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LACUNARY INVARIANT STATISTICAL EQUIVALENCE FOR DOUBLE SET SEQUENCES

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Abstract. In this paper, we introduce the notions of asymptotical strong σ_2-equivalence, asymptotical σ_2-statistical equivalence, asymptotical lacunary strong σ_2-equivalence and asymptotical lacunary σ_2-statistical equivalence in the Wijsman sense for double set sequences. Also, we investigate some relations between these new asymptotical equivalence notions.

1. Introduction

Long after the notion of convergence for double sequences was introduced by Pringsheim [1], this notion was extended to the notion of statistical convergence by Móricz [2] and Mursaleen and Edely [3] in the same year, to the notion of lacunary statistical convergence by Patterson and Savaş [4] and to the notion of double σ-convergent lacunary statistical sequence by Savaş and Patterson [5]. Moreover, for double sequences, the notion of asymptotical equivalence was introduced by Patterson [6].

Over the years, on the various convergence notions for set sequences have been studied by many authors (see, [7–9]). One of them, discussed in this paper, is the notion of convergence in the Wijsman sense [10]. Using the notions of statistical convergence, double lacunary sequence and invariant mean, this notion was extended to the notions of convergence for double set sequences by some authors [11–13]. Furthermore, for double set sequences, the notions of asymptotical equivalence in the Wijsman sense were introduced by Nuray et al. [14] and then these notions were studied by some authors [15–17]. In this paper, using the notion of invariant

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mean, we study on new asymptotical equivalence notions in the Wijsman sense for double set sequences. More information on the notions of asymptotical equivalence for set sequences can be found in [18, 19].

2. Basic Definitions and Notations

In this section, let us remind the basic notions necessary for a better understanding of our paper.

Definition 1. *[1] A double sequence (x_{jk}) is called convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$, whenever $j, k > N_\varepsilon$. It is denoted by $P - \lim_{j,k\to\infty} x_{kj} = L$ or $\lim_{j,k\to\infty} x_{jk} = L$.*

Definition 2. *[3] A double sequence (x_{jk}) is called statistically convergent to L if for every $\varepsilon > 0$,*

$$P - \lim_{m,n\to\infty} \frac{1}{mn}\Big|\big\{(j,k) : \, j \le m, k \le n, \, |x_{jk} - L| \ge \varepsilon\big\}\Big| = 0.$$

For a metric space (Y, d), $\mu(y, B)$ denote the distance from y to B where

$$\mu(y, B) = \inf_{b\in B} d(y, b)$$

for any $y \in Y$ and any nonempty $B \subseteq Y$.

Throughout this study, (Y, d) will be considered as a metric space and B, B_{jk}, D_{jk} will be considered as any nonempty closed subsets of Y.

Definition 3. *[13] A double set sequence $\{B_{jk}\}$ is called convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P - \lim_{j,k\to\infty} \mu(y, B_{jk}) = \mu(y, B).$$

Let σ be a mapping such that $\sigma : \mathbb{N} \to \mathbb{N}$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞, the space of real bounded sequences, is called an invariant mean (or a σ-mean) if it satisfies the following conditions:

(1) $\psi(x_s) \ge 0$, when the sequence (x_s) has $x_s \ge 0$ for all s,
(2) $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
(3) $\psi(x_{\sigma(s)}) = \psi(x_s)$ for all $(x_s) \in \ell_\infty$.

The mapping σ is assumed to be one-to-one and such that $\sigma^j(s) \ne s$ for all $j, s \in \mathbb{N}$, where $\sigma^j(s)$ denotes the j th iterate of the mapping σ at s. Thus ψ extends the limit functional on c, the space of convergent sequences, in the sense that $\psi(x_s) = \lim x_s$ for all $(x_s) \in c$.

Definition 4. *[12] A double set sequence $\{B_{jk}\}$ is called invariant convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} \mu(y, B_{\sigma^j(s)\sigma^k(t)}) = \mu(y, B), \quad \textit{uniformly in } s, t.$$

Definition 5. *[12] A double set sequence $\{B_{jk}\}$ is called strong invariant convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{n,m\to\infty}\frac{1}{nm}\sum_{j,k=1,1}^{n,m}\left|\mu(y,B_{\sigma^j(s)\sigma^k(t)})-\mu(y,B)\right|=0,\quad \textit{uniformly in } s,t.$$

Definition 6. *[12] A double set sequence $\{B_{jk}\}$ is called invariant statistically convergent to the set B in the Wijsman sense if for every $\varepsilon > 0$ and each $y \in Y$,*

$$P-\lim_{n,m\to\infty}\frac{1}{nm}\left|\left\{(j,k): j\le n, k\le m, \left|\mu(y,B_{\sigma^j(s)\sigma^k(t)})-\mu(y,B)\right|\ge\varepsilon\right\}\right|=0,$$

uniformly in s,t.

A double sequence $\theta_2=\{(j_r,k_u)\}$ is called a double lacunary sequence if there exist increasing sequences (j_r) and (k_u) of the integers such that

$$j_0=0,\ h_r=j_r-j_{r-1}\to\infty \ \text{ and } \ k_0=0,\ \bar{h}_u=k_u-k_{u-1}\to\infty \ \text{ as } \ r,u\to\infty.$$

In general, the following notations is used for any double lacunary sequence:

$$h_{ru}=h_r\bar{h}_u,\quad I_{ru}=\{(j,k): j_{r-1}<j\le j_r \ \text{ and } \ k_{u-1}<k\le k_u\},$$

$$q_r=\frac{j_r}{j_{r-1}} \ \text{ and } \ q_u=\frac{k_u}{k_{u-1}}.$$

Throughout this study, $\theta_2=\{(j_r,k_u)\}$ will be considered as a double lacunary sequence.

Definition 7. *[12] A double set sequence $\{B_{jk}\}$ is called lacunary invariant convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{(j,k)\in I_{ru}}\mu(y,B_{\sigma^j(s)\sigma^k(t)})=\mu(y,B),\quad \textit{uniformly in } s,t.$$

Definition 8. *[12] A double set sequence $\{B_{jk}\}$ is called lacunary strong invariant convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{(j,k)\in I_{ru}}\left|\mu(y,B_{\sigma^j(s)\sigma^k(t)})-\mu(y,B)\right|=0,\quad \textit{uniformly in } s,t.$$

Definition 9. *[12] A double set sequence $\{B_{jk}\}$ is called lacunary invariant statistically convergent to the set B in Wijsman sense if for every $\varepsilon > 0$ and each $y \in Y$,*

$$P-\lim_{r,u\to\infty}\frac{1}{h_{ru}}\left|\left\{(j,k)\in I_{ru}: \left|\mu(y,B_{\sigma^j(s)\sigma^k(t)})-\mu(y,B)\right|\ge\varepsilon\right\}\right|=0,$$

uniformly in s,t.

The term $\mu_y\left(\frac{B_{jk}}{D_{jk}}\right)$ is defined as follows:

$$\mu_y\left(\frac{B_{jk}}{D_{jk}}\right)=\begin{cases}\dfrac{\mu(y,B_{jk})}{\mu(y,D_{jk})} & , \quad y\notin B_{jk}\cup D_{jk}\\ \lambda & , \quad y\in B_{jk}\cup D_{jk}.\end{cases}$$

Definition 10. *[14] Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are called asymptotically equivalent of multiplicity λ in the Wijsman sense if for each $y\in Y$,*

$$P-\lim_{j,k\to\infty}\mu_y\left(\frac{B_{jk}}{D_{jk}}\right)=\lambda.$$

It is denoted by $B_{jk}\overset{W_2^\lambda}{\sim}D_{jk}$ and simply called asymptotically equivalent in the Wijsman sense if $\lambda=1$.

As an example to asymptotically equivalent double set sequences, the following sequences can be considered:

$$B_{jk}=\{(a,b)\in\mathbb{R}^2 : a^2+b^2-2jkb=0\}$$

and

$$D_{jk}=\{(a,b)\in\mathbb{R}^2 : a^2+b^2+2jkb=0\}.$$

Since

$$P-\lim_{j,k\to\infty}\mu_y\left(\frac{B_{jk}}{D_{jk}}\right)=1$$

for every $y\in\mathbb{R}^2$, the double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are asymptotically equivalent in the Wijsman sense, i.e., $B_{jk}\overset{W_2}{\sim}D_{jk}$.

3. Main results

In this section, for double set sequences, we introduce the notions of asymptotical σ_2-equivalence, asymptotical strong σ_2-equivalence, asymptotical σ_2-statistical equivalence, asymptotical lacunary σ_2-equivalence, asymptotical strong lacunary σ_2-equivalence and asymptotical lacunary σ_2-statistical equivalence in the Wijsman sense. Also, we investigate some relations between some of these new asymptotical equivalence notions.

Definition 11. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically σ_2-equivalent of multiplicity λ in the Wijsman sense if for each $y\in Y$,*

$$P-\lim_{n,m\to\infty}\frac{1}{nm}\sum_{j,k=1,1}^{n,m}\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)=\lambda,\quad \textit{uniformly in } s,t.$$

This type of equivalence is denoted by $B_{jk}\overset{W_{\sigma_2}^\lambda}{\sim}D_{jk}$ and simply called asymptotically σ_2-equivalent in the Wijsman sense if $\lambda=1$.

Definition 12. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically strong σ_2-equivalent of multiplicity λ in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{n,m\to\infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| = 0, \quad \textit{uniformly in } s,t.$$

This type of equivalence is denoted by $B_{jk} \overset{[W^{\lambda}_{\sigma_2}]}{\sim} D_{jk}$ and simply called asymptotically strong σ_2-equivalent in the Wijsman sense if $\lambda = 1$.

The set of all asymptotically strong σ_2-equivalent double set sequences of multiplicity λ in the Wijsman sense is denoted by $\{[W^{\lambda}_{\sigma_2}]\}$.

Definition 13. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically σ_2-statistical equivalent of multiplicity λ in the Wijsman sense if for every $\varepsilon > 0$ and each $y \in Y$,*

$$P-\lim_{n,m\to\infty} \frac{1}{nm} \left| \left\{ (j,k) : j \leq n, k \leq m, \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| = 0,$$

uniformly in s,t. This type of equivalence is denoted by $B_{jk} \overset{WS^{\lambda}_{\sigma_2}}{\sim} D_{jk}$ and simply called asymptotically σ_2-statistical equivalent in the Wijsman sense if $\lambda = 1$.

The set of all asymptotically σ_2-statistical equivalent double set sequences of multiplicity λ in the Wijsman sense is denoted by $\{WS^{\lambda}_{\sigma_2}\}$.

Definition 14. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically lacunary σ_2-equivalent of multiplicity λ in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(j,k)\in I_{ru}} \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) = \lambda, \quad \textit{uniformly in } s,t.$$

This type of equivalence is denoted by $B_{jk} \overset{W^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk}$ and simply called asymptotically lacunary σ_2-equivalent in the Wijsman sense if $\lambda = 1$.

Definition 15. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically lacunary strong σ_2-equivalent of multiplicity λ in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(j,k)\in I_{ru}} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| = 0, \quad \textit{uniformly in } s,t.$$

This type of equivalence is denoted by $B_{jk} \overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim} D_{jk}$ and simply called asymptotically lacunary strong σ_2-equivalent in the Wijsman sense if $\lambda = 1$.

Example 1. *Let $Y = \mathbb{R}^2$ and double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ be defined as following:*

$$B_{jk} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : a^2 + (b+1)^2 = \frac{1}{jk}\right\} & ; \quad \text{if } (j,k) \in I_{ru},\ j \text{ and } k \text{ are} \\ & \qquad \text{square integers,} \\ \{(2,0)\} & ; \quad \text{otherwise.} \end{cases}$$

and

$$D_{jk} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : a^2 + (b-1)^2 = \frac{1}{jk}\right\} & ; \quad \text{if } (j,k) \in I_{ru},\ j \text{ and } k \text{ are} \\ & \qquad \text{square integers,} \\ \{(2,0)\} & ; \quad \text{otherwise.} \end{cases}$$

In this case, the double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are asymptotically lacunary strong σ_2-equivalent in the Wijsman sense.

The set of all asymptotically lacunary strong σ_2-equivalent double set sequences of multiplicity λ in the Wijsman sense is denoted by $\{[W^{\lambda}_{\theta\sigma_2}]\}$.

Definition 16. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically lacunary σ_2-statistical equivalent of multiplicity λ in the Wijsman sense if for every $\varepsilon > 0$ and each $y \in Y$,*

$$P - \lim_{r,u\to\infty} \frac{1}{h_{ru}} \left| \left\{ (j,k) \in I_{ru} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| = 0,$$

uniformly in s,t. This type of equivalence is denoted by $B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk}$ and simply called asymptotically lacunary σ_2-statistical equivalent in the Wijsman sense if $\lambda = 1$.

Example 2. *Let $Y = \mathbb{R}^2$ and double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ be defined as following:*

$$B_{jk} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : (a-j)^2 + (b+k)^2 = 4\right\} & ; \quad \text{if } (j,k) \in I_{ru},\ j \text{ and } k \text{ are} \\ & \qquad \text{square integers,} \\ \{(-2,1)\} & ; \quad \text{otherwise.} \end{cases}$$

and

$$D_{jk} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : (a+j)^2 + (b-k)^2 = 4\right\} & ; \quad \text{if } (j,k) \in I_{ru},\ j \text{ and } k \text{ are} \\ & \qquad \text{square integers,} \\ \{(-2,1)\} & ; \quad \text{otherwise.} \end{cases}$$

In this case, the double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are asymptotically lacunary σ_2-statistical equivalent in the Wijsman sense.

The set of all asymptotically lacunary σ_2-statistical equivalent double set sequences of multiplicity λ in the Wijsman sense is denoted by $\{WS^{\lambda}_{\theta\sigma_2}\}$.

Theorem 1.
(i) *If* $B_{jk} \overset{[W^{\lambda}_{\theta\sigma 2}]}{\sim} D_{jk}$, *then* $B_{jk} \overset{WS^{\lambda}_{\theta\sigma 2}}{\sim} D_{jk}$.
(ii) *If for each* $y \in Y$ $\sup_{j,k,s,t} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) \right| < \infty$ *and* $B_{jk} \overset{WS^{\lambda}_{\theta\sigma 2}}{\sim} D_{jk}$, *then* $B_{jk} \overset{[W^{\lambda}_{\theta\sigma 2}]}{\sim} D_{jk}$.

Proof. (i) Let $B_{jk} \overset{[W^{\lambda}_{\theta\sigma 2}]}{\sim} D_{jk}$. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\begin{aligned}
\sum_{(j,k)\in I_{ru}} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| &\geq \sum_{\substack{(j,k)\in I_{ru} \\ \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon}} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \\
&\geq \varepsilon \left| \left\{ (j,k) \in I_{ru} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right|
\end{aligned}$$

for all s,t, which gives the result.

(ii) Let $B_{jk} \overset{WS^{\lambda}_{\theta\sigma 2}}{\sim} D_{jk}$. Also, suppose that $\sup_{j,k,s,t} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) \right| < \infty$ for each $y \in Y$. Then, there exists an $M > 0$ such that for each $y \in Y$

$$\left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \leq M$$

for all j,k and s,t. Thus, for every $\varepsilon > 0$ and each $y \in Y$ we have

$$\begin{aligned}
&\frac{1}{h_{ru}} \sum_{(j,k)\in I_{ru}} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \\
&= \frac{1}{h_{ru}} \sum_{\substack{(j,k)\in I_{ru} \\ \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon}} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \\
&\quad + \frac{1}{h_{ru}} \sum_{\substack{(j,k)\in I_{ru} \\ \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| < \varepsilon}} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \\
&\leq \frac{M}{h_{ru}} \left| \left\{ (j,k) \in I_{ru} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| + \varepsilon
\end{aligned}$$

for all s,t, which gives the result. $\square$

With a technique similar to that of Theorem 1, the following theorem can be proved.

Theorem 2.
(i) *If* $B_{jk} \overset{[W_{\sigma_2}^{\lambda}]}{\sim} D_{jk}$, *then* $B_{jk} \overset{WS_{\sigma_2}^{\lambda}}{\sim} D_{jk}$.
(ii) *If for each* $y \in Y$ $\sup_{j,k,s,t} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)\right| < \infty$ *and* $B_{jk} \overset{WS_{\sigma_2}^{\lambda}}{\sim} D_{jk}$, *then* $B_{jk} \overset{[W_{\sigma_2}^{\lambda}]}{\sim} D_{jk}$.

Theorem 3. *If* $\liminf_r q_r > 1$ *and* $\liminf_u q_u > 1$ *for any* $\theta_2 = \{(j_r, k_u)\}$, *then*

$$B_{jk} \overset{WS_{\sigma_2}^{\lambda}}{\sim} D_{jk} \quad \textit{implies} \quad B_{jk} \overset{WS_{\theta\sigma_2}^{\lambda}}{\sim} D_{jk}.$$

Proof. Let $B_{jk} \overset{WS_{\sigma_2}^{\lambda}}{\sim} D_{jk}$. Also, suppose that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$. Then, there exist $\eta, \rho > 0$ such that $q_r \geq \eta + 1, q_u \geq \rho + 1$ for all $r, u > 1$, which implies that

$$\frac{h_{ru}}{j_r k_u} \geq \frac{\eta\rho}{(\eta+1)(\rho+1)}.$$

Thus, for every $\varepsilon > 0$ and each $y \in Y$ we have

$$\begin{aligned}
&\frac{1}{j_r k_u}\left|\left\{(j,k) : j \leq j_r, k \leq k_u, \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right| \\
&\qquad \geq \frac{1}{j_r k_u}\left|\left\{(j,k) \in I_{ru} : \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right| \\
&\qquad = \frac{h_{ru}}{j_r k_u}\frac{1}{h_{ru}}\left|\left\{(j,k) \in I_{ru} : \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right| \\
&\qquad \geq \frac{\eta\rho}{(\eta+1)(\rho+1)}\frac{1}{h_{ru}}\left|\left\{(j,k) \in I_{ru} : \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

for all s, t, which gives the result. □

Theorem 4. *If* $\limsup_r q_r < \infty$ *and* $\limsup_u q_u < \infty$ *for any* $\theta_2 = \{(j_r, k_u)\}$, *then*

$$B_{jk} \overset{WS_{\theta\sigma_2}^{\lambda}}{\sim} D_{jk} \quad \textit{implies} \quad B_{jk} \overset{WS_{\sigma_2}^{\lambda}}{\sim} D_{jk}.$$

Proof. Let $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$. Then, there exist $\alpha, \beta > 0$ such that $q_r < \alpha, q_u < \beta$ for all $r, u > 1$. Also, suppose that $B_{jk} \overset{WS_{\theta\sigma_2}^{\lambda}}{\sim} D_{jk}$ and

$\delta > 0$. Then, there exist $n_0, m_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$, each $y \in Y$ and all $j \geq n_0, k \geq m_0$

$$\mathcal{S}_{jk} := \frac{1}{h_{jk}} \left| \left\{ (j,k) \in I_{jk} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| < \delta$$

for all s, t. We can also find an $M > 0$ such that $\mathcal{S}_{jk} < M$ for all $j, k = 1, 2, \ldots$.

Now, let n and m be any integers satisfying $j_{r-1} < n \leq j_r, k_{u-1} < m \leq k_u$ where $r > n_0, u > m_0$. Then, for every $y \in Y$ we have

$$\begin{aligned}
&\frac{1}{nm} \left| \left\{ (j,k) : j \leq n, k \leq m, \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\leq \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) : j \leq j_r, k \leq k_u, \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&= \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{11} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\quad + \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{12} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\quad + \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{21} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\quad + \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{22} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\quad \vdots \\
&\quad + \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{ru} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&= \frac{j_1 k_1}{j_{r-1}k_{u-1}} \mathcal{S}_{11} + \frac{j_1(k_2 - k_1)}{j_{r-1}k_{u-1}} \mathcal{S}_{12} + \frac{(j_2 - j_1)k_1}{j_{r-1}k_{u-1}} \mathcal{S}_{21} + \frac{(j_2 - j_1)(k_2 - k_1)}{j_{r-1}k_{u-1}} \mathcal{S}_{22} \\
&\quad \vdots \\
&\quad + \frac{(j_{n_0} - j_{n_0-1})(k_{m_0} - k_{m_0-1})}{j_{r-1}k_{u-1}} \mathcal{S}_{n_0 m_0} \\
&\quad \vdots
\end{aligned}$$

$$+\frac{(j_r - j_{r-1})(k_u - k_{u-1})}{j_{r-1}k_{u-1}}\mathcal{S}_{ru}$$

$$\leq \left\{ \sup_{1\leq j,\, 1\leq k} \mathcal{S}_{jk} \right\} \frac{j_{n_0}k_{m_0}}{j_{r-1}k_{u-1}} + \left\{ \sup_{j\geq n_0,\, k\geq m_0} \mathcal{S}_{jk} \right\} \frac{(j_r - j_{n_0})(k_u - k_{m_0})}{j_{r-1}k_{u-1}}$$

$$\leq M\frac{j_{n_0}k_{m_0}}{j_{r-1}k_{u-1}} + \delta\,\alpha\,\beta$$

for all s, t, which gives the result. □

Theorem 5. *If*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty \quad and \quad 1 < \liminf_u q_u \leq \limsup_u q_u < \infty$$

for any $\theta_2 = \{(j_r, k_u)\}$, then

$$B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk} \quad \text{if and only if} \quad B_{jk} \overset{WS^{\lambda}_{\sigma_2}}{\sim} D_{jk}.$$

Proof. The proof is obvious from Theorem 3 and Theorem 4. □

With techniques similar to that of Theorem 3, Theorem 4 and Theorem 5, the following theorems can be respectively proved.

Theorem 6. *If $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$ for any $\theta_2 = \{(j_r, k_u)\}$, then*

$$B_{jk} \overset{[W^{\lambda}_{\sigma_2}]}{\sim} D_{jk} \quad \text{implies} \quad B_{jk} \overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim} D_{jk}.$$

Theorem 7. *If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$ for any $\theta_2 = \{(j_r, k_u)\}$, then*

$$B_{jk} \overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim} D_{jk} \quad \text{implies} \quad B_{jk} \overset{[W^{\lambda}_{\sigma_2}]}{\sim} D_{jk}.$$

Theorem 8. *If*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty \quad and \quad 1 < \liminf_u q_u \leq \limsup_u q_u < \infty$$

for any $\theta_2 = \{(j_r, k_u)\}$, then

$$B_{jk} \overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim} D_{jk} \quad \text{if and only if} \quad B_{jk} \overset{[W^{\lambda}_{\sigma_2}]}{\sim} D_{jk}.$$

4. Conclusion

When $(\sigma(s), \sigma(t)) = (s + 1, t + 1)$, from Definitions 11-16 we get the definitions of asymptotical almost equivalence, asymptotical strong almost equivalence, asymptotical almost statistical equivalence, asymptotical lacunary almost equivalence, asymptotical lacunary strong almost equivalence and asymptotical lacunary almost statistical equivalence in the Wijsman sense for double set sequences. So, the analogues of Theorem 1-8 can also be obtained between these definitions, which have not been appeared anywhere by this time.

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AN INVESTIGATION ON THE TRIPLE IDEAL CONVERGENT SEQUENCES IN FUZZY METRIC SPACES

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Abstract. The notion of ideal convergence is a process of generalizing of statistical convergence which is dependent on the idea of the ideal $\mathcal{I}$ of subsets of the set positive integer numbers. In this study we also present the concept of ideal convergence for triple sequences in fuzzy metric spaces (FMS) in the manner of George and Veeramani and the terms of ideal Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence in FMS and examine their some properties.

1. Introduction and Literature Review

Statistical convergence for real sequence was rst introduced by Fast [4] in 1951. Since then statistical convergence was investigated by more and more researchers. The concept of $\mathcal{I}$-convergence, and interesting generalization of statistical convergence [4], was first presented by Kostyrko et al. [20] with use of the ideal $\mathcal{I}$ of subsets of the set of natural numbers $\mathbb{N}$ and further studies done in [27]. The study of ideal convergence in triple sequence has been initiated by Şahiner and Tripathy [31]. More analysis in this field and more implications of these statistical convergence and ideal convergence can be seen in [1, 11–13, 15, 22, 24–26, 28, 32–36, 39, 40].

After Zadeh's leading work in 1965, fuzzy set theory has been widely applied into practical problems. Fuzzy set theory is a very effective set for modelling uncertainty and vagueness in various problems that arise in some fields. Many authors have defined several concepts of FMS in different ways [3, 5, 16–18, 21, 23]. In [5, 6], George and Veeramani first investigated and presented the notion of fuzzy metric space with the use of continuous t-norms. Lately, several convergences in fuzzy metric spaces were studied by Gregori et al. [7–10].

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Generally, statistically convergent sequences fulfills most of the features of ordinary convergent sequences in metric spaces. For example, a statistically convergent sequence is statistically Cauchy ([29]) in an arbitrary metric space. Concordantly, we introduce studying $\mathcal{I}$-Cauchy and $\mathcal{I}$-convergence concepts of triple sequences on FMS.

Here, as it can be recalled the following basic concepts from [2, 5, 18, 38] needed in the course of the paper.

Definition 1. *The* 3*-tuple* $(X, \mathcal{M}, *)$ *is said to be a FMS if* X *is a nonempty set,* $*$ *is a continuous* t*-norm and* $\mathcal{M}$ *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following cases for all* $x, y, z \in X$ *and* $s, t > 0$:

Case 1. $\mathcal{M}(x, y, t) > 0$;
Case 2. $\mathcal{M}(x, y, t) = 1$ *iff* $x = y$;
Case 3. $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
Case 4. $\mathcal{M}(x, y, t) * \mathcal{M}(y, z, s) \leq \mathcal{M}(x, z, t + s)$;
Case 5. $\mathcal{M}(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous.*

Definition 2. *Let* $(X, \mathcal{M}, *)$ *be a FMS. We define open ball* $B_{\mathcal{M}}(x, r, t)$ *with centre* $x \in X$ *and radius* r, $0 < r < 1$, $t > 0$ *as*

$$B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) > 1 - r\}.$$

Let $(X, \mathcal{M}, *)$ be a FMS. We have

$$\tau_{\mathcal{M}} = \{A \subset X : x \in A \text{ iff there exists } t > 0, r \in (0, 1) \text{ such that } B_{\mathcal{M}}(x, r, t) \subset A\}.$$

Hence $\tau_{\mathcal{M}}$ is a topology on X. George and Veeramani [5] proved that $\{B_{\mathcal{M}}(x, r, t) : x \in X, t > 0, r \in (0, 1)\}$ forms a base of a topology $\tau_{\mathcal{M}}$ in X.

Definition 3. *Let* $(X, \mathcal{M}, *)$ *be a FMS. If for every* $r \in (0, 1)$ *and* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\mathcal{M}(x_n, x_0, t) > 1 - r$ *for all* $n > n_0$, *then a sequence* $\{x_n\}$ *converges to* x_0.

Definition 4. *A sequence* $\{x_n\}$ *in a FMS* $(X, \mathcal{M}, *)$ *is called to be a Cauchy sequence if for all* ε, $0 < \varepsilon < 1$ *and* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\mathcal{M}(x_n, x_m, t) > 1 - \varepsilon$ *for every* $n, m \geq n_0$.

Definition 5. *When every Cauchy sequence is convergent, a FMS is called to be complete.*

Definition 6. *([4]) Let* $A \subset \mathbb{N}$, *put* $A_n = \{k \in A : k \leq n\}$, $\forall n \in \mathbb{N}$. *Then*

$$\overline{\delta}(A) := \limsup_{n\to\infty} \frac{|A_n|}{n} \text{ and } \underline{\delta}(A) := \liminf_{n\to\infty} \frac{|A_n|}{n}$$

are called upper and lower asymptotic density of the set A, respectively. When $\overline{\delta}(A) = \underline{\delta}(A)$,

$$\delta(A) := \lim_{n\to\infty} \frac{|A_n|}{n}$$

is called to be an asymptotic density of A. All the three densities, if they exist, are in $[0,1]$.

Utilizing above information, we recall that a sequence $(x_k)_{k\in\mathbb{N}}$ is statistical convergent to x, if for all $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}\right) = 0.$$

If $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to x, we show st-$\lim x_k = x$.

The terms of statistical convergence and statistical Cauchy for sequences in FMS have been investigated by Li et al. [19].

Definition 7. *Let $(X, \mathcal{M}, *)$ be a FMS. if for all $r \in (0,1)$ and $t > 0$*

$$\delta\left(\{n \in \mathbb{N} : \mathcal{M}\left(x_n, x_0, t\right) > 1 - r\}\right) = 1,$$

then a sequence $\{x_n\}$ in X is called statistically convergent to $x_0 \in X$

Definition 8. *Let $(X, \mathcal{M}, *)$ be a FMS. If for every $r \in (0,1)$ and $t > 0$, there exists $N_0 \in \mathbb{N}$ such that*

$$\delta(\{k \in \mathbb{N} : \mathcal{M}(x_k, x_{N_0}, t) > 1 - r\}) = 1.$$

A sequence $\{x_n\}$ in X is called a statistically Cauchy sequence.

Also, Şahiner et al. [30] investigated the statistical convergence for triple sequence. A function $x : \mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is said to be a real triple sequence. A triple sequence (x_{nkl}) in $\mathbb{R}$ is called to be converge if there exists a point ℓ such that for all $\varepsilon > 0$, there exists a positive integer n_0 such that $|x_{nkl} - \ell| < \varepsilon$ for all $n, k, l \geq n_0$.

Definition 9. *If*

$$\delta_3(A) = \lim_{n,k,l\to\infty} \frac{|A_{nkl}|}{nkl}$$

exists, then a subset A of $\mathbb{N}^3$ is called to have natural density $\delta_3(A)$. From here, if for every $\varepsilon > 0$

$$\delta_3\left(\left\{(n,k,l) \in \mathbb{N}^3 : |x_{nkl} - \ell| \geq \varepsilon\right\}\right) = 0,$$

then a real triple sequence $x = (x_{nkl})$ is called to be statistically convergent to ℓ

Then, we give the terms of lacunary statistical convergence and lacunary statistical Cauchy for triple sequences in FMS as follows.

Definition 10. *Let $(X, \mathcal{M}, *)$ be a FMS and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. A triple sequence $\{x_{jkl}\}$ is called to be lacunary statistically convergent to $\ell \in X$, written as sts_{θ_3}-$\lim x_{jkl} = \ell$, if, for all $r \in (0,1)$ and $t > 0$,*

$$\lim_{r,s,t} \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : \mathcal{M}\left(x_{jkl}, \ell, t\right) > 1 - r\}\right| = 1.$$

Definition 11. *Let $(X, \mathcal{M}, *)$ be a FMS and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. A triple sequence $\{x_{jkl}\}$ in X is said to be lacunary statistically Cauchy sequence, if, for all $\alpha \in (0,1)$ and $t > 0$, there exists $\mathcal{M}, \mathcal{M}', \mathcal{M}'' \in \mathbb{N}$ such that for all $j, p \geq \mathcal{M}''$, $k, q \geq \mathcal{M}'$, $l, r \geq \mathcal{M}$,*

$$\delta_{\theta_3}\left(\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, x_{pqr}, t\right) > 1 - \alpha\right\}\right) = 1.$$

We recall the following some notations used in [20, 27].

Definition 12. *A class $\mathcal{I} \subset 2^R$ of subsets of a nonempty set R is called to be an ideal in R if (**i**) $\emptyset \in \mathcal{I}$; (**ii**) $M, N \in \mathcal{I}$ imply $M \cup N \in \mathcal{I}$; (**iii**) $M \in \mathcal{I}$, $N \subset M$ imply $N \in \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ in R is called an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletons, that is, $\{x\} \in I$ for each $x \in R$.*

Lemma 1. *Let $\mathcal{I}$ be a proper ideal in R, so $R \notin \mathcal{I}$, $R \neq \emptyset$. Then the class of sets*

$$\mathcal{F}(\mathcal{I}) = \{A \subset R : \exists M \in \mathcal{I} : A = R \backslash M\}$$

is a filter in R. It is said to be the filter associated with the ideal $\mathcal{I}$.

Definition 13. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and (X, ρ) be a metric space. The sequence $x = (x_n)$ in X is called to be $\mathcal{I}$-convergence to $\xi \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \varepsilon\} \in \mathcal{I}$.*

Definition 14. *A sequence $x = \{x_n\}_{n \in \mathbb{N}}$ in X is called to be $\mathcal{I}^*$-convergent to $\xi \in X$ iff there exists a set*

$$K \in \mathcal{F}(\mathcal{I}), \ K = \{k_1 < k_2 < ... < k_p < ...\} \subset \mathbb{N}$$

such that $\lim\limits_{p \to \infty} \rho\left(x_{k_p}, \xi\right) = 0$.

Definition 15. *([27]) Let (X, ρ) be a linear metric space. If for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x_N) \geq \varepsilon\} \in \mathcal{I}$, a sequence $x = (x_n)$ in X is called an $\mathcal{I}$-Cauchy sequence in X*

Definition 16. *([27]) Let (X, ρ) be a linear metric space. If there exists a set $K = \{k_1 < k_2 < ... < k_p < ...\} \subset \mathbb{N}$, $K \in \mathcal{F}(\mathcal{I})$ such that $\lim\limits_{p,r \to \infty} \rho\left(x_{k_p}, x_{k_r}\right) = 0$, a sequence $x = (x_n)$ in X is called to be $\mathcal{I}^*$-Cauchy sequence .*

In 2008, the term of ideal convergence for triple sequences used first time by Şahiner and Tripathy [31] in 2008.

Definition 17. *A real triple sequence (x_{nkl}) is called to be $\mathcal{I}$-convergent to ℓ if for every $\varepsilon > 0$,*

$$\left\{(n,k,l) \in \mathbb{N}^3 : |x_{nkl} - \ell| \geq \varepsilon\right\} \in \mathcal{I}_3.$$

In this case, one writes $\mathcal{I}_3\text{-}\lim x_{nkl} = \ell$.

Throughout the paper we consider the ideals of $2^{\mathbb{N}}$ by $\mathcal{I}$; the ideals of $2^{\mathbb{N}^2}$ by $\mathcal{I}_2$ and the ideals of $2^{\mathbb{N}^3}$ by $\mathcal{I}_3$.

2. $\mathcal{I}_3$-convergence in FMS

The terms of ideal convergence of triple sequences with a FMS were presented in this section.

Definition 18. *Let $\mathcal{I}_3$ be a nontrivial ideal of $\mathbb{N}^3$ and $(X, \mathcal{M}, *)$ be a FMS. A triple sequence $x = \{x_{jkl}\}$ of elements of X is said to be $\mathcal{I}_3$-convergent to $\ell \in X$ if, for each $r \in (0,1)$ and each $t > 0$,*

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, t\right) > 1 - r\right\} \in \mathcal{F}\left(\mathcal{I}_3\right).$$

In this stution we prefer to write as $\mathcal{I}_3^{\mathcal{M}}\text{-}\lim x = \ell$.

