^2-({\sigma^I}_s)^4{\sigma_I}^2\theta_I{\sigma^I}_p+\frac{({\sigma^I}_s)^4{\sigma_I}^2({\sigma^I}_p)^2}{4}+({\sigma^s}_s)^2({\sigma^I}_s)^2{\sigma_I}^2{\theta_I}^2-({\sigma^s}_s)^2{\sigma^I}_s{\sigma_I}^2\theta_I{\sigma^I}_p \\
&+2{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p-2{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p+\frac{({\sigma^s}_s)^2({\sigma^I}_s)^2{\sigma_I}^2({\sigma^I}_p)^2}{4}-\frac{2({\sigma^I}_p)^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}
\end{aligned}
$$

$$
\begin{aligned}
\rho_4 =\ & \frac{({\sigma^I}_s)^4{\theta_I}^2}{({\sigma^s}_s)^4}+\frac{({\sigma^I}_s)^3\theta_I{\sigma^I}_p}{({\sigma^s}_s)^4}+\frac{({\sigma^I}_s)^2({\sigma^I}_p)^2}{4({\sigma^s}_s)^4}-\frac{2{\theta_I}^2({\sigma^I}_s)^2}{({\sigma^s}_s)^2}+\frac{{\sigma^I}_s{\sigma^I}_p\theta_I}{({\sigma^s}_s)^2}+\frac{{\sigma^I}_p({\sigma^I}_s)^2\theta_I}{({\sigma^s}_s)^2}-\frac{{\sigma^I}_s({\sigma^I}_p)^2}{2({\sigma^s}_s)^2} \\
&+\frac{2\lambda_1\lambda_2{\sigma^I}_s\theta_I}{{\sigma^s}_s}-\frac{\lambda_1{\sigma^I}_s{\sigma^I}_p}{{\sigma^s}_s}+\frac{{\lambda_1}^2({\sigma^I}_s)^2{\theta_I}^2}{({\sigma^s}_s)^2}-\frac{\lambda_2({\sigma^I}_s)^2\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}-\frac{\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s}+\frac{({\sigma^I}_s)^2({\sigma^I}_p)^2}{4({\sigma^s}_s)^2}+\frac{{\sigma^I}_s({\sigma^I}_p)^2}{2{\sigma^s}_s}
\end{aligned}
$$

$$
\begin{aligned}
\rho_5 =\ & \frac{({\sigma^I}_s)^3\sigma_I\theta_I{\sigma^I}_p}{({\sigma^s}_s)^2}-\frac{2({\sigma^I}_s)^4\sigma_I{\theta_I}^2}{({\sigma^s}_s)^2}+\frac{({\sigma^I}_s)^4\sigma_I{\sigma^I}_p\theta_I}{({\sigma^s}_s)^2}-\frac{({\sigma^I}_s)^3\sigma_I({\sigma^I}_p)^2}{2({\sigma^s}_s)^2}+2{\theta_I}^2\sigma_I({\sigma^I}_s)^2-\theta_I({\sigma^I}_s)^2\sigma_I{\sigma^I}_p+\frac{({\sigma^I}_p)^2({\sigma^I}_s)^2\sigma_I}{2}-2{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I\lambda_1 \\
&-2({\sigma^I}_s)^2\sigma_I{\theta_I}^2\lambda_2+{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p+{\sigma^s}_s{\sigma^I}_s\sigma_I\theta_I{\sigma^I}_p\lambda_1+({\sigma^I}_s)^2\sigma_I{\sigma^I}_p\lambda_2\theta_I-\frac{({\sigma^I}_s)^2\sigma_I({\sigma^I}_p)^2}{2} \\
&-\frac{{\sigma^s}_s{\sigma^I}_s\sigma_I({\sigma^I}_p)^2}{2}-2{\sigma^s}_s\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p+\frac{2\lambda_2{\sigma^I}_s\theta_I{\sigma^I}_p}{{\sigma^s}_s},\ {\sigma^I}_p={\sigma^s}_p
\end{aligned}
$$

$$u_C=1-u_B-u_S$$

$$
{u_S}^*=\frac{{\sigma^I}_p}{{\sigma^s}_s}-\left(\frac{-\lambda_1{\sigma^s}_s-\lambda_2{\sigma^I}_s\theta_2+{\sigma^I}_s{\sigma^I}_p+{\sigma^I}_s(\theta_I-\frac{{\sigma^I}_p}{2})}{y({\sigma^s}_s)^2}\right)zh_z
\tag{4.12}
$$