Theorem 1. *Let $(X, \mathcal{M}, *)$ be a FMS. Then, for each $r \in (0,1)$ and each $t > 0$, the following expression were equivalent:*
(i) $\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, t\right) > 1 - r\right\} \in \mathcal{F}\left(\mathcal{I}_3\right).$
(ii) $\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, t\right) \leq 1 - r\right\} \in \mathcal{I}_3.$

Theorem 2. *Let $x = \{x_{jkl}\}$ be a triple sequence in a FMS $(X, \mathcal{M}, *)$. When $x = \{x_{jkl}\}$ is $\mathcal{I}_3$-convergent to ℓ_1 and ℓ_2, $\ell_1 = \ell_2$.*

Proof. Assume that $\mathcal{I}_3^{\mathcal{M}}\text{-}\lim x = \ell_1$ and $\mathcal{I}_3^{\mathcal{M}}\text{-}\lim x = \ell_2$. Let ℓ_1 and ℓ_2 be two distinct points in X and $t > 0$. In that case $0 < \mathcal{M}\left(\ell_1, \ell_2, t\right) < 1$. Let $1 - \varepsilon \in \left(\mathcal{M}\left(\ell_1, \ell_2, t\right), 1\right).$ For each $1 - s \in (1 - \varepsilon, 1)\,,$ there exists $1 - s$ such that $(1 - s) * (1 - s) \geq 1 - \varepsilon$. Let

$$K_{\ell_1} = \left\{y \in X : \mathcal{M}\left(\ell_1, y, \frac{t}{2}\right) > 1 - s\right\}$$

and

$$K_{\ell_2} = \left\{y \in X : \mathcal{M}\left(\ell_2, y, \frac{t}{2}\right) > 1 - s\right\}.$$

We claim that $K_{\ell_1} \cap K_{\ell_2} = \emptyset$. Really, if there exists $z \in K_{\ell_1} \cap K_{\ell_2}$, then we get

$$\begin{aligned} \mathcal{M}\left(\ell_1, \ell_2, t\right) &\geq \mathcal{M}\left(\ell_1, z, \frac{t}{2}\right) * \mathcal{M}\left(z, \ell_2, \frac{t}{2}\right) \\ &\geq (1 - s) * (1 - s) \geq 1 - \varepsilon \\ &> \mathcal{M}\left(\ell_1, \ell_2, t\right) \end{aligned}$$

which is a contradiction. Since

$$\left\{y \in X : \mathcal{M}\left(\ell_2, y, \frac{t}{2}\right) > 1 - s\right\} \subset \left\{x \in X : \mathcal{M}\left(x, \ell_1, \frac{t}{2}\right) \leq 1 - s\right\},$$

it follows that

$$\begin{aligned} &\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell_2, \frac{t}{2}\right) > 1 - s\right\} \\ &\subseteq \left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell_1, \frac{t}{2}\right) \leq 1 - s\right\}. \end{aligned} \tag{1}$$

By (1), we get

$$\left\{(j,k,l)\in\mathbb{N}^3:\mathcal{M}\left(x_{jkl},\ell_2,\frac{t}{2}\right)>1-s\right\}\in\mathcal{F}(\mathcal{I}_3)$$
$$\subseteq\left\{(j,k,l)\in\mathbb{N}^3:\mathcal{M}\left(x_{jkl},\ell_1,\frac{t}{2}\right)\leq 1-s\right\}\in\mathcal{I}_3.$$

which is a contradiction. Therefore, we conclude that $\mathcal{I}_3^{\mathcal{M}}$-lim must be unique. So the desired result has been obtained. □

Theorem 3. *Let $(X,\mathcal{M},*)$ be a FMS and $\mathcal{I}_3$ be an admissible ideal. When triple sequence $x=\{x_{jkl}\}$ in X is convergent to ℓ, $x=\{x_{jkl}\}$ ideal converges to ℓ.*

Proof. Let $\lim\{x_{jkl}\}=\ell$. Let $r\in(0,1)$ and $t>0$. Then there exists a positive integer n_0 such that

$$\mathcal{M}(x_{jkl},\ell,t)>1-r$$

for all $j>n_0, k>n_0, l>n_0$. Since

$$\begin{aligned}K_{\mathcal{M}}&=\left\{(j,k,l)\in\mathbb{N}^3:\mathcal{M}(x_{jkl},\ell,\varepsilon)\leq 1-r\right\}\\&\subseteq\mathbb{N}^3-\{(j_{n_0+1},k_{n_0+1},l_{n_0+1}),(j_{n_0+2},k_{n_0+2},l_{n_0+2}),...\}\end{aligned}$$

and the ideal $\mathcal{I}_3$ is admissible, this implies that $K_{\mathcal{M}}\in\mathcal{I}_3$. Therefore

$$\left\{(j,k,l)\in\mathbb{N}^3:\mathcal{M}(x_{jkl},\ell,\varepsilon)>1-r\right\}\in\mathcal{F}(\mathcal{I}_3),$$

that is $\mathcal{I}_3$-$\lim x=\ell$. We complete the proof. □

We gave the term of $\mathcal{I}_3^*$-convergence of triple sequences with a FMS.

Definition 19. *Let $(X,\mathcal{M},*)$ be a FMS. We say that a triple sequence $x=\{x_{jkl}\}$ in X is said to be $\mathcal{I}_3^*$-convergence to $\ell\in X$ if there exists a subset $K=\{(j_m,k_m,l_m):j_1<j_2<...;k_1<k_2<...;l_1<l_2<....\}$ of $\mathbb{N}^3$ such that $K\in\mathcal{F}(\mathcal{I}_3)$ (i.e. $\mathbb{N}^3\backslash K\in\mathcal{I}_3$) and $\{x_{j_mk_ml_m}\}$ converges to ℓ.*

In this stution we prefer to write $\mathcal{I}_3^{*\text{-}\mathcal{M}}\lim x=\ell$.

Theorem 4. *Let $(X,\mathcal{M},*)$ be a FMS and $\mathcal{I}_3$ be an admissible ideal. If $\mathcal{I}_3^{*\text{-}\mathcal{M}}\lim x=\ell$, then $\mathcal{I}_3^{\mathcal{M}}\lim x=\ell$.*

Proof. Let $x=\{x_{jkl}\}$ be an $\mathcal{I}_3^*$-convergence to $\ell\in X$. Then by definition,

$$K=\{(j_m,k_m,l_m):j_1<j_2<...;k_1<k_2<...;l_1<l_2<...\}$$

of $\mathbb{N}^3$, $K\in\mathcal{F}(\mathcal{I}_3)$ such that $\{x_{j_mk_ml_m}\}$ converges to ℓ, so there exists $N\in\mathbb{N}$ such that all $r\in(0,1)$ and $t>0$,

$$\mathcal{M}(x_{j_mk_ml_m},\ell,t)>1-r,\ \forall m>N.$$

Since $\mathcal{I}_3$ is an admissible and

$$\{(j_m,k_m,l_m)\in K:\mathcal{M}(x_{j_mk_ml_m},\ell,t)\leq 1-r\}$$

is contained in $\{j_1 < j_2 < ... < j_{N-1}; k_1 < k_2 < ... < k_{N-1}; l_1 < l_2 < .... < l_{N-1}\}$, we get

$$\{(j_m, k_m, l_m) \in K : \mathcal{M}(x_{j_m k_m l_m}, \ell, t) \leq 1 - r\} \in \mathcal{I}_3.$$

In this case, when we let $H = \mathbb{N}^3 \backslash K$ it is obvious that $H \in \mathcal{I}_3$ and

$$\begin{aligned} &\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}(x_{jkl}, \ell, t) \leq 1 - r\} \subset H \cup \\ &\qquad \{j_1 < j_2 < ... < j_{N-1}; k_1 < k_2 < ... < k_{N-1}; l_1 < l_2 < .... < l_{N-1}\}. \end{aligned} \tag{2}$$

Hence

$$\{j_1 < j_2 < ... < j_{N-1}; k_1 < k_2 < ... < k_{N-1}; l_1 < l_2 < .... < l_{N-1}\} \in \mathcal{I}_3.$$

This means that

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}(x_{jkl}, \ell, t) > 1 - r\right\} \in \mathcal{F}(\mathcal{I}_3),$$

so, $\{x_{jkl}\}$ is $\mathcal{I}_3$-convergent to ℓ. Hence the proof is complete. $\square$

In the example given below, the inverse of Theorem 4 is generally not provided.

Example 1. *Let $a * b = ab$ and for all $a, b \in [0,1]$. If for every $x, y \in \mathbb{R}$ and $t > 0$*

$$\mathcal{M}(x, y, t) = \frac{t}{t + |x - y|},$$

then $(\mathbb{R}, \mathcal{M}, *)$ is a FMS with the usual metric $|.|$.

Let $\mathbb{N}^3 = \cup_{i,j,l} \Delta_{ijl}$ be a decomposition of $\mathbb{N}^3$ such that, for any $(m,n,o) \in \mathbb{N}^3$, each Δ_{ijl} contains infinitely many (i,j,l)'s where $i \geq m$, $j \geq n$, $l \geq o$ and $\Delta_{ijl} \cap \Delta_{mno} = \emptyset$ for $(i,j,l) \neq (m,n,o)$. Now we define a sequence $x_{mno} = \frac{1}{ijl}$ if $(m,n,o) \in \Delta_{ijl}$. It is immediate to see that $\{x_{mno}\}$ is not $\mathcal{I}_3^*$-convergence to 0, but $\{x_{jkl}\}$ is $\mathcal{I}_3$-convergence to 0.

The following definition was needed to prove that an $\mathcal{I}_3$-convergence come across with an $\mathcal{I}_3^*$-convergence for admissible ideals with property (AP3).

Definition 20. *An admissible ideal $\mathcal{I}_3 \subset 2^{\mathbb{N}^3}$ is said to satisfy the condition (AP3) if for every sequence $(A_j)_{j \in \mathbb{N}}$ of pairwise disjoint sets from $\mathcal{I}_3$ there are sets $B_j \subset \mathbb{N}$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is a finite set for every $j \in \mathbb{N}$ and $\cup_{j \in \mathbb{N}} B_j \in \mathcal{I}_3$.*

Theorem 5. *Let $(X, \mathcal{M}, *)$ be a FMS and $\mathcal{I}_3$ satisfy the condition (AP3). Then $\mathcal{I}_3$-convergence and $\mathcal{I}_3^*$-convergence coincide.*

Proof. Let $x = \{x_{jkl}\}$ be an $\mathcal{I}_3^*$-convergence. Then, by Theorem 4, this sequence is $\mathcal{I}_3$-convergence where $\mathcal{I}_3$ need not have the (AP3) condition. Then, it is sufficient to prove that $x = (x_{jkl})$ in X is a $\mathcal{I}_3^*$-convergence to $\ell \in X$ under assumption that (x_{jkl}) is an $\mathcal{I}_3$-convergence to $\ell \in X$. Hence by definition, for all $r \in (0,1)$ and $t > 0$,

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, t\right) > 1 - r\right\} \in \mathcal{F}(\mathcal{I}_3).$$

Let

$$K_s = \left\{(j,k,l) \in \mathbb{N}^3 : 1 - \frac{1}{s+1} > \mathcal{M}\left(x_{jkl}, \ell, t\right) > 1 - \frac{1}{s}\right\}.$$

Then, for $t > 0$ and each $s = 1, 2, ...$, we have that $\{K_1, K_2, ...\}$ is countable and $K_s \in \mathcal{I}_3$, and $K_i \cap K_j = \emptyset$ for $i \neq j$. By the property (AP3), there is countable class of sets $\{B_1, B_2, ...\} \in \mathcal{I}_3$ such that $K_i \Delta B_i$ is a finite set for every $i \in \mathbb{N}$ and $B = \cup_{i\in\mathbb{N}} B_i \in \mathcal{I}_3$. From the definition of the associate filter $\mathcal{F}(\mathcal{I}_3)$ there is a set $A \in \mathcal{F}(\mathcal{I}_3)$ such that $A = \mathbb{N}^3 \backslash B$. To prove the theorem we should aim that the subsequence $\{x_{jkl}\}_{(j,k,l)\in A}$ converges to ℓ. Let $\mu \in (0,1)$ and each $t > 0$. Put $q = 1, 2, ...$ such that $\frac{1}{q} < \mu$. So

$$\begin{aligned}
&\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, t\right) \leq 1 - \mu\right\} \\
&\subset \left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, t\right) \leq 1 - \frac{1}{q}\right\} \\
&\subset \cup_{i=1}^{q+1} K_i.
\end{aligned}$$

Since $K_i \Delta B_i$, $i = 1, 2, ..., q+1$ are finite, there exists $(j_0, k_0, l_0) \in \mathbb{N}^3$ such that

$$\begin{aligned}
&\cup_{i=1}^{q+1} B_i \cap \left\{(j_0, k_0, l_0) : j \geq j_0,\ k \geq k_0 \text{ and } l \geq l_0\right\} \\
&= \cup_{i=1}^{q+1} K_i \cap \left\{(j_0, k_0, l_0) : j \geq j_0,\ k \geq k_0 \text{ and } l \geq l_0\right\}.
\end{aligned} \tag{3}$$

If $j \geq j_0$, $k \geq k_0$, $l \geq l_0$ and $(j,k,l) \in A$ then $(j,k,l) \notin \cup_{i=1}^{q+1} B_i$. Therefore, by (3), we have $(j,k,l) \notin \cup_{i=1}^{q+1} K_i$. Thus, $j \geq j_0$, $k \geq k_0$, $l \geq l_0$ and $(j,k,l) \in A$, we have

$$\mathcal{M}(x_{jkl}, \ell, t) > 1 - \mu.$$

Since $\mu \in (0,1)$ is arbitrary, this shows that $\mathcal{I}_3^*\text{-}\lim x_{jkl} = \ell$. □

3. $\mathcal{I}_3$- and $\mathcal{I}_3^*$-Cauchy sequences on FMS

Now, the terms of $\mathcal{I}_3$-Cauchy sequence and $\mathcal{I}_3^*$-Cauchy sequence was presented in FMS.

Definition 21. *Let $(X, \mathcal{M}, *)$ be a FMS. A triple sequence $\{x_{jkl}\}$ in X is called $\mathcal{I}_3$-Cauchy sequence if for every $\alpha \in (0,1)$ and $t > 0$, there exists N_1, N_2 and N_3 such that for all $j, p \geq N_1$, $k, q \geq N_2$, $l, r \geq N_3$,*

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, x_{pqr}, t\right) > 1 - \alpha\right\} \in \mathcal{F}(\mathcal{I}_3).$$

In this case, it is stated that $\{x_{jkl}\}$ is $\in \mathcal{I}_3^{\mathcal{M}}$-Cauchy.

Proceeding similarly, we get the following consequence.

Corollary 1. *When a triple sequence in a FMS is Cauchy, it is $\mathcal{I}_3^{\mathcal{M}}$-Cauchy.*

Definition 22. *Let $(X, \mathcal{M}, *)$ be a FMS. A triple sequence $x = \{x_{jkl}\}$ in X is called to be $\mathcal{I}_3^*$-Cauchy sequence in X if there exists a subset $K = \{(j_m, k_m, l_m) :$*

$j_1 < j_2 < ...; k_1 < k_2 < ...; l_1 < l_2 < ....\}$ of $\mathbb{N}^3$ such that $K \in \mathcal{F}(\mathcal{I}_3)$ and $\{x_{j_m k_m l_m}\}$ is a Cauchy sequence in X, i.e., there exists $N \in \mathbb{N}$ such that

$$\mathcal{M}(x_{jkl}, x_{pqr}, t) > 1 - \alpha$$

whenever $j \geq p \geq N$, $k \geq q \geq N$, $l \geq r \geq N$.

Here we can say that $\{x_{jkl}\}$ is $\in \mathcal{I}_3^{*\text{-}\mathcal{M}}$-Cauchy.

Since the next theorems are respectively analogues to Theorems 4 and 5, it can be proved on same methods.

Theorem 6. *Let $(X, \mathcal{M}, *)$ be a FMS and $\mathcal{I}_3$ be an admissible ideal. When a triple sequence $\{x_{jkl}\}$ is $\mathcal{I}_3^{*\text{-}\mathcal{M}}$-Cauchy, it is $\mathcal{I}_3^{\mathcal{M}}$-Cauchy.*

Theorem 7. *Let $(X, \mathcal{M}, *)$ be a FMS and $\mathcal{I}_3$ satisfy the condition (AP3). When a triple sequence $\{x_{jkl}\}$ is $\mathcal{I}_3^{\mathcal{M}}$-Cauchy, it is also $\mathcal{I}_3^{*\text{-}\mathcal{M}}$-Cauchy.*

Therefore, we now present the following theorem.

Theorem 8. *Let $\{x_{jkl}\}$ be a triple sequence in a FMS $(X, \mathcal{M}, *)$ and $\mathcal{I}_3$ be an arbitrary admissible ideal with property (AP3). Then $\mathcal{I}_3^{\mathcal{M}}\text{-}\lim x = \ell$ implies that $\{x_{jkl}\}$ is an $\mathcal{I}_3^{\mathcal{M}}$-Cauchy sequence.*

Proof. Let $\mathcal{I}_3^{\mathcal{M}}\text{-}\lim x = \ell$. Then for every $r \in (0,1)$ and $t > 0$,

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, t\right) > 1 - r\right\} \in \mathcal{F}\left(\mathcal{I}_3\right).$$

Let $\alpha \in (0,1)$ and $t > 0$. Then there exists $\alpha_1 \in (0, \alpha)$ such that $(1 - \alpha_1) * (1 - \alpha_1) > 1 - \alpha$. According to Theorem 5 and Definition 20, there exists a subset $A = \{(j_m, k_m, l_m) : j_1 < j_2 < ...; k_1 < k_2 < ...; l_1 < l_2 < ....\}$ of $\mathbb{N}^3$ such that $A \in \mathcal{F}(\mathcal{I}_3)$ and $\{x_{j_m k_m l_m}\}$ converges to ℓ. Thus there exists $N \in \mathbb{N}$ such that

$$\mathcal{M}(x_{j_m k_m l_m}, \ell, \frac{t}{2}) > 1 - \alpha_1 \text{ for every } m > N.$$

Let $(p,q,r) \in \left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, \frac{t}{2}\right) > 1 - \alpha_1\right\}$. Then

$$\begin{aligned} \mathcal{M}(x_{pqr}, x_{j_m k_m l_m}, t) &\geq \mathcal{M}(x_{pqr}, \ell, t/2) * \mathcal{M}(x_{j_m k_m l_m}, \ell, t/2) \\ &\geq (1 - \alpha_1) * (1 - \alpha_1) > 1 - \alpha. \end{aligned}$$

Hence $(p,q,r) \in \left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, x_{j_m k_m l_m}, \frac{t}{2}\right) > 1 - \alpha_1\right\}$. It follows that

$$\begin{aligned} &\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, \frac{t}{2}\right) > 1 - \alpha_1\right\} \\ &\subseteq \left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, x_{j_m k_m l_m}, t\right) > 1 - \alpha_1\right\}. \end{aligned} \tag{4}$$

Since $\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, \ell, \frac{t}{2}\right) > 1 - \alpha_1\right\} \in \mathcal{F}(\mathcal{I}_3)$ and (4), we get that

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}\left(x_{jkl}, x_{j_m k_m l_m}, t\right) > 1 - \alpha_1\right\} \in \mathcal{F}\left(\mathcal{I}_3\right).$$

This indicate that the triple sequence $\{x_{jkl}\}$ in X is an $\mathcal{I}_3^{\mathcal{M}}$-Cauchy sequence. $\square$

Remark 1. *But the converse of the above theorem is not necessarily true, i.e.* $\mathcal{I}_3^{\mathcal{M}}$*-Cauchy sequence does not imply* $\mathcal{I}_3^{\mathcal{M}}\text{-}\lim x = \ell$*. This can be illustrated by the example given below.*

Example 2. *Let* $\mathcal{I}_3^{\mathcal{M}}(\delta) = \left\{A \subset \mathbb{N}^3 : \delta_3(A) = 0\right\}$ *and* $X = \{x_{jkl} : (j,k,l) \in \mathbb{N}^3\}$*, where* $x_{jkl} = 1 - \frac{1}{(j+1)(k+1)(l+1)}$ $(j,k,l \in \mathbb{N})$ *and* $a*b = \min\{a,b\}$ *for all* $a, b \in [0,1]$*, and let* $\mathcal{M}$ *be a fuzzy set on* $X^2 \times (0,\infty)$ *define as follows* $\mathcal{M}(x,y,t)$ *to be* 1 *for* $x = y$ *and* $\min\{x,y\}$ *otherwise, for all* $x, y \in X$ *and* $t > 0$*. Hence* $(X, \mathcal{M}, *)$ *is a FMS and triple sequence* $\{x_{jkl}\}$ *in* $(X, \mathcal{M}, *)$ *is* $\mathcal{I}_3^{\mathcal{M}}$*-Cauchy, but it is not* $\mathcal{I}_3^{\mathcal{M}}$*-convergent.*

Let $\alpha \in (0,1)$ *and* $t > 0$*. Therefore there exists* $p, q, r \in \mathbb{N}$ *such that* $\frac{1}{(p+1)(q+1)(r+1)} < \alpha$*. Hence*

$$\mathcal{M}(x_{jkl}, x_{pqr}, t) = x_{pqr} = 1 - \frac{1}{(p+1)(q+1)(r+1)} > 1 - \alpha$$

for all $j > p$, $k > q$, $l > r$*. Thus*

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}(x_{jkl}, x_{pqr}, t) > 1 - \alpha\right\} \in \mathcal{F}(\mathcal{I}_3(\delta)).$$

which means that $\{x_{jkl}\}$ *is* $\mathcal{I}_3^{\mathcal{M}}$*-Cauchy sequence. Let* $\ell \in X$*. Then there exists* $p, q, r \in \mathbb{N}$ *such that* $\ell = x_{pqr} = 1 - \frac{1}{(p+1)(q+1)(r+1)}$*. Now, fix* $t_0 = \alpha_0 = \frac{1}{3(p+1)(q+1)(r+1)}$*. Then*

$$\mathcal{M}(x_{jkl}, \ell, t_0) = \mathcal{M}(x_{jkl}, x_{pqr}, t_0) = x_{pqr} = 1 - \frac{1}{(p+1)(q+1)(r+1)} \leq 1 - \alpha_0$$

for all $j > p$, $k > q$, $l > r$*. Hence*

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}(x_{jkl}, \ell, t_0) \leq 1 - \alpha_0\right\} \in \mathcal{F}(\mathcal{I}_3(\delta)),$$

which implies that

$$\left\{(j,k,l) \in \mathbb{N}^3 : \mathcal{M}(x_{jkl}, \ell, t_0) > 1 - \alpha_0\right\} \in \mathcal{I}_3(\delta).$$

So $\{x_{jkl}\}$ *is not* $\mathcal{I}_3^{\mathcal{M}}$*-convergent.*

As a note, all these findings imply the similar theorems for ideal and statistically convergence and Cauchy sequences which are investigated in [19] and [26].

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Q-MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS RELATED WITH JANOWSKI FUNCTION

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Abstract. In the present paper, we introduce and explore certain new classes of meromorphic functions related to closed-to-convexity and q-calculus. Such results as coefficient estimates, grow the property and partial sums are derived. It is important to mention that our results are generalization of number of existing results in literature.

1. Introduction

Let $\sum_1$ denote the class of meromorphic functions of the form:

$$f(\omega)=\frac{1}{\omega}+\sum_{t=1}^{\infty}a_t\omega^t, \tag{1}$$

which are analytic in the punctured open unit disc $U^*=\{\omega:\omega\in\mathbb{C}$ and $0<\{\omega\}<1\}=U\backslash\{0\}$, where $U=U^*\cup\{0\}$.

In Geometric Function Theory, several subclasses of the meromorphic functions have already been examined and investigated through many perceptions, see([9,10, 12,18,21,22]). Ismail et al. [8] were the first to use the q-derivative operator Δ_q in order to study a certain q-analogue of the class T^* of starlike functions in U. Certain basic properties of the q-close-to-convex functions were studied by Raghavendar and Swaminathan [28], Aral et al. [2] successfully studied the applications of the

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q-calculus in operator theory. In fact, they found significant application of the q-calculus mainly in the Geometric Function Theory. Moreover, the generalized q-hypergeometric function was first introduced by Srivastava [26], see also([1, 3, 5, 6, 14, 16, 20]).

A function $f \in \sum_1$ is said to be meromorphic starlike of order α defined as:

$$f \in \sum^{MS}(\alpha) \Leftrightarrow \operatorname{Re}\left(\frac{\omega f'(\omega)}{f(\omega)}\right) < -\alpha \quad (0 \leq \alpha < 1;\ \omega \in U^*). \tag{2}$$

A related class of meromorphic convex function $\sum^{MC}(\alpha)$ is defined as:

$$f \in \sum^{MC}(\alpha) \Leftrightarrow \operatorname{Re}\left(1 + \frac{\omega f''(\omega)}{f'(\omega)}\right) < -\alpha \quad (\ \omega \in U^*). \tag{3}$$

By $\sum^{MK}(\alpha)$, we mean $f \in \sum_1$ and the class of all close-to-convex functions which satisfies the condition

$$\operatorname{Re}\left(\frac{\omega f'(\omega)}{g(\omega)}\right) < -\alpha, \quad \text{where } g \in \sum^{MS}(\alpha). \tag{4}$$

The study of operators plays main role in the theory of geometric functions. Many differential and integral operators can be written in terms of convolution of certain holomorphic functions.

For $g(\omega) = \frac{1}{\omega} + \sum\limits_{t=0}^{\infty} b_t \omega^t \in \sum_1$ and f given in (1). The Convolution (Hadamard product) is denoted by $f * g$ and defined as:

$$(f * g)(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} a_t b_t \omega^t = (g * f)(\omega). \tag{5}$$

A function h analytic in U and of the form

$$h(\omega) = 1 + \sum_{t=1}^{\infty} r_t \omega^t,$$

A given function Ψ with $\Psi(0) = 1$ is said to belong to the class $S^*[A, B]$ if and only if

$$\Psi(\omega) \prec \frac{1 + A\omega}{1 + B\omega} \qquad (-1 \leq B < A \leq 1).$$

This class was presented and studied by Janowski [11]. By taking $A = 1$ and $B = -1$, we obtain the class P of functions with a positive real part. It is important to mention that $\Psi(\omega) \in S^*[A, B]$ if and only if there exists $r \in P$ such that

$$\Psi(\omega) = \frac{(A+1) R(\omega) - (A-1)}{(B+1) R(\omega) - (B-1)} \qquad (-1 \leq B < A \leq 1).$$

Motivated by the works of Srivastava et al. see([7, 17, 19, 23, 25, 27])also see([4,13,15,24,29]). In this paper, we shall consider new subfamilies of q meromorphic close-to-convex functions with respect to Janowski functions.

Throughout in this paper, we assume

$$0 \leq \eta < 1, -1 \leq B < A \leq 1, 0 \leq q < 1,\ \omega \in U^*,\ f, g \in \sum_1,$$

$$\begin{aligned} \Lambda(t, \eta, A, q) &= \left[|b_t| \left| (2[t]_q \eta + 2(1-\eta) + \eta(A+1)) - (A+1)(1-\eta) \right| \right], \\ \Lambda(t, B, q) &= [t]_q (2 + B + 1), \end{aligned}$$

and

$$\gamma(\eta, A, B, q) = |(B+1) + (A+1)\eta - (A+1)(1-\eta)q| + 2(1-\eta)(1-q),$$

unless otherwise mentioned.

Definition 1. *(see [9] and [10]) The q-derivative (q-difference) Δ_q of a function f is defined in a given subset of $\mathbb{C}$ by*

$$\left(\Delta_q f\right)(\omega) = \begin{cases} \frac{f(\omega) - f(q\omega)}{(1-q)\omega} & (\ \omega \neq 0), \\ f^{'}(0) & (\ \omega = 0), \end{cases}$$

where $0 < q < 1$. This implies the following.

$$\lim_{q \to 1^-} \left(\Delta_q f\right)(\omega) = \lim_{q \to 1^-} \frac{f(\omega) - f(q\omega)}{(1-q)\omega} = f^{'}(\omega),$$

provided that $f^{'}(0)$ exists.

The function $\Delta_q f$ has Maclaurin's series representation

$$(\Delta_q f)(\omega) = \sum_{t=0}^{\infty} [t]_q a_t \omega^{t-1},$$

where $q \in (0, 1)$ and define the q-number $[\gamma]_q$ by

$$[\gamma]_q = \begin{cases} \frac{1-q^\gamma}{1-q} & (\ \gamma \in \mathbb{C}), \\ \sum_{k=0}^{t-1} q^\gamma = 1 + q^2 + q^3 + \ldots + q^{t-1} & (t \in\ \mathbb{N}). \end{cases}$$

For more details about q-derivatives, we refer the reader to (see [6]).

Definition 2. *For $f \in \sum_1$, let the q-derivative operator (q-difference operator) be defined by*

$$(\Delta_q f)(\omega) = \frac{f(q\omega) - f(\omega)}{(q-1)\omega} = -\frac{1}{q\omega^2} + \sum_{t=0}^{\infty} [t]_q a_t \omega^{t-1} \qquad (\omega \in U^*). \tag{6}$$

Similarly

$$(\Delta_q g)(\omega) = \frac{g(q\omega) - g(\omega)}{(q-1)\omega} = -\frac{1}{q\omega^2} + \sum_{t=0}^{\infty} [t]_q b_t \omega^{t-1} \qquad (\omega \in U^*). \tag{7}$$

Definition 3. *A function $f \in \sum_1$ is said to belong to the class $f \in T^*_{(q,\eta)}[A,B]$ if and only if*

$$\left| \frac{(B-1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A-1)}{(B+1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Where $g \in \sum^{MS}(\alpha)$, It is easily observed that

$$\lim_{q\to 1^-} T^*_{(q,0)}[A,B] = S_q^{MK}[A,B],$$

secondly we have

$$\lim_{q\to 1^-} T^*_{(q,0)}[1,-1] = S_q^{MK},$$

where $S_q^{MK}[A,B]$ is the well-known function of meromorphic close-to-convex function.

2. Main Results

2.1. **Coefficient estimates.**

Theorem 1. *A function $f \in \sum_1$ of the form given by (1) is in the class $T^*_{(q,\eta)}[A,B]$ if it satisfies the following condition.*

$$\sum_{t=1}^{\infty}(\Lambda(t,B,q)\,|a_t|\,q + (t,\eta,A,q)\,|b_t|\,q) \leq \gamma(\eta,A,B,q), \tag{8}$$

where

$$\Lambda(t,B,q) = [t]_q(2+B+1), \tag{9}$$

$$\Lambda(t,\eta,A,q) = [|b_t|\,|(2[t]_q\eta + 2(1-\eta) + \eta(A+1)) - (A+1)(1-\eta)|] \tag{10}$$

and

$$\gamma(\eta,A,B,q) = |(B+1) + (A+1)\eta - (A+1)(1-\eta)q| + 2(1-\eta)(1-q). \tag{11}$$

Proof. Assuming that (8) holds, it suffices to show that

$$\left| \frac{(B-1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A-1)}{(B+1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Consider we have

$$\left| \frac{(B-1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A-1)}{(B+1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A+1)} - \frac{1}{1-q} \right|$$

which implies

$$= \left| \frac{-(B-1)\omega\Delta_q f(\omega) - (A-1)\left[(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)\right]}{-(B+1)\omega\Delta_q f(\omega) - (A+1)\left[(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)\right]} - 1 \right| + \frac{q}{1-q}$$

Thus

$$2\left|\frac{\omega\Delta_q f(\omega)+(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}{-(B+1)\omega\Delta_q f(\omega)-(A+1)\left[(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)\right]}\right|+\frac{q}{1-q}$$

Using (1), (6) and (7) in above equation.

$$\left|\frac{\begin{array}{c}2(1-\eta)(q-1)\\ +2\sum_{t=1}^{\infty}\left[[t]_q(a_t+\eta b_t)+(1-\eta)b_t\right]q\omega^{t+1}\end{array}}{\begin{array}{c}(B+1)+(A+1)\eta-(A+1)(1-\eta)q\\ -\sum_{t=1}^{\infty}\left[[t]_q((B+1)a_t+\eta(A+1)b_t)-(A+1)(1-\eta)b_t\right]q\omega^{t+1}\end{array}}\right|\leq 1,$$

we get

$$\begin{aligned}&\sum_{t=1}^{\infty}[t]_q\,|a_t|\,(2+B+1)q+\sum_{t=1}^{\infty}\left[|b_t|\,|(2[t]_q\eta+2(1-\eta)+\eta(A+1))-(A+1)(1-\eta)|\right]q\\ &\leq |(B+1)+(A+1)\eta-(A+1)(1-\eta)q|+2(1-\eta)(1-q).\end{aligned}\tag{12}$$

The last expression become

$$\sum_{t=1}^{\infty}\Lambda(t,B,q)\,|a_t|\,q+\sum_{t=1}^{\infty}\Lambda(t,\eta,A,q)\,|b_t|\,q\ \leq\gamma(\eta,A,B,q).$$

This complete the proof of Theorem 2.1.

□

Corollary 1. *If a function $f\in\sum_1$ of the form given by (1) is in the class $T^*_{(q,\eta)}[A,B]$, then*

$$|a_t|\leq\frac{\gamma(\eta,A,B,q)}{\Lambda(t,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(t,B,q)}\,|b_t|\qquad(t\in N),\tag{13}$$

with equality for each t, we define the function of the form

$$f_t(\omega)=\frac{1}{\omega}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(t,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(t,B,q)}\,|b_t|\right)\omega^t,$$

where $\Lambda(t,B,q)$, $\Lambda(t,\eta,A,q)$ and $\gamma(\eta,A,B,q)$ are given by (9), (10) and (11) respectively.

2.2. **Distortion inequalities.**

Theorem 2. *If $f\in T^*_{(q,\eta)}[A,B]$, then*

$$\begin{aligned}&\frac{1}{r}-\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)r\leq|f(\omega)|\\ &\leq\frac{1}{r}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)r\ \ (|\omega|=r),\end{aligned}$$

where equality holds for the function

$$f(\omega)=\frac{1}{\omega}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)\omega.$$

Proof. Let $f \in T^{*}_{(q,\eta)}[A,B]$. Then in view of Theorem (2.1), we have

$$\Lambda(1,B,q)\sum_{t=1}^{\infty}|a_t| \le \sum_{t=1}^{\infty}\Lambda(t,B,q)\,|a_t| \quad \le \gamma(\eta,A,B,q)-\sum_{t=1}^{\infty}\Lambda(1,\eta,A,q)\,|b_t|\,,$$

which yields

$$|f(\omega)| \le \frac{1}{r}+\sum_{t=1}^{\infty}|a_t|\,r^{t} \le \frac{1}{r}+r\sum_{t=1}^{\infty}|a_t| \le \frac{1}{r}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)r. \tag{14}$$

Similarly, we have

$$|f(\omega)| \ge \frac{1}{r}-\sum_{t=1}^{\infty}|a_t|\,r^{t} \ge \frac{1}{r}-r\sum_{t=1}^{\infty}|a_t| \ge \frac{1}{r}-\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)r \tag{15}$$

which is required. □

Theorem 3. *If $f \in T^{*}_{(q,\eta)}[A,B]$, then*

$$\begin{aligned}&\frac{1}{r^2}-2\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)\\ &\le |f(\omega)| \le \frac{1}{r^2}+2\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)\quad (|\omega|=r)\,,\end{aligned}$$

where equality holds for the function

$$f(\omega)=\frac{1}{\omega}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)\omega.$$

Proof. Let $f \in T^{*}_{(q,\eta)}[A,B]$. Then in view of theorem (2.1), we have

$$\Lambda(1,B,q)\sum_{t=1}^{\infty}|a_t| \le \sum_{t=1}^{\infty}\Lambda(t,B,q)\,|a_t| \quad \le \gamma(\eta,A,B,q)-\sum_{t=1}^{\infty}\Lambda(1,\eta,A,q)\,|b_t|\,.$$

Differentiate (14) and (15), we get

$$\left|f^{'}(\omega)\right| \le -\frac{1}{r^2}+\sum^{\infty} t\,|a_t|\,r^{t-1} \le -\frac{1}{r^2}+\sum_{t=1}^{\infty}|a_t| \le -\frac{1}{r^2}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right). \tag{16}$$

Similarly, we have

$$\left|f^{'}(\omega)\right| \ge -\frac{1}{r^2}-\sum^{\infty} t\,|a_t|\,r^{t-1} \ge -\frac{1}{r^2}-\sum_{t=1}^{\infty}|a_t| \ge -\frac{1}{r^2}-\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right). \tag{17}$$

Comparing (16) and (17).