$$
{u_B}^*=\frac{{\sigma^I}_p}{\sigma_I}-\frac{{\sigma^I}_p{\sigma^I}_s}{{\sigma^s}_s\sigma_I}+\frac{[({\sigma^I}_s)^2\theta_I-{\sigma^I}_s\sigma_I{\sigma^I}_p-{\sigma^I}_s\lambda_1{\sigma^s}_s-({\sigma^I}_s)^2\lambda_2\theta_I+\frac{({\sigma^I}_s)^2{\sigma^I}_p}{2}+\frac{{\sigma^s}_s{\sigma^I}_s{\sigma^I}_p}{2}]}{h({\sigma^s}_s)^2}zh_z+\frac{\theta_I zh_z}{h\sigma_I},
\tag{4.13}
$$

$${\sigma^I}_p={\sigma^s}_p$$

We will now solve (4.11) for $h$ and substitute into (4.12) and (4.13) to obtain the optimal investment strategies.

## 5. Explicit Solution of the optimal investment strategies for The CRRA Utility Function

Assume the investor takes a power utility function

$$
U(x)=\frac{y^p}{p},\quad p<1,\ \ p\neq 0
\tag{5.1}
$$

The relative risk aversion of an investor with utility described in (5.1) is constant and (5.1) is a CRRA utility.
From (3.5) we have $h(T,r,z)=(V')^{-1}(z)$ and from (5.1), we have

$$h(T,r_R,z)=z^{\frac{1}{p-1}}$$

We assume a solution to (4.11) with the following form

$$h(t,r,z)=g(t,r)\left[z^{\frac{1}{p-1}}\right]+v(t),\qquad v(T)=0,\ \ g(T,s)=1.$$

Then

$$
h_t=g_t z^{\frac{1}{p-1}}+v',\ \ h_z=-\frac{g}{1-p}z^{\left(\frac{1}{p-1}-1\right)},\ \ h_{r_R z}=-\frac{g_{r_R}}{1-p}z^{\left(\frac{1}{p-1}-1\right)}
\tag{5.2}
$$

$$h_{zz} = \frac{(2-p)g}{(1-p)^2} z^{\left(\frac{1}{p-1}-1\right)},\ h_{r_R} = g_{r_R} z^{\frac{1}{p-1}},\ h_{r_R r_R} = g_{r_R r_R} z^{\frac{1}{p-1}}.$$

Substituting (5.2) into (4.11), we have

$$\left\{ \begin{array}{l} g_t + (a - br_R) g_{r_R} - \frac{g_{r_R r_R} \sigma_{r_R}{}^2}{2} + \frac{g(\frac{\rho_5}{2}+\rho_1)}{1-p} - \frac{g\rho_5}{2} - g\rho_1 + \frac{2g(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{1-p} \\ - \frac{(2-p)g(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{(1-p)^2} \end{array} \right\} z^{\frac{1}{p-1}} + v^I(t) - k - \frac{1}{2} v\rho_5 - v\rho_1 = 0$$

$$+ y^I(t) - \rho_1 y(t) - \rho_0 = 0 \tag{5.3}$$

Splitting (5.3), we have

$$v^I(t) - (\frac{1}{2}\rho_5 + \rho_1) v(t) - k = 0$$

$$g_t + (a - br_R) g_{r_R} - \frac{g_{r_R r_R} \sigma_{r_R}{}^2}{2} + \frac{g(\frac{\rho_5}{2}+\rho_1)}{1-p} - \frac{g\rho_5}{2} - g\rho_1 + \frac{2g(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{1-p} - \frac{(2-p)g(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{(1-p)^2} = 0 \tag{5.4}$$

Considering the boundary condition,

$$v(T) = 0,$$

yields the solution

$$v(t) = -\frac{k}{\rho_*}(1 - e^{-\rho_*(T-t)}),$$

where $\rho_3 = 0,\ \rho_* = \frac{1}{2}\rho_5 + \rho_1$
Next, obtain the solution of (5.4), by assuming, a solution of the form

$$g(t, r_R) = M(t)\ e^{N(t) r_R} M(T) = 1,\ N(T) = 0$$

$$g_{r_R} = M(t) N(t) e^{N(t) r_R},\ g_{r_R r_R} = M(t) N^2(t) e^{N(t) r_R}\ and\ g_t = r_R M(t) N^I(t)\ e^{N(t) r_R} + M^I(t)\ e^{N(t) r_R} \tag{5.5}$$