We have thus completed the proof of Theorem 2.4. □

2.3. **Partial sums.**
In this section, we examine the ratio of a function of the form (1) to its sequence of partial sums

$$f_t(\omega) = \frac{1}{\omega} + \sum_{t=1}^{k} a_t \omega^t,$$

when the coefficients of f are sufficiently small to satisfy condition (8). We will determine sharp lower bounds for

$$\mathrm{Re}\left(\frac{f(\omega)}{f_\nu(\omega)}\right), \quad \mathrm{Re}\left(\frac{f_\nu(\omega)}{f(\omega)}\right), \quad \mathrm{Re}\left(\frac{(\Delta_q f)(\omega)}{(\Delta_q f_\nu)(\omega)}\right) \quad \text{and} \quad \mathrm{Re}\left(\frac{(\Delta_q f_\nu)(\omega)}{(\Delta_q f)(\omega)}\right).$$

Theorem 4. *If f of the form (1) satisfies condition (8), then*

$$\mathrm{Re}\left(\frac{f(\omega)}{f_\nu(\omega)}\right) \geq 1 - \frac{1}{\kappa_{\nu+1}} \qquad (\omega \in U^*), \tag{18}$$

and

$$\mathrm{Re}\left(\frac{f_\nu(\omega)}{f(\omega)}\right) \geq \frac{\kappa_{\nu+1}}{1+\kappa_{\nu+1}} \qquad (\omega \in U^*), \tag{19}$$

where

$$\kappa_\nu = \frac{\gamma(\eta, A, B, q)}{\Lambda(t, B, q)} - \frac{\Lambda(t, \eta, A, q)}{\Lambda(t, B, q)} |b_t|. \tag{20}$$

Proof. In order to prove inequality (18), we set

$$\begin{aligned} \kappa_{\nu+1}\left[\frac{f(\omega)}{f_\nu(\omega)} - \left(1 - \frac{1}{\kappa_{\nu+1}}\right)\right] &= \frac{1 + \sum\limits_{t=1}^{\nu} a_t \omega^{t-1} + \kappa_{\nu+1} \sum\limits_{t=\nu+1}^{\infty} a_t \omega^{t+1}}{1 + \sum\limits_{t=1}^{\nu} a_t \omega^{t+1}} \\ &= \frac{1 + h_1(\omega)}{1 + h_2(\omega)}. \end{aligned}$$

Let

$$\frac{1 + h_1(\omega)}{1 + h_2(\omega)} = \frac{1 + g(\omega)}{1 - g(\omega)}.$$

Finally, to prove the inequality in (18), we get

$$\sum_{t=1}^{\nu} \left(1 - \kappa_{\nu+1}\right) |a_t| + \sum_{t=\nu+1}^{\infty} \left(\kappa_{\nu+1} - \kappa_t\right) |a_t| \geq 0.$$

The proof of inequality in (18) is now completed.
Similarly, we set

$$1 + \kappa_\nu \left[\frac{f_\nu(\omega)}{f(\omega)} - \left(\frac{\kappa_{\nu+1}}{1+\kappa_{\nu+1}}\right)\right] = \frac{1 + \sum\limits_{t=1}^{\nu} a_t \omega^{t-1} - \kappa_{\nu+1} \sum\limits_{t=\nu+1}^{\infty} a_t \omega^{t-1}}{1 + \sum\limits_{t=1}^{\nu} a_t \omega^{t-1}}$$

$$= \frac{1+g(\omega)}{1-g(\omega)}.$$

We have completed the proof of (19), which complete the proof of Theorem 2.5. □

Theorem 5. *If f of the form (1) satisfies condition (8), then*

$$\operatorname{Re}\left(\frac{(\Delta_q f)(\omega)}{(\Delta_q f_\nu)(\omega)}\right) \geq 1 - \frac{[\nu+1]_q}{\kappa_{\nu+1}} \qquad (\omega \in U^*), \tag{21}$$

and

$$\operatorname{Re}\left(\frac{(\Delta_q f_\nu)(\omega)}{(\Delta_q f)(\omega)}\right) \geq \frac{\kappa_{\nu+1}}{\kappa_{\nu+1} + [\nu+1]_q} \qquad (\omega \in U^*), \tag{22}$$

where κ_ν is given by (20).

The proof of Theorem 2.6, is similar to that of Theorem 2.5.

2.4. **Radius of starlikeness.**

In the next theorem we find the radius of q-starlikeness for the class $T^*_{(q,\eta)}[A,B]$.

Theorem 6. *Let the function f given by (1) be in the class $T^*_{(q,\eta)}[A,B]$. Then f is meromorphic starlike of order α in $|\omega| \leq r$, where*

$$r = inf_{t\geq 1}\left[\frac{(1-\alpha)\Lambda(t,B,q)}{(n+2-\alpha)\left[\gamma(\eta,A,B,q) - \Lambda(t,\eta,A,q)\,|b_t|\right]}\right]^{\frac{1}{t+1}},$$

Proof. In order to prove above result, we must show that

$$\left|\frac{f'(\omega)}{f(\omega)} + 1\right| \leq 1-\alpha \quad (0 \leq \alpha < 1) \text{ and } |\omega| \leq r_1,$$

we have

$$\left|\frac{f'(\omega)}{f(\omega)} + 1\right| = \frac{\sum_{t=1}^{\infty}(t+1)a_t\omega^t}{\frac{1}{\omega} + \sum_{t=1}^{\infty} a_t\omega^t}.$$

$$\leq \frac{\sum_{t=1}^{\infty}(t+1)\,|a_t|\,|\omega|^{t+1}}{1 - \sum_{t=1}^{\infty}|a_t|\,|\omega|^{t+1}}. \tag{23}$$

Since the appropriate condition for a function f to be in the class $\sum^{MS}(\alpha)$ is given by

$$\sum_{t=1}^{\infty}(t+\alpha)\,|a_t| < 1-\alpha \quad (0 \leq \alpha < 1;\ \omega \in U^*). \tag{24}$$

Hence (23) holds true if

$$\sum_{t=1}^{\infty}(t+1)\,|a_t|\,|\omega|^{t+1} \leq (1-\alpha)\left(1 - \sum_{t=1}^{\infty}|a_t|\,|\omega|^{t+1}\right). \tag{25}$$

The inequality in (25) can be written as:

$$\sum_{t=1}^{\infty}\left(\frac{t+2-\alpha}{1-\alpha}\right)|a_t|\,|\omega|^{t+1}\leq 1. \tag{26}$$

With the aid of (8), inequality (26) is true if

$$\left(\frac{t+2-\alpha}{1-\alpha}\right)|\omega|^{t+1}\leq\frac{\Lambda(t,B,q)}{\gamma(\eta,A,B,q)-\Lambda(t,\eta,A,q)\,|b_t|}. \tag{27}$$

Solving (27) for $|\omega|$, we have

$$|\omega|=\left[\frac{(1-\alpha)\Lambda(t,B,q)}{(n+2-\alpha)\left[\gamma(\eta,A,B,q)-\Lambda(t,\eta,A,q)\,|b_t|\right]}\right]^{\frac{1}{t+1}}. \tag{28}$$

In view of (28) the proof of our theorem is now completed. □

Definition 4. *A function $f\in\sum_1$ is said to belong to the class $f\in T^*_{(q,\eta,1)}[A,B]$ if and only if*

$$\mathrm{Re}\left(\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A+1)}\right)\geq 0.$$

We call $T^*_{(q,\eta,1)}[A,B]$ the class of q close-to-convex function of Type 1 related with the Janowski functions.

Definition 5. *A function $f\in\sum_1$ is said to belong to the class $f\in T^*_{(q,\eta,2)}[A,B]$ if and only if*

$$\left|\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

We call $T^*_{(q,\eta,2)}[A,B]$ the class of q close-to-convex function of Type 2 related with the Janowski functions.

Definition 6. *A function $f\in\sum_1$ is said to belong to the class $f\in T^*_{(q,\eta,3)}[A,B]$ if and only if*

$$\left|\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A+1)}-1\right|<1.$$

We call $T^*_{(q,\eta,3)}[A,B]$ the class of q close-to-convex function of Type 3 related with the Janowski functions.

For Special Cases.

(1) For $\eta=0$ and $g(\omega)=f(\omega)$ then $T^*_{(q,0,)}[A,B]$, $T^*_{(q,0,1)}[A,B]$, $T^*_{(q,0,2)}[A,B]$ and $T^*_{(q,0,3)}[A,B]$ classes reduced to $S^*_q[A,B]$, $S^*_{(q,1)}[A,B]$, $S^*_{(q,2)}[A,B]$ and $S^*_{(q,3)}[A,B]$ studied by Srivastava et al [17, 27].

(2) For $\eta = 0$, $g(\omega) = f(\omega)$, $A = 1 - 2\alpha$ and $B = -1$ in $T^*_{(q,0,)}[A,B]$, $T^*_{(q,0,1)}[A,B]$, $T^*_{(q,0,2)}[A,B]$ and $T^*_{(q,0,3)}[A,B]$ we get the classes S^*_q, $S^*_{(q,1)}(\alpha)$, $S^*_{(q,2)}(\alpha)$ and $S^*_{(q,3)}(\alpha)$, which was introduced and studied by Wongsaijai and Sukantamala (see [30]).

2.5. Main Results and Their Demonstration.

We first derive the presence results for the succeeding generalized q-starlike functions:

$$T^*_{(q,\eta,1)}[A,B], \quad T^*_{(q,\eta,2)}[A,B] \quad \textit{and} \quad T^*_{(q,\eta,3)}[A,B],$$

which are associated with the Janowski functions.

Theorem 7. *If* $-1 \leq B < A < 1$*, then*

$$T^*_{(q.\eta,3)}[A,B] \subset T^*_{(q.\eta,2)}[A,B] \subset T^*_{(q,\eta,1)}[A,B].$$

Proof. First of all, we suppose that $f \in T^*_{(q.\eta,3)}[A,B]$. Then, by Definition 2.10, we have

$$\left| \frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - 1 \right| < 1,$$

so that

$$\left| \frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega D_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - 1 \right| + \frac{q}{1-q} < 1 + \frac{q}{1-q}. \tag{29}$$

By using the triangle inequality and equation (29), we find that

$$\left| \frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}. \tag{30}$$

The last expression in (30) now implies that $f \in T^*_{(q.\eta,2)}[A,B]$, that is, that

$$T^*_{(q,3)}[A,B] \subset T^*_{(q,2)}[A,B].$$

Next, we let $f \in T^*_{(q,\eta,2)}[A,B]$, so that

$$f \in T^*_{(q,\eta,2)}[A,B] \Longleftrightarrow \left| \frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

As we know

$$\begin{aligned} \frac{1}{1-q} &> \left| \frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - \frac{1}{1-q} \right| \\ &= \left| \frac{1}{1-q} - \frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega D_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} \right|, \end{aligned}$$

we have

$$\operatorname{Re}\left(\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega D_q g(\omega)}-(A+1)}\right)>0 \qquad (\omega\in U^*). \tag{31}$$

This last equation now shows that $f\in T^*_{(q,\eta,1)}[A,B]$, that is, that

$$T^*_{(q.\eta,2)}[A,B]\subset T^*_{(q.\eta,1)}[A,B].$$

We have thus completed the proof of Theorem 2.11. □

Theorem 8. *Let $f\in\sum_1$, then $f\in T^*_{(q,\eta,2)}[A,B]$ if and only if*

$$\left|\frac{f(q\omega)}{(1-\eta)g(\omega)+\eta g(q\omega)}-\frac{\varkappa}{(B-1)q+B+3}\right|\leqq\frac{(A+1)(1-q)}{(B-1)q+B+3},$$

where

$$\varkappa=(A-1)q^2+(B-A+2)q+B+1.$$

Proof. Let

$$\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}=\left(\frac{1}{1-q}\right)\left(1-\frac{f(q\omega)}{(1-\eta)g(\omega)+\eta g(q\omega)}\right).$$

Using Definition 2.9 of the class $T^*_{(q,\eta,2)}[A,B]$ associated with the Janowski functions.

$$\left|\frac{(B-1)\left(\frac{1}{1-q}\right)\left(1-\frac{f(q\omega)}{(1-\eta)g(\omega)+\eta g(q\omega)}\right)-(A-1)}{(B+1)\left(\frac{1}{1-q}\right)\left(1-\frac{f(q\omega)}{(1-\eta)g(\omega)+\eta g(q\omega)}\right)-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

We have thus completed the proof of Theorem 2.12. □

Corollary 2. *It is worth mentioning that the classes*

$$T^*_{(q,\eta,1)}[A,B],\quad T^*_{(q,\eta,2)}[A,B]\quad\text{and}\quad T^*_{(q,\eta,3)}[A,B].$$

of the generalized q closed-to-convex functions of Type1, Type 2, and Type3, respectively, satisfy the following properties:

$$\cap_{q\in(0,1)}T^*_{(q,\eta,1)}[A,B]=\cap_{q\in(0,1)}T^*_{(q,\eta,2)}[A,B]=T^*[A,B].$$

and

$$\cap_{q\in(0,1)}T^*_{(q,\eta,1)}[A,B]=\cap_{q\in(0,1)}T^*_{(q,\eta,3)}[A,B]\subset T^*[A,B].$$

Let L be a subset of $\sum_1$ consisting of functions with a negative coefficient, that is,

$$f(\omega)=\frac{1}{\omega}-\sum_{t=1}^{\infty}|a_t|\,\omega^t \qquad (a_t\geq 0).$$

We also let

$$LT^*_{(q,\eta,t)}[A,B]=T^*_{(q,\eta,t)}[A,B]\cap L \quad (t=1,2,3).$$

Theorem 9. *For $-1 \leq B < A < 1$, then*

$$LT^{*}_{(q,\eta,1)}[A,B] = LT^{*}_{(q,\eta,2)}[A,B] = LT^{*}_{(q,\eta,3)}[A,B].$$

Proof. In view of Theorem 2.11, it is sufficient here to show that

$$LT^{*}_{(q,\eta,1)}[A,B] \subset LT^{*}_{(q.\eta,3)}[A,B].$$

Indeed, if we assume that , $f \in LT^{*}_{(q.\eta,1)}[A,B]$, then we have

$$\operatorname{Re}\left(\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A+1)}\right) \geq 0,$$

so that

$$\operatorname{Re}\left(\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A+1)}-1\right) \geq -1.$$

After a simple calculation, we thus find that

$$2\left|\frac{-\omega\Delta_q f(\omega)+(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}{(B+1)\omega\Delta_q f(\omega)-(A+1)\left[(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)\right]}\right| \geq -1$$

Using (1), (6) and (7) in above equation.

$$\left|\frac{\begin{array}{c}2(2\eta-1)-2(1-\eta)q+\\ 2\sum_{t=1}^{\infty}\left[[t]_q(a_t-\eta b_t)-(1-\eta)b_t\right]q\omega^{t+1}\end{array}}{\begin{array}{c}-(B+1)+(A+1)\eta-(A+1)(1-\eta)q\\ +\sum_{t=1}^{\infty}\left[[t]_q((B+1)a_t-\eta(A+1)b_t)-(A+1)(1-\eta)b_t\right]q\omega^{t+1}\end{array}}\right| < 1$$

This implies we get

$$\begin{aligned}&\sum_{t=1}^{\infty}|a_t|\,[t]_q(2-(B+1))q+\sum_{t=1}^{\infty}\left[\begin{array}{c}-2[t]_q\eta+(A+1)\eta\\ +(A+1)(1-\eta)-2(1-\eta)\end{array}\right]|b_t|\,q\\ &\leq |(B+1)-(A+1)\eta-(A+1)(1-\eta)q|+2(1-\eta)q-2(2\eta-1),\end{aligned}$$

which satisfies $T^{*}_{(q,\eta,3)}[A,B]$. By Definition 2.10, the proof of Theorem 2.14 is completed. □

3. Conclusion

In our current investigation, we have presented and studied thoroughly some new subclasses of q meromorphic close-to-convex functions, which is connected with the Janowski functions. Then we discussed some interesting properties and characteristics of these new subclasses, including distortion theorem, radius problem and partial sum. Some special cases have been discussed as applications of our main results. The technique and ideas of this paper may stimulate further research in this dynamic field.

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MOTIONS ON CURVES AND SURFACES USING GEOMETRIC ALGEBRA

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Abstract. Geometric algebra is a useful tool to overcome some problems in kinematics. Thus, the geometric algebra has attracted the attention of many researchers. In this paper, quaternion operators on curves and surfaces in Euclidean 3-space are defined by using geometric algebra. These operators generate the curves or the surfaces from the points, curves or surfaces. Using quaternion operators, we obtain motions that have orbits along the generated curve or surface. Also, these motions are expressed as 1-parameter or 2-parameter homothetic motions.

1. Introduction

Kinematics is a research field of geometry to describe the motion of points, lines and other geometric objects. Thus, kinematics is used in many fields such as physics, mechanics, robotics and neuroscience. Homothetic motion is one of the most commonly researched topic in kinematics. 1-parameter and 2-parameter homothetic motions were researched in Euclidean 3-space $\mathbb{E}^3$ [1, 2]. Yaylı gave homothetic motions in Euclidean 4-space with Hamilton operators [3].

Sir William Rowan Hamilton [4] interpreted the quaternions as an extension to the complex numbers in 1843. K. Shoemake defined the system of rotation in $\mathbb{E}^3$ by using quaternions [5]. Quaternions are more useful than Euler angles and matrices in representing of rotations of vectors. Therefore, quaternions have been used in many fields such as computer graphics, robotics and control theory.

Some problems and difficulties have been encountered in modeling of the mathematics of 3-dimensional (3D) kinematics. These difficulties have been tried to overcome by using quaternions. Bayro-Corrochano [6] used geometric algebra for the mathematical model of 3D kinematics of eye movements. Then, Leclercq at

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al. modeled some movements in 3D kinematics such as rotations, translations and screw movements [7]. In [8], an isomorphism was given between the algebra of split semi-quaternions and the Clifford algebra $Cl_{1,0,1}$. Moreover, semi-Euclidean planar motion was defined by using the algebra of split semi-quaternions.

Some surfaces were obtained by quaternions or homothetic motions in [9-15]. Some results have been achieved about these surfaces using quaternions. Also, using quaternions in the shape operator expressed by Darboux frame, we defined the quaternionic shape operator [16]. Moreover, we used the quaternionic shape operator in researching of the differential properties of surfaces.

In this study, we define quaternion operators using curves and surfaces in $\mathbb{E}^3$. These operators have allowed us to obtain a quaternionic or a homothetic motion on each curve and surface in $\mathbb{E}^3$. These motions have orbits along curves or surfaces. Quaternion operator with curve orbit converts a point to a curve or a curve to a curve. This operator is expressed as 1-parameter homothetic motion. Similarly, quaternion operator with surface orbit converts a point to a surface, a curve to a surface, or a surface to a surface. Moreover, quaternion operator with surface orbit is expressed as 2-parameter homothetic motion. Finally, we give some applications of the quaternion operators.

2. Preliminaries

In this section, definitions and some algebraic properties of the concepts real quaternions, homothetic motions and geometric algebra will be given to provide a background.

The set $H = \{q = a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in R\}$ of real quaternions is equal to the 4-dimensional vector space R^4. Quaternions have a basis $\{1, i, j, k\}$ shortly given with some properties as

$$i^2 = j^2 = k^2 = ijk = -1.$$

The set of real quaternion is associative and not commutative algebra. 1 is identity element of H. Scalar and vector component of q are $S(q) = a_0 \in R$ and $V(q) = a_1 i + a_2 j + a_3 k \in \mathbb{E}^3$, respectively. We can write quaternion q as $q = S(q) + V(q)$. If $S(q) = 0$, q is called pure quaternion. Quaternion product $*$ of $q = S(q) + V(q)$ and $p = S(p) + V(p)$ is defined as

$$q * p = S(q)S(p) - V(q) \cdot V(p) + S(q)V(p) + S(p)V(q) + V(q) \times V(p). \qquad (1)$$

Conjugate, norm, modulus and inverse of q is

$$\bar{q} = a_0 - a_1 i - a_2 j - a_3 k,$$

$$N_q = \bar{q} * q = a_0^2 + a_1^2 + a_2^2 + a_3^2,$$

$$|q| = \sqrt{N_q},$$

$$q^{-1} = \frac{\bar{q}}{N_q}, \qquad N_q \neq 0,$$

respectively. If $N_q = 1$, q is called unit quaternion. A unit quaternions can be written in the trigonometric form as $q = \cos\theta + \sin\theta\boldsymbol{v}$, where $\boldsymbol{v} \in \mathbb{E}^3$ and $\|\boldsymbol{v}\| = 1$. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be unit vectors in $\mathbb{E}^3$ (i.e., pure quaternions), and $\theta = \arccos(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)$, Thus, the unit quaternion q can be given as

$$q = \boldsymbol{v}_2 * \boldsymbol{v}_1^{-1} = \cos\theta + \sin\theta\boldsymbol{v}, \tag{2}$$

where $\boldsymbol{v} = \dfrac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|}$. $\|\ \|$ is the modulus in $\mathbb{E}^3$. Unit quaternion $q = \cos\theta + \sin\theta\boldsymbol{v}$ rotates the vector $\boldsymbol{v}_1$ to the vector $\boldsymbol{v}_2$ around the axis vector $\boldsymbol{v}$, see Figure 1. For further information about real quaternions, see [3-5, 17].

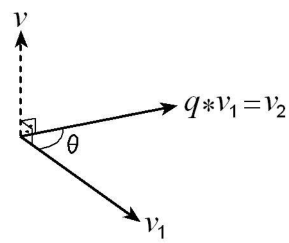

FIGURE 1. Rotation with unit quaternion

Let $p = a_0 + a_1 i + a_2 j + a_3 k$ be a unit quaternion and w be a pure quaternion (i.e., vector in $\mathbb{E}^3$). Linear mapping ϕ can be defined as

$$\phi : \mathbb{E}^3 \to \mathbb{E}^3, \quad \phi(\boldsymbol{w}) = p * \boldsymbol{w} * p^{-1}. \tag{3}$$

Matrix corresponding to the linear mapping ϕ can be given as

$$R = \begin{bmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0a_3 + 2a_1a_2 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 + a_2^2 - a_1^2 - a_3^2 & 2a_2a_3 - 2a_0a_1 \\ 2a_1a_3 - 2a_0a_2 & 2a_0a_1 + 2a_2a_3 & a_0^2 + a_3^2 - a_2^2 - a_1^2 \end{bmatrix},$$

where R is orthogonal since $RR^T = I$ and $\det R = 1$. Thus, ϕ represents a rotation in $\mathbb{E}^3$. If unit quaternion p is in the form

$$p = \cos\theta + \sin\theta\boldsymbol{v}, \tag{4}$$

then $\phi(\boldsymbol{w})$ rotates the vector $\boldsymbol{w}$ by 2θ [5].

1-parameter homothetic motion in $\mathbb{E}^3$ can be given as

$$y(t) = h(t)A(t)\boldsymbol{x}(t) + \boldsymbol{c}(t), \tag{5}$$

where $\boldsymbol{y}$ and $\boldsymbol{x}$ are the position vectors of the same point in the fixed space $R^\imath$ and the moving space R, respectively. h, A and $\boldsymbol{c}$ are homothetic scalar, orthogonal matrix and translation vector, respectively. And "t" is homothetic parameter [1, 2].

Similarly, 2-parameter homothetic motion in $\mathbb{E}^3$ can be given as

$$y(t,s) = h(t,s)A(t,s)\boldsymbol{x}(t,s) + \boldsymbol{c}(t,s), \tag{6}$$

where $\boldsymbol{y}$ and $\boldsymbol{x}$ are the position vectors of the same point in the fixed space R' and the moving space R, respectively. h, A and $\boldsymbol{c}$ are homothetic scalar, orthogonal matrix and translation vector, respectively. And "t and s" are homothetic parameters [1, 2].

The geometric product of two unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as $\boldsymbol{a} * \boldsymbol{b}$ and can be expressed as a sum of its symmetric and antisymmetric parts

$$\boldsymbol{a} * \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}, \tag{7}$$

where the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ and the outer product $\boldsymbol{a} \times \boldsymbol{b}$ are defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = \frac{1}{2}(\boldsymbol{a} * \boldsymbol{b} + \boldsymbol{b} * \boldsymbol{a}), \tag{8}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \frac{1}{2}(\boldsymbol{a} * \boldsymbol{b} - \boldsymbol{b} * \boldsymbol{a}). \tag{9}$$

The inner product of two vectors is the standard scalar or dot product which results in a scalar. The outer or wedge product of two vectors is a new quantity we call a bivector. We think of a bivector as a directed area in the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, formed by sweeping $\boldsymbol{a}$ along $\boldsymbol{b}$ [6].

3. Quaternion Operators

In this part, we have defined quaternion operators by geometric algebra. By using this operator, we have obtained some results on the curves and surfaces.

Definition 1. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors in $\mathbb{E}^3$. By using the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ and the vectorial product $\boldsymbol{a} \times \boldsymbol{b}$, quaternion operator can be defined as*

$$Q = \frac{1}{\|\boldsymbol{a}\|^2}\left(\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}\right). \tag{10}$$

The quaternion operator Q converts the vector $\boldsymbol{a}$ to the vector $\boldsymbol{b}$ around the axis vector $\boldsymbol{a} \times \boldsymbol{b}$ in the plane formed by $\boldsymbol{a}$ and $\boldsymbol{b}$ as

$$\begin{aligned} Q * \boldsymbol{a} &= \frac{1}{\|\boldsymbol{a}\|^2}\left(\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}\right) * \boldsymbol{a} \qquad (11) \\ &= \frac{1}{\|\boldsymbol{a}\|^2}\left(-\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \boldsymbol{a} + \left(\boldsymbol{a} \cdot \boldsymbol{b}\right)\boldsymbol{a} + \left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{a}\right) \\ &= \frac{1}{\|\boldsymbol{a}\|^2}\left(\left(\boldsymbol{a} \cdot \boldsymbol{b}\right)\boldsymbol{a} + \left(\boldsymbol{a} \cdot \boldsymbol{a}\right)\boldsymbol{b} - \left(\boldsymbol{b} \cdot \boldsymbol{a}\right)\boldsymbol{a}\right) \\ &= \frac{1}{\|\boldsymbol{a}\|^2}\|\boldsymbol{a}\|^2\boldsymbol{b} \end{aligned}$$

$$= \boldsymbol{b},$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are pure quaternion. Using $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos\theta$ and $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta$ in Eq. (10), we get

$$\begin{aligned} Q &= \frac{1}{\|\boldsymbol{a}\|^2} (\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}) \\ &= \frac{1}{\|\boldsymbol{a}\|^2} \left(\|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos\theta + \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta \frac{\boldsymbol{a} \times \boldsymbol{b}}{\|\boldsymbol{a} \times \boldsymbol{b}\|} \right) \\ &= \frac{\|\boldsymbol{b}\|}{\|\boldsymbol{a}\|} (\cos\theta + \sin\theta \boldsymbol{v}) \\ &= hq. \end{aligned}$$

where $q = \cos\theta + \sin\theta\boldsymbol{v}$, $h = \dfrac{\|\boldsymbol{b}\|}{\|\boldsymbol{a}\|}$ and $\boldsymbol{v} = \dfrac{\boldsymbol{a} \times \boldsymbol{b}}{\|\boldsymbol{a} \times \boldsymbol{b}\|}$. Thus, quaternion operator Q can be given as $Q = hq$.

Hence Eq. (11) can be expressed as

$$Q * \boldsymbol{a} = hq * \boldsymbol{a}.$$

$Q * \boldsymbol{a} = hq * \boldsymbol{a}$ can be given in Figure 2.

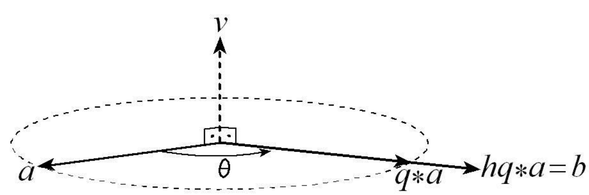

FIGURE 2. Quaternion operator

3.1. Quaternion Operator with Curve Orbit.

Theorem 1. *Let $\alpha(t)$ and P be a curve and a point in $\mathbb{E}^3$, respectively. Quaternion operator can be given as*

$$Q(t) = \frac{1}{\|P\|^2} (P \cdot \alpha(t) + P \times \alpha(t)). \tag{12}$$

$Q(t)$ generates the curve $\alpha(t)$ from the point P as

$$Q(t) * P = \alpha(t), \tag{13}$$

where $\alpha(t)$ is the orbit of $Q(t) * P$ and P, $\alpha(t)$ are pure quaternions.

Proof. The quaternion product of quaternion operator $Q(t)$ and the pure quaternion P can be given as

$$\begin{aligned} Q(t) * P &= \frac{1}{\|P\|^2}\left(P \cdot \alpha(t) + P \times \alpha(t)\right) * P \qquad (14) \\ &= \frac{1}{\|P\|^2}\left(\left(P \cdot \alpha(t)\right) P + \left(P \times \alpha(t)\right) \times P\right) \\ &= \frac{1}{\|P\|^2}\left(P \cdot P\right) \alpha(t) \\ &= \frac{1}{\|P\|^2}\|P\|^2 \alpha(t) \\ &= \alpha(t). \end{aligned}$$

Quaternion operator $Q(t)$ generates the curve $\alpha(t)$ from the point P. □

Remark 1. *Using* $P \cdot \alpha(t) = \|P\|\|\alpha(t)\| \cos\theta(t)$ *and* $\|P \times \alpha(t)\| = \|P\|\|\alpha(t)\| \sin\theta(t)$, *the quaternion operator* $Q(t)$ *can be given by unit quaternion* $q(t) = \cos\theta(t) + \sin\theta(t)\boldsymbol{v}(t)$, *where* $\boldsymbol{v}(t) = \dfrac{P \times \alpha(t)}{\|P \times \alpha(t)\|}$ *is rotation axis, as*

$$\begin{aligned} Q(t) &= \frac{1}{\|P\|^2}(P \cdot \alpha(t) + P \times \alpha(t)) \\ &= \frac{1}{\|P\|^2}(P \cdot \alpha(t) + \|P \times \alpha(t)\| \frac{P \times \alpha(t)}{\|P \times \alpha(t)\|}) \\ &= \frac{\|\alpha(t)\|}{\|P\|}(\cos\theta(t) + \sin\theta(t)\boldsymbol{v}(t)) \\ &= \frac{\|\alpha(t)\|}{\|P\|} q(t). \qquad (15) \end{aligned}$$

Thus, Eq. (14) can be given as

$$Q(t) * P = \frac{\|\alpha(t)\|}{\|P\|} q(t) * P. \qquad (16)$$

Theorem 2. $Q(t) * P$ *given in Eq. (13) can be expressed by 1-parameter homothetic motion in* $\mathbb{E}^3$ *as*

$$Q(t) * P = h(t)R(t)P$$

where $R(t)$ *is the orthogonal matrix satisfying* $R(t)P = q(t) * P$, $q(t) = \dfrac{Q(t)}{|Q(t)|}$, $h(t) = \dfrac{\|\alpha(t)\|}{\|P\|}$ *is a homothetic scalar and* t *is homothetic parameter.*

Proof. If we take the unit quaternion $p = \cos\theta + \sin\theta\boldsymbol{v}$ in Eq. (3) as $q_1(t) = \cos\frac{\theta(t)}{2} + \sin\frac{\theta(t)}{2}\boldsymbol{v}(t)$, we get the orthogonal matrix corresponding to the mapping

ϕ as

$$R = \begin{bmatrix} (\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}(2v_1^2 - 1) & -2\sin\frac{\theta}{2}(\cos\frac{\theta}{2}v_3 - \sin\frac{\theta}{2}v_1v_2) & 2\sin\frac{\theta}{2}(\cos\frac{\theta}{2}v_2 + \sin\frac{\theta}{2}v_1v_3) \\ 2\sin\frac{\theta}{2}(\cos\frac{\theta}{2}v_3 + \sin\frac{\theta}{2}v_1v_2) & \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}(2v_2^2 - 1) & 2\sin\frac{\theta}{2}(\sin\frac{\theta}{2}v_2v_3 - \cos\frac{\theta}{2}v_1) \\ 2\sin\frac{\theta}{2}(\sin\frac{\theta}{2}v_1v_3 - \cos\frac{\theta}{2}v_2) & 2\sin\frac{\theta}{2}(\cos\frac{\theta}{2}v_1 + \sin\frac{\theta}{2}v_2v_3) & \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}(2v_3^2 - 1) \end{bmatrix},$$

where $\boldsymbol{v}(t) = (v_1(t), v_2(t), v_3(t))$. In this case, matrix $R(t)$ performs a rotation by angle $2\dfrac{\theta(t)}{2} = \theta(t)$ of the vector P around the axis $\boldsymbol{v}(t)$. Thus, we can give the equalities

$$q(t) * P = \phi(P) = R(t)P. \tag{17}$$

Using these equations and $h(t) = \dfrac{\|\alpha(t)\|}{\|P\|}$, we get

$$\begin{aligned} Q(t) * P &= h(t)q(t) * P \\ &= h(t)R(t)P. \end{aligned} \tag{18}$$

It means that $Q(t) * P$ can be expressed as 1-parameter homothetic motion $Q(t) * P = h(t)R(t)P$ in $\mathbb{E}^3$.

If we take the point P on the curve $\alpha(t)$ as $P = \alpha(t_0)$, then $Q(t) * \alpha(t_0) = h(t)R(t)\alpha(t_0)$ can be given in Figure 3.

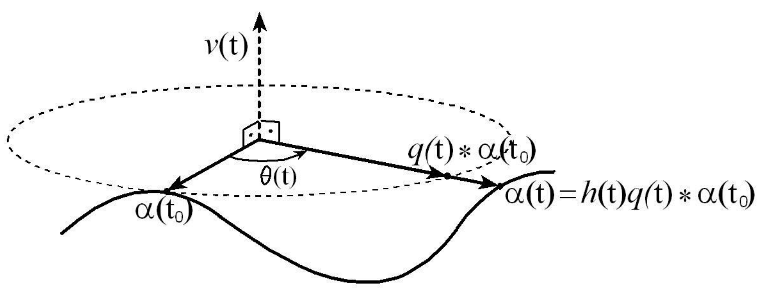

FIGURE 3. Quaternion operator with curve orbit

□

Corollary 1. *If we take the curve $\alpha(t)$ on a surface $M(t,s)$, then orbit of motions obtained in Theorem 1 and Theorem 2 can be confined on $M(t,s)$. Thus, these operators can allow us to obtain a 1-parameter motion on every surface in $\mathbb{E}^3$.*

Proposition 1. *Let $\alpha(t)$ and $\beta(t)$ be curves in $\mathbb{E}^3$. Quaternion operator $Q(t)$ can be given as*

$$Q(t) = \frac{1}{\|\alpha(t)\|^2}(\alpha(t) \cdot \beta(t) + \alpha(t) \times \beta(t)), \tag{19}$$

where $\alpha(t)$ and $\beta(t)$ are position vectors in $\mathbb{E}^3$. This quaternion operator converts the curve $\alpha(t)$ to the curve $\beta(t)$ as

$$Q(t) * \alpha(t) = \beta(t), \tag{20}$$

where the curve $\beta(t)$ is the orbit of $Q(t) * \alpha(t)$. Moreover, $Q(t) * \alpha(t)$ can be given by 1-parameter homothetic motion as

$$Q(t) * \alpha(t) = h(t)R(t)\alpha(t), \tag{21}$$

where $R(t)$ is the orthogonal matrix satisfying $R(t)\alpha(t) = q(t) * \alpha(t)$, $q(t) = \dfrac{Q(t)}{|Q(t)|}$, $h(t) = \dfrac{\|\beta(t)\|}{\|\alpha(t)\|}$ is a homothetic scalar and t is homothetic parameter.

3.2. **Quaternion Operator with Surface Orbit.**

Theorem 3. *Let $M(t,s)$ and P be a surface and a point in $\mathbb{E}^3$, respectively. Quaternion operator $Q(t,s)$ can be defined as*

$$Q(t,s) = \frac{1}{\|P\|^2}(P \cdot M(t,s) + P \times M(t,s)), \tag{22}$$

where $M(t,s)$ and P are position vectors in $\mathbb{E}^3$. The operator $Q(t,s)$ generates the surface $M(t,s)$ from the point P as

$$Q(t,s) * P = M(t,s), \tag{23}$$

where $M(t,s)$ is the orbit of the $Q(t,s) * P$ and P, $M(t,s)$ are pure quaternions.

Proof. The proof of this theorem is similar to the proof of Theorem 1. □

Remark 2. *By using $P \cdot M(t,s) = \|P\|\|M(t,s)\| \cos\theta(t,s)$ and $\|P \times M(t,s)\| = \|P\|\|M(t,s)\| \sin\theta(t,s)$, the quaternion operator $Q(t,s)$ with unit quaternion $q(t,s) = \cos\theta(t,s) + \sin\theta(t,s)\boldsymbol{v}(t,s)$ can be given as*

$$Q(t,s) = \frac{\|M(t,s)\|}{\|P\|}(\cos\theta(t,s) + \sin\theta(t,s)\boldsymbol{v}(t,s)) \tag{24}$$

where $\boldsymbol{v}(t,s) = \dfrac{P \times M(t,s)}{\|P \times M(t,s)\|}$. Eq. (23) can be expressed as

$$Q(t,s) * P = \frac{\|M(t,s)\|}{\|P\|} q(t,s) * P. \tag{25}$$

Theorem 4. *$Q(t,s) * P$ given in Eq. (25) can be given by 2-parameter homothetic motion in $\mathbb{E}^3$ as*

$$Q(t,s) * P = h(t,s)R(t,s)P, \tag{26}$$

where $R(t,s)$ is the orthogonal matrix satisfying $R(t,s)P = q(t,s) * P$, $q(t,s) = \dfrac{Q(t,s)}{|Q(t,s)|}$, $h(t,s) = \dfrac{\|M(t,s)\|}{\|P\|}$ is a homothetic scalar, and t, s are homothetic parameters.

Proof. The proof of this theorem is similar to the proof of Theorem 2. □

If we take the point P on the surface $M(t,s)$ as $P = M(t_0, s_0)$, then $Q(t,s) * M(t_0, s_0) = h(t,s)R(t,s)M(t_0, s_0)$ can be given in Figure 4.

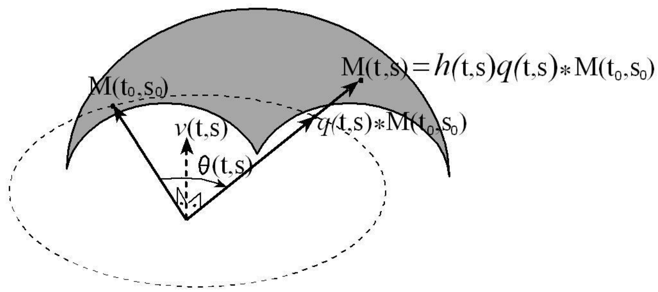

FIGURE 4. Quaternion operator with surface orbit

Proposition 2. *Let $M(t,s)$ and $\alpha(t)$ be a surface and a curve in $\mathbb{E}^3$, respectively. Quaternion operator $Q(t,s)$ can be defined as*

$$Q(t,s) = \frac{1}{\|\alpha(t)\|^2}(\alpha(t) \cdot M(t,s) + \alpha(t) \times M(t,s)) \tag{27}$$

where $M(t,s)$ and $\alpha(t)$ are pure quaternions. The operator $Q(t,s)$ generates the surface $M(t,s)$ from the curve $\alpha(t)$ as

$$Q(t,s) * \alpha(t) = M(t,s) \tag{28}$$

where $Q(t,s) * \alpha(t)$ has the surface orbit $M(t,s)$.

Corollary 2. *$Q(t,s) * \alpha(t)$ given in Eq. (28) can be given by 2-parameter homothetic motion in $\mathbb{E}^3$ as*

$$Q(t,s) * \alpha(t) = h(t,s)R(t,s)\alpha(t), \tag{29}$$

where $R(t,s)$ is the orthogonal matrix satisfying $R(t,s)\alpha(t) = q(t,s) * \alpha(t)$, $q(t,s) = \dfrac{Q(t,s)}{|Q(t,s)|}$, $h(t,s) = \dfrac{\|M(t,s)\|}{\|\alpha(t)\|}$ is a homothetic scalar, and t, s are homothetic parameters.

Proposition 3. *Let $M(t,s)$ and $N(t,s)$ be surfaces in $\mathbb{E}^3$. Quaternion operator can be defined as*

$$Q(t,s) = \frac{1}{\|M(t,s)\|^2}(M(t,s) \cdot N(t,s) + M(t,s) \times N(t,s)), \tag{30}$$

where $M(t,s)$ and $N(t,s)$ are pure quaternions. The operator $Q(t,s)$ generates the surface $N(t,s)$ from the surface $M(t,s)$ as

$$Q(t,s) * M(t,s) = N(t,s), \tag{31}$$

where $Q(t,s) * M(t,s)$ has the surface orbit $N(t,s)$.

Corollary 3. *$Q(t,s)*M(t,s)$ given in Eq. (31) can be given by 2-parameter homothetic motion in $\mathbb{E}^3$ as*

$$Q(t,s)*M(t,s)=h(t,s)R(t,s)M(t,s), \tag{32}$$

where $R(t,s)$ is the orthogonal matrix satisfying $R(t,s)M(t,s)=q(t,s)*M(t,s)$, $q(t,s)=\dfrac{Q(t,s)}{|Q(t,s)|}$, $h(t,s)=\dfrac{\|N(t,s)\|}{\|M(t,s)\|}$ is a homothetic scalar, and t, s are homothetic parameters.

3.3. **Applications of Quaternion Operators.**

Example 1. *Let $\alpha(t)=(\cos t,\sin t,0)$ and $\beta(t)=(\cos t,\sin t,t)$ be curves in $\mathbb{E}^3$. The quaternion operator $Q(t)$ can be given as*

$$\begin{aligned} Q(t) &= \frac{1}{\|\alpha(t)\|^2}(\alpha(t)\cdot\beta(t)+\alpha(t)\times\beta(t)) \\ &= 1+t(\sin t,-\cos t,0). \end{aligned} \tag{33}$$

The operator $Q(t)$ converts $\alpha(t)$ to $\beta(t)$ as

$$\begin{aligned} Q(t)*\alpha(t) &= (1+t(\sin t,-\cos t,0))*(\cos t,\sin t,0) \\ &= (\cos t,\sin t,0)+(0,0,t) \\ &= (\cos t,\sin t,t) \\ &= \beta(t). \end{aligned} \tag{34}$$

By using $\|\alpha(t)\|=1$, $\|\beta(t)\|=\sqrt{1+t^2}$ and $v(t)=(\sin t,-\cos t,0)$, the quaternion operator can be given by unit quaternion $q(t)=\cos\theta(t)+\sin\theta(t)v(t)$ as

$$\begin{aligned} Q(t) &= \frac{\|\beta(t)\|}{\|\alpha(t)\|}q(t) \\ &= \sqrt{1+t^2}(\cos\theta(t)+\sin\theta(t)(\sin t,-\cos t,0)) \end{aligned} \tag{35}$$

where $\theta(t)=\arccos\left(\dfrac{1}{\sqrt{1+t^2}}\right)$. For $p(t)=\cos\frac{\theta(t)}{2}+\sin\frac{\theta(t)}{2}v(t)$, the corresponding matrix $R(t)$ to the linear mapping ϕ can be obtained as

$$R(t)=\begin{bmatrix} \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}(\cos 2t) & -\sin^2\frac{\theta}{2}\sin 2t & -\sin\theta\cos t \\ -\sin^2\frac{\theta}{2}\sin 2t & \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\cos 2t & -\sin\theta\sin t) \\ \sin\theta\cos t & \sin\theta\sin t & \cos\theta \end{bmatrix}. \tag{36}$$

Thus, Eq. (34) can be given by the 1-parameter homothetic motion as

$$Q(t)*\alpha(t)=\sqrt{1+t^2}R(t)\alpha(t). \tag{37}$$

Example 2. *Let $\alpha(t)$ be center curve of the tube surface $Tube(t,\theta)$. The tube surface $Tube(t,\theta)$ can be given by the surface $S(t,\theta)=\cos\theta N(t)+\sin\theta B(t)$ as*

$$\begin{aligned} Tube(t,\theta) &= \alpha(t)+r(t)(\cos\theta N(t)+\sin\theta B(t)) \\ &= \alpha(t)+r(t)S(t,\theta). \end{aligned}$$

In the study of canal surfaces with quaternions [9] Corollary 1, the unit quaternion $q(t,\theta)=\cos\theta+\sin\theta T(t)$ generates the surface $S(t,\theta)$ from the normal vector $N(t)$, where $\{T(t),N(t),B(t)\}$ is the Frenet frame of $\alpha(t)$. Using definition of quaternion operator, unit quaternion operator $q(t,\theta)$ can be obtained as

$$\begin{aligned} Q(t,\theta) &= \frac{1}{\|N(t)\|^2}(N(t)\cdot S(t,\theta)+N(t)\times S(t,\theta)) \\ &= (N(t)\cdot(\cos\theta N(t)+\sin\theta B(t))+N(t)\times(\cos\theta N(t)+\sin\theta B(t))) \\ &= (\cos\theta N(t)\cdot N(t)+\sin\theta N(t)\times B(t)) \\ &= (\cos\theta\|N(t)\|^2+\sin\theta T(t)) \\ &= \cos\theta+\sin\theta T(t) \\ &= q(t,\theta). \end{aligned}$$

where $\|N(t)\|=1$. Thus, quaternion operator $q(t,\theta)$ generates the surface $S(t,\theta)$ from the normal vector $N(t)$ as

$$\begin{aligned} q(t,\theta)*N(t) &= (\cos\theta+\sin\theta T(t))*N(t) \\ &= \cos\theta N(t)+\sin\theta T(t)\times N(t) \\ &= \cos\theta N(t)+\sin\theta B(t) \\ &= S(t,\theta), \end{aligned}$$

where $S(t,\theta)$ is the surface orbit of $q(t,\theta)*N(t)$. Thus, tube surface can be given by quaternion product

$$Tube(t,\theta)=\alpha(t)+r(t)q(t,\theta)*N(t).$$

4. Conclusions

In this paper, we define quaternion operators using geometric algebra and classify these operators according to their orbits (i.e., curves or surfaces). Quaternion operator with curve orbit generates a curve from a point or a curve. This operator is given as 1-parameter homothetic motion. Similarly, quaternion operator with surface orbit generates a surface from a point, a curve or a surface. Quaternion operator with surface orbit is also expressed as 2-parameter homothetic motion. Thus, quaternion operators can form a homothetic and a quaternionic motion on every surface and curve in $\mathbb{E}^3$. Finally, we give some examples of the quaternion operators.

Author Contribution Statements The authors contributed equally to this work.

Declaration of Competing Interests The authors have no competing interests to declare.

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A NUMERICAL METHOD ON BAKHVALOV SHISHKIN MESH FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH A BOUNDARY LAYER

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Abstract. We construct a finite difference scheme for a first-order linear singularly perturbed Volterra integro-differential equation (SPVIDE) on Bakhvalov-Shishkin mesh. For the discretization of the problem, we use the integral identities and deal with the emerging integrals terms with interpolating quadrature rules which also yields remaining terms. The stability bound and the error estimates of the approximate solution are established. Further, we demonstrate that the scheme on Bakhvalov-Shishkin mesh is $O(N^{-1})$ uniformly convergent, where N is the mesh parameter. The numerical results are also provided for a couple of examples.

1. Introduction

In this present work, we are specifically consider the following class of the singularly perturbed linear Volterra integro-differential equations (SPVIDEs)

$$Lu := \varepsilon u' + a(x)u + \lambda \int_0^x K(x,t)u(t)dt = f(x), \quad x \in I = [0,\ell], \tag{1}$$

subject to

$$u(0) = A, \tag{2}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter. We assume $a(x) \geq \alpha > 0$, $f(x)(x \in I)$ and $K(x,t)((x,t) \in I \times I)$ are sufficiently smooth functions such

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that the initial layer for the solution $u(x)$ occurs at $x = 0$ for small values of ε. Volterra integro-differential equations (VIDEs) are an important class of equations which are extensively used to model many sciencitific problems such as population dynamics [13], filament streching [5] and epidemics [37]. Many techniques have been introduced to solve VIDEs analytically. The variational iteration method, the Adomian decomposition method and the homotopy perturba-tion method are some well-known analytical methods to solve VIDEs([40], [9], [17]). Recently, a new approach on the variational analytical method has been introduced to solve Volterra-Fredholm Integral equations which does not require construction of the variational principle [18]. Further, a finite difference scheme is utilized to examine the numerical solutions of a non-linear VIDE in [11].

Singularly perturbed differential equations, which have the highest order derivative term multi- plied with a small positive number ε, possess solutions with interior or boundary layers. Boundary layers are regions where rapid changes occur which makes solving such problems more challenging. Since standard schemes fail to give the accurate results for problems with boundary layer for small ε values, numerical solutions of such problems have been of interest to many researchers([12], [15], [16], [22], [28], [29], [31], [34], [38], [35]). Singularly perturbed Volterra integro-differential equations (SPVIDEs) have been widely used to model problems in many science fields such as epidemic dynamics, synchronous control systems, filament stretching and heat transfer ([6], [7], [14], [20], [21], [32], [33]). A review on the literature of the SPVIDEs was given in [25]. Further, asymptotic expansions derivation of the solutions to SPVIDEs are studied in [6], [7], [25]. In [32] a problem of nonlinear SPVIDE modelling the elongation ratio of filament is studied and the qualitative properties of the solution is discussed under some physically interesting assumptions. In [5], a specific integro-differential equation with a boundary layer which describes filament stretching process is considered and the leading order behavior of the problem is examined by an asymptotic method. Singularly perturbed integro differential equations have been also an interest to many researchers. In [23] and [24], the numerical solutions of singularly perturbed integro-differential and integro-differential-algebraic equations are analyzed by the implicit Runge-Kutta methods. An exponential finite difference method is applied for the inner and outer layers and a type of implicit Runge-Kutta method is performed to obtain the outer layer solutions of SPVIDEs in [36]. A finite Legendre expansion is constructed to solve different kinds of integral equations and integro-differential equations [26]. In [19], tension spline collocation methods are utilized to numerically discretize singularly perturbed Volterra integral and integro-differential equations. In [39], the authors present different types of exponential schemes to solve SPVIDEs and the stability analysis of the schemes is examined. Fitted difference schemes are also proven to provide accurate results in the solution process of different types of singularly perturbed problems. In [2], an exponentially fitted difference method is designed on a uniform mesh to solve linear SPVIDEs. First-order convergent finite difference schemes are developed to solve linear first order SPVIDEs with delay in [4], [27]. In [3], using a fitted difference operator a second-order difference scheme is constructed on a piecewise uniform mesh to solve linear SPVIDEs.

In this present work, we mainly construct a uniform convergent difference scheme on a Bakhvalov-Shishkin mesh for the problem (1)-(2). Bakhvalov-Shishkin mesh is a mixed version of the Shishkin mesh and Bakhvalov mesh which are known to yield accurate results for singularly perturbed problems with boundary layers. In [30], the author demonstrated that the results from an upwind difference scheme on Bakhvalov-Shishkin mesh applied to a linear convection-diffusion equation are more accurate than the results from the upwind scheme on a Shishkin mesh. Further, a finite difference scheme on Bakhvalov-Shishkin mesh is utilized to deal with a singularly perturbed boundary value problem in [10].

The rest of the paper is organized in the following order. In Section 2, the asymptotic estimates on the exact solution to (1)-(2) are established. In Section 3, we define the Bakhvalov-Shishkin mesh points according to the boundary layer conditions of the problem (1)-(2) and derive a finite difference scheme utilizing the integral identities with exponential basis functions and then applying interpolating quadrature rules provided in [1] to the integral terms. In Section 4, we establish the stability bounds and the error estimates of the numerical solution and as a result we show that the scheme demonstrates $O(N^{-1})$ uniform convergence with respect to the perturbation parameter. We also provide the numerical results in Section 5.

2. Asymptotic Behavior of the Solution

In the following lemma, we establish a priori estimates for the asymptotic behavior of the solution to the problem (1)-(2).

Lemma 1. *Let $a, f \in C(I)$ and $K \in C(I \times I)$. The solution u to the problem* (1)-(2) *holds*

$$\|u\|_\infty \leq C, \tag{3}$$

where

$$C = (A + \alpha^{-1}\|f\|_\infty)e^{\lambda \bar{K} \alpha^{-1} \ell},$$

and $\bar{K} = \max_{I\times I} |K(x,t)|$. In addition, if $a, f \in C^1(I)$ and $K \in C^1(I \times I)$ with

$$\Big|\frac{\partial}{\partial x}K(x,t)\Big| \leq \bar{K}_1 < \infty, \tag{4}$$

then the solution $u(x)$ satisfies

$$|u'(x)| \leq C\Big(1 + \frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\Big), x \in I. \tag{5}$$

Proof. To establish the first estimate given in (3) we start by rewriting (1) as

$$\varepsilon u' + a(x)u = F(x), \tag{6}$$

where

$$F(x)=f(x)-\lambda\int_0^x K(x,t)u(t)dt. \tag{7}$$

Solving the equation (6) with $u(0)=A$ yields

$$u(x)=Ae^{-\frac{1}{\varepsilon}\int_0^x a(s)ds}+\frac{1}{\varepsilon}\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds}d\xi,$$

and further we calculate

$$|u(x)| \quad \leq |A|e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds}+\frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds}d\xi.$$

Since we have $a(x)\geq\alpha>0$, it follows

$$\begin{aligned}|u(x)| &\leq |A|e^{-\frac{1}{\varepsilon}\int_0^x \alpha ds}+\frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha ds}d\xi\\ &=|A|e^{-\frac{\alpha x}{\varepsilon}}+\frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{\alpha(x-\xi)}{\varepsilon}}d\xi.\end{aligned} \tag{8}$$

Here, by the definition of $F(x)$ in (7), we get

$$|F(x)|\leq\|f\|_\infty+\lambda\bar{K}\int_0^x|u(t)|dt. \tag{9}$$

Substituting (9) into (8) yields

$$\begin{aligned}|u(x)| &\leq |A|e^{-\frac{\alpha x}{\varepsilon}}+\frac{1}{\varepsilon}\int_0^x\Big(\|f\|_\infty+\lambda\bar{K}\int_0^\xi|u(t)|dt\Big)e^{-\frac{\alpha(x-\xi)}{\varepsilon}}d\xi\\ &=|A|e^{-\frac{\alpha x}{\varepsilon}}+\frac{1}{\varepsilon}\|f\|_\infty\int_0^x e^{-\frac{\alpha(x-\xi)}{\varepsilon}}d\xi+\frac{\lambda\bar{K}}{\varepsilon}\int_0^x\int_0^\xi|u(t)|dte^{-\frac{\alpha(x-\xi)}{\varepsilon}}d\xi\end{aligned}$$

We integrate by parts the last term with double integral here

$$\begin{aligned}|u(x)| &\leq |A|e^{-\frac{\alpha x}{\varepsilon}}+\alpha^{-1}\|f\|_\infty\Big(1-e^{-\frac{\alpha x}{\varepsilon}}\Big)+\alpha^{-1}\lambda\bar{K}\Big(1-e^{-\frac{\alpha x}{\varepsilon}}\Big)\int_0^x|u(t)|dt\\ &\leq |A|+\alpha^{-1}\|f\|_\infty+\alpha^{-1}\lambda\bar{K}\int_0^x|u(t)|dt.\end{aligned} \tag{10}$$

An application of the Gronwall's inequality to (10) provides

$$\begin{aligned}|u(x)| &\leq \Big(|A| + \alpha^{-1}\|f\|_{\infty}\Big)e^{\alpha^{-1}\lambda\bar{K}x}\\ &\leq \Big(|A| + \alpha^{-1}\|f\|_{\infty}\Big)e^{\alpha^{-1}\lambda\bar{K}\ell},\end{aligned}$$

which leads to the desired result in (3).

For the next estimate provided in (5), we first differentiate the equation (1) and have

$$\varepsilon u'' + a'(x)u + a(x)u' + \lambda K(x,x)u + \lambda\int_0^x \frac{\partial}{\partial x}K(x,t)u(t)dt = f'(x).$$

Then, letting

$$v(x) = u'(x),$$

and

$$g(x) = f'(x) - a'(x)u - \lambda K(x,x)u - \lambda\int_0^x \frac{\partial}{\partial x}K(x,t)u(t)dt, \tag{11}$$

we have

$$\varepsilon v' + a(x)v = g(x). \tag{12}$$

In a similar manner to the previous work above, we solve (12)

$$v(x) = v(0)e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x g(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds}d\xi.$$

Then, we have

$$\begin{aligned}|v(x)| &\leq |v(0)|e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x |g(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds}d\xi\\ &\leq |v(0)|e^{-\frac{1}{\varepsilon}\int_0^x \alpha ds} + \frac{1}{\varepsilon}\int_0^x |g(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha ds}d\xi\\ &\leq |v(0)|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^x |g(\xi)|e^{-\frac{\alpha(x-\xi)}{\varepsilon}}d\xi.\end{aligned} \tag{13}$$

Here, by the formula of $g(x)$ given in (11), from (3) and knowing that $a, f \in C^1(I)$, $K \in C^1(I \times I)$ and from (4) we obtain

$$\begin{aligned} |g(x)| &\leq ||f'||_\infty + ||a'||_\infty |u| + \lambda \bar{K}|u| + \lambda \bar{K}_1 \int_0^x |u(t)|dt \\ &\leq ||f'||_\infty + C\Big(||a'||_\infty + \lambda \bar{K}_1 + \ell\Big), \end{aligned} \tag{14}$$

which implies $||g||_\infty \leq C_*$ for a $C_* \in \mathbb{R}$. Hence, utilizing this estimate on $g(x)$ in (13) provides

$$\begin{aligned} |v(x)| &\leq |v(0)|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\|g\|_\infty \int_0^x e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi \\ &\leq |v(0)|e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1}C_*(1 - e^{-\frac{\alpha x}{\varepsilon}}). \end{aligned} \tag{15}$$

On the other hand, inserting $x = 0$ in (1) and since $a, f \in C^1(I)$ it follows that

$$|v(0)| = |u'(0)| = \frac{1}{\varepsilon}|f(0) - Aa(0)| \leq \frac{c}{\varepsilon}.$$

Substituting this into (15) yields

$$|v(x)| \leq \frac{c}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1}C_*(1 - e^{-\frac{\alpha x}{\varepsilon}}),$$

which provides the desired result. □

3. Difference Scheme

3.1. **Notation.** Before we proceed to the definition of the mesh points and discretization of the problem we provide the notation we use throughout the paper. Let $\bar{\omega}_h = \{0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = \ell\}$ denote a non-uniform mesh on $[0, \ell]$. For each $i = 0, \cdots, N$, let $h_i = x_i - x_{i-1}$ denote the step size. For any continuous mesh function v_i defined on ω_h we use the notation

$$v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}$$

for backward difference.

3.2. **Discretization.** In this section, we construct our difference scheme based on Bakhvalov-Shishkin mesh. According to this mesh construction, we divide the domain into two subintervals $[0, \sigma]$ and $[\sigma, \ell]$, where σ is the transition parameter. For a positive even discretization parameter N, we determine the transition parameter σ as

$$\sigma = \min\Big\{\frac{\ell}{2}, \varepsilon\alpha^{-1}\ln N\Big\}. \tag{16}$$

We assume $\varepsilon \ll N^{-1}$ as it is used in practice. We define a set of mesh points as the following

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon \ln[1-2(1-N^{-1})\frac{i}{N}], & x_i \in [0,\sigma], i = 0,1,\cdots,\frac{N}{2}, \\ \sigma + \left(i-\frac{N}{2}\right)h, \quad h = \frac{2(\ell-\sigma)}{N}, & x_i \in [\sigma,\ell], i = \frac{N}{2}+1,\cdots,N. \end{cases} \tag{17}$$

To derive the difference approximation, we use the following integral identity

$$\chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} Lu(x)\varphi_i(x)dx = \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx, \tag{18}$$

with the exponential basis function

$$\varphi_i(x) = e^{-\frac{a_i}{\varepsilon}(x_i-x)}, \quad i = 1,\cdots,N,$$

where

$$\chi_i = h_i{}^{-1}\int_{x_{i-1}}^{x_i}\varphi_i(x)dx = \frac{1-e^{-a_i\rho_i}}{a_i\rho_i}, \quad \rho_i = \frac{h_i}{\varepsilon}.$$

We remark that φ_i solves the equation

$$\begin{aligned} &-\varepsilon\varphi_i(x) + a_i\varphi_i(x) = 0, \quad x_{i-1} \le x \le x_i \\ &\varphi_i(x_i) = 1. \end{aligned} \tag{19}$$

To obtain the difference scheme from (18), we proceed by evaluating the integrals term by term applying the interpolating quadrature rules with weight functions and obtain the remainder terms as provided in [1]. In the following, we handle the differential term on the left-hand side of (18),

$$\begin{aligned} \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i}\Big[\varepsilon u'(x) + a(x)u(x)\Big]\varphi_i(x)dx &= \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i}[\varepsilon u'(x) + a_i u(x)]\varphi_i(x)dx \\ &\quad + \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i}[a(x)-a_i]u(x)\varphi_i(x)dx \\ &= \varepsilon\theta_i u_{\bar{x},i} + a_i u_i + R_i^{(1)}, \end{aligned} \tag{20}$$

where

$$\theta_i = \frac{a_i\rho_i e^{-a_i\rho_i}}{1-e^{-a_i\rho_i}}, \tag{21}$$

and

$$R_i^{(1)} = \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i}[a(x)-a_i]u(x)\varphi_i(x)dx. \tag{22}$$

Further, applying the first quadrature rules provided in [1] to the integral term in (18) twice we obtain

$$\chi_i^{-1}h_i^{-1}\lambda\int_{x_{i-1}}^{x_i}\varphi_i(x)\int_0^x K(x,t)u(t)dtdx = \lambda\int_0^{x_i} K(x_i,t)u(t)dt + R_i^{(2)}, \tag{23}$$

where

$$R_i^{(2)} = \lambda \int_{x_{i-1}}^{x_i} \frac{\partial}{\partial \xi}\Big(\int_0^{\xi} K(\xi,t)u(t)dt\Big)\Big[T_0(x-\xi) - h_i^{-1}(x-x_{i-1})\Big]d\xi, \tag{24}$$

and $T_0(\lambda) = 1$ for $\lambda \geq 0$ and $T_0(\lambda) = 0$ for $\lambda < 0$. Here, we apply the composite right-side rectangle rule to the integral term in the right-hand side of (27) and get

$$\lambda \int_0^{x_i} K(x_i,t)u(t)dt = \lambda \sum_{j=1}^{i} h_j K(x_i,x_j)u_j + R_i^{(3)}, \tag{25}$$

where

$$R_i^{(3)} = -\lambda \sum_{j=1}^{i} \int_{x_{j-1}}^{x_j} (\xi - x_{j-1}) \frac{\partial}{\partial \xi}\Big(K(x_i,\xi)u(\xi)\Big)d\xi. \tag{26}$$

Then, inserting (25) in (23) provides

$$\chi_i^{-1}h_i^{-1}\lambda \int_{x_{i-1}}^{x_i} \varphi_i(x) \int_0^x K(x,t)u(t)dtdx = \lambda \sum_{j=1}^{i} h_j K(x_i,x_j)u_j + R_i^{(2)} + R_i^{(3)}. \tag{27}$$

On the other hand, the right-hand side of (18) gets the in the form

$$\chi_i^{-1}h_i^{-1} \int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx = f_i + R_i^{(4)}, \tag{28}$$

where

$$R_i^{(4)} = \chi_i^{-1}h_i^{-1} \int_{x_{i-1}}^{x_i} [f(x) - f(x_i)]\varphi_i(x)dx. \tag{29}$$

Inserting the relations (20), (27) and (28) in (18), we obtain the difference problem for the problem (1)-(2) as

$$\begin{aligned} &\varepsilon\theta_i u_{\bar{x},i} + a_i u_i + \lambda \sum_{j=1}^{i} h_j K(x_i,x_j)u_j = f_i - R_i, \quad i = 1,2,\cdots,N, \\ &u_0 = A, \end{aligned} \tag{30}$$

where

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)} - R_i^{(4)}. \tag{31}$$

As a result, neglecting the error term R_i in (30) provides the following difference scheme

$$L_N y_i := \varepsilon\theta_i y_{\bar{x},i} + a_i y_i + \lambda \sum_{j=1}^{i} h_j K(x_i,x_j)y_j = f_i, \quad i = 1,2,\cdots,N, \tag{32}$$

$$y_0 = A, \tag{33}$$

where θ_i defined by (21).

4. Stability, Error Estimates and Convergence Results

Here, we establish the stability bound and the error estimates of the approximate solution y. Further, the convergence of the difference scheme provided in (32)-(33) is analyzed.

Lemma 2. *Assume that $|F_i| \leq \mathcal{F}_i$ and $\mathcal{F}_i$ be a non-decreasing function. The solution to the problem*

$$\ell_N v_i := \varepsilon\theta_i v_{\bar{x},i} + a_i v_i = F_i, \quad 1 \leq i \leq N,$$
$$v_0 = A.$$
$$|v_i| \leq |A| + \alpha^{-1}\mathcal{F}_i, \quad 1 \leq i \leq N.$$

Proof. The proof follows from the maximum principle for difference operators. Details can be found in [27]. □

Lemma 3. *Let y_i be the solution of the problem* (32)-(33). *Then, y_i satisfies*

$$\|y\|_\infty \leq C_0\big(|A| + \|f\|_\infty\big). \tag{34}$$

Proof. The difference scheme equation given in (32) can be rewritten in the form

$$\theta_i\varepsilon y_{\bar{x},i} + a_i y_i = F_i, \tag{35}$$

where

$$F_i = f_i - \lambda\sum_{j=1}^{i} h_j K(x_i, x_j) y_j.$$

For F_i, we have the estimate

$$\begin{aligned}|F_i| &\leq |f_i| + \lambda\Big|\sum_{j=1}^{i} h_j K(x_i, x_j) y_j\Big| \\ &\leq |f_i| + \lambda\bar{K}\sum_{j=1}^{i} h_j |y_j| \\ &\leq \|f\|_\infty + \lambda\bar{K}\sum_{j=1}^{i} h_j |y_j|.\end{aligned}$$

Then, applying Lemma 2 to (35) and utilizing this estimate provide

$$|y_i| \leq |A| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}\lambda\bar{K}\sum_{j=1}^{i} h_j |y_j|. \tag{36}$$

Further, applying the difference analogue of the Gronwall's inequality to (36) we have

$$|y_i| \leq \Big(|A| + \alpha^{-1}\|f\|_\infty\Big)e^{\alpha^{-1}\lambda\bar{K}\ell},$$

which yields the result in (34). □

The error of the difference problem is given by the solution to the problem

$$L_N z_i = R_i, \quad 1 \le i \le N, \tag{37}$$

$$z_0 = 0. \tag{38}$$

Lemma 4. *Suppose that z_i be the solution of (37)-(38). Then, z_i holds the estimate*

$$\|z\|_\infty \le C\|R\|_\infty. \tag{39}$$

Proof. The result follows from Lemma 3 taking $A = 0$ and $f = R$. □

Lemma 5. *Let $a, f \in C^1(I)$ and $K \in C^1(I \times I)$ with*

$$\bar{K} = \max_{I\times I} |K(x,t)|, \tag{40}$$

$$\Big|\frac{\partial}{\partial x} K(x,t)\Big| \le \bar{K}_1 < \infty, \tag{41}$$

and

$$\Big|\frac{\partial}{\partial t} K(x,t)\Big| \le \bar{K}_2 < \infty. \tag{42}$$

Then, the truncation error R_i satisfies the estimate

$$\|R\|_\infty \le CN^{-1}. \tag{43}$$

Proof. To establish the estimate given in (43), we proceed by bounding each term in R_i provided in (31). For $R_i^{(1)}$, we have

$$|R_i^{(1)}| \le \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} |(a'(s)(x - x_i))u(x)|\varphi_i(x)dx,$$

where $s \in [x, x_i]$ comes from the Mean Value Theorem. Then, since $a \in C^1(I)$ and from (3) we get

$$|R_i^{(1)}| \le C_1 h_i. \tag{44}$$

Further, for $R_i^{(2)}$ we take into account of (40), (41) and $|T_0(\lambda)| \le 1$, so

$$\begin{aligned} |R_i^{(2)}| &\le \lambda \int_{x_{i-1}}^{x_i} \Big|\big(1 + h_i^{-1}(x - x_i)\big)\frac{\partial}{\partial \xi}\Big(\int_0^\xi K(\xi,t)u(t)dt\Big)\Big| d\xi \\ &\le 2\lambda \int_{x_{i-1}}^{x_i} \Big|\frac{\partial}{\partial \xi}\Big(\int_0^\xi K(\xi,t)u(t)dt\Big)\Big| d\xi. \end{aligned} \tag{45}$$

Then, applying the Leibnitz formula to (45) yields

$$\begin{aligned} |R_i^{(2)}| &\le 2\lambda\left(\int_{x_{i-1}}^{x_i} \Big|K(\xi,\xi)\Big|\Big|u(\xi)\Big|\right) + \int_{x_{i-1}}^{x_i}\int_0^\xi \Big|\frac{\partial}{\partial \xi} K(\xi,t)u(t)\Big| dt d\xi\Bigg) \\ &\le 2\lambda(C\bar{K} + C\bar{K}_1)h_i \\ &\le C_2 h_i. \end{aligned} \tag{46}$$

On the other hand, by the Leibnitz formula and from (40), (42) and (5) we have

$$\begin{aligned}
|R_i^{(3)}| &\le \lambda \sum_{j=1}^{i} \int_{x_{j-1}}^{x_j} \left(\Big| \frac{\partial}{\partial \xi} K(x_i,\xi) u(\xi) \Big| + \Big| K(x_i,\xi) u'(\xi) \Big| \right) d\xi \\
&\le \lambda \sum_{j=1}^{i} \left(C\bar{K}_2 + \bar{K} \int_{x_{j-1}}^{x_j} \Big(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha\xi}{\varepsilon}} \Big) \right) d\xi \qquad (47) \\
&= \lambda \sum_{j=1}^{i} \left(C\bar{K}_2 h_j + \bar{K} h_j + \alpha^{-1} \bar{K} \Big(e^{-\frac{\alpha x_{j-1}}{\varepsilon}} - e^{-\frac{\alpha x_j}{\varepsilon}} \Big) \right).
\end{aligned}$$

Then, by the Mean Value Theorem applied to the exponential term in (47) with $s \in [x_{j-1}, x_j]$ it follows that

$$\begin{aligned}
|R_i^{(3)}| &\le \lambda \sum_{j=1}^{i} \Big(C\bar{K}_2 h_j + \bar{K} h_j + \alpha^{-1} \bar{K} h_j e^{-\frac{\alpha s}{\varepsilon}} \Big) \\
&\le C_3^* i |h^*|,
\end{aligned} \qquad (48)$$

where $h^* = \max\limits_{1 \le j \le i} h_j$. Lastly, for $R_i^{(4)}$, similarly to the work above and since $f \in C^1(I)$ we have

$$\begin{aligned}
|R_i^{(4)}| &\le \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} |f'(s)(x - x_i)| \varphi_i(x) dx \\
&\le C_4 h_i,
\end{aligned} \qquad (49)$$

where $s \in [x_{i-1}, x_i]$ by the Mean Value Theorem.

Further in the proof, we need to evaluate each estimate above on the sub-intervals $[0, \sigma]$ and $[\sigma, \ell]$. For this, we first establish the bounds on the step-size h_i on each interval. In the first sub-interval $[0, \sigma]$ with $\sigma \le \frac{\ell}{2}$,

$$x_i = -\alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i}{N}], \quad i = 1, \cdots, N/2$$

and hence,

$$h_i = -\alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i}{N}] + \alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i-1}{N}].$$

Then, we apply the Mean Value Theorem to h_i with $i_* \in [i-1, i]$ and get

$$h_i \le \alpha^{-1} \varepsilon \frac{2(1 - N^{-1}) N^{-1}}{1 - 2 i_* (1 - N^{-1}) N^{-1}} \le C N^{-1}. \qquad (50)$$

In the second sub-interval $[\sigma, \ell]$, we have

$$x_i = \sigma + \big(i - \frac{N}{2} \big) h, \quad i = N/2 + 1, \cdots, N,$$

where $\sigma \leq \frac{\ell}{2}$ and

$$h_i = \frac{2(\ell-\sigma)}{N} \leq CN^{-1}. \tag{51}$$

Inserting the bounds (50) and (51) in (44), (46), (48) and (49), we have

$$|R_i^{(k)}| \leq CN^{-1}, \quad k = 1,2,3,4.$$

which implies the desired result (43). $\square$

Theorem 1. *Let u be the exact solution of* (1)-(2) *and y be the solution of* (32)-(33)*. If the assumptions on the functions a, f and K from Lemma 5 hold, then*

$$\|y-u\|_\infty \leq CN^{-1}.$$

Proof. The proof follows from Lemma 4 and Lemma 5. $\square$

5. Algorithm and Numerical Results

In this section, we present the numerical results on an example with an exact solution and an example with an unknown solution. The results include graphs of the approximate solutions, error estimates and the convergence values of the approximate solution to the exact solution. In our algorithm, we consider the following elimination method

$$y_i^{(n)} = \frac{1}{\varepsilon\theta_i + h_i a_i}\Big[\varepsilon\theta_i y_{i-1}^{(n} + h_i\big(f_i - \lambda\sum_{j=1}^{i} h_j K(x_i,x_j) y_j^{(n-1)}\big)\Big], \tag{52}$$

$$y_0^{(n)} = A, \tag{53}$$

$$y_i^{(0)} = A. \tag{54}$$

where $y_i^{(0)}$ is the initial process.

Example 1. *We study the following initial value problem*

$$\varepsilon u'(x) + u(x) + \int_0^x xu(t)dt = 2\varepsilon(x-1) + (x-1)^2 - \varepsilon x e^{-\frac{x}{\varepsilon}} + \frac{x(x-1)^3}{3} + (\varepsilon - 1 + x)e^{-x} + (\varepsilon - \frac{2}{3})x, \quad 0 \leq x \leq 2,$$

$$u(0) = 1.$$

The exact solution of this problem is

$$u(x) = e^{-\frac{x}{\varepsilon}} + (x-1)^2 - e^{-x}.$$

The exact error is calculated by the formula

$$e_\varepsilon^N = \|y^N - u\|_\infty,$$

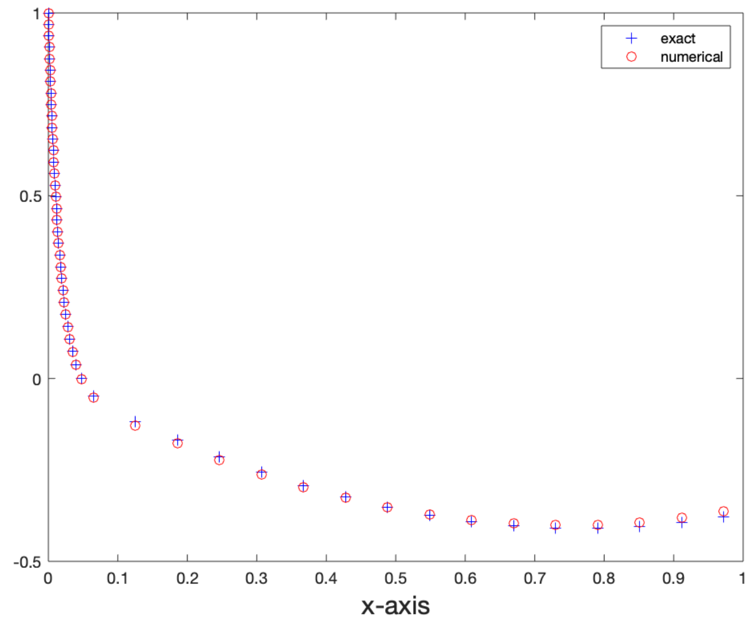

FIGURE 1. The figure depicts the graphs for the exact solution and the approximate solution for $N = 32$.

where y^N is the numerical approximation of u for different N and ε values. We compute the convergence rate by

$$r^N = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

In Table 1, we provide the errors e^N, e^{2N} and the convergence rates of the approximate solution for various N and $\varepsilon = 2^{-i}$ values.