Substituting (5.5) into (5.4), we have

$$\begin{aligned} & N_t r_R + \frac{M_t}{M} + Na - Nbr_R + \frac{1}{2}N^2 k_1 r_R + \frac{1}{2}N^2 k_2 + \frac{\rho_5}{2(1-p)} + \frac{\rho_1}{1-p} - \frac{1}{2}\rho_5 - \frac{1}{2}\rho_1 \\ + \ & \frac{2(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{1-p} - \frac{(2-p)(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{(1-p)^2} = 0, \rho_3 \end{aligned}$$

Splitting (5.6), we have

$$\begin{array}{l} \frac{M_t}{M} + Na + \frac{1}{2}N^2 k_1 + \frac{\rho_5}{2(1-p)} + \frac{\rho_1}{1-p} - \frac{1}{2}\rho_5 - \frac{1}{2}\rho_1 \\ + \frac{2(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{1-p} - \frac{(2-p)(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{(1-p)^2} = 0 \end{array} \tag{5.6}$$

$$N_t - Nb + \frac{1}{2}N^2 k_1 = 0 \tag{5.7}$$

Solving (5.6) and (5.7), we obtain

$$N(t) = \frac{2b[t-T]}{k_1}$$

$$M(t) = c_1 e^{\left\{a \pm (a^2 - 2k_2 H)^{\frac{1}{2}} k_2{}^{-1} t\right\}},\ c_1 = e^c,$$

$$H = \frac{\rho_5}{2(1-p)} + \frac{\rho_1}{1-p} - \frac{1}{2}\rho_5 - \frac{1}{2}\rho_1 + \frac{2(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{1-p} - \frac{(2-p)(2\theta_I{}^2 \frac{1}{2}(\sigma^I{}_p)^2 - \theta_I \sigma^I{}_p + \rho_2 + \rho_4)}{(1-p)^2} M(T) = 1$$

where

$$d_1 = \frac{4b}{2k_1}$$

$$d_2 = 0$$

$$g(r_R,t)=\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}exp\frac{2b(t-T)}{k_1}r_R$$

Therefore, the solution of (4.11) becomes

$$h(t,r_R,z)=\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}z^{\frac{1}{p-1}}-\frac{k}{\rho_*}(1-e^{-\rho_*(T-t)}),$$

where $\rho_3=0,\ \rho_*=\frac{1}{2}\rho_5+\rho_1$

**Theorem 5.1.** *Let the optimal investment strategies for cash, bond and stock be given as follows* $u_C{}^*=1-u_B{}^*-u_S{}^*$.*Then* $N(t)=\frac{2b[t-T]}{k_1}$ *with* $d_1=\frac{4b}{2k_1}$ *and* $d_2=0$.

*Proof.* Let

$$\begin{aligned} u_S{}^* &= \frac{\sigma^I{}_p}{\sigma^s{}_s}-\left(\frac{-\lambda_1\sigma^s{}_s-\lambda_2\sigma^I{}_s\theta_2+\sigma^I{}_s\sigma^I{}_p+\sigma^I{}_s(\theta_I-\frac{\sigma^I{}_p}{2})}{\sigma^s{}_s}\right)\frac{1}{p-1} \\ &\times \frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}-\frac{\rho_*z^{\frac{1}{p-1}}}{k(1-e^{-\rho_*(T-t)})}\right)\end{aligned} \tag{5.8}$$