Example 2. *Consider the following test problem*

$$\varepsilon u' + (x+1)u + \int_0^x xt(x-t)^2 u(t)dt = x - e^{2x}, \quad 0 \le x \le 2,$$
$$u(0) = 1.$$

The exact solution to this problem is not known. To compute the approximate solution and estimate the errors, we utilize the double mesh principle, that is calculating the error of the approximate solution on mesh size N with the approximate solution

TABLE 1. Errors e^N, e^{2N}, and rate of convergence r for Example 1.

ε		$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
	e^N	0.065 181 2	0.036 774 9	0.018 718 3	0.009 187 0	0.004 326 4
2^{-12}	e^{2N}	0.029 862 0	0.016 962 6	0.008 791 5	0.004 253 9	0.001 871 4
	r	1.126 145 3	1.116 368 7	1.090 268 5	1.110 475 2	
	e^N	0.065 374 3	0.036 984 5	0.018 926 7	0.009 393 2	0.004 531 1
2^{-18}	e^{2N}	0.030 078 8	0.017 176 5	0.009 001 5	0.004 462 0	0.002 076 7
	r	1.119 980	1.106 485 5	1.072 189 3	1.073 941 7	
	e^N	0.065 377 7	0.036 987 8	0.018 929 9	0.009 396 4	0.004 534 3
2^{-24}	e^{2N}	0.030 082 1	0.017 179 8	0.009 004 8	0.004 465 2	0.002 080 0
	r	1.119 883 7	1.106 332 4	1.071 911 6	1.073 390 5	

computed on double mesh $2N$, namely

$$e_\varepsilon^N = \|y^N - y^{2N}\|_\infty,$$

where y^N is the approximate solution on mesh N and y^{2N} is the approximate solution on mesh $2N$. The convergence rate is calculated as it is in Example 1.

In Table 2, the errors and the convergence rates of the approximate solution for various N and $\varepsilon = 2^{-i}$ values are presented.

TABLE 2. Errors e^N, e^{2N}, and rate of convergence r for Example 2.

ε		$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
	e^N	0.031 218 4	0.015 601 2	0.007 796 0	0.003 895 5	0.001 946 6
2^{-12}	e^{2N}	0.015 601 2	0.007 796 0	0.003 895 5	0.001 946 6	0.000 972 9
	r	1.000 741 7	1.000 858 3	1.000 922 3	1.000 865 3	
	e^N	0.031 249 5	0.015 624 6	0.007 812 2	0.003 906 1	0.001 953 0
2^{-18}	e^{2N}	0.015 624 6	0.007 812 2	0.003 906 1	0.001 953 0	0.000 488 2
	r	1.000 012 1	1.000 014 6	1.000 017 2	1.000 019 8	
	e^N	0.031 250 0	0.015 625 0	0.007 812 5	0.003 906 3	0.001 953 1
2^{-24}	e^{2N}	0.015 625 0	0.007 812 5	0.003 906 3	0.001 953 1	0.000 976 6
	r	1.000 000 2	1.000 000 2	1.000 000 3	1.000 000 3	

6. CONCLUSION

To sum up, we constructed a finite difference scheme on a Bakhvalov-Shishkin mesh to obtain the numerical solution of an initial value problem for a linear first-order singularly perturbed Volterra integro-differential equation with a boundary layer. We proved that the method is first-order uniformly convergent with respect to the perturbation parameter. As we can see in Table 1, Table 2 and Figure 1, the numerical results of the test problems are also consistent with the analysis on the error estimates and convergence order and hence, it is confirmed that the convergence order of the scheme $O(N^{-1})$. For future work, we suggest that this difference scheme method on Bakhvalov-Shishkin mesh can be applied to the singularly perturbed linear or non-linear problems with delay to obtain accurate numerical solutions. Further, our proposed scheme can be modified to handle integro-differential equations with fractal derivatives which are studied in [8].

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ON A NONLINEAR FUZZY DIFFERENCE EQUATION

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Abstract. In this paper we investigate the existence, the boundedness and the asymptotic behavior of the positive solutions of the fuzzy difference equation

$$z_{n+1} = \frac{Az_{n-1}}{1 + z_{n-2}^p},\ n \in \mathbb{N}_0$$

where (z_n) is a sequence of positive fuzzy numbers, A and the initial conditions z_{-j} $(j = 0, 1, 2)$ are positive fuzzy numbers and p is a positive integer.

1. Introduction

Over the last two decades, a lot of study has been published on difference equations and systems. One reason for this is that such equations and systems have high applicability both in mathematics and other sciences such as population biology, economics, probability theory, genetics, psychology etc., (see, e.g., [2, 6, 14, 15] and the references therein). In this way, many real life problems are modeled by means of difference equations and systems. In some cases, however, measurements or data on a problem may reveal uncertainty or the problem considered may require subjective interpretations. In such cases, a fuzzy difference equation model can be established using notion of fuzzy set. In this way, the uncertainty is modeled.

Fuzzy set theory has recently become a popular subject due to the increasing number of applications in technology, mathematics and other sciences. The part that we are interested in is, of course, that the notion of fuzzy set can be easily

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applied to difference equations. With this application, a powerful method for determining the behavior of solutions of difference equations emerges. Some studies using the method will be summarized below.

In [3], Deeba et al. studied the fuzzy analog of a difference equation which arises in population genetics. More precisely they studied the first order difference equation

$$x_{n+1} = wx_n + q, \ n \in \mathbb{N}_0 \tag{1}$$

where (x_n) is a sequence of fuzzy numbers and w, q, x_0 are fuzzy numbers. Also, they discussed the fuzzy nonlinear difference equation

$$x_{n+1} = f(x_n, w, q), \ n \in \mathbb{N}_0 \tag{2}$$

where (x_n) is a sequence of fuzzy numbers, w, q, x_0 are fuzzy numbers and $f : \mathbb{R}_a^+ \times \mathbb{R}_a^+ \times \mathbb{R}_a^+ \to \mathbb{R}_a^+$, $\mathbb{R}_a^+$ is the set of all real numbers greater or equal to a, is a continuous and nondecreasing function in its arguments.

In [4], Deeba and Korvin studied the second order difference equation

$$x_{n+1} = x_n - abx_{n-1} + c, \ n \in \mathbb{N}_0 \tag{3}$$

where (x_n) is a sequence of fuzzy numbers and a, b, c, x_0, x_{-1} are fuzzy numbers. This equation is a linearized model of a nonlinear model which determines the carbondioxide (CO_2) level in the blood.

In [12], Papaschinopoulos and Papadopoulos studied the existence, the boundedness and the asymptotic behavior of the positive solutions of the fuzzy difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-m}}, \ n \in \mathbb{N}_0 \tag{4}$$

where (x_n) is a sequence of fuzzy numbers and A and the initial conditions x_{-j} $(j = 0, 1, \ldots, m)$ are fuzzy numbers for $m \in \mathbb{N}_1$.

For more works on fuzzy difference equations, see the references [9, 11, 13] and the references cited therein.

In [5], El-Owaidy et al. investigated the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}, \ n \in \mathbb{N}_0 \tag{5}$$

where the nonnegative parameters and nonnegative initial conditions.

Moreover, in [7], Gümüş and Soykan investigated the behavior of solutions of the system of difference equations

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_{n-2}^p}, \ v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_{n-2}^p}, \ n \in \mathbb{N}_0 \tag{6}$$

where the positive parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p$ and the initial conditions u_{-i}, v_{-i} for $i = 0, 1, 2$ are positive real numbers. Note that system (6) can be reduced to the following system of difference equations

$$x_{n+1} = \frac{r x_{n-1}}{1 + y_{n-2}^p},\ y_{n+1} = \frac{s y_{n-1}}{1 + x_{n-2}^p},\ n \in \mathbb{N}_0 \tag{7}$$

by the change of variables $u_n = (\beta_1/\gamma_1)^{1/p} x_n$ and $v_n = (\beta/\gamma)^{1/p} y_n$ with $r = \alpha/\beta$ and $s = \alpha_1/\beta_1$. So, in order to study system (6), they investigated system (7).

In this paper we investigate the existence, the boundedness and the asymptotic behavior of the positive solutions of the fuzzy difference equation

$$z_{n+1} = \frac{A z_{n-1}}{1 + z_{n-2}^p},\ n \in \mathbb{N}_0 \tag{8}$$

where (z_n) is a sequence of positive fuzzy numbers, A and the initial conditions z_{-j} $(j = 0, 1, 2)$ are positive fuzzy numbers and p is a positive integer.

2. Preliminaries

In this section, we give some definitions which will be used in this paper. For more details see [1, 8, 10, 16].

Definition 1. *Consider a fuzzy subset of the real line $A : \mathbb{R} \to [0,1]$. Then we say A is a fuzzy number if it is satisfies the following properties*

(a) *A is normal, i.e., $\exists\ x_0 \in \mathbb{R}$ with $A(x_0) = 1$,*
(b) *A is fuzzy convex, i.e., $A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\}$, $\forall t \in [0,1]$ and $x_1, x_2 \in \mathbb{R}$,*
(c) *A is upper semicontinuous on $\mathbb{R}$,*
(d) *A is compactly supported, i.e., $\overline{\{x \in \mathbb{R} : A(x) > 0\}}$ is compact.*

Let us denote by $\mathbb{R}_F$ the space of all fuzzy numbers. For $0 < \alpha \leq 1$ and $A \in \mathbb{R}_F$, we denote α-cuts of fuzzy number A by $[A]_\alpha = \{x \in \mathbb{R}, A(x) \geq \alpha\}$ and $[A]_0 = \overline{\{x \in \mathbb{R}, A(x) \geq 0\}}$. We call $[A]_0$, the support of fuzzy number A and denote it by *supp* (A).

The fuzzy number A is called positive if *supp*$(A) \subset (0, \infty)$. We denote by $\mathbb{R}_F^+$, the space of all positive fuzzy numbers.

Definition 2. *(a) Let A, B be any fuzzy numbers with $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$ and $[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$ for $\alpha \in (0,1]$. We define $\|A\|$ on the fuzzy numbers space as follow;*

$$\|A\| = \sup \max\{|A_{l,\alpha}|, |A_{r,\alpha}|\},$$

where sup *is taken for all $\alpha \in (0,1]$. Then from the above we take the following metric*

$$D(A,B) = \sup\{\max\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}\},$$

where sup *is taken for all $\alpha \in (0,1]$.*

(b) Let (x_n) be a sequence of positive fuzzy numbers and x is a fuzzy number. Then we say that

$$\lim_{n\to\infty} x_n = x \quad \textit{iff} \quad \lim_{n\to\infty} D(x_n, x) = 0.$$

The following lemma and definition are given in [8]:

Lemma 1. *Let X, Y be fuzzy numbers and $[X]_\alpha = [X_{l,\alpha}, X_{r,\alpha}]$, $[Y]_\alpha = [Y_{l,\alpha}, Y_{r,\alpha}]$ for $\alpha \in (0,1]$ be the α-cuts of X, Y, respectively. Let Z be a fuzzy number such that $[Z]_\alpha = [Z_{l,\alpha}, Z_{r,\alpha}]$ for $\alpha \in (0,1]$. Then, $MIN\{X,Y\} = Z$ (resp. $MAX\{X,Y\} = Z$) if and only if $\min\{X_{l,\alpha}, Y_{l,\alpha}\} = Z_{l,\alpha}$ and $\min\{X_{r,\alpha}, Y_{r,\alpha}\} = Z_{r,\alpha}$ (resp. $\max\{X_{l,\alpha}, Y_{l,\alpha}\} = Z_{l,\alpha}$ and $\max\{X_{r,\alpha}, Y_{r,\alpha}\} = Z_{r,\alpha}$).*

Definition 3. *(a) We say that a sequence of positive fuzzy numbers (x_n) is bounded and persistent if there exist $n_0 \in \mathbb{N}$ and positive fuzzy numbers C, D such that $MIN\{x_n, C\} = C$ and $MIN\{x_n, D\} = D$ for $n \geq n_0$.*

(b) We say that (x_n) for $n \in \mathbb{N}_0$ is an unbounded sequence if the $||x_n||$ for $n \in \mathbb{N}_0$ is an unbounded sequence.

We need the following lemma which has been proved in [12].

Lemma 2. *Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function and A, B, C be fuzzy numbers. Then, $[f(A,B,C)]_\alpha = f([A]_\alpha, [B]_\alpha, [C]_\alpha)$ for $\alpha \in (0,1]$.*

3. Main Results

In this section, we prove our main results. Firstly, we will study the existence of the positive solutions of equation (8). We say (z_n) is a positive solution of equation (8) if (z_n) is a sequence of positive fuzzy numbers which satisfies equation (8).

Theorem 1. *Consider equation (8) where A is a positive fuzzy number. Then for any positive fuzzy numbers z_{-j} $(j = 0,1,2)$ there exists a unique positive solution (z_n) of (8) with the initial conditions z_{-j} $(j = 0,1,2)$.*

Proof. Suppose that there exists a sequence of fuzzy numbers (z_n) satisfying (8) with the initial conditions z_{-j} $(j = 0,1,2)$. Consider the α-cuts

$$\begin{aligned} [z_n]_\alpha &= [L_{n,\alpha}, R_{n,\alpha}], \\ [A]_\alpha &= [A_{l,\alpha}, A_{r,\alpha}], \end{aligned} \tag{9}$$

for $n = -2, -1, 0, ...$ and $\alpha \in (0,1]$. Then from (8)-(9) and Lemma 2 it follows that

$$\begin{aligned} [z_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] \\ &= \left[\frac{Az_{n-1}}{1 + z_{n-2}^p}\right]_\alpha = \frac{[A]_\alpha [z_{n-1}]_\alpha}{[1 + z_{n-2}^p]_\alpha} \\ &= \frac{[A_{l,\alpha} L_{n-1,\alpha}, A_{r,\alpha} R_{n-1,\alpha}]}{[1 + L_{n-2,\alpha}^p, 1 + R_{n-2,\alpha}^p]} = \left[\frac{A_{l,\alpha} L_{n-1,\alpha}}{1 + R_{n-2,\alpha}^p}, \frac{A_{r,\alpha} R_{n-1,\alpha}}{1 + L_{n-2,\alpha}^p}\right], \end{aligned}$$

from which we get

$$L_{n+1,\alpha} = \frac{A_{l,\alpha} L_{n-1,\alpha}}{1 + R_{n-2,\alpha}^p},\ R_{n+1,\alpha} = \frac{A_{r,\alpha} R_{n-1,\alpha}}{1 + L_{n-2,\alpha}^p},\ \alpha \in (0,1]. \tag{10}$$

for $n \in \mathbb{N}_0$. Then it is clear that for any $(L_{j,\alpha}, R_{j,\alpha})$, $j = -2, -1, 0$ there exists a unique solution $(L_{n,\alpha}, R_{n,\alpha})$ with the initial conditions $(L_{j,\alpha}, R_{j,\alpha})$, $j = -2, -1, 0$ for $\alpha \in (0, 1]$.

Now, we prove that $[L_{n,\alpha}, R_{n,\alpha}]$ for $\alpha \in (0, 1]$ where $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of the system (10) with the initial conditions $(L_{j,\alpha}, R_{j,\alpha})$, $j = -2, -1, 0$ determines the solution (z_n) of (8) with the initial conditions z_{-j} $(j = 0, 1, 2)$ such that

$$[z_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}],\ \alpha \in (0, 1],\ n = -2, -1, 0, ... \tag{11}$$

Since A and z_{-j} $(j = 0, 1, 2)$ are positive fuzzy numbers for any $\alpha_1, \alpha_2 \in (0, 1]$, $\alpha_1 \leq \alpha_2$ we get,

$$\begin{gathered} 0 < A_{l,\alpha_1} \leq A_{l,\alpha_2} \leq A_{r,\alpha_2} \leq A_{r,\alpha_1}, \\ 0 < L_{-2,\alpha_1} \leq L_{-2,\alpha_2} \leq R_{-2,\alpha_2} \leq R_{-2,\alpha_1}, \\ 0 < L_{-1,\alpha_1} \leq L_{-1,\alpha_2} \leq R_{-1,\alpha_2} \leq R_{-1,\alpha_1}, \\ 0 < L_{0,\alpha_1} \leq L_{0,\alpha_2} \leq R_{0,\alpha_2} \leq R_{0,\alpha_1}. \end{gathered} \tag{12}$$

We prove by the induction that

$$L_{n,\alpha_1} \leq L_{n,\alpha_2} \leq R_{n,\alpha_2} \leq R_{n,\alpha_1},\ n \in \mathbb{N}. \tag{13}$$

From (12) we have that (13) hold for $n = -2, -1, 0$. Suppose that (13) is valid for $n \leq k$, $k \in \{1, 2, ...\}$. Then from (10), (12) and (13) for $n \leq k$ it follows that

$$\begin{aligned} L_{k+1,\alpha_1} &= \frac{A_{l,\alpha_1} L_{k-1,\alpha_1}}{1 + R^p_{k-2,\alpha_1}} \leq \frac{A_{l,\alpha_2} L_{k-1,\alpha_2}}{1 + R^p_{k-2,\alpha_2}} = L_{k+1,\alpha_2}, \\ L_{k+1,\alpha_2} &= \frac{A_{l,\alpha_2} L_{k-1,\alpha_2}}{1 + R^p_{k-2,\alpha_2}} \leq \frac{A_{r,\alpha_2} R_{k-1,\alpha_2}}{1 + L^p_{k-2,\alpha_2}} = R_{k+1,\alpha_2} \end{aligned}$$

and

$$R_{k+1,\alpha_2} = \frac{A_{r,\alpha_2} R_{k-1,\alpha_2}}{1 + L^p_{k-2,\alpha_2}} \leq \frac{A_{r,\alpha_1} R_{k-1,\alpha_1}}{1 + L^p_{k-2,\alpha_1}} = R_{k+1,\alpha_1}.$$

Therefore (13) is satisfied. Moreover from (10) we get

$$L_{1,\alpha} = \frac{A_{l,\alpha} L_{-1,\alpha}}{1 + R^p_{-2,\alpha}},\ R_{1,\alpha} = \frac{A_{r,\alpha} R_{-1,\alpha}}{1 + L^p_{-2,\alpha}},\ \alpha \in (0, 1]. \tag{14}$$

Then since A and z_{-j} $(j = 0, 1, 2)$ are positive fuzzy numbers, we have that $A_{l,\alpha}$, $A_{r,\alpha}$, $L_{-1,\alpha}$, $R_{-1,\alpha}$, $L_{-2,\alpha}$ and $R_{-2,\alpha}$ are left continuous. So, from (14) we see that $L_{1,\alpha}$ and $R_{1,\alpha}$ are also left continuous. Working inductively we can easily prove that $L_{n,\alpha}$ and $R_{n,\alpha}$ are left continuous for $n \in \mathbb{N}$.

Now, we prove that $\overline{\cup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]}$ is compact. It is sufficient to prove that $\cup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]$ is bounded. Let $n = 1$, since A and z_{-j} $(j = 0, 1, 2)$ are positive fuzzy numbers there exist constants $M_A, N_A, M_{-2}, N_{-2}, M_{-1}, N_{-1}, M_0, N_0 > 0$

such that

$$\begin{array}{c}[A_{l,\alpha}, A_{r,\alpha}] \subset [M_A, N_A],\\ [L_{-2,\alpha}, R_{-2,\alpha}] \subset [M_{-2}, N_{-2}],\\ [L_{-1,\alpha}, R_{-1,\alpha}] \subset [M_{-1}, N_{-1}],\\ [L_{0,\alpha}, R_{0,\alpha}] \subset [M_0, N_0].\end{array} \tag{15}$$

Therefore, from (14)-(15) we can easily prove that

$$[L_{1,\alpha}, R_{1,\alpha}] \subset \left[\frac{M_A M_{-1}}{1+N_{-2}^p}, \frac{N_A N_{-1}}{1+M_{-2}^p}\right],$$

for $\alpha \in (0,1]$ from which it is clear that

$$\cup_{\alpha\in(0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset \left[\frac{M_A M_{-1}}{1+N_{-2}^p}, \frac{N_A N_{-1}}{1+M_{-2}^p}\right], \tag{16}$$

for $\alpha \in (0,1]$. Also, (16) implies that $\overline{\cup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]}$ is compact and

$$\overline{\cup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]} \subset (0,\infty).$$

Working inductively we can easily see that

$$\overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \text{ is compact, } \overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subset (0,\infty) \text{ for } n \in \mathbb{N}_1. \tag{17}$$

Therefore, using (13), (17) and since $L_{n,\alpha}$, $R_{n,\alpha}$ are left continuous we see that $[L_{n,\alpha}, R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers (z_n) such that (11) holds.

We prove now that (z_n) is the solution of equation (8) with the initial conditions z_{-j} $(j = 0,1,2)$. Since

$$[z_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[\frac{A_{l,\alpha} L_{n-1,\alpha}}{1+R_{n-2,\alpha}^p}, \frac{A_{r,\alpha} R_{n-1,\alpha}}{1+L_{n-2,\alpha}^p}\right] = \left[\frac{A z_{n-1}}{1+z_{n-2}^p}\right]_\alpha,$$

for all $\alpha \in (0,1]$, we have that (z_n) is the solution of equation (8) with the initial conditions z_{-j} $(j = 0,1,2)$.

Suppose that there exists another solution $(\widetilde{z}_n)$ of equation (8) with the initial conditions z_{-j} $(j = 0,1,2)$. Then arguing as above we can easily show that

$$[\widetilde{z}_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}] \text{ for } \alpha \in (0,1] \text{ and } n \in \mathbb{N}_0. \tag{18}$$

Then from (11) and (18) we have that $[z_n]_\alpha = [\widetilde{z}_n]_\alpha$ for $\alpha \in (0,1]$ and $n = -2,-1,0,...$ from which it holds $z_n = \widetilde{z}_n$ for $n = -2,-1,0,....$ Thus, the proof is completed. □

To study the boundedness of the positive solutions of equation (8), we need the following theorem which has been proved in [7]:

Theorem 2. *Assume that $r, s \in (1,\infty)$, then there exist unbounded solutions of system (7).*

In the following lemma, we will study the boundedness and persistence of the positive solutions of system (7):

Lemma 3. *Assume that $r, s \in (0,1)$, then every positive solution of system (7) is bounded and persists.*

Proof. Assume that $r, s \in (0,1)$. From system (7), we have that

$$0 < x_{n+1} = \frac{r x_{n-1}}{1 + y_{n-2}^p} < r x_{n-1} < x_{n-1}, \tag{19}$$

and

$$0 < y_{n+1} = \frac{s y_{n-1}}{1 + x_{n-2}^p} < s y_{n-1} < y_{n-1}, \tag{20}$$

for $n \in \mathbb{N}_0$. From (19) and (20), we have by induction

$$0 < x_{2n-i} < x_{-i} \text{ and } 0 < y_{2n-i} < y_{-i}, \tag{21}$$

for $n \in \mathbb{N}_1$ where x_{-i}, y_{-i} $(i = 0, 1)$ are the initial conditions. This completes the proof. $\square$

Theorem 3. *Consider equation (8). Then the following statements are true:*

(i) *If $A_{r,\alpha} < 1$ for all $\alpha \in (0,1]$, then every positive solution of equation (8) is bounded and persists.*

(ii) *If there exists an $\overline{\alpha} \in (0,1]$ such that $A_{l,\overline{\alpha}} > 1$, then the equation (8) has unbounded solutions.*

Proof. (i) Let (z_n) be a positive solution of equation (8) such that (11) holds. From (10) and Lemma 3, we get

$$[L_{n,\alpha}, R_{n,\alpha}] \subset [0, T_\alpha], \tag{22}$$

for $n \in \mathbb{N}_1$ where $T_\alpha = \max\{R_{-1,\alpha}, R_{0,\alpha}\}$. Since (z_n) are positive fuzzy numbers, there exists a constant $T > 0$ such that $T_\alpha \leq T$ for all $\alpha \in (0,1]$. Therefore, $[L_{n,\alpha}, R_{n,\alpha}] \subset [0,T]$ for $n \in \mathbb{N}_1$ from which we get $\cup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}] \subset [0,T]$ for $n \in \mathbb{N}_1$ so $\overline{\cup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \subseteq [0,T]$. This completes the proof of (i).

(ii) Suppose that there exists an $\overline{\alpha} \in (0,1]$ such that $A_{l,\overline{\alpha}} > 1$. If $A_{l,\overline{\alpha}} = r$, $A_{r,\overline{\alpha}} = s$, $L_{n,\overline{\alpha}} = x_n$ and $R_{n,\overline{\alpha}} = y_n$ for $n = -2, -1, ...$, then we can apply Theorem 2 to system (10). If there exists an $\overline{\alpha} \in (0,1]$ such that $r = A_{l,\overline{\alpha}} > 1$, then there exist solutions (x_n, y_n) of system (10) where $\overline{\alpha} = \alpha$ with initial conditions (x_{-j}, y_{-j}) for $j = 0, 1, 2$ such that

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = \infty. \tag{23}$$

Moreover, if $x_{-j} < y_{-j}$ $(j = 0, 1, 2)$, we can find positive fuzzy numbers z_{-j} $(j = 0, 1, 2)$ such that

$$[z_j]_\alpha = [L_{j,\alpha}, R_{j,\alpha}], \tag{24}$$

for $\alpha \in (0,1]$ and

$$[z_j]_{\overline{\alpha}} = [L_{j,\overline{\alpha}}, R_{j,\overline{\alpha}}] = [x_j, y_j], \tag{25}$$

for $j=-2,-1,0$. Let (z_n) be a positive solution of equation (8) with the initial conditions z_{-j} $(j=0,1,2)$ and $[z_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}]$ for $\alpha\in(0,1]$. Since (24) and (25) hold and $(L_{n,\alpha},R_{n,\alpha})$ satisfies system (10) we have

$$[z_n]_{\overline{\alpha}}=[L_{n,\overline{\alpha}},R_{n,\overline{\alpha}}]=[x_n,y_n]. \tag{26}$$

Thefore, from (23), (26) and since

$$||z_n||=\sup\max\{|L_{n,\alpha}|,|R_{n,\alpha}|\}\geq\max\{|L_{n,\overline{\alpha}}|,|R_{n,\overline{\alpha}}|\}=R_{n,\overline{\alpha}},$$

where sup is taken for all $\alpha\in(0,1]$, it is clear that solution (z_n) is unbounded. This completes the proof of (ii). □

In the last theorem, we will study the convergence of the positive solutions of equation (8). We need the following theorem which has been obtained from Theorem 10 in [7]:

Theorem 4. *If* $r,s\in(0,1)$, *then every positive solution* (x_n,y_n) *of system (7) converges to* $(0,0)$ *as* $n\to\infty$.

Theorem 5. *If* $A_{r,\alpha}<1$ *for all* $\alpha\in(0,1]$, *then every positive solution* (z_n) *of equation (8) converges to* 0 *as* $n\to\infty$.

Proof. Let (z_n) be a positive solution of equation (8) such that (9) holds with $A_{r,\alpha}<1$ for all $\alpha\in(0,1]$. Then, we can apply Theorem 4 to system (10). So, we get

$$\lim_{n\to\infty}L_{n,\alpha}=\lim_{n\to\infty}R_{n,\alpha}=0. \tag{27}$$

Therefore, from (27) we get

$$\lim_{n\to\infty}D(z_n,0)=\lim_{n\to\infty}\sup\{\max\{|L_{n,\alpha}-0|,|R_{n,\alpha}-0|\}\}=0.$$

□

4. Numerical Examples

In this section, we give two numerical examples for the solutions of equation (8) regard to the different values of A for $p=1$ with the inital conditions z_{-j} $(j=0,1,2)$ are satisfied

$$z_{-2}(x)=\begin{cases}20x-8, & 0.40\leq x\leq 0.45,\\ 10-20x, & 0.45\leq x\leq 0.50,\end{cases}$$

$$z_{-1}(x)=\begin{cases}\frac{5x-0.50}{2}, & 0.10\leq x\leq 0.50,\\ \frac{4.50-5x}{2}, & 0.50\leq x\leq 0.90,\end{cases} \tag{28}$$

$$z_{0}(x)=\begin{cases}20x-6, & 0.30\leq x\leq 0.35,\\ 8-20x, & 0.35\leq x\leq 0.40.\end{cases}$$

From (28), we get $[z_{-2}]_\alpha = \left[\frac{\alpha+8}{20}, \frac{10-\alpha}{20}\right]$, $[z_{-1}]_\alpha = \left[\frac{2\alpha+0.50}{5}, \frac{4.50-2\alpha}{5}\right]$, $[z_0]_\alpha = \left[\frac{\alpha+6}{20}, \frac{8-\alpha}{20}\right]$ for all $\alpha \in (0,1]$.

Example 1. *Consider equation (8) for $p = 1$ where z_n is a sequence of positive fuzzy numbers, the initial conditions z_{-j} $(j = 0,1,2)$ are satisfied (28) and A is satisfied*

$$A = \begin{cases} 4x-1, & \frac{1}{4} \le x \le \frac{1}{2}, \\ 3-4x, & \frac{1}{2} \le x \le \frac{3}{4}. \end{cases} \tag{29}$$

From (29), we get $[A]_\alpha = \left[\frac{\alpha+1}{4}, \frac{3-\alpha}{4}\right]$ for all $\alpha \in (0,1]$. There exists a unique solution of equation (8) by Theorem 1. Since $A_{r,\alpha} < 1$ for all $\alpha \in [0,1]$, then by Theorem 5, the positive solution (z_n) of equation (8) converges to 0 as $n \to \infty$, see Figures 1-3.

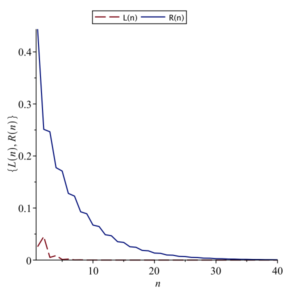

FIGURE 1. $\alpha = 0.1$

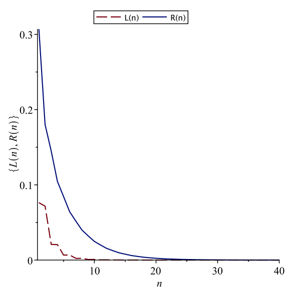

FIGURE 2. $\alpha = 0.5$

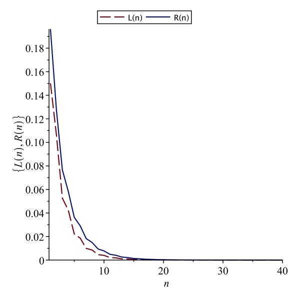

FIGURE 3. $\alpha = 0.9$

Example 2. *Consider equation (8) for $p = 1$ where z_n is a sequence of positive fuzzy numbers, the initial conditions z_{-j} $(j = 0,1,2)$ are satisfied (28) and A is satisfied*

$$A = \begin{cases} x-2, & 2 \le x \le 3, \\ 4-x, & 3 \le x \le 4. \end{cases} \tag{30}$$

From (30), we get $[A]_\alpha = [\alpha+2, 4-\alpha]$ for all $\alpha \in (0,1]$. There exists a unique positive solution of equation (8) by Theorem 1. It is easy to see that for all $\alpha \in (0,1]$, we have $A_{l,\alpha} > 1$. So, by case (ii) in Theorem 3, equation (8) has unbounded solutions, see Figures 4-6.

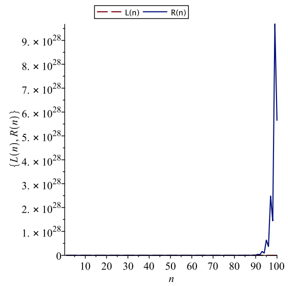

Figure 4. $\alpha = 0.1$

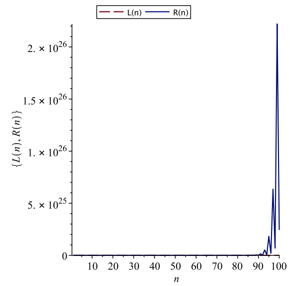

Figure 5. $\alpha = 0.5$

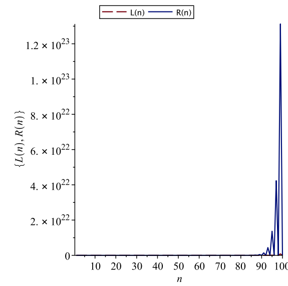

Figure 6. $\alpha = 0.9$

5. Conclusion

In this study, we investigated behavior of the fuzzy difference equation $z_{n+1} = Az_{n-1}/(1 + z_{n-2}^p)$, where (z_n) is a sequence of positive fuzzy numbers, A and the initial conditions z_{-j} $(j = 0, 1, 2)$ are positive fuzzy numbers and p is a positive integer. We have shown that, under certain conditions, the positive solutions of this equation converge to zero. Also, we have considered the case where the solutions are unbounded. Finally, we have supported our theoretical results.

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SIMILAR AND SELF-SIMILAR NULL CARTAN CURVES IN MINKOWSKI-LORENTZIAN SPACES

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Abstract. In this paper, differential invariants of null Cartan curves are studied in (n+2) dimensional Lorentzian similarity geometry. The fundamental theorem for a null Cartan curve in similarity geometry is investigated and the characterization of all self-similar null Cartan curves parameterized by de Sitter parameter in Minkowski space-time is given.

1. Introduction

A similarity transformation of Euclidean space, which consists of a rotation, a translation and an isotropic scaling, is an automorphism preserving the angles and ratios between lengths. The structure consisting of unchanging geometric properties under the similarity transformation is called similarity geometry. The whole Euclidean geometry can be considered as a class of similarity geometry. The similarity transformations are studied in most areas of the pure and applied mathematics. For example, S. Li [23] presented a system for matching and pose estimation of 3D space curves under the similarity transformation. Brook et al. [5] discussed various problems of image processing and analysis by using the similarity transformation. Sahbi [26] investigated a method for shape description based on kernel principal component analysis (KPCA) in the similarity invariance of KPCA. On the other hand, the self-similar objects, whose images under the similarity map are themselves, have had a wide range of applications in areas such as fractal geometry, dynamical systems, computer networks, statistical physics and so on. The Cantor set, the von Koch snowflake curve and the Sierpinski gasket are some of most famous examples of such objects (see [14, 21]). Recently, the self-similarity started playing a role in algebra as well, first of all in group theory [17].

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Bonnor [4] introduced the Cartan frame to study the behaviors of a null curve and proved the fundamental existence and congruence theorems in Minkowski space-time. Bejancu [2] represented a method for the general study of the geometry of null curves in Lorentz manifolds and, more generally, in semi-Riemannian manifolds (see also the book [11]). Ferrandez, Gimenez and Lucas [15] gave a reference along a null curve in an n-dimensional Lorentzian space. They showed the fundamental existence and uniqueness theorems and described the null helices in higher dimensions. Cöken and Ciftci [10] studied null curves in the Minkowski space-time and characterized pseudo-spherical null curves and Bertrand null curves.

The study of the geometry of null curves has a growing importance in the mathematical physics. The null curves are useful to find the solution of some equations in the classical relativistic string theory (see [6,19,20]) Moreover, there exists a geometric particle model associated with the geometry of null curves in the Minkowski space-time (see [16, 24]).

Berger [3] represented the broad content of similarity transformations of Euclidean space. Encheva and Georgiev [12, 13] studied the differential geometric invariants of curves according to a similarity in the Euclidean n-space. Chou and Qu [9] showed that the motions of curves in two, three and n-dimensional (n > 3) similarity geometries correspond to the Burgers hierarchy, Burgers-mKdV hierarchy and a multi-component generalization of these hierarchies. Şimşek and Özdemir [27] introduced the geometry of non-lightlike curves in the n-dimensional Lorentzian similarity geometry. Ateş et.al. [1] studied the similarity invariants of Frenet curves by considering the parametrization of any spherical indicatrix curve in Eucliedan space $\mathbb{E}^n$. Kamishima [22] examined the properties of compact Lorentzian similarity manifolds using developing maps and holonomy representations. The main idea of this paper is to study the differential geometry of a null curve under the similarity mapping.

The scope of paper is as follows. First, we give basic information about null Cartan curves. Then, we introduce a new parameter, which is called de Sitter parameter that is invariant under the similarity transformation. We represent the differential geometric invariants of a null Cartan curve, which are called shape Cartan curvatures, in (n+2)-dimensional Lorentzian similarity geometry. We prove the uniqueness theorem which states that two null Cartan curves are equivalent according to a similarity mapping. Furthermore, we show the existence theorem that is a process for constructing a null Cartan curve by the shape Cartan curvatures under some initial conditions. Lastly, we obtain the equations of all self-similar null Cartan curves parameterized by the de Sitter parameter in Minkowski spacetime.

2. Preliminaries

Let $\mathbf{u} = (u_1, u_2, \ldots, u_{n+2})$, $\mathbf{v} = (v_1, v_2, \ldots, v_{n+2})$ be two arbitrary vectors in Minkowski-Lorentzian space $\mathbb{M}^{n+2}$. The Lorentzian inner product of $\mathbf{u}$ and $\mathbf{v}$ can be stated as $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} I^* \mathbf{v}^T$ where $I^* = diag(-1, 1, \ldots, 1)$. We say that a vector

$\mathbf{u}$ in $\mathbb{M}^{n+2}$ is called spacelike, null (lightlike) or timelike if $\mathbf{u}\cdot\mathbf{u}>0$, $\mathbf{u}\cdot\mathbf{u}=0$ or $\mathbf{u}\cdot\mathbf{u}<0$, respectively. The norm of the vector $\mathbf{u}$ is defined by $\|\mathbf{u}\|=\sqrt{|\mathbf{u}\cdot\mathbf{u}|}$. The *pseudohyperbolic space* (or anti-de Sitter space) is defined by

$$\mathbb{H}_0^{n+1}(r)=\left\{\mathbf{u}\in\mathbb{M}^{n+2}:\mathbf{u}\cdot\mathbf{u}=-r^2\right\}$$

and *pseudo-sphere* (or *de Sitter space)* is defined by

$$\mathbb{S}_1^{n+1}(r)=\left\{\mathbf{u}\in\mathbb{M}^{n+2}:\mathbf{u}\cdot\mathbf{u}=r^2\right\},\ \ (\ [25])\,.$$

A basis $\mathbf{B}=\{\mathbf{L},\mathbf{N},\mathbf{W}_1,\ldots,\mathbf{W}_n\}$ is said pseudo-orthonormal if it satisfies the following conditions:

$$\mathbf{L}\cdot\mathbf{L}=\mathbf{N}\cdot\mathbf{N}=0,\quad \mathbf{L}\cdot\mathbf{N}=1$$
$$\mathbf{L}\cdot\mathbf{W}_i=\mathbf{N}\cdot\mathbf{W}_i=\mathbf{W}_i\cdot\mathbf{W}_j=0,\ i\neq j$$
$$\mathbf{W}_i\cdot\mathbf{W}_i=1$$

where $i,j\in\{1,\ldots,n\}$ ([11]) .

Now, we consider the mapping $\mathcal{A}:\left(\bar{\mathbf{L}},\bar{\mathbf{N}},\bar{\mathbf{W}}_1,\cdots,\bar{\mathbf{W}}_n\right)\to(\mathbf{L},\mathbf{N},\mathbf{W}_1,\cdots,\mathbf{W}_n)$ of one pseudo-orthonormal basis onto another at any point P in $\mathbb{M}^{n+2}$, which is given as either

$$\begin{bmatrix}\mathbf{L}\\ \mathbf{N}\\ \mathbf{W}_1\\ \mathbf{W}_2\\ \vdots\\ \mathbf{W}_{n-1}\\ \mathbf{W}_n\end{bmatrix}=\begin{bmatrix}\lambda & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ -\frac{1}{2}\lambda\left(\varepsilon_1^2+\varepsilon_2^2+\ldots+\varepsilon_n^2\right) & \lambda^{-1} & -\epsilon_1 & \epsilon_2 & -\epsilon_3 & \epsilon_4 & \cdots & -\epsilon_{n-1} & \epsilon_n\\ \lambda\varepsilon_1\cos\theta+\lambda\varepsilon_2\sin\theta & 0 & \cos\theta & -\sin\theta & 0 & 0 & \cdots & 0 & 0\\ \lambda\varepsilon_1\sin\theta-\lambda\varepsilon_2\cos\theta & 0 & \sin\theta & \cos\theta & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \lambda\varepsilon_{n-1}\cos\theta+\lambda\varepsilon_n\sin\theta & 0 & 0 & 0 & 0 & \cdots & 0 & \cos\theta & -\sin\theta\\ \lambda\varepsilon_{n-1}\sin\theta-\lambda\varepsilon_n\cos\theta & 0 & 0 & 0 & 0 & \cdots & 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\bar{\mathbf{L}}\\ \bar{\mathbf{N}}\\ \bar{\mathbf{W}}_1\\ \bar{\mathbf{W}}_2\\ \vdots\\ \bar{\mathbf{W}}_{n-1}\\ \bar{\mathbf{W}}_n\end{bmatrix}\qquad(1)$$

when n is even, or the same matrix with the additional row (column) $\begin{matrix}0 & 0 & \cdots & 1\end{matrix}$ when n is odd, where λ, ε_i $(1\leq i\leq n)$ and θ are real constants and $\lambda\neq 0$. The image of pseudo-orthonormal basis under the mapping $\mathcal{A}$ is a pseudo-orthonormal basis. Moreover, we have $\mathcal{A}^T\mathbf{J}^*\mathcal{A}=\mathbf{J}^*$, $\det\mathcal{A}=1$ where

$$\mathbf{J}^*=\begin{bmatrix}0 & 1 & 0 & 0 & \cdots & 0\\ 1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

and the orientation is preserved by (1). Bonnor [4] defined the mapping $\mathcal{A}$ as a *null rotation.* A null rotation at P is equivalent to a Lorentzian transformation between two sets of natural coordinate functions whose values coincide at P.

A curve locally parameterized by $\gamma:J\subset\mathbb{R}\to\mathbb{M}^{n+2}$ is called a *null curve* if $\frac{d}{dt}\gamma(t)\neq 0$ is a null vector for all t. We know that a null curve $\gamma(t)$ satisfies $\frac{d^2}{dt^2}\gamma(t)\cdot\frac{d^2}{dt^2}\gamma(t)\geq 0$ (see [11]). If the acceleration vector of the null curve is a unit vector, that is, $\frac{d^2}{dt^2}\gamma(t)\cdot\frac{d^2}{dt^2}\gamma(t)=1$, then, null curve $\gamma(t)$ in $\mathbb{M}^{n+2}$ is said to be parameterized by *pseudo-arc*. If the acceleration vector of the null curve is not a unit vector, then the pseudo-arc parametrization becomes as the following

$$s=\int_{t_0}^{t}\left(\frac{d^2}{du^2}\gamma(u)\cdot\frac{d^2}{du^2}\gamma(u)\right)^{1/4}du \quad ([4,10]). \tag{2}$$

A null curve $\gamma(t)$ in $\mathbb{M}^{n+2}$ with $\frac{d^2}{dt^2}\gamma(t)\cdot\frac{d^2}{dt^2}\gamma(t)\neq 0$ is a *Cartan curve* if

$$\left\{\tfrac{d}{dt}\gamma(t),\tfrac{d^2}{dt^2}\gamma(t),\tfrac{d^3}{dt^3}\gamma(t),\cdots,\tfrac{d^{n+1}}{dt^{n+1}}\gamma(t)\right\}$$

is linearly independent for any t.

There exists a unique *Cartan frame* $C_\gamma:=\{\mathbf{L},\mathbf{N},\mathbf{W}_1,\cdots,\mathbf{W}_n\}$ of the Cartan curve parameterized by a pseudo arc-parameter s such that the following equations are satisfied

$$\begin{aligned}
\gamma'(s)&=\mathbf{L}(s),\\
\mathbf{L}'(s)&=\mathbf{W}_1(s),\\
\mathbf{N}'(s)&=\kappa(s)\mathbf{W}_1(s)+\tau(s)\mathbf{W}_2(s),\\
\mathbf{W}_1'(s)&=-\kappa(s)\mathbf{L}(s)-\mathbf{N}(s),\\
\mathbf{W}_2'(s)&=-\tau(s)\mathbf{L}(s)+\kappa_3\mathbf{W}_3(s),\\
\mathbf{W}_i'(s)&=-\kappa_i(s)\mathbf{W}_{i-1}(s)+\kappa_{i+1}\mathbf{W}_{i+1}(s)\quad i\in\{3,\ldots,n-1\},\\
\mathbf{W}_n'(s)&=-\kappa_n(s)\mathbf{W}_{n-1}(s)
\end{aligned}\tag{3}$$

where $\mathbf{N}$ is a null vector called *null transversal vector field*, and C_γ is pseudo-orthonormal, $\{\gamma',\gamma'',\ldots,\gamma^{(i+2)}\}$ and $\{\mathbf{L},\mathbf{N},\mathbf{W}_1,\ldots,\mathbf{W}_i\}$ have the same orientation for $2\leq i\leq n-1$, C_γ is positively oriented and the differentiation with respect to s is denoted by prime "$\prime$". The functions κ, τ, κ_j $(3\leq j\leq n)$ are called the *Cartan curvatures* of $\gamma(s)$ and are given as

$$\begin{aligned}
\kappa(s)&=\frac{1}{2}\left(\gamma^{(3)}(s)\cdot\gamma^{(3)}(s)\right)\\
\tau^2(s)&=\gamma^{(4)}(s)\cdot\gamma^{(4)}(s)-\left(\gamma^{(3)}(s)\cdot\gamma^{(3)}(s)\right)^2\\
\kappa_j^2(s)&=\frac{D_j(s)\,D_{j+2}(s)}{D_{j+1}^2(s)}
\end{aligned}\tag{4}$$

for the pseudo-arc parameter s, where D_j denotes the j-th order main determinant of the matrix of the metric with respect to $\{\gamma',\gamma'',\ldots,\gamma^{(n+2)}\}$. We know that $\tau<0$, $\kappa_i>0$ $(3\leq i\leq n-1)$, and $\kappa_n>0$ or $\kappa_n<0$ according to $\{\gamma',\gamma'',\ldots,\gamma^{(n+2)}\}$ is positively or negatively oriented, respectively. More information about the geometry of null curves can be found in the papers [2], [4], [11] and [15].

3. Geometric Invariants of Null Curves Under a Similarity Map

In this section, we introduce the similarity geometry of null curves in $\mathbb{M}^{n+2}$. A *null-similarity (n-similarity)* $f : \mathbb{M}^{n+2} \to \mathbb{M}^{n+2}$ is determined by

$$f(x) = \mu \mathcal{A}(x) + \mathbf{C}, \tag{5}$$

where $\mu > 0$ is a real constant, $\mathcal{A}$ is a null rotation and $\mathbf{C}$ is a translation vector. The n-similarity transformations form a group under the composition of maps and denoted by $\mathbf{Simn}(\mathbb{M}^{n+2})$. The n-similarity transformations in $\mathbb{M}^{n+2}$ preserve the orientation.

Let $\gamma(t) : J \subset \mathbb{R} \to \mathbb{M}^{n+2}$ be a null curve. The image of γ under $f \in \mathbf{Simn}(\mathbb{M}^{n+2})$ is denoted by β. Then, the null curve β can be stated as

$$\beta(t) = \mu \mathcal{A}(\gamma(t)) + \mathbf{b}, \qquad t \in J. \tag{6}$$

The pseudo-arc length function β starting at $t_0 \in J$ is

$$s^*(t) = \int_{t_0}^{t} \left(\frac{d^2}{du^2}\beta(u) \cdot \frac{d^2}{du^2}\beta(u) \right)^{1/4} du = \sqrt{\mu}\, s(t) \tag{7}$$

where $s \in I \subset \mathbb{R}$ is pseudo-arc parameter of $\gamma : I \to \mathbb{M}^{n+2}$. We can compute the Cartan curvatures $\kappa_\beta\left(\sqrt{\mu}s\right)$ and $\tau_\beta\left(\sqrt{\mu}s\right)$ of β by using (4) as

$$\kappa_\beta = \frac{1}{\mu}\kappa_\gamma, \quad \tau_\beta = \frac{1}{\mu}\tau_\gamma, \text{ and } \kappa_{i\beta} = \frac{1}{\sqrt{\mu}}\kappa_{i\alpha}, \quad 3 \le i \le n. \tag{8}$$

We define $\mathbf{W}_1$-indicatrix γ_{W_1} of the null curve γ parameterized by $\gamma_{W_1}(s) = \mathbf{W}_1(s)$. The $\mathbf{W}_1$-indicatrix is a pseudo-spherical non-null curve lies on the de Sitter (n+1)-space $S_1^{n+1}(1)$. If we state the arc-length parameter of γ_{W_1} as σ_γ, we can find $d\sigma_\gamma = \sqrt{2\,|\kappa_\gamma|}ds$. The arc-length element $d\sigma_\gamma$ is invariant under the n-similarity transformation since the equality $d\sigma_\beta = d\sigma_\gamma$ can be easily found, where σ_β is the de Sitter parameter of β. The parameter σ_γ is called *de Sitter parameter* of γ. Therefore, we reparametrize a null curve by the de Sitter parameter so that we can study the differential geometry of a null curve under the n-similarity transformation.

The derivative formulas of γ and C_γ with respect to σ_γ are given by

$$\frac{d\gamma}{d\sigma_\gamma} = \frac{1}{\sqrt{2\,|\kappa_\gamma|}}\mathbf{L}, \qquad \frac{d^2\gamma}{d\sigma_\gamma^2} = \frac{-d\,|\kappa_\gamma|}{2\,|\kappa_\gamma|\,d\sigma_\gamma}\frac{d\gamma}{d\sigma_\gamma} + \frac{1}{2\,|\kappa_\gamma|}\mathbf{W}_1 \tag{9}$$

and

$$\begin{aligned} \frac{d\mathbf{L}}{d\sigma_\gamma} &= \frac{1}{\sqrt{2\,|\kappa_\gamma|}}\mathbf{W}_1 \\ \frac{d\mathbf{N}}{d\sigma_\gamma} &= \frac{\kappa_\gamma}{\sqrt{2\,|\kappa_\gamma|}}\mathbf{W}_1 + \frac{\tau_\gamma}{\sqrt{2\,|\kappa_\gamma|}}\mathbf{W}_2 \end{aligned}$$

$$\begin{aligned}
\frac{d\mathbf{W}_1}{d\sigma_\gamma} &= -\frac{\kappa_\gamma}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{L} - \frac{1}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{N} \\
\frac{d\mathbf{W}_2}{d\sigma_\gamma} &= -\frac{\tau_\gamma}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{L} + \frac{\kappa_{3\gamma}}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{W}_3 \\
\frac{d\mathbf{W}_3}{d\sigma_\gamma} &= -\frac{\kappa_{3\gamma}}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{W}_2 + \frac{\kappa_{4\gamma}}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{W}_3 \\
&\vdots \\
\frac{d\mathbf{W}_n}{d\sigma_\gamma} &= -\frac{\kappa_n}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{W}_{n-1}
\end{aligned} \tag{10}$$

Similarly, we can find the same formulas (9) and (10) for the null curve β.

Now, we construct a new frame corresponding to n-similarity transformation for a null curve. Let's define the functions

$$\tilde{\kappa}_\gamma := \frac{-d\left|\kappa_\gamma\right|}{2\left|\kappa_\gamma\right| d\sigma_\gamma},\ \ \tilde{\tau}_\gamma := \frac{\tau_\gamma}{2\left|\kappa_\gamma\right|} \text{ and } \tilde{\kappa}_{i\gamma} = \frac{\kappa_{i\gamma}}{\sqrt{2\left|\kappa_\gamma\right|}},\ \ 3 \leq i \leq n,$$

which are invariant under the n-similarity since we get the equalities

$$\tilde{\kappa}_\beta = \tilde{\kappa}_\gamma,\ \ \tilde{\tau}_\beta = \tilde{\tau}_\gamma \text{ and } \tilde{\kappa}_{i\beta} = \tilde{\kappa}_{i\gamma}.$$

If we set $\mathbf{L}^{sim} = \sqrt{2\left|\kappa_\gamma\right|}\mathbf{L}$, then we get a unit spacelike vector

$$\mathbf{W}_1^{sim} = \frac{d\mathbf{L}^{sim}}{d\sigma_\gamma} = \frac{d\left|\kappa_\gamma\right|}{\sqrt{2\left|\kappa_\gamma\right|}d\sigma_\gamma}\mathbf{L} + \mathbf{W}_1$$

such that $\mathbf{L}^{sim} \cdot \mathbf{W}_1^{sim} = 0$. From [2], we know that there exists a null vector $\mathbf{N}^{sim}$ satisfying

$$\mathbf{L}^{sim} \cdot \mathbf{N}^{sim} = 1, \qquad \mathbf{N}^{sim} \cdot \mathbf{W}_1^{sim} = 0,$$

in the space spanned by $\{\gamma',\ \gamma'',\ \gamma'''\}$ such that $\mathbf{N}^{sim}$ can be given in the form

$$\mathbf{N}^{sim} = \frac{1}{\mathbf{L}^{sim} \cdot \mathbf{V}}\left(\mathbf{V} - \frac{\mathbf{V} \cdot \mathbf{V}}{2\mathbf{L}^{sim} \cdot \mathbf{V}}\mathbf{L}^{sim}\right)$$

where $\mathbf{V} \in span\left\{\gamma',\ \gamma'',\ \gamma'''\right\}$. Choosing $\mathbf{V} = \frac{1}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{N} + \tilde{\kappa}_\gamma\mathbf{W}_1$ bring about

$$\mathbf{N}^{sim} = \frac{1}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{N} + \tilde{\kappa}_\gamma\mathbf{W}_1 - \frac{\tilde{\kappa}_\gamma^2\sqrt{2\left|\kappa_\gamma\right|}}{2}\mathbf{L},$$

which satisfies the relations $\mathbf{N}^{sim} \cdot \mathbf{N}^{sim} = 0$, $\mathbf{L}^{sim} \cdot \mathbf{N}^{sim} = 1$ and $\mathbf{N}^{sim} \cdot \mathbf{W}_1^{sim} = 0$. Moreover, If we choose $\mathbf{W}_i^{sim} = \mathbf{W}_i$, $2 \leq i \leq n$, then

$$C_\gamma^{sim} := \left\{\mathbf{L}^{sim}, \mathbf{N}^{sim}, \mathbf{W}_1^{sim}, \mathbf{W}_2^{sim}, \ldots, \mathbf{W}_n^{sim}\right\}$$

becomes a pseudo-orthonormal frame of γ under the n-similarity map. Then, the derivative formulas of C_γ^{sim} are computed as

$$\frac{d}{d\sigma_\gamma}\left(C_\gamma^{sim}\right)^T = P\left(C_\gamma^{sim}\right)^T \tag{11}$$

where

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \tilde{\xi}_\gamma & \tilde{\tau}_\gamma & 0 & 0 & \dots & 0 \\ -\tilde{\xi}_\gamma & -1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -\tilde{\tau}_\gamma & 0 & 0 & 0 & \tilde{\kappa}_{3\gamma} & 0 & \dots & 0 \\ 0 & 0 & 0 & -\tilde{\kappa}_{3\gamma} & 0 & \tilde{\kappa}_{4\gamma} & \dots & 0 \\ 0 & 0 & 0 & 0 & -\tilde{\kappa}_{4\gamma} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\tilde{\kappa}_{n\gamma} & 0 \end{bmatrix} \tag{12}$$

where $\tilde{\xi}_\gamma = \pm\frac{1}{2} - \frac{\tilde{\kappa}_\gamma^2}{2} + \frac{d\tilde{\kappa}_\gamma}{d\sigma_\gamma}$.

Let's consider the pseudo-orthogonal frame $C_\gamma^H := \left\{\mathbf{H}_1^\gamma, \mathbf{H}_2^\gamma, \mathbf{H}_3^\gamma, ..., \mathbf{H}_{n+2}^\gamma\right\}$ of γ where

$$\mathbf{H}_1^\gamma = \tfrac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{L}^{sim},\ \mathbf{H}_2^\gamma = \tfrac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{N}^{sim},\ \mathbf{H}_3^\gamma = \tfrac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{W}_1^{sim}, \dots,\ \mathbf{H}_{n+2}^\gamma = \tfrac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{W}_n^{sim}.$$

Since, from (5), we can obtain $f(\mathbf{H}_i^\gamma) = \mathbf{H}_i^\beta$, $i = 1, \cdots n+2$, the pseudo-orthogonal frame C_γ^H is invariant according to n-similarity map. Then, using (9) and (11), we get the derivative formulas of C_γ^H as the following

$$\frac{d}{d\sigma}\left(C_\gamma^H\right)^T = \tilde{P}\left(C_\gamma^H\right)^T \tag{13}$$

where

$$\tilde{P} = \begin{bmatrix} \tilde{\kappa}_\gamma & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \tilde{\kappa}_\gamma & \tilde{\xi}_\gamma & \tilde{\tau}_\gamma & 0 & 0 & \dots & 0 \\ -\tilde{\xi}_\gamma & -1 & \tilde{\kappa}_\gamma & 0 & 0 & 0 & \dots & 0 \\ -\tilde{\tau}_\gamma & 0 & 0 & \tilde{\kappa}_\gamma & \tilde{\kappa}_{3\gamma} & 0 & \dots & 0 \\ 0 & 0 & 0 & -\tilde{\kappa}_{3\gamma} & \tilde{\kappa}_\gamma & \tilde{\kappa}_{4\gamma} & \dots & 0 \\ 0 & 0 & 0 & 0 & -\tilde{\kappa}_{4\gamma} & \tilde{\kappa}_\gamma & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\tilde{\kappa}_{n\gamma} & \tilde{\kappa}_\gamma \end{bmatrix}.$$

We can consider the equation (13) as the Frenet-Serret equation of a null Cartan curve γ according to the pseudo-orthogonal moving frame C_γ^H under the group $\mathbf{Simn}\left(\mathbb{M}^{n+2}\right)$. As a result, the following theorem is obtained.

Theorem 1. *Let $\gamma : I \to \mathbb{M}^{n+2}$ be a null Cartan curve with pseudo-de Sitter parameter σ_γ and $\{\kappa_\gamma, \tau_\gamma, \tilde{\kappa}_{i\gamma}\ \ (3 \le i \le n)\}$ be Cartan curvatures of γ with the Cartan frame C_γ. Then, the functions*

$$\tilde{\kappa}_\gamma := \frac{-d\,|\kappa_\gamma|}{2\,|\kappa_\gamma|\,d\sigma_\gamma},\ \ \tilde{\tau}_\gamma := \frac{\tau_\gamma}{2\,|\kappa_\gamma|}\ \text{ and } \tilde{\kappa}_{i\gamma} = \frac{\kappa_{i\gamma}}{\sqrt{2\,|\kappa_\gamma|}},\ 3 \le i \le n, \tag{14}$$

and the pseudo-orthogonal frame C_γ^H are invariant under the n-similarity transformation in $\mathbb{M}^{n+2}$ and the derivative formulas of C_γ^H with respect to σ_γ are given by the equation (13).

Definition 1. *The functions $\tilde{\xi}_\gamma$ $\tilde{\tau}_\gamma$ and $\tilde{\kappa}_{i\gamma}$ $(3 \le i \le n)$ are called* shape Cartan curvatures *of a null Cartan curve γ and the pseudo-orthonormal frame C_γ^{sim} are called* shape Cartan frame *of γ.*

4. The Fundamental Theorem for a Null Curve

The existence and uniqueness theorems were shown by [2], [15] and [4] for a null Cartan curve under the Lorentz transformations. This notion can be extended with respect to $\mathbf{Simn}\left(\mathbb{M}^{n+2}\right)$ for the null Cartan curves parameterized by de Sitter parameter.

Theorem 2. *Let $\gamma, \beta : I \to \mathbb{M}^{n+2}$ be two null Cartan curves parameterized by the same de Sitter parameter σ, where $I \subset \mathbb{R}$ is an open interval. Suppose that γ and β have the same shape Cartan curvatures; namely, $\tilde{\kappa}_\gamma = \tilde{\kappa}_\beta$, $\tilde{\tau}_\gamma = \tilde{\tau}_\beta$ and $\tilde{\kappa}_{i\gamma} = \tilde{\kappa}_{i\beta}$ $(3 \le i \le n)$ for all $\sigma \in I$. Then, there exists a $f \in \boldsymbol{Simn}\left(\mathbb{M}^{n+2}\right)$ such that $\beta = f \circ \gamma$.*

Proof. Let κ_γ, τ_γ, $\kappa_{i\gamma}$ and κ_β, τ_β, $\kappa_{i\beta}$ $(3 \le i \le n)$ be the Cartan curvatures and also s and s^* be the pseudo-arc length parameters of γ and β, respectively. Using the equality $\tilde{\kappa}_\gamma = \tilde{\kappa}_\beta$, we get $|\kappa_\gamma| = \mu\,|\kappa_\beta|$ for some real constant $\mu > 0$. Then, the equality $\tilde{\tau}_\gamma = \tilde{\tau}_\beta$ imply $\tau_\gamma = \mu\tau_\beta$. Therefore, we find $ds = \dfrac{1}{\sqrt{\mu}}ds^*$ from the definition of de Sitter parameter σ.

There exists a Lorentzian motion $\varphi = \mathcal{A} \circ T$ of $\mathbb{M}^{n+2}$ satisfying the equality $\varphi\left(\gamma\left(\sigma_0\right)\right) = \beta\left(\sigma_0\right)$ for any fixed $\sigma_0 \in I$, where $\mathcal{A}$ is a null rotation and T is a translation map, such that φ maps the pseudo-orthonormal frame C_γ^{sim} to pseudo-orthonormal frame C_β^{sim}. Therefore, the map $g = \mu\varphi : \mathbb{M}^{n+2} \to \mathbb{M}^{n+2}$ is a n-similarity transformation of $\mathbb{M}^{n+2}$. Let's define a function $\Phi : I \to \mathbb{R}$ as the following

$$\Phi\left(\sigma\right) = \left\| \frac{d}{d\sigma} g\left(\gamma\left(\sigma\right)\right) - \frac{d}{d\sigma}\beta\left(\sigma\right) \right\|^2 \qquad \text{for } \forall\sigma \in I.$$

Taking derivative of this function with respect to σ, we conclude that

$$\frac{d\Phi}{d\sigma} = 2g\left(\frac{d^2\gamma}{d\sigma^2}\right) \cdot g\left(\frac{d\gamma}{d\sigma}\right) - 2\left[g\left(\frac{d^2\gamma}{d\sigma^2}\right) \cdot \frac{d\beta}{d\sigma}\right]$$

$$-2\frac{d^2\beta}{d\sigma^2}\cdot g\left(\frac{d\gamma}{d\sigma}\right)+2\left[\frac{d^2\beta}{d\sigma^2}\cdot\frac{d\beta}{d\sigma}\right].$$

Then, we obtain the following equation

$$\frac{d\Phi}{d\sigma}=0$$

since we have

$$\frac{d\gamma}{d\sigma}=\frac{1}{2\left|\kappa_\gamma\right|}\mathbf{L}_\gamma^{sim},\qquad \frac{d^2\gamma}{d\sigma^2}=\frac{\tilde{\kappa}_\gamma}{2\left|\kappa_\gamma\right|}\mathbf{L}_\gamma^{sim}+\frac{1}{2\left|\kappa_\gamma\right|}\mathbf{W}_{1\gamma}^{sim}$$

from (9). Also, we can find

$$\frac{d}{d\sigma}g\left(\gamma\left(\sigma_0\right)\right)=g\left(\frac{1}{2\left|\kappa_\gamma\right|}\mathbf{L}_\gamma^{sim}\left(\sigma_0\right)\right)=\frac{1}{2\left|\kappa_\beta\right|}\mathbf{L}_\beta^{sim}\left(\sigma_0\right)=\frac{d}{d\sigma}\beta\left(\sigma_0\right),$$

which implies $\Phi\left(\sigma_0\right)=0$. This means that

$$\frac{d}{d\sigma}g\left(\gamma\left(\sigma\right)\right)=\frac{d}{d\sigma}\beta\left(\sigma\right)$$

or equivalently $\beta\left(\sigma\right)=g\left(\gamma\left(\sigma\right)\right)+\mathbf{b}$ for all σ where $\mathbf{b}$ is a constant vector. Thus, the image of γ under the n-similarity $f=E\circ g$ is the null Cartan curve β, where $E:\mathbb{M}^{n+2}\to\mathbb{M}^{n+2}$ is a translation determined by $\mathbf{b}$. □

The following theorem shows that every two functions determine a null Cartan curve according to a n-similarity under some initial conditions.

Theorem 3. *Let $z_i:I\to\mathbb{R}$, $i=1,2,\dots n$ be smooth functions and $\mathbf{L}^{0sim}$, $\mathbf{N}^{0sim}$, $\mathbf{W}_1^{0sim}$, $\mathbf{W}_2^{0sim},\dots,\mathbf{W}_n^{0sim}$ be a pseudo-orthonormal frame at a point x_0 in the Minkowski space $\mathbb{M}^{n+2}$. According to a n-similarity, there exists a unique null Cartan curve $\gamma:I\to\mathbb{M}^{n+2}$ parameterized by the de Sitter parameter σ such that γ satisfies the following conditions:*

(i) There exists $\sigma_0\in I$ such that $\gamma\left(\sigma_0\right)=x_0$ and the shape Cartan frame of γ at x_0 is $\mathbf{L}^{0sim}$, $\mathbf{N}^{0sim}$, $\mathbf{W}_1^{0sim}$, $\mathbf{W}_2^{0sim},\dots,\mathbf{W}_n^{0sim}$.

(ii) $\tilde{\kappa}_\gamma\left(\sigma\right)=z_1\left(\sigma\right)$, $\tilde{\tau}_\gamma\left(\sigma\right)=z_2\left(\sigma\right)$ and $\tilde{\kappa}_{i\gamma}\left(\sigma\right)=z_i\left(\sigma\right)$ ($3\le i\le n$) for all $\sigma\in I$.

Proof. Let us consider the following system of differential equations with respect to a matrix-valued function $\mathbf{K}\left(\sigma\right)=\left(\mathbf{L}^{sim},\mathbf{N}^{sim},\mathbf{W}_1^{sim},\mathbf{W}_2^{sim},\dots,\mathbf{W}_n^{sim}\right)^T$

$$\frac{d\mathbf{K}}{d\sigma}\left(\sigma\right)=\mathbf{M}\left(\sigma\right)\mathbf{K}\left(\sigma\right)\tag{15}$$

with a given matrix

$$\mathbf{M}(\sigma)=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0\\ 0 & 0 & z & z_2 & 0 & 0 & \dots & 0\\ -z & -1 & 0 & 0 & 0 & 0 & \dots & 0\\ -z_2 & 0 & 0 & 0 & z_3 & 0 & \dots & 0\\ 0 & 0 & 0 & -z_3 & 0 & z_4 & \dots & 0\\ 0 & 0 & 0 & 0 & -z_4 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & -z_n & 0 \end{bmatrix},$$

where $z(\sigma)=\pm\frac{1}{2}-\frac{z_1^2}{2}+\frac{dz_1}{d\sigma}$. The system (15) has a unique solution $\mathbf{K}(\sigma)$ which satisfies the initial conditions

$$\mathbf{K}(\sigma_0)=\left(\mathbf{L}^{0sim},\mathbf{N}^{0sim},\mathbf{W}_1^{0sim},\mathbf{W}_2^{0sim},\dots,\mathbf{W}_n^{0sim}\right)^T$$

Then, we can write

$$\begin{aligned}\frac{d}{d\sigma}\left(\mathbf{J}^*\mathbf{K}^T\mathbf{J}^*\mathbf{K}\right)&=\mathbf{J}^*\frac{d}{d\sigma}\mathbf{K}^T\mathbf{J}^*\mathbf{K}+\mathbf{J}^*\mathbf{K}^T\mathbf{J}^*\frac{d}{d\sigma}\mathbf{K}\\ &=\mathbf{J}^*\mathbf{K}^T\left(\mathbf{M}^T\mathbf{J}^*+\mathbf{J}^*\mathbf{M}\right)\mathbf{K}=0\end{aligned}$$

since we have the equation $\mathbf{M}^T\mathbf{J}^*+\mathbf{J}^*\mathbf{M}=[0]_{(n+2)\times(n+2)}$. Also, we have

$$\mathbf{J}^*\mathbf{K}^T(\sigma_0)\,\mathbf{J}^*\mathbf{K}(\sigma_0)=\mathbf{I}$$

where $\mathbf{I}$ is the unit matrix since $\mathbf{L}^{0sim}$, $\mathbf{N}^{0sim}$, $\mathbf{W}_1^{0sim}$, $\mathbf{W}_2^{0sim},\dots,\mathbf{W}_n^{0sim}$ is the pseudo-orthonormal (n+2)-frame. As a result, we find $\mathbf{J}^*\mathbf{X}^T(\sigma)\,\mathbf{J}^*\mathbf{X}(\sigma)=\mathbf{I}$ for all $\sigma\in I$. This means that the vector fields $\mathbf{L}^{sim}$, $\mathbf{N}^{sim}$, $\mathbf{W}_1^{sim}$, $\mathbf{W}_2^{sim},\dots,\mathbf{W}_n^{sim}$ form a pseudo-orthonormal frame field in $\mathbb{M}^{n+2}$.

Let $\gamma: I\to\mathbb{M}^{n+2}$ be a null curve given by

$$\gamma(\sigma)=x_0+\frac{1}{2}\int_{\sigma_0}^{\sigma}e^{2\int z_1(\sigma)d\sigma}\mathbf{L}^{sim}(\sigma)\,d\sigma,\qquad \sigma\in I. \tag{16}$$

By the equality (15), we get that $\gamma(\sigma)$ is a similar null Cartan curve with the curvatures $\tilde{\kappa}_\gamma(\sigma)=z_1(\sigma)$, $\tilde{\tau}_\gamma(\sigma)=z_2(\sigma)$ and $\tilde{\kappa}_{i\gamma}(\sigma)=z_i(\sigma)$ $(3\le i\le n)$ in $\mathbb{M}^{n+2}$. Also, we find $\sqrt{\frac{1}{2}e^{2\int z_1(\sigma)d\sigma}}d\sigma=ds$ by using (2) and (15), where s is a pseudo-arc parameter; thus, σ is the de Sitter parameter of the null Cartan curve γ. Besides, the pseudo-orthonormal (n+2)-frame $\mathbf{L}^{sim}$, $\mathbf{N}^{sim}$, $\mathbf{W}_1^{sim}$, $\mathbf{W}_2^{sim},\dots,\mathbf{W}_n^{sim}$ is a shape Cartan frame of the null Cartan curve γ under the n-similarity transformation. □

Remark 1. *In case of $\tilde{\kappa}_\gamma(\sigma)=0$, the Cartan curvature $|\kappa_\gamma|=d$ is a positive real constant. Then, the parametrization of a null curve $\gamma: I\to\mathbb{M}^{n+2}$ with $\tilde{\kappa}_\gamma(\sigma)=0$*

with respect to de Sitter parameter σ *is given by*

$$\gamma(\sigma) = x_0 + \frac{1}{2d}\int_{\sigma_0}^{\sigma} \mathbf{L}^{sim}(\sigma)\, d\sigma, \qquad \sigma \in I \tag{17}$$

from the equation (9) *and* (16)*.*

Example 1. *Let's choose* $\tilde{\kappa}_\gamma = -\tanh\left(\frac{\sigma}{2}\right)$ *and* $\tilde{\tau}_\gamma = 4$ *for the shape Cartan curvatures of a null curve* $\gamma : I \to \mathbb{M}^4$ *with the initial conditions*

$$\begin{aligned} \mathbf{L}^{0sim} &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), \ \mathbf{N}^{0sim} = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), \\ \mathbf{W}_1^{0sim} &= \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \ \mathbf{W}_2^{0sim} = \left(0, \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right). \end{aligned} \tag{18}$$

Then, we get $\tilde{\xi}_\gamma = 0$ *and the system* (15) *determines a null vector* $\mathbf{L}^{sim}$ *given by*

$$\mathbf{L}^{sim}(\sigma) = \frac{1}{\sqrt{2}}(\cosh(2\sigma), 0, \cosh(2\sigma), 0) \tag{19}$$

with $\mathbf{L}^{sim}(0) = \mathbf{L}^{0sim}$*, in* $\mathbb{M}^4$*. The parametrization of null Cartan curve* γ *is found as*

$$\gamma(\sigma) = \frac{2\sqrt{2}}{3}\left(3\sigma + \frac{12e^{2\sigma} + 21e^{\sigma} + 11}{(1+e^{\sigma})^3}, 0, 3\sigma + \frac{12e^{2\sigma} + 21e^{\sigma} + 11}{(1+e^{\sigma})^3}, 0\right)$$

for any $\sigma \in I$ *by solving the equation* (16)*.*

5. Self-similar Null Cartan Curves

In this section, the concept of self-similarity is applied to null Cartan curves. Self-similar spacelike and timelike curves were studied by [27] and were defined as curves with constant p-shape curvatures. This idea can be extended to a null Cartan curve $\gamma : I \to \mathbb{M}^{n+2}$; that is, γ is called *self-similar* if shape Cartan curvatures of γ are constant.