and

$$\begin{aligned} u_B{}^* &= \frac{\sigma^I{}_p}{\sigma_I}-\frac{\sigma^I{}_p\sigma^I{}_s}{\sigma^s{}_s\sigma_I}+\frac{\left[(\sigma^I{}_s)^2\theta_I-\sigma^I{}_s\sigma_I\sigma^I{}_p-\sigma^I{}_s\lambda_1\sigma^s{}_s-(\sigma^I{}_s)^2\lambda_2\theta_I+\frac{(\sigma^I{}_s)^2\sigma^I{}_p}{2}+\frac{\sigma^s{}_s\sigma^I{}_s\sigma^I{}_p}{2}\right]}{(\sigma^s{}_s)^2} \\ &\times \frac{1}{p-1}\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}-\frac{\rho_*z^{\frac{1}{p-1}}}{k(1-e^{-\rho_*(T-t)})}\right) \\ &+ \frac{\theta_I}{p-1}\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm\left(a^2-2k_2H\right)^{\frac{1}{2}}k_2{}^{-1}t\right\}}}-\frac{\rho_*z^{\frac{1}{p-1}}}{\sigma_Ik(1-e^{-\rho_*(T-t)})}\right),\end{aligned} \tag{5.9}$$

$$\rho_3=0,\ \rho_*=\frac{1}{2}\rho_5+\rho_1,\ \sigma^I{}_p=\sigma^s{}_p$$

Then

$$H=\frac{\rho_5}{2(1-p)}+\frac{\rho_1}{1-p}-\frac{1}{2}\rho_5-\frac{1}{2}\rho_1+\frac{2(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p}-\frac{(2-p)(2\theta_I{}^2\frac{1}{2}(\sigma^I{}_p)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2} \tag{5.10}$$

$$\begin{aligned}\rho_1 &= \frac{3}{2}(\sigma^I{}_p)^2+\lambda_1\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I+\lambda_2(\sigma^I{}_s)^2\theta_I{}^2\sigma_I-\frac{1}{2}\lambda_1\sigma^s{}_s\sigma^I{}_s\sigma_I\sigma^I{}_p-\frac{1}{2}\lambda_2(\sigma^I{}_s)^2\theta_I\sigma_I\sigma^I{}_p-\frac{1}{2}(\sigma^I{}_s)^2\theta_I\sigma_I\sigma^I{}_p \\ &- \frac{1}{2}\sigma^s{}_s\sigma^I{}_p\sigma^I{}_s\sigma_I\theta..._I+\frac{1}{4}(\sigma^I{}_s)^2(\sigma^I{}_p)^2\sigma_I+\frac{1}{4}\sigma^s{}_s(\sigma^I{}_s)^2\sigma^I{}_s\sigma_I+\lambda_1\sigma^s{}_p+\frac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2} \\ &+ \frac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}-\frac{\theta_I\lambda_2(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{(\sigma^s{}_s)^2}-\frac{(\sigma^I{}_p)^2\sigma^I{}_s}{2\sigma^s{}_s}-u_p.\end{aligned}$$

And

$$\begin{aligned}\rho_2 &= \frac{\lambda_1\sigma^I{}_s\sigma^I{}_p}{2\sigma^s{}_s}-\frac{2\lambda_1\lambda_2\sigma^I{}_s\theta_I}{\sigma^s{}_s}-\frac{\lambda_2{}^2(\sigma^I{}_s)^2\theta_I{}^2}{(\sigma^s{}_s)^2}+\frac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2}+\frac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}-\frac{3(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4\sigma^s{}_s{}^2} \\ &- \frac{3}{2}\frac{\theta_I(\sigma^I{}_s)^2\sigma^I{}_p}{(\sigma^s{}_s)^2}-\frac{\theta_1\sigma^I{}_p\sigma^I{}_s}{2\sigma^s{}_s}-\frac{\theta_I{}^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2}+\frac{\theta_I\sigma^I{}_s\lambda_1}{\sigma^s{}_s}+\frac{(\sigma^I{}_p)^2(\sigma^I{}_s)^2}{2(\sigma^s{}_s)^2}+\frac{(\sigma^s{}_p)^2\sigma^I{}_s}{4\sigma^s{}_s}.\end{aligned}$$

$$\begin{aligned}\rho_3 &= (\sigma^I{}_s)^4\sigma_I\theta_I{}^2-(\sigma^I{}_s)^4\sigma_I{}^2\theta_I\sigma^I{}_p+\frac{(\sigma^I{}_s)^4\sigma_I{}^2(\sigma^I{}_p)^2}{4}+(\sigma^s{}_s)^2(\sigma^I{}_s)^2\sigma_I{}^2\theta_I{}^2-(\sigma^s{}_s)^2\sigma^I{}_s\sigma_I{}^2\theta_I\sigma^I{}_p \\ &+ 2\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p-2\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p+\frac{(\sigma^s{}_s)^2(\sigma^I{}_s)^2\sigma_I{}^2(\sigma^I{}_p)^2}{4}-\frac{2(\sigma^I{}_p)^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2},\end{aligned}$$

$$\begin{aligned}\rho_4 \;=\;& \frac{(\sigma^I{}_s)^4\theta_I{}^2}{(\sigma^s{}_s)^4}+\frac{(\sigma^I{}_s)^3\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^4}+\frac{(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4(\sigma^s{}_s)^4}-\frac{2\theta_I{}^2(\sigma^I{}_s)^2}{(\sigma^s{}_s)^2}+\frac{\sigma^I{}_s\sigma^I{}_p\theta_I}{(\sigma^s{}_s)^2}+\frac{\sigma^I{}_p(\sigma^I{}_s)^2\theta_I}{(\sigma^s{}_s)^2}-\frac{\sigma^I{}_s(\sigma^I{}_p)^2}{2(\sigma^s{}_s)^2}+\frac{2\lambda_1\lambda_2\sigma^I{}_s\theta_I}{\sigma^s{}_s}\\ &-\;\frac{\lambda_1\sigma^I{}_s\sigma^I{}_p}{\sigma^s{}_s}+\frac{\lambda_1{}^2(\sigma^I{}_s)^2\theta_I{}^2}{(\sigma^s{}_s)^2}-\frac{\lambda_2(\sigma^I{}_s)^2\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2}-\frac{\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}+\frac{(\sigma^I{}_s)^2(\sigma^I{}_p)^2}{4(\sigma^s{}_s)^2}+\frac{\sigma^I{}_s(\sigma^I{}_p)^2}{2\sigma^s{}_s}\end{aligned}$$

and

$$\begin{aligned}\rho_5 =\;& \frac{(\sigma^I{}_s)^3\sigma_I\theta_I\sigma^I{}_p}{(\sigma^s{}_s)^2}-\frac{2(\sigma^I{}_s)^4\sigma_I\theta_I{}^2}{(\sigma^s{}_s)^2}+\frac{(\sigma^I{}_s)^4\sigma_I\sigma^I{}_p\theta_I}{(\sigma^s{}_s)^2}-\frac{(\sigma^I{}_s)^3\sigma_I(\sigma^I{}_p)^2}{2(\sigma^s{}_s)^2}+2\theta_I{}^2\sigma_I(\sigma^I{}_s)^2\\ &-\theta_I(\sigma^I{}_s)^2\sigma_I\sigma^I{}_p+\frac{(\sigma^I{}_p)^2(\sigma^I{}_s)^2\sigma_I}{2}-2\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\lambda_1-2(\sigma^I{}_s)^2\sigma_I\theta_I{}^2\lambda_2+\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p\\ &+\sigma^s{}_s\sigma^I{}_s\sigma_I\theta_I\sigma^I{}_p\lambda_1+(\sigma^I{}_s)^2\sigma_I\sigma^I{}_p\lambda_2\theta_I-\frac{(\sigma^I{}_s)^2\sigma_I(\sigma^I{}_p)^2}{2}-\frac{\sigma^s{}_s\sigma^I{}_s\sigma_I(\sigma^I{}_p)^2}{2}-2\sigma^s{}_s\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p+\frac{2\lambda_2\sigma^I{}_s\theta_I\sigma^I{}_p}{\sigma^s{}_s}.\end{aligned} \tag{5.11}$$

It therefore follows that

$$N(t)=\frac{2b[t-T]}{k_1}$$

$$d_1=\frac{4b}{2k_1}$$

$$d_2=0.$$

□

**Remark 5.1**
If we let $\sigma^I{}_p=\sigma^I{}_p=\theta_I=0$, the optimal strategies (5.8) and (5.9) would be of the form of the [7]
Recall from [7], the coefficients $d_1,\ d_2$ degenerates to $\frac{4b}{2k_1}$ and zero, in the absence of the coefficient of the CRRA (i.e, as $p\to 0$), however, in this work, even in the presence of the coefficient of CRRA the coefficients $d_1,\ d_2$ are already degenerate. We therefore, conclude that, under the inflationary market, the CRRA utility function has little or no effect on the investment strategy.
The associated optimal investment strategy for a logarithmic utility function, as $p\to 0$ is given by