A null curve is called a *null helix* if it has the constant Cartan curvatures which are not all zero in $\mathbb{M}^{n+2}$. A null helix is automotically a self-similar null Cartan curve in $\mathbb{M}^{n+2}$. Thus, null helices can be considered as a subclass of self-similar null Cartan curves.

5.1. Self-similar null Cartan curves in $\mathbb{M}^4$.

Now, we determine the parametrizations of all self-similar null Cartan curves by means of the constant shape Cartan curvatures in the Minkowski space-time. They can be examined by separating into four different cases as follows. For each case, we choose the initial conditions (18) in the example 1.

- **Case 1:** Let's take $\tilde{\kappa}_{\gamma_1} = 0$ and $\tilde{\tau}_{\gamma_1} = 0$, which means $\tilde{\xi}_{\gamma_1} = \pm\frac{1}{2}$.
- **Case 2:** Let's take $\tilde{\kappa}_{\gamma_1} = 0$ and $\tilde{\tau}_{\gamma_1} = a \neq 0$, which means $\tilde{\xi}_{\gamma_1} = \pm\frac{1}{2}$.

- **Case 3:** Let's take $\tilde{\kappa}_{\gamma_1} = b \neq 0$ and $\tilde{\tau}_{\gamma_1} = 0$, which means $\tilde{\xi}_{\gamma_1}$ is also a constant different from 0 and $\pm\frac{1}{2}$.
- **Case 4:** Let's take $\tilde{\kappa}_{\gamma_1} = b \neq 0$ and $\tilde{\tau}_{\gamma_1} = a \neq 0$, which means $\tilde{\xi}_{\gamma_1}$ is also a constant different from 0 and $\pm\frac{1}{2}$.

All the cases above correspond to the following two general cases:

- **GCase 1:** Let's take $\tilde{\xi}_{\gamma_1} = c \neq 0$ and $\tilde{\tau}_{\gamma_1} = 0$, such that it corresponds to the Case 1 and Case 3.
- **GCase 2:** Let's take $\tilde{\xi}_{\gamma_1} = c \neq 0$ and $\tilde{\tau}_{\gamma_1} = a \neq 0$, such that it corresponds to the Case 2 and Case 4.

GCase 1: Using the equation (15) , we obtain the following differential equation

$$\left(\mathbf{L}^{sim}\right)''' + 2c\left(\mathbf{L}^{sim}\right)' = 0$$

and by solving this equation, we conclude that if $c > 0$

$$\mathbf{L}^{sim}(\sigma) = \frac{1}{\sqrt{2}}\left(1, 0, \cos\left(\sqrt{2c}\sigma\right), \sin\left(\sqrt{2c}\sigma\right)\right)$$

and if $c < 0$

$$\mathbf{L}^{sim}(\sigma) = \frac{1}{\sqrt{2}}\left(\cosh\left(\sqrt{-2c}\sigma\right), \sinh\left(\sqrt{-2c}\sigma\right), 1, 0\right).$$

If the **Case 1** is valid**,** we use the equation (17) so that we get the following parametrization of the self-similar null Cartan curve

$$\gamma_1(\sigma) = \frac{1}{2\sqrt{2}d}\left(\sigma, 0, \frac{\sin\left(\sqrt{2c}\sigma\right)}{\sqrt{2c}}, -\frac{\cos\left(\sqrt{2c}\sigma\right)}{\sqrt{2c}}\right)$$

when $c > 0$ and

$$\gamma_2(\sigma) = \frac{1}{2\sqrt{2}d}\left(\frac{\sinh\left(\sqrt{-2c}\sigma\right)}{\sqrt{-2c}}, \frac{\cosh\left(\sqrt{-2c}\sigma\right)}{\sqrt{-2c}}, \sigma, 0\right)$$

when $c < 0$.

Since $\tilde{\xi}_{\gamma_1} = c = \pm\frac{1}{2}$ for the Case 1, we obtain

$$\gamma_1(\sigma) = \frac{1}{2\sqrt{2}d}\left(\sigma, 0, \sin(\sigma), -\cos(\sigma)\right)$$

for $c = 1/2$ and

$$\gamma_2(\sigma) = \frac{1}{2\sqrt{2}d}\left(\sinh(\sigma), \cosh(\sigma), \sigma, 0\right)$$

for $c = -1/2$.

If the **Case 3** is valid**,** we use the equation (16) so that we get the following parametrization of the self-similar null Cartan curve

$$\gamma_3(\sigma) = \begin{array}{l} \frac{1}{2\sqrt{2}}(\frac{e^{2b\sigma}}{2b}, 0, \frac{e^{2b\sigma}}{4b^2+2c}\left(2b\cos\left(\sqrt{2c}\sigma\right)+\sqrt{2c}\sin\left(\sqrt{2c}\sigma\right)\right), \\ \frac{e^{2b\sigma}}{4b^2+2c}\left(2b\sin\left(\sqrt{2c}\sigma\right)-\sqrt{2c}\cos\left(\sqrt{2c}\sigma\right)\right)) \end{array}$$

when $c>0$ and

$$\gamma_4(\sigma) = \begin{array}{l} \frac{1}{4\sqrt{2}}(\frac{\cosh(m_1\sigma)+\sinh(m_1\sigma)}{m_1} - \frac{\cosh(m_2\sigma)+\sinh(m_2\sigma)}{m_2}, \\ \frac{\cosh(m_1\sigma)+\sinh(m_1\sigma)}{m_1} + \frac{\cosh(m_2\sigma)+\sinh(m_2\sigma)}{m_2}, \frac{e^{2b\sigma}}{b}, 0) \end{array}$$

when $c<0$, where $m_1 = 2b+\sqrt{-2c}$, $m_2 = 2b-\sqrt{-2c} \neq 0$ and $c = \pm\frac{1}{2} - \frac{b^2}{2}$.

GCase 2: Using the equation (15) , we obtain the following differential equation

$$\left(\mathbf{L}^{sim}\right)^{iv} + 2c\left(\mathbf{L}^{sim}\right)'' - a^2\mathbf{L}^{sim} = 0$$

and by solving this equation, we conclude that

$$\mathbf{L}^{sim}(\sigma) = \frac{1}{\sqrt{2}}\left(\cosh(q_1\sigma), \sinh(q_1\sigma), \cos(q_2\sigma), \sin(q_2\sigma)\right)$$

where $q_1 = \sqrt{-c+\sqrt{c^2+a^2}}$, $q_2 = \sqrt{c+\sqrt{c^2+a^2}}$. and $c = \pm\frac{1}{2} - \frac{b^2}{2}$.

If the **Case 2** is valid**,** we use the equation (17) so that we get the following parametrization of the self-similar null Cartan curve

$$\gamma_5(\sigma) = \frac{1}{2d\sqrt{2}}\left(\frac{\sinh(q_1\sigma)}{q_1}, \frac{\cosh(q_1\sigma)}{q_1}, \frac{\sin(q_2\sigma)}{q_2}, -\frac{\cos(q_2\sigma)}{q_2}\right).$$

If the **Case 4** is valid**,** we use the equation (16) so that we get the following parametrization of the self-similar null Cartan curve

$$\begin{array}{l} \gamma_6(\sigma) = \frac{1}{4\sqrt{2}}(\frac{\cosh(n_1\sigma)+\sinh(n_1\sigma)}{n_1} + \frac{\cosh(n_2\sigma)+\sinh(n_2\sigma)}{n_2}, \\ \quad \frac{\cosh(n_1\sigma)+\sinh(n_1\sigma)}{n_1} - \frac{\cosh(n_2\sigma)+\sinh(n_2\sigma)}{n_2}, \\ \quad \frac{4be^{2b\sigma}\cos(q_2\sigma)+2q_2e^{2b\sigma}\sin(q_2\sigma)}{4b^2+q_2^2}, \frac{-2q_2e^{2b\sigma}\cos(q_2\sigma)+4be^{2b\sigma}\sin(q_2\sigma)}{4b^2+q_2^2}) \end{array}$$

where $n_1 = 2b+q_1 \neq 0$, $n_2 = 2b-q_1 \neq 0$.

From the above calculations, we obtain the following result.

Theorem 4. *Let γ be a null Cartan curve in $\mathbb{M}^4$. Then γ is a self-similar null Cartan curve if and only if it is congruent to one of the curves γ_1, γ_2 γ_3, γ_4, γ_5 and γ_6.*

In [4], null helices were defined and found their parametrizations in $\mathbb{M}^4$. When we compare the parametrizations of null helices with self-similar null curves, we conclude that null helices are a special class of self-similar null Cartan curves in $\mathbb{M}^4$. For example, a null helix satisfying $\tau \neq 0$ are expressed by

$$\alpha(s) = \sqrt{\frac{1}{v^2 + r^2}} \left(\frac{1}{v} \sinh vs, \frac{1}{v} \cosh vs, \frac{1}{r} \sin rs, -\frac{1}{r} \cos rs \right) \tag{20}$$

where $v = \sqrt{\sqrt{\kappa^2 + \tau^2} - \kappa}$ and $r = \sqrt{\sqrt{\kappa^2 + \tau^2} + \kappa}$ and this curve is a kind of self-similar null Cartan curve γ_5 (see also [10] for null helices).

In [10], Theorem 3.2 says that a null Cartan curve γ lies on $\mathbb{S}_1^3(r)$ iff $\tau_\gamma \neq 0$ is a constant in $\mathbb{M}^4$. Then, we conclude that the self-similar null Cartan curve lying on $\mathbb{S}_1^3(r)$ is similar to γ_5 because of the definitions $\tilde{\kappa}_\gamma$ and $\tilde{\tau}_\gamma$. On the other hand, in [7], Theorem 3.10 states that there are no null curves lying on $\mathbb{H}_0^3(r)$ in $\mathbb{M}^4$, which means that there is no a self-similar (similar) null Cartan curve lying on $\mathbb{H}_0^3(r)$.

6. Concluding Remarks

In the current paper, the similarity geometry of a null Cartan curve in Minkowski-Lorentzian spaces was investigated and self-similar null Cartan curves were studied in Minkowski space-time. Next study will be about self-similar null Cartan curves in Lorentzian space forms like null helices studied in [15].

The motions of curves in $\mathbb{E}^2$, $\mathbb{E}^3$ and $\mathbb{E}^n$ $(n > 3)$ yield the mKdV hierarchy, Schrödinger hierarchy and a multi- component generalization of mKdV-Schrödinger hierarchies, respectively. KS. Chou and C. Qu [9] showed that the motions of curves in two-, three- and n-dimensional $(n > 3)$ similarity geometries correspond to the Burgers hierarchy, Burgers-mKdV hierarchy and a multi-component generalization of these hierarchies by using the similarity invariants of curves in comparison with its invariants under the Euclidean motion. Also, they [8] found that many 1+1-dimensional integrable equations like KdV, Burgers, Sawada-Kotera, Harry-Dym hierarchies and Camassa-Holm equations arise from motions of plane curves in centro-affine, similarity, affine and fully affine geometries. The motion of curves on two-dimensional surfaces in $\mathbb{E}_1^3$ was considered by Gürses [18]. Therefore, the motion of Lorentzian similar (null and nonnull) curves in Lorentzian-Minkowski similarity geometries will be investigated as well.

Declaration of Competing Interests The author declares that he has no competing interest.

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EXPONENTIAL STABILITY OF A TIMOSHENKO TYPE THERMOELASTIC SYSTEM WITH GURTIN-PIPKIN THERMAL LAW AND FRICTIONAL DAMPING

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Abstract. In this paper we consider a linear thermoelastic system of Timoshenko type where the heat conduction is given by the linearized law of Gurtin-Pipkin. An existence and uniqueness result is proved by the use of a semigroup approach. We establish an exponential stability result without any assumption on the wave speeds once here we have a fully damped system.

1. Introduction

In the present paper we investigate the well-posedness and the asymptotic behavior of the following Timoshenko type system

$$\begin{cases} \rho_1 u_{tt} = \kappa\left(u_x+\varphi\right)_x & in\ (0,\pi)\times\mathbb{R}_+, \\ \rho_2\varphi_{tt} = b\varphi_{xx} - \kappa\left(u_x+\varphi\right) + \delta\theta - \tau\varphi_t & in\ (0,\pi)\times\mathbb{R}_+, \\ c\theta_t = -q_x - \delta\varphi_t & in\ (0,\pi)\times\mathbb{R}_+, \end{cases} \tag{1}$$

where u is the transverse displacement of a beam of length π, φ is the rotation angle of filament, θ is the temperature variation from an equilibrium reference value and q is the heat flux. The coefficients $\rho_1, \rho_2, c, \kappa, \tau$ are positive and present the mass density, the polar moment of inertia of a cross section, the specific heat constant, the shear modulus and the intensity of the frictional damping respectively, $b = EI$ is the product of Young's modulus of elasticity and the moment of inertia of a cross section, β and δ are coupling constants that are different from zero but their signs does not matter in the analysis.

To render the system (1) determined an additional equation relating q and θ is needed. In the classical theory of thermoelasticity the constitutive equation for the

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heat flux is expressed through Fourier's law of heat conduction

$$q = -k\theta_x, \tag{2}$$

where $k > 0$ represents the coefficient of the thermal conductivity of the material.

In 1921, Timoshenko [32] introduced a shear deformation and a rotational inertia into the derivation of the vibrating beam theory. He modelled the transverse vibrations of a beam by the conservative system

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } (0,L)\times(0,\infty) \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), & \text{in } (0,L)\times(0,\infty). \end{cases} \tag{3}$$

In the last three decades, the system (3) has been intensively studied for possible damping mechanisms. Muñoz Rivera and Racke [25] introduced a thermal damping by coupling system (3) with the classical heat equation. They proved that the system

$$\begin{cases} \rho_1\varphi_{tt} = k\left(\varphi_x + \psi\right)_x, \\ \rho_2\psi_{tt} = b\psi_{xx} - k\left(\varphi_x + \psi\right) + \gamma\theta_x, \\ c\theta_t = \kappa\theta_{xx} - \gamma\psi_{tx}. \end{cases} \tag{4}$$

(of course with some boundary and initial conditions), is exponentially stable if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}. \tag{5}$$

If (5) does not hold Guesmia *et al.* [17] established a polynomial decay result provided that the initial data are regular enough.

Almeida Junior *et al.* [1] considered the thermal coupling of the system (3) in shear force

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x + \sigma\theta_x = 0 & in\ (0,L)\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) - \sigma\theta = 0 & in\ (0,L)\times\mathbb{R}_+, \\ \rho_3\theta_t - \gamma\theta_{xx} + \sigma\left(\varphi_x + \psi\right)_t = 0 & in\ (0,L)\times\mathbb{R}_+, \end{cases} \tag{6}$$

subjected to either the boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi(t,0) = \psi(t,L) = \theta(t,0) = \theta(t,L) = 0, \tag{7}$$

or

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \theta_x(t,0) = \theta_x(t,L) = 0, \tag{8}$$

and proved that the solution is exponentially stable if and only if

$$\chi = \frac{\kappa}{\rho_1} - \frac{b}{\rho_2} = 0. \tag{9}$$

Otherwise, when (9) does not hold, the authors showed that the system is polynomially stable with a rate of decay $t^{-1/4}$ for the boundary conditions (7) and an optimal rate of decay $t^{-1/2}$ for the boundary conditions (8). Recently [18] reached the rate $t^{-1/2}$ for the boundary conditions (7) and

$$\varphi_x(t,0) = \varphi_x(t,L) = \psi_(t,0) = \psi(t,L) = \theta_x(t,0) = \theta_x(t,L) = 0.$$

Alves *et al.* [2] improve the results of [1] for the case of different wave speeds and obtained the same rate of decay $t^{-1/2}$ independently of the boundary conditions. Later, Alves *et al.* [3] extended the results of [1] to the non-homogeneous case with the boundary conditions (7). Precisely, they established an exponential stability provided that the non-homogeneous wave speeds satisfy the condition

$$\frac{\kappa(x)}{\rho_1(x)} = \frac{b(x)}{\rho_2(x)}, \quad x \in I \subset (0, L), \tag{10}$$

in an open subinterval I of $(0, L)$. When (10) does not hold they obtained a polynomial stability result with a rate of decay depending on the regularity of the initial data.

Recently, Jorge-Silva and Racke [19] considered (6) with Cattaneo's law and proved that there is non exponential stability no matter if (9) holds which confirms the result of [10].

We recall that the model using Fourier's law (2) leads to a parabolic equation. Consequently, the heat propagates with an infinite speed, that is, any thermal disturbance produced at some point in the body has an instantaneous effect elsewhere in the body. To overcome this physical paradox, many theories were developed. Green and Naghdi [12–14] expanded three new theories based on an entropy equality rather than the entropy inequality. They called them thermoelasticity of type I, type II and type III respectively. In each of these theories the equation for the heat flux is given by a different constitutive assumption. The constitutive equation for the heat flux in the type III theory is given by

$$q = -f_1\alpha_x - f_2\theta_x,$$

where

$$\alpha = \alpha_0(x) + \int_0^t \theta(x, \tau)\, d\tau$$

is the thermal displacement and f_1, f_2 are two positive constants.

In the framework of the thermoelasticity of type III, Messaoudi and Said-Houari [24] considered the following Timoshenko type system

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 & in\ (0,1) \times (0, +\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0 & in\ (0,1) \times (0, +\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \beta\psi_{ttx} + \kappa\theta_{txx} = 0 & in\ (0,1) \times (0, +\infty), \end{cases}$$

and showed that the solution (φ, ψ, θ) decays exponentially provided that $\frac{K}{\rho_1} = \frac{b}{\rho_2}$. The case of non equal speeds was examined by Messaoudi and Fareh [23]. They established a polynomial rate of decay. Fatori *et al.* [9] show that the optimal rate in this case is $t^{-1/2}$.

Santos and Almeida Júnior [30] extended the results of [23,24] to the Timoshenko system with thermoelastic effect acting on a shear force

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x + \sigma\theta_{tx} = 0 & in\ (0, L) \times (0, +\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) - \sigma\theta_t = 0 & in\ (0, L) \times (0, +\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \sigma\left(\varphi_x + \psi\right)_t - \gamma\theta_{txx} = 0 & in\ (0, L) \times (0, +\infty). \end{cases}$$

The second theory proposed to overcome the paradox of infinite speed was developed by Lord and Shulman [21]. They suggested to replace Fourier's law (2) by Cattaneo's one

$$\tau_0 q_t + q + k\theta_x = 0,$$

where the positive constant τ_0 represents the time lag in the response of the heat flux to the temperature gradient and is referred to as the thermal relaxation time. According to this theory, the system becomes fully hyperbolic, as a result the heat propagates with a finite speed and is viewed as a wave-like propagation rather than a diffusion phenomenon. A wave-like thermal disturbance is referred to as a second sound (where the first sound being the usual sound) and a nonclassical theory predicting the occurrence of such disturbances are known as thermoelasticity with finite wave speeds or second sound thermoelasticity.

Fernández Sare and Racke [10] considered the following Timoshenko type system with second sound thermoelasticity

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa\theta_x = 0, \end{cases} \tag{11}$$

and proved that the solution of (11) is no longer exponentially stable even if $\frac{\rho_1}{k} = \frac{\rho_2}{b}$. However, the incorporation of the frictional damping $\mu\varphi_t$ into the first equation of (11) produces an exponential stability independently of the wave speeds [22].

Santos *et al.* [31] introduced the stability number

$$\chi_0 = \left(\tau - \frac{\rho_1}{\rho_3\kappa}\right)\left(\rho_2 - \frac{b\rho_1}{\kappa}\right) - \frac{\tau\rho_1\delta^2}{\kappa\rho_3},$$

and proved that the solution of (11) is exponentially stable provided that $\chi_0 = 0$.

It is worth noting that the type III thermoelasticity and the second sound thermoelasticity are unable to describe the memory effect which reigns in some materials, particularly at a low temperature. This fact leads to the look for a more general constitutive assumption relating the heat flux to the thermal memory. Gurtin and Pipkin [16] assumed that the heat flux depends on the integrated history of the temperature gradient, and established a general nonlinear theory for which thermal disturbances propagate with a finite speed. In accordance with this theory, the linearized constitutive equation for q is given by

$$q = -\int_{-\infty}^{t} k\left(t - s\right)\theta_x\left(x, s\right)ds, \tag{12}$$

where $k(s)$ is the heat conductivity relaxation kernel. The presence of the convolution term (12) renders the Timoshenko system coupled with the heat equation into a fully hyperbolic system, which allows the heat to propagate with a finite speed and admits to describe the memory effect of the heat conduction.

In the context of Gurtin-Pipkin theory Pata and Vuk [26] studied the linear thermoelastic system

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - \theta_x(x,t), \\ \theta_t(x,t) = -u_{tx}(x,t) - q_x(x,t), \end{cases}$$

where the heat flux q is given by (12). They proved, under some assumptions on $\mu(s) = -k'(s)$, that the solution of the system decays exponentially. Fatori and Muñoz Rivera [8] considered the system

$$\begin{cases} u_{tt} - au_{xx} + \alpha\theta_x = 0 \; in \; (0,L) \times \mathbb{R}_+ \\ \theta_t - k * \theta_{xx} + \alpha u_{xt} = 0 \; in \; (0,L) \times \mathbb{R}_+, \end{cases}$$

where

$$(k * \theta_{xx})(t) = \int_0^t k(t-\tau)\,\theta_{xx}(\tau)\,d\tau,$$

and established an exponential decay result provided that the kernel k is positive definite and decays exponentially.

Concerning Timoshenko systems coupled with the heat equation in the framework of Gurtin-Pipkin's theory, Dell'Oro an Pata [7] analyzed the following system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ \rho_3\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\,\theta_{xx}(t-s)\,ds + \delta\psi_{tx} = 0, \end{cases} \tag{13}$$

and proved that the semigroup associated with the solution of the system (13) is exponentially stable if and only if

$$\chi_g = \left[\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g(0)}\right]\left[\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right] - \frac{\beta}{g(0)}\frac{\rho_1\delta^2}{\rho_3\kappa b} = 0.$$

Closely related to Timoshenko's beam theory, Raposo [29] investigated the laminated Timoshenko system

$$\begin{cases} \rho_1 u_{tt} - \kappa(u_x - \psi)_x + \alpha u_t = 0 & in \; (0,L) \times \mathbb{R}_+, \\ \rho_2(s-\psi)_{tt} - b(s-\psi)_{xx} + \kappa(\psi - u_x) + \beta(s-\psi)_t = 0 & in \; (0,L) \times \mathbb{R}_+, \\ \rho_2 s_{tt} - bs_{xx} + 3\kappa(\psi - u_x) + 4\delta s + 4\gamma s_t = 0 & in \; (0,L) \times \mathbb{R}_+, \end{cases} \tag{14}$$

and obtained an exponential stability result. Regarding the damping by the heat conduction, Liu and Zhao [20] showed that the laminated beam coupled with the heat equation modelled via Fourier's law of the heat conduction is exponentially stable provided that the wave speeds are equal. Apalara [4] obtained the same result by coupling the laminated beam with the heat equation moddeled via Cattaneo's law, provided that the equal wave speeds is replaced by a relation between the coefficients of the system. Choucha *et al.* [5] added a distributed delay and proved the exponential and the polynomial stability for the equal and the non-equal wave speeds respectively. They also kept the same results in the presence of a viscoelastic damping and a distributed delay [6].

In view of the aforementioned studies we can summarized the stability results for Timoshenko systems coupled with thermal effects as follows:

i) A fully damped Timoshenko system with parabolic thermal effects is exponentially stable regardless any restriction on the wave speeds.
ii) A Timoshenko system damped only by thermal effects is exponentially stable if and only if the coefficients of the system satisfy a stability condition (equal wave speeds, in the case of the classical parabolic heat equation).

To the best of my knowledge there is no results concerning the fully damped Timoshenko system with hyperbolic thermal dissipation. One can expected that this leads to an exponential stability. In the present paper we give a positive answer to this concern.

It should be noted here, that replacing the parabolic heat conduction by a hyperbolic type one is not obviously profitable, first, because the system becomes fully hyperbolic and therefore it loses the exponential decay reached with one dissipation when (5) holds, (see [10, 28]), secondly, because the dissipative effects due to the hyperbolic type heat conduction are generally weaker than those induced by Fourier's law.

In the present paper we consider the fully damped case of (13) and prove the exponential stability of the solution without any condition. The importance of our result manifested from the fact that the case of equal speeds is purely mathematical, since it is physically never satisfied [15]. Therefore, the stability result obtained without any restriction on the coefficients is more realistic than that obtained with a stability condition.

Note that the presence of the convolution term in the constitutive equation for q renders the family operators mapping the initial value $(u_0, u_1, \varphi_0, \varphi_1, \theta_0)$ into the solution (u, φ, θ) not match the semigroup properties. This is due to the fact that the solution value of θ at time t depends on the whole function up to time t.

In order to overcome this difficulty we introduce the new variables

$$\theta^t(x,s) = \theta(x, t-s),\ s \geq 0,$$

and

$$\eta(x,s) = \eta^t(x,s) = \int_0^s \theta^t(x,\tau)d\tau,\ s \geq 0,$$

which denote the past history and the summed past history of θ up to t, respectively.

Clearly $\eta^t(x,s)$ satisfies the boundary conditions

$$\eta(0,s) = \eta(\pi,s) = 0.$$

Moreover, we assume that $k(\infty) = 0$ and $\eta(x,0) = \lim\limits_{s \longrightarrow 0^+} \eta^t(x,s) = 0$, then

$$q = -\int_{-\infty}^{t} k(t-s)\,\theta_x(x,s)\,ds = \int_0^{\infty} k'(s)\,\eta_x^t(x,s)\,ds.$$

Further, we have

$$\eta_t(x,s) = \theta - \eta_s(x,s). \tag{15}$$

Setting $\mu(s) = -k'(s)$, the system (1) and equations (12), (15) become

$$\begin{cases} \rho_1 u_{tt} = \kappa(u_{xx} + \varphi_x) - \beta\theta_x & in\ (0,\pi)\times\mathbb{R}_+, \\ \rho_2\varphi_{tt} = b\varphi_{xx} - \kappa(u_x + \varphi) + \delta\theta - \tau\varphi_t & in\ (0,\pi)\times\mathbb{R}_+, \\ c\theta_t = \displaystyle\int_0^{\infty} \mu(s)\,\eta_{xx}^t(s)\,ds - \beta u_{xt} - \delta\varphi_t & in\ (0,\pi)\times\mathbb{R}_+, \\ \eta_t^t(s) = \theta - \eta_s^t(s) & in\ (0,\pi)\times\mathbb{R}_+\times\mathbb{R}_+. \end{cases} \tag{16}$$

The system (16) is complemented with the boundary conditions

$$\begin{array}{c} u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0, \\ \eta(0,s) = \eta(\pi,s) = 0, \forall t \in \mathbb{R}^+,\ \eta(x,0) = 0,\ \forall x \in (0,\pi), \end{array} \tag{17}$$

and the initial data

$$\begin{array}{c} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \varphi(x,0) = \varphi_0(x), \\ \varphi_t(x,0) = \varphi_1(x),\ \theta(x,0) = \theta_0(x),\ \eta^0(x,s) = \eta_0(x,s). \end{array} \tag{18}$$

Regarding the memory kernel μ, we assume the following set of hypotheses:

(h1) $\mu \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$,
(h2) $\mu(s) \geq 0$, $\mu'(s) \leq 0\ \ \forall s \geq 0$,
(h3) $\int_0^{\infty} \mu(s)\,ds = k_0 > 0$,
(h4) there exists $\xi > 0$, such that $\mu'(s) \leq -\xi\mu(s)$, $\forall s \geq 0$.

The rest of the paper is organized as follows: in Section 2, we introduce some functional preliminaries. Section 3 is devoted to the proof of an existence and uniqueness result. In Section 4, we state and prove our stability result.

2. Functional Setting

Let $A = -D^2$ be the operator defined over $L^2(0,\pi)$. It is well known that the operator A with the Dirichlet boundary conditions is a self-adjoint and positive operator with domain $D(A) = H^2 \cap H_0^1$. Thus, it is possible to define the powers A^{α} of A for $\alpha \in \mathbb{R}$, and the Hilbert space $V_\alpha = D\left(A^{\alpha/2}\right)$ endowed with the inner product

$$\langle u, v\rangle_\alpha = \left\langle A^{\alpha/2}u, A^{\alpha/2}v\right\rangle$$

and the associated norm denoted by $\|u\|_\alpha$. In particular, $V_0 = L^2$, $V_{-1} = H^{-1}$, $V_1 = H_0^1$ and

$$\left\langle A^{1/2}u, A^{1/2}v\right\rangle = \langle Du, Dv\rangle,\ \forall u, v \in H_0^1.$$

For $\alpha_1 > \alpha_2$ the injection $V_{\alpha_1} \hookrightarrow V_{\alpha_2}$ is continuous.

Furthermore, we introduce the weighted Hilbert space

$$\mathcal{M}_1 = L^2_\mu\left((0,+\infty); H^1_0(0,\pi)\right)$$

with the inner product

$$\langle \eta, \zeta\rangle_{\mathcal{M}_1} = \int_0^\infty \mu(s)\langle \eta(s), \zeta(s)\rangle_1 \, ds$$

and the norm

$$\|\eta\|^2_{\mathcal{M}_1} = \int_0^\infty \mu(s)\|D\eta(s)\|^2 \, ds.$$

We shall also need to define the spaces

$$\mathcal{M}_0 = L^2_\mu\left((0,+\infty); L^2(0,\pi)\right)$$

and

$$\begin{aligned}\mathcal{K} &= H^1_\mu\left((0,+\infty); H^1_0(0,\pi)\right)\\ &= \{\eta/\eta, \eta_s \in \mathcal{M}_1\}.\end{aligned}$$

The following lemma will be useful in the proof of our main result.

Lemma 1. *Let $v \in L^2(0,\pi)$ be given and*

$$\overline{v} = \frac{1}{\pi}\int_0^\pi v(x)\, dx$$

the mean value of v. Then,

$$\|Dv\|_{-1} = \|v - \overline{v}\|. \tag{19}$$

Proof. We have

$$\|Dv\|_{-1} = \sup_{\|D\psi\|=1} |\langle Dv, \psi\rangle| = \sup_{\|D\psi\|=1} |\langle v, D\psi\rangle| \le \|v\|.$$

Let $\psi(x) = \frac{1}{\|v\|}\int_0^x v(y)\, dy$, then $\|D\psi\| = 1$ and

$$|\langle Dv, \psi\rangle| = \|v\| \le \|Dv\|_{-1}.$$

Therefore,

$$\|Dv\|_{-1} = \|v\|.$$

Suppose that $\overline{v} = 0$, then

$$\|Dv\|_{-1} = \|v - \overline{v}\|.$$

If $\overline{v} \neq 0$, then

$$\|Dv\|_{-1} = \|D(v - \overline{v})\|_{-1} = \|v - \overline{v}\|.$$

$\square$

3. Well Posedness

In this section we prove that the problem determined by (16)-(18) has a unique solution. The main tools of the proof are the Lumer-Phillips and the Lax-Milgram theorems. First we need to rewrite the problem in the semigroups setting.

Let $\mathcal{H}$ be the Hilbert space

$$\mathcal{H} = H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times L^2 \times \mathcal{M}_1$$

endowed with the inner product

$$\begin{aligned} \langle U, U^* \rangle = \quad & \kappa \int_0^\pi (u_x + \varphi)(u_x^* + \varphi^*)\, dx + \rho_1 \int_0^\pi v v^* dx + b \int_0^\pi \varphi_x \varphi_x^* dx \\ & + \rho_2 \int_0^\pi \phi \phi^* dx + c \int_0^\pi \theta \theta^* dx + \int_0^\infty \int_0^\pi \mu(s)\, \eta_x(s)\, \eta_x^*(s)\, dx ds \end{aligned}$$

and the associated norm

$$\|U\|_{\mathcal{H}}^2 = \kappa \|u_x + \varphi\|^2 + \rho_1 \|v\|^2 + b\|\varphi_x\|^2 + \rho_2 \|\phi\|^2 + c\|\theta\|^2 + \|\eta\|_{\mathcal{M}_1}^2 .$$

We note that by virtue of the inequalities

$$\begin{aligned} u_x^2 &\le 2(u_x + \varphi)^2 + 2\varphi^2, \\ (u_x + \varphi)^2 &\le 2u_x^2 + 2\varphi^2, \end{aligned}$$

the above norm in $\mathcal{H}$ is equivalent to the usual norm. Therefore, we use either of the norms indifferently.

To rewrite the system (16) in the semigroup setting we introduce the new variables $v = u_t$ and $\phi = \varphi_t$, then the system (16) becomes

$$\begin{cases} u_t = v \\ v_t = \frac{\kappa}{\rho_1}(u_{xx}(x,t) + \varphi_x(x,t)) - \frac{\beta}{\rho_1}\theta_x(x,t) \\ \varphi_t = \phi \\ \phi_t = \frac{b}{\rho_2}\varphi_{xx}(x,t) - \frac{\kappa}{\rho_2}(u_x(x,t) + \varphi(x,t)) + \frac{\delta}{\rho_2}\theta(x,t) - \frac{\tau}{\rho_2}\phi(x,t) \\ \theta_t(x,t) = \frac{1}{c}\int_0^\infty \mu(s)\,\eta_{xx}^t(x,s)\, ds - \frac{\beta}{c} v_x(x,t) - \frac{\delta}{c}\phi(x,t) \\ \eta_t^t(x,s) = \theta(x,t) - \eta_t^t(x,s) \end{cases}$$

and the problem (16)–(18) rewritten

$$\begin{cases} \dfrac{d}{dt} U = \mathcal{A} U, \ t > 0, \\ U(0) = U_0, \end{cases} \tag{20}$$

where, $\mathcal{A}$ is the operator defined by

$$\mathcal{A}U=\begin{pmatrix} v \\ \dfrac{\kappa}{\rho_1}u_{xx}+\dfrac{\kappa}{\rho_1}\varphi_x-\dfrac{\beta}{\rho_1}\theta_x \\ \phi \\ \dfrac{b}{\rho_2}\varphi_{xx}-\dfrac{\kappa}{\rho_2}u_x-\dfrac{\kappa}{\rho_2}\varphi+\dfrac{\delta}{\rho_2}\theta-\dfrac{\tau}{\rho_2}\phi \\ \dfrac{1}{c}\displaystyle\int_0^\infty \mu(s)\,\eta_{xx}(s)\,ds-\dfrac{\beta}{c}v_x-\dfrac{\delta}{c}\phi \\ \theta-\eta_s \end{pmatrix}$$

with domain

$$D(\mathcal{A}):=\left\{\begin{array}{c} U\in\mathcal{H}; u,\varphi\in H^2, v,\theta\in H_0^1, \phi\in H_*^1, \eta\in H_\mu^1\left((0,+\infty);H_0^1\right), \\ \displaystyle\int_0^\infty \mu(s)\,\eta_{xx}(s)\,ds\in L^2, \eta(0)=0 \end{array}\right\}.$$

Before stating the main result of this section let us recall the following theorems.

Theorem 1. (Lumer-Phillips) *[27,33] Let $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}$ be a densely defined operator. Then $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$ if and only if*

i) $\mathcal{A}$ is dissipative;

ii) there exists a constant $\lambda>0$ such that $\lambda I-\mathcal{A}$ is onto.

Theorem 2. *[33] Let $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}$ be the infinitesimal generator of a C_0-semigroup $\{S(t);t\geq 0\}$. Then, for each $\xi\in D(\mathcal{A})$ and each $t\geq 0$, we have $S(t)\xi\in D(\mathcal{A})$ and the mapping*

$$\begin{aligned} S:[0,+\infty[\, &\longrightarrow \mathcal{H} \\ t &\longrightarrow S(t)\xi \end{aligned}$$

is of class C^1 on $[0,+\infty[$ and satisfies

$$\frac{d}{dt}\left(S(t)\,\xi\right)=\mathcal{A}S(t)\,\xi=S(t)\,\mathcal{A}\xi.$$

Our main result reads as follows:

Theorem 3. *Suppose that μ satisfies the hypotheses (h1)-(h4), then for any $U_0=(u_0,u_1,\varphi_0,\varphi_1,\theta_0,\eta_0)^T\in\mathcal{H}$ the problem (20) has a unique solution $U\in C\left((0,+\infty);\mathcal{H}\right)$. Moreover, if $U_0=(u_0,u_1,\varphi_0,\varphi_1,\theta_0,\eta_0)^T\in D(\mathcal{A})$ then the solution U satisfies*

$$U\in C\left((0,+\infty);D(\mathcal{A})\right)\cap C^1\left((0,+\infty);\mathcal{H}\right).$$

Proof. First, we prove that $\mathcal{A}$ is dissipative. Indeed, for every $U\in D(\mathcal{A})$ we have

$$\begin{aligned}\langle\mathcal{A}U,U\rangle=\kappa\int_0^\pi (v_x+\phi)(u_x+\varphi)\,dx+\int_0^\pi(\kappa u_{xx}+\kappa\varphi_x-\beta\theta_x)\,v dx+b\int_0^\pi\phi_x\varphi_x dx \\ +\int_0^\pi(b\varphi_{xx}-\kappa u_x-\kappa\varphi+\delta\theta-\tau\phi)\,\phi dx\end{aligned}$$

$$+\int_0^\pi \left(\int_0^\infty \mu(s)\,\eta^t_{xx}(s)\,ds - \beta v_x - \delta\phi\right)\theta dx$$
$$+\int_0^\pi\int_0^\infty \mu(s)\left(\theta_x - \eta^t_{xs}\right)\eta_x(s)\,dsdx,$$
$$= -\tau\int_0^\pi \phi^2 dx - \frac{1}{2}\int_0^\infty \mu(s)\,\frac{d}{ds}\left\|\eta_x(s)\right\|^2 ds.$$

For the second term in the right-hand side, we have

$$\int_0^\infty \mu(s)\frac{d}{ds}\|\eta_x(s)\|^2 ds = \mu(s)\|\eta_x(s)\|^2\Big|_0^\infty - \int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds.$$

Since $\mu(s)\,\|\eta_x(s)\|^2$ and $\mu(s)\,\|\eta_{xs}(s)\|^2$ belong to $L^1(\mathbb{R}^+)$ and $\eta_x(0)=0$, hence

$$\lim_{s\to 0}\mu(s)\|\eta_x(s)\|^2 = \lim_{s\to 0}\mu(s)\left\|\int_0^s \eta_{xs}(\tau)d\tau\right\|^2,$$
$$\leq \limsup_{s\to 0}\left(\int_0^s \mu(s)^{1/2}\left\|\eta_{xs}(\tau)\right\| d\tau\right)^2.$$

The Cauchy-Schwarz inequality, leads to

$$\lim_{s\to 0}\mu(s)\|\eta_x(s)\|^2 \leq \limsup_{s\to 0} s\int_0^s \mu(\tau)\left\|\eta_{xs}(\tau)\right\|^2 d\tau = 0.$$

Therefore,

$$\int_0^\infty \mu(s)\frac{d}{ds}\|\eta_x(s)\|^2 ds = \lim_{s\to\infty}\mu(s)\|\eta_x(s)\|^2 - \int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds.$$

The left-hand side of the last equation is bounded, and from (h2) both terms on the right-hand side are positive. Then, the limit in the right hand side exists and is finite, and therefore equals zero. Thus,

$$\langle \mathcal{A}U, U\rangle = -\tau\int_0^\pi \phi^2 dx + \frac{1}{2}\int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds \leq 0,$$

which proves the dissipativeness of $\mathcal{A}$. Next, we show that $\mathcal{A}$ is maximal. Let $U^* = (u^*, v^*, \varphi^*, \phi^*, \theta^*, \eta^*)^T \in \mathcal{H}$, and find $U = (u, v, \varphi, \phi, \theta, \eta)^T \in D(\mathcal{A})$ such that

$$(I-\mathcal{A})\,U = U^*, \tag{21}$$

which reads in components

$$u - v = u^*, \tag{22}$$
$$\rho v - \kappa u_{xx} - \kappa\varphi_x + \beta\theta_x = \rho_1 v^*, \tag{23}$$
$$\varphi - \phi = \varphi^*, \tag{24}$$
$$(\rho_2+\tau)\,\phi - b\varphi_{xx} + \kappa u_x + \kappa\varphi - \delta\theta = \rho_2\phi^*, \tag{25}$$
$$c\theta - \int_0^\infty \mu(s)\,\eta^t_{xx}(s)\,ds + \beta v_x + \delta\phi = c\theta^*, \tag{26}$$
$$\eta - \theta + \eta_s = \eta^*. \tag{27}$$

Solving equation (27) gives

$$\eta(s) = \left(1 - e^{-s}\right)\theta + \int_0^s e^{y-s}\eta^*(y)\,dy. \tag{28}$$

Substituting (22), (24) and (28) into (23), (25) and (26) we get

$$\begin{cases} \kappa u_{xx} + \kappa\varphi_x - \beta\theta_x - \rho_1 u = -\rho_1\left(u^* + v^*\right), \\ b\varphi_{xx} - \kappa u_x - (\kappa + \rho_2 + \tau)\varphi + \delta\theta = -(\rho_2 + \tau)\varphi^* - \rho_2\phi^*, \\ c_\mu\theta_{xx} - c\theta - \beta u_x - \delta\varphi = -(c\theta^* + \beta u^* + \delta\varphi^*) - \int_0^\infty \mu(s)\left(\int_0^s e^{y-s}\eta^*_{xx}(y)\,dy\right)ds \end{cases} \tag{29}$$

where,

$$c_\mu = \int_0^\infty \mu(s)\left(1 - e^{-s}\right)ds$$

is a positive constant. The last term in the right-hand side of the third equation of (29) belongs to H^{-1}. Indeed, let $\psi \in H_0^1$ such that $\|\psi_x\| \leq 1$, then

$$\begin{aligned} \left|\left\langle \int_0^\infty \mu(s)\left(\int_0^s e^{y-s}\eta^*_{xx}(y)\,dy\right)ds, \psi\right\rangle\right| &= \left|\left\langle \int_0^\infty \mu(s)\left(\int_0^s e^{y-s}\eta^*_{x}(y)\,dy\right)ds, \psi_x\right\rangle\right| \\ &\leq \int_0^\infty \mu(s)\,e^{-s}\left(\int_0^s e^{y}\left\|\eta^*_x(y)\right\|dy\right)ds \\ &= \int_0^\infty e^y\left\|\eta^*_x(y)\right\|\int_y^\infty \mu(s)\,e^{-s}ds dy \\ &\leq \int_0^\infty \mu(y)\,e^y\left\|\eta^*_x(y)\right\|\int_y^\infty e^{-s}ds dy \\ &= \int_0^\infty \mu(y)\left\|\eta^*_x(y)\right\|dy < \infty. \end{aligned}$$

At this point we multiply the equations $(29)_1$,$(29)_2$ and $(29)_3$ by $\widetilde{u}, \widetilde{\varphi}$ and $\widetilde{\theta}$ respectively, integrating over $(0, \pi)$ and summing up, we obtain

$$B\left(U, \widetilde{U}\right) = L\left(\widetilde{U}\right), \tag{30}$$

where

$$\begin{aligned} B\left(U, \widetilde{U}\right) := {}& \kappa\int_0^\pi u_x\widetilde{u}_x dx - \kappa\int_0^\pi \varphi_x\widetilde{u}dx + \beta\int_0^\pi \theta_x\widetilde{u}dx + \rho_1\int_0^\pi u\widetilde{u}dx \\ &+ b\int_0^\pi \varphi_x\widetilde{\varphi}_x dx + \kappa\int_0^\pi u_x\widetilde{\varphi}dx + (\kappa + \rho_2 + \tau)\int_0^\pi \varphi\widetilde{\varphi}dx \\ &- \delta\int_0^\pi \theta\widetilde{\varphi}dx + c_\mu\int_0^\pi \theta_x\widetilde{\theta}_x dx + c\int_0^\pi \theta\widetilde{\theta}dx + \beta\int_0^\pi u_x\widetilde{\theta}dx + \delta\int_0^\pi \varphi\widetilde{\theta}dx, \end{aligned}$$

and

$$L\left(\widetilde{U}\right):=\rho_1\int_0^\pi\left(u^*+v^*\right)\widetilde{u}dx+\left(\rho_2+\tau\right)\int_0^\pi\varphi^*\widetilde{\varphi}dx-\rho_2\int_0^\pi\phi^*\widetilde{\varphi}dx$$
$$+\int_0^\pi\left(c\theta^*+\beta u^*+\delta\varphi^*\right)\widetilde{\theta}dx+\int_0^\pi\widetilde{\theta}\int_0^\infty\mu\left(s\right)\left(\int_0^s e^{y-s}\eta_{xx}^*\left(y\right)dy\right)dsdx.$$

Clearly, $B\left(\cdot,\cdot\right)$ is a bounded bilinear form over $\mathcal{W}=H_0^1\times H_*^1\times H_0^1$ and L is a bounded linear form. Furthermore, we have

$$\begin{aligned}B\left(U,U\right)= \quad & \kappa\int_0^\pi u_x^2dx-\kappa\int_0^\pi\varphi_x udx+\beta\int_0^\pi\theta_x udx+\rho_1\int_0^\pi u^2dx+b\int_0^\pi\varphi_x^2dx\\ &+\kappa\int_0^\pi u_x\varphi dx+\left(\kappa+\rho_2+\tau\right)\int_0^\pi\varphi^2dx-\delta\int_0^\pi\theta\varphi dx+c_\mu\int_0^\pi\theta_x^2dx\\ &+c\int_0^\pi\theta^2dx+\beta\int_0^\pi u_x\theta dx+\delta\int_0^\pi\varphi\theta dx,\end{aligned}$$

$$\begin{aligned}B(U,U)=\kappa\int_0^\pi\left(u_x+\varphi\right)^2dx+\rho_1\int_0^\pi u^2dx+b\int_0^\pi\varphi_x^2dx\\ +\left(\rho_2+\tau\right)\int_0^\pi\varphi^2dx+c_\mu\int_0^\pi\theta_x^2dx+c\int_0^\pi\theta^2dx.\end{aligned}$$

Therefore, there exists a positive constant α such that

$$B\left(U,U\right)\geq\alpha\left\|U\right\|^2.$$

Thus, $B\left(\cdot,\cdot\right)$ is coercive and by means of the Lax-Milgram theorem, the problem (30) has a unique solution

$$(u,\varphi,\theta)\in\mathcal{W}.$$

Moreover, taking $\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right)=\left(\widetilde{u},0,0\right)$ in (30) we get

$$\kappa\int_0^\pi u_x\widetilde{u}_xdx=\int_0^\pi\left(\kappa\varphi_x-\beta\theta_x-\rho_1u+\rho_1\left(u^*+v^*\right)\right)\widetilde{u}dx,\ \forall\widetilde{u}\in H_0^1. \tag{31}$$

Using standard arguments of elliptic equations we infer that

$$u\in H^2(0,\pi)\cap H_0^1(0,\pi),$$

with

$$\kappa u_{xx}=-\kappa\varphi_x+\beta\theta_x+\rho_1u-\rho_1\left(u^*+v^*\right),$$

which solves $(29)_1$. Similarly, by choosing $\left(\widetilde{u},\widetilde{\varphi},\widetilde{\theta}\right)=\left(0,\widetilde{\varphi},0\right)$, we obtain

$$b\int_0^\pi\varphi_x\widetilde{\varphi}_xdx=-\int_0^\pi\left(\kappa\left(u_x+\varphi\right)+\left(\rho_2+\tau\right)\left(\varphi-\varphi^*\right)-\delta\theta-\rho_2\phi^*\right)\widetilde{\varphi}dx,\ \forall\widetilde{\varphi}\in H_*^1.$$

Let $\Psi \in H_0^1(0,\pi)$ and set

$$\widetilde{\Psi}(x,t) = \Psi(x,t) - \int_0^\pi \Psi(x,t)\,dx.$$

Clearly $\widetilde{\Psi} \in H_*^1(0,\pi)$. Plugging $\widetilde{\Psi}$ in (31) and recalling that

$$\kappa(u_x+\varphi) + (\rho_2+\tau)(\varphi-\varphi^*) - \delta\theta - \rho_2\phi^* \in L_*^2(0,\pi),$$

we arrive at

$$b\int_0^\pi \varphi_x\Psi_x dx = \int_0^\pi (\kappa(u_x+\varphi) + (\rho_2+\tau)(\varphi-\varphi^*) - \delta\theta - \rho_2\phi^*)\,\Psi dx,\ \forall\Psi \in H_0^1(0,\pi).$$

Thus, by virtue of the theory of elliptic equations, $\varphi \in H^2(0,\pi) \cap H_*^1(0,\pi)$ with

$$\varphi_{xx} = \frac{-1}{b}\left(\kappa(u_x+\varphi) + (\rho_2+\tau)(\varphi-\varphi^*) - \delta\theta - \rho_2\phi^*\right).$$

Then, φ solves $(29)_2$.

Substituting u, φ, θ just obtained in (22), (24) and (28), we infer that

$$v \in H_0^1(0,\pi), \phi \in H_*^1(0,\pi) \quad \text{and} \quad \eta \in H_\mu^1\left((0,+\infty); H_0^1(0,\pi)\right).$$

Moreover, (26) implies that

$$\int_0^\infty \mu(s)\,\eta_{xx}^t(s)\,ds \in L^2(0,\pi).$$

Finally we have

$$\eta_s(s) = e^{-s}\theta + \eta^*(s) - \int_0^s e^{y-s}\eta^*(y)\,dy \in \mathcal{M}_0$$

and $\eta(0) = 0$, which proves that the solution U of (21) belongs to $D(\mathcal{A})$. Hence, Lumer-Phillips theorem ensures that the problem (20) has a unique solution $U(x,t) = e^{\mathcal{A}t}U_0(x)$. This completes the proof of Theorem 3. □

4. Asymptotic Behavior

In this section we establish an exponential rate of decay for the solution of the system (16)-(18). The following Lemma gives a sufficient condition for a C_0−semigroup in order to be exponentially stable.

Lemma 2. *[11] Let $S(t)$ be a contraction semigroup on $\mathcal{H}$, and let $\mathcal{A}$ be its infinitesimal generator. If the operator $i\beta I - \mathcal{A}$ is bounded below as $\beta \in \mathbb{R}$, that is there exists $\lambda > 0$ such that*

$$\inf_{\beta\in\mathbb{R}} \|(i\beta I - \mathcal{A})U\| \geq \lambda\|U\|,\ \forall U \in D(\mathcal{A}),$$

then $S(t)$ is exponentially stable.

The main result of this paper reads as follows:

Theorem 4. *Assume that the memory kernel μ satisfies the hypotheses (h1)–(h5). Then the semigroup $S(t) = e^{\mathcal{A}t}$ associated to the problem (16)-(18) is exponentially stable.*

Proof. The proof will be done by a contradiction argument. Suppose that the assertion is false. Then there exist a sequence $(\lambda_n) \subset \mathbb{R}$ and a sequence $(U_n) \subset D(\mathcal{A})$, of unit norm

$$\begin{aligned}\kappa \|Du_n + \varphi_n\|^2 + \rho_1 \|v_n\|^2 + b \|D\varphi_n\|^2 + \rho_2 \|\phi_n\|^2 + c \|\theta_n\|^2 \\ + \int_0^\infty \mu(s) \|D\eta_n(s)\|^2 \, ds = 1,\end{aligned}$$

such that

$$\lim_{n \longrightarrow \infty} \|(i\lambda_n I - \mathcal{A}) U_n\| = 0,$$

which reads in components as

$$i\lambda_n u_n - v_n \longrightarrow 0 \ \ in \ \ H_0^1, \tag{32}$$

$$i\rho_1\lambda_n v_n - \kappa D^2 u_n - \kappa D\varphi_n + \beta D\theta_n \longrightarrow 0 \ \ in \ \ L^2, \tag{33}$$

$$i\lambda_n\varphi_n - \phi_n \longrightarrow 0 \ \ in \ \ H_*^1, \tag{34}$$

$$i\rho_2\lambda_n\phi_n - bD^2\varphi_n + \kappa Du_n + \kappa\varphi_n + \tau\phi_n - \delta\theta_n \longrightarrow 0 \ \ in \ \ L_*^2, \tag{35}$$

$$ic\lambda_n\theta_n - \int_0^\infty \mu(s) D^2\eta_n(s) \, ds + \beta Dv_n + \delta\phi_n \longrightarrow 0 \ \ in \ \ L^2, \tag{36}$$

$$i\lambda_n\eta_n - \theta_n + D_s\eta_n \longrightarrow 0 \ \ in \ \ \mathcal{M}_1. \tag{37}$$

Note that since the norm in $\mathcal{H}$ is equivalent to the usual norm, then there exists $\gamma > 0$ such that for any $U \in D(\mathcal{A})$ of unit norm, we have

$$\|Du_n\|^2 + \|v_n\|^2 + \|\varphi_n\|^2 + \|D\varphi_n\|^2 + \|\phi_n\|^2 + \|\theta_n\|^2 + \int_0^\infty \mu(s) \|D\eta_n(s)\|^2 \, ds = \gamma. \tag{38}$$

First we have

$$\operatorname{Re}\langle (i\lambda_n I - \mathcal{A}) U_n, U_n \rangle = \tau \int_0^\pi \phi_n^2 dx - \frac{1}{2} \int_0^\infty \mu'(s) \|D\eta_n(s)\|^2 ds \longrightarrow 0.$$

Thus,

$$\|\phi_n\| \longrightarrow 0 \tag{39}$$

and

$$\|\eta_n\|_{\mathcal{M}_1}^2 \leq -\frac{1}{\xi} \int_0^\infty \mu'(s) \|D\eta_n(s)\|^2 ds \longrightarrow 0. \tag{40}$$

Moreover, from (34) we have

$$\varphi_n \sim \frac{1}{\lambda_n}\phi_n \longrightarrow 0 \ in \ L^2. \tag{41}$$

The injection $L^2 \hookrightarrow H^{-1}$ is continuous, hence (33) holds in H^{-1} instead of L^2 and

$$i\rho_1\lambda_n v_n \sim \kappa D^2 u_n + \kappa D\varphi_n - \beta D\theta_n \ \ in \ \ H^{-1}.$$

On the other hand we have

$$\left\|\kappa\left(D^2u_n+D\varphi_n\right)-\beta D\theta_n\right\|_{-1}=\sup_{\|D\psi\|\le 1}\left|\left\langle\kappa\left(D^2u_n+D\varphi_n\right)-\beta D\theta_n,\psi\right\rangle\right|,$$
$$\le\|\kappa\left(Du_n+\varphi_n\right)-\beta\theta_n\|\,.\sup_{\|D\psi\|\le 1}\|D\psi\|,$$
$$\le\kappa\,\|Du_n+\varphi_n\|+|\beta|\,\|\theta_n\|\le\sqrt{2}.$$

Therefore,

$$|\lambda_n|\,\|v_n\|_{-1}\le C_1, \tag{42}$$

for a positive constant C_1 independent of $n\in\mathbb{N}$.
Similarly, we get

$$\left\|\int_0^\infty\mu\left(s\right)D^2\eta_n\left(s\right)ds\right\|_{-1}\le\int_0^\infty\mu\left(s\right)\|D\eta_n\left(s\right)\|\,ds,$$
$$\le\sqrt{\int_0^\infty\mu\left(s\right)ds}\left(\int_0^\infty\mu\left(s\right)\int_0^\pi|D\eta_n|^2\left(s\right)dxds\right)^{1/2},$$

then

$$\left\|\int_0^\infty\mu\left(s\right)D^2\eta_n\left(s\right)ds\right\|_{-1}\le\sqrt{\int_0^\infty\mu\left(s\right)ds}\,\|\eta_n\|_{\mathcal{M}_1}\longrightarrow 0.$$

Note that (36) holds with H^{-1} instead of L^2, hence

$$\|ic\lambda_n\theta_n+\beta Dv_n\|_{-1}\longrightarrow 0\,. \tag{43}$$

Since

$$\|Dv_n\|_{-1}=\sup_{\|D\psi\|\le 1}|\langle Dv_n,\psi\rangle|\le\|v_n\|<\infty,$$

Dv_n is bounded in H^{-1}, then

$$\|c\lambda_n\theta_n\|_{-1}\le C_2,$$

for a positive constant C_2 independent of $n\in\mathbb{N}$.
Next, we need to show that $\|\theta_n\|\longrightarrow 0$. Exploiting the continuous embedding of $\mathcal{M}_1$ into $\mathcal{M}_0$, (37) holds in $\mathcal{M}_0$ instead of $\mathcal{M}_1$. Let (ξ_n) be the sequence $\xi_n=s\theta_n$. Clearly $\xi_n\in\mathcal{M}_0$. Indeed, from (h2), $\mu(s)$ goes to zero exponentially fast, then

$$\int_0^\infty s^2\mu\left(s\right)\int_0^\pi|\theta_n|^2\,dxds=\|\theta_n\|^2\int_0^\infty s^2\mu\left(s\right)ds=C_3<\infty.$$

Multiplying (37) by ξ_n in $\mathcal{M}_0$ we get

$$\langle i\lambda_n\eta_n,\xi_n\rangle_0-\langle\theta_n,\xi_n\rangle_0+\langle D_s\eta_n,\xi_n\rangle_0\longrightarrow 0. \tag{44}$$

For the first term we have

$$|\langle i\lambda_n\eta_n,\xi_n\rangle_0|=|\lambda_n|\int_0^\infty s\mu\left(s\right)\int_0^\pi\eta_n\theta_n dxds.$$

Then, using Hölder inequality we get

$$
\begin{aligned}
\left|\langle i\lambda_n\eta_n,\xi_n\rangle_0\right| &\leq |\lambda_n|\,\|\theta_n\|_{-1}\int_0^\infty s\mu\left(s\right)\|D\eta_n\left(s\right)\|\,ds,\\
&\leq |\lambda_n|\,\|\theta_n\|_{-1}\sqrt{\int_0^\infty s^2\mu\left(s\right)ds\int_0^\infty \mu\left(s\right)\|D\eta_n\left(s\right)\|^2\,ds},\\
&\leq C_2\sqrt{C_3}\,\|\eta_n\|_1\longrightarrow 0.
\end{aligned}
$$

From (h4) we infer that $\lim\limits_{s\to+\infty}s^2\mu(s)=0$, then, again (h4) and integration by parts yield

$$
-\int_0^\infty s^2\mu'\left(s\right)ds=2\int_0^\infty s\mu\left(s\right)ds=C_4<\infty.
$$

For the third term of (44) we have,

$$
\begin{aligned}
\left|\langle D_s\eta_n,\xi_n\rangle_0\right| &=\left|\int_0^\infty s\mu\left(s\right)\frac{d}{ds}\int_0^\pi \eta_n\theta_n dxds\right|,\\
&=\left|\int_0^\infty \mu\left(s\right)\int_0^\pi \eta_n\theta_n dxds+\int_0^\infty s\mu'\left(s\right)\int_0^\pi \eta_n\theta_n dxds\right|,
\end{aligned}
$$

then,

$$
\begin{aligned}
\left|\langle D_s\eta_n,\xi_n\rangle_0\right| &\leq \|\theta_n\|\left[\int_0^\infty \mu\left(s\right)\|\eta_n\|\,ds-\int_0^\infty s\mu'\left(s\right)\|\eta_n\|\,ds\right],\\
&\leq \int_0^\infty \mu\left(s\right)\|\eta_n\|\,ds-\int_0^\infty s\mu'\left(s\right)\|\eta_n\|\,ds.
\end{aligned}
$$

Using the Cauchy-Schwarz and Poincaré's inequalities we conclude that

$$
\begin{aligned}
\int_0^\infty \mu\left(s\right)\|\eta_n\|\,ds &\leq \sqrt{\int_0^\infty \mu\left(s\right)ds}\sqrt{\int_0^\infty \mu\left(s\right)\|\eta_n\|\,ds},\\
&\leq \sqrt{\int_0^\infty \mu\left(s\right)ds}\,\|\eta_n\|_0\,,\\
&\leq C_P\sqrt{\int_0^\infty \mu\left(s\right)ds}\,\|\eta_n\|_1\longrightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_0^\infty s\mu'\left(s\right)\|\eta_n\|\,ds &=\int_0^\infty s\sqrt{-\mu'\left(s\right)}\sqrt{-\mu'\left(s\right)}\,\|\eta_n\|\,ds\\
&\leq \left(-\int_0^\infty s^2\mu'\left(s\right)ds\right)^{1/2}\left(-\int_0^\infty \mu'\left(s\right)\|\eta_n\|^2\,ds\right)^{1/2}\\
&\leq \left(-C_4C_P\int_0^\infty \mu'\left(s\right)\|D\eta_n\|^2\,ds\right)^{1/2}\longrightarrow 0.
\end{aligned}
$$

Thus, (44) is reduced to

$$\|\theta_n\|^2 \int_0^\infty s\mu(s)\,ds = \langle \theta_n, \xi_n \rangle_0 \longrightarrow 0,$$

that is,

$$\|\theta_n\|^2 = \frac{2\langle \theta_n, \xi_n \rangle_0}{C_4} \longrightarrow 0. \tag{45}$$

Removing the terms that tend to 0 from (35), then multiplying by φ_n we obtain

$$i\rho_2\lambda_n \langle \phi_n, \varphi_n \rangle + b\,\|D\varphi_n\|^2 + \kappa \langle Du_n, \varphi_n \rangle \longrightarrow 0\,. \tag{46}$$

We point out that

$$\langle Du_n, \varphi_n \rangle \leq \|Du_n\|\,\|\varphi_n\| \longrightarrow 0$$

and

$$i\lambda_n \langle \phi_n, \varphi_n \rangle \sim \|\varphi_n\|^2 \longrightarrow 0.$$

Therefore,

$$\|D\varphi_n\| \longrightarrow 0. \tag{47}$$

Multiplying (32) by $\rho_1 v_n$ and (33) by u_n we get

$$i\rho_1\lambda_n \langle u_n, v_n \rangle - \rho_1 \|v_n\|^2 \longrightarrow 0, \tag{48}$$

and

$$i\rho_1\lambda_n \langle v_n, u_n \rangle + \kappa \|Du_n\|^2 \longrightarrow 0. \tag{49}$$

Adding (48) to the complex conjugate of (49), we get

$$\kappa \|Du_n\|^2 - \rho_1 \|v_n\|^2 \longrightarrow 0. \tag{50}$$

Combining (38), (39), (40), (41), (45),(47), and (50) we obtain

$$\left(1 + \frac{\rho_1}{\kappa}\right) \|v_n\|^2 \to \gamma. \tag{51}$$

We complete the proof by showing that (51) leads to a contradiction.
Since $A^{-1}Dv_n$ is bounded in H_0^1 (recall that $A = -D^2$), from (43) we have

$$\left\langle ic\lambda_n\theta_n + \beta Dv_n, A^{-1}Dv_n \right\rangle = \left\langle ic\lambda_n\theta_n, A^{-1}Dv_n \right\rangle + \beta \|Dv_n\|_{-1}^2 \longrightarrow 0. \tag{52}$$

On the other hand, from (45) we have

$$\begin{aligned} \left|\left\langle ic\lambda_n\theta_n, A^{-1}Dv_n \right\rangle\right| &= \left|\left\langle ic\lambda_n\theta_n, A^{-1/2}v_n \right\rangle\right| \\ &\leq c\,|\lambda_n| \left\|A^{-1/2}v_n\right\| \|\theta_n\| = c\,|\lambda_n|\,\|v_n\|_{-1}\,\|\theta_n\| \\ &\leq cC_1 \|\theta_n\| \longrightarrow 0. \end{aligned}$$

Thus, (52) leads to

$$\|Dv_n\|_{-1} \longrightarrow 0.$$

From (19) we infer that

$$\|v_n - \overline{v}_n\| = \|Dv_n\|_{-1} \longrightarrow 0.$$

Therefore,

$$\|v_n - \overline{v}_n\|^2 = \|v_n\|^2 - \pi \left|\overline{v}_n\right|^2 \longrightarrow 0. \tag{53}$$

The comparison of (51) and (53) leads to

$$\left|\overline{v}_n\right| \longrightarrow \sqrt{\frac{\kappa\gamma}{\pi\left(\kappa+\rho_1\right)}}.$$

Thus, there exists a subsequence $(\overline{v}_n)$ that converges to $\overline{v}$, such that

$$\left|\overline{v}\right| = \sqrt{\frac{\kappa\gamma}{\pi\left(\kappa+\rho_1\right)}}. \tag{54}$$

Using (53) again we conclude that there exists a subsequence of (v_n) which converges to $\overline{v}$ in $L^2(0,\pi)$. Exploiting the continuous embedding of $L^2(0,\pi)$ into $H^{-1}(0,\pi)$, one can deduce that

$$v_n \longrightarrow \overline{v}, \qquad \text{in } H^{-1}(0,\pi). \tag{55}$$

At this point we distinguish two cases. Suppose that (λ_n) is unbounded, then we can choose a subsequence (λ_n) such that $|\lambda_n| \longrightarrow \infty$ and from (42) we have

$$v_n \longrightarrow 0 \; in \; H^{-1}(0,\pi).$$

From the uniqueness of the limit we conclude that $\overline{v} = 0$, which is incompatible with (54).

Conversely, assume that (λ_n) is bounded, again, there exists a subsequence (λ_n) that converges to some $\lambda \in \mathbb{R}$. In this case we have

$$\lim_{n\longrightarrow\infty} \left\|\left(i\lambda I - \mathcal{A}\right) U_n\right\| = 0,$$

and (32)-(37) hold with λ instead of λ_n. In particular

$$i\lambda u_n - v_n \longrightarrow 0 \; in \; H_0^1(0,\pi).$$

Since (u_n) is bounded in $H_0^1(0,\pi)$, we conclude that there exists $v^* \in H_0^1(0,\pi)$ and a subsequence (v_n) that converges weakly to v^* in $H_0^1(0,\pi)$. From the uniqueness of the limit we infer that $v^* = \overline{v}$, which is in contradiction with $v^* \in H_0^1(0,\pi)$, since $\overline{v}$ is a non-zero constant function, and therefore cannot be in $H_0^1(0,\pi)$. This completes the proof of Theorem 4. □

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APPLICATION OF THE RATIONAL (G'/G)-EXPANSION METHOD FOR SOLVING SOME COUPLED AND COMBINED WAVE EQUATIONS

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Abstract. In this paper, we explore the travelling wave solutions for some nonlinear partial differential equations by using the recently established rational (G'/G)-expansion method. We apply this method to the combined KdV-mKdV equation, the reaction-diffusion equation and the coupled Hirota-Satsuma KdV equations. The travelling wave solutions are expressed by hyperbolic functions, trigonometric functions and rational functions. When the parameters are taken as special values, the solitary waves are also derived from the travelling waves. We have also given some figures for the solutions.

1. Introduction

In the past decades, the travelling wave solutions of nonlinear partial differential equations (NLPDEs) play an effective role in physics, engineering and applied mathematics. The mathematical models of these subjects give important information about the behaviour of the physical event. Therefore, it is very important to obtain the traveling wave solutions of NLPDEs [32]. The NLPDEs have interesting structures that deals with many phenomena in physics, chemistry and engineering, for example; in fluid flow, plasma waves, mechanics, solid state physics, oceanic phenomena, atmospheric phenomena and so on. Many researchers have been proposed various different methods to find solutions for nonlinear partial differential equations and nonlinear fractional differential equations [36–40]. Such as the inverse scattering transform method [1], the Hirota's bilinear method [2], truncated

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Painlevé expansion method [3], the tanh-function expansion method [4], the Jacobi elliptic function expansion method [5], the homogeneous balance method [6–8], the trial function method [9], the exp-function method [10, 34], differential transform method [33], the Bäcklund transform method [11], the generalized Riccati equation method [12–15], the sub-ODE method [17–20], the original (G'/G)-expansion method [16, 29], the double $(G'/G$,1/G)-expansion method [35] etc.. Since there is not a common method that can be used to solve all types of nonlinear evolution equations.

Some researchers established several powerful and direct methods. Wang et al. [16] first introduced the (G'/G)-expansion method to find travelling wave solutions of nonlinear evolution equations. Later Islam et al. [21] proposed the rational (G'/G)-expansion method which aims to derive closed form travelling wave solutions. In this paper we use the rational (G'/G)- expansion method and apply for the combined KdV-mKdV equation, the reaction-diffusion equation, and the coupled Hirota-Satsuma KdV equations. We derived abundant solutions for each equation that is different from the solutions in the literature.

2. Description of the Method

Suppose that $u = u(x,t)$ is an unknown function depends on the x and t variables and we define the polynomial P in $u(x,t)$ and its various order partial derivatives and nonlinear terms as

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0. \tag{1}$$

We use the following steps, to solve Eq.(1) by means of the rational (G'/G)-expansion method.

Step 1: We assign a new variable $U(\xi)$ in terms of x and t variables and a new transformation:

$$u(x,t) = U(\xi) \quad , \quad \xi = x - st + \xi_0 \tag{2}$$

where is ξ_0 a constant and s is the velocity of the wave. The transformation in Eq.(2) transforms Eq.(1) into an ordinary differential equation (ODE) for $u = U(\xi)$.

$$Q(U, U', -sU', U'', s^2U'', -sU'', ...) = 0 \tag{3}$$

where U and its derivatives with respect to ξ are the elements of the Q polynomial of $U(\xi)$.

Step 2: Next we integrate Eq.(3) one or twice as possible. Suppose that the solution of Eq.(3) can be written in the following form

$$u(\xi) = \frac{\sum_{j=0}^{n} aj(G'/G)^j}{\sum_{j=0}^{n} bj(G'/G)^j} \tag{4}$$

where a_j and $b_j\,(j=0,1,2,...,n)\,,(a_n\neq 0,\ b_n\neq 0)$ are arbitrary coefficient to be found later. Next we write, the $G=G(\xi)$ function, which satisfies the following second order ODE;

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0 \tag{5}$$

where λ and μ are real constants. We convert Eq.(5) into (G'/G) form,

$$\frac{d}{d\xi}(G'/G)=-\left(G'/G\right)^2-\lambda\left(G'/G\right)-\mu. \tag{6}$$

From Eq.(5) or Eq.(6) the solution for (G'/G) as follows

$$(G'/G)=\begin{cases} -\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\cosh\left((\frac{\sqrt{\lambda^2-4\mu}}{2})\xi\right)+c_2\sinh\left((\frac{\sqrt{\lambda^2-4\mu}}{2})\xi\right)}{c_1\sinh\left((\frac{\sqrt{\lambda^2-4\mu}}{2})\xi\right)+c_2\cosh\left((\frac{\sqrt{\lambda^2-4\mu}}{2})\xi\right)}\right) & ;\ \lambda^2-4\mu>0\ , \\ -\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\cos\left((\frac{\sqrt{4\mu-\lambda^2}}{2})\xi\right)+c_2\sin\left((\frac{\sqrt{4\mu-\lambda^2}}{2})\xi\right)}{c_1\sin\left((\frac{\sqrt{4\mu-\lambda^2}}{2})\xi\right)+c_2\cos\left((\frac{\sqrt{4\mu-\lambda^2}}{2})\xi\right)}\right) & ;\ \lambda^2-4\mu<0\ , \\ -\frac{\lambda}{2}+\frac{c_2}{c_1+c_2\xi} & ;\ \lambda^2-4\mu=0\ , \end{cases} \tag{7}$$

where c_1 and c_2 are constants.

Step 3: To determine the value of n, which is the degree of $U(\xi)$, in Step 2, we apply the homogeneous balance method, that is balancing between the highest order nonlinear terms and the highest order derivatives in Eq.(3). The degree of other terms in Eq.(3) can be written as in the following form [21]

$$deg\left[\frac{d^m u(\xi)}{d\xi^m}\right]=n+m\ ,\ deg\left[u^m\left(\frac{d^l u(\xi)}{d\xi^l}\right)^p\right]=mn+p\,(n+l)$$

where deg$[U(\xi)]$ is the degree of $U(\xi)$.

Step 4: After determining the value of n, we substitute Eq.(4) along with Eq.(5) into Eq.(3). Equating the coefficients of (G'/G) to zero, gives a system of algebraic equations. In order to solve these equations we use the computer software programme such as Maple or Matematica. If there is a possible solution, we obtain values for a_i, b_i, λ, μ and s $(i=0,1,2,...,n)$.

Step 5: Finally we substitute the values of a_i, $b_i\,(i=0,1,2,...,n)$, λ, μ, s and the solutions given in Eq.(7), into Eq.(4), hence the solutions of the nonlinear Eq.(1) are derived.

3. Application of the Method

Example 1. *The combined KdV-mKdV equation*

The KdV and mKdV equations are widely studied popular soliton equations. The nonlinear terms appearing in the KdV and mKdV equations often exist in applied science and engineering, such as in plasma physics, ocean dynamics and quantum field theory [22–24]. If we combine the quadratic nonlinear term of the KdV equation and the qubic nonlinear term of the mKdV equation, then we get the combined KdV-mKdV equation or the Gardner equation [25]

$$u_t + \alpha u\ u