$$u_S{}^* = \frac{\sigma^I{}_p}{\sigma^s{}_s}+\left(\frac{-\lambda_1\sigma^s{}_s-\lambda_2\sigma^I{}_s\theta_2+\sigma^I{}_s\sigma^I{}_p+\sigma^I{}_s(\theta_I-\frac{\sigma^I{}_p}{2})}{\sigma^s{}_s}\right)\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}t\right\}}}-\frac{\rho_*}{zk(1-e^{-\rho_*(T-t)})}\right)$$

$$\begin{aligned}u_B{}^* \;=\;& \frac{\sigma^I{}_p}{\sigma_I}-\frac{\sigma^I{}_p\sigma^I{}_s}{\sigma^s{}_s\sigma_I}-\frac{\left[(\sigma^I{}_s)^2\theta_I-\sigma^I{}_s\sigma_I\sigma^I{}_p-\sigma^I{}_s\lambda_1\sigma^s{}_s-(\sigma^I{}_s)^2\lambda_2\theta_I+\frac{(\sigma^I{}_s)^2\sigma^I{}_p}{2}+\frac{\sigma^s{}_s\sigma^I{}_s\sigma^I{}_p}{2}\right]}{(\sigma^s{}_s)^2}\\ &\times\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}t\right\}}}-\frac{\rho_*}{zk(1-e^{-\rho_*(T-t)})}\right)\\ &+\frac{\theta_I}{p-1}\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}t\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}T\right\}}}\left(\frac{e^{\left\{a\pm\left(a^2-2k_2\rho_1\right)^{\frac12}k_2{}^{-1}T\right\}}}{e^{\left\{a\pm(a^2-2k_2\rho_1)^{\frac12}k_2{}^{-1}t\right\}}}-\frac{\rho_*}{z\sigma_I k\left(1-e^{-\rho_*(T-t)}\right)}\right),\end{aligned}$$

$$\rho_3=0,\ \rho_*=\frac12\rho_5+\rho_1,\ \sigma^I{}_p=\sigma^s{}_p,$$

$$H=\frac{\rho_5}{2(1-p)}+\frac{\rho_1}{1-p}-\frac12\rho_5-\frac12\rho_1+\frac{2(2\theta_I{}^2\frac12\left(\sigma^I{}_p\right)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{1-p}-\frac{(2-p)(2\theta_I{}^2\frac12\left(\sigma^I{}_p\right)^2-\theta_I\sigma^I{}_p+\rho_2+\rho_4)}{(1-p)^2}.$$

# 6. Discussion and conclusion

## 6.1. Discussion

From Proposition 5.1, we deduced that in the absence of inflation, proportions of the pension wealth invested in stock and bond would be at least at minimal returns, and the optimal investment strategy, with CRRA utility function, would be constant. From (5.10) and (5.11), we observe that the optimal investment process is lumped with a lot of inflation radicals. More so, from remark 5.1, we discovered that the CRRA utility function does not have much effect on inflation and its effect on wealth investment. From the analysis, we see that the returns on investment of the pension wealth will reduce drastically, therefore, the contributor require the extra measure to dampen the effect of inflation on the investment strategy. From this analysis, we deduce also that the more the returns on optimal investment degenerates, which is as a result of inflation-affected optimal investment strategy, the more the price of stock becomes non-increasing, then the need for more wealth investment in both stock and bond becomes necessary, in order to recover for the lost times, and pull down the price of stock, hence the need for an amortization fund by the plan member becomes necessary.

### 6.2. Conclusion

The optimal investment strategy for a prospective investor in a DC pension scheme, under the inflationary market, with stochastic salary, under the affine interest rate model has been studied. Relevant to this work, the CRRA utility function was used and we obtained the optimal investment strategies for cash, bond and stock using the Legendre transform and dual theory. More so, the effects of inflation parameters and the coefficient of CRRA utility function were analyzed, with insignificant input on the investment strategy. We conclude, therefore, inflation has significant negative effect on optimal investment strategy, particularly, the CRRA utility function is not constant with the investment strategy.

### 6.3. Recommendation

From the result obtained in this work, we recommend the investigation of the effect of extra contribution on optimal investment strategy, in DC pension scheme, under inflationary market.

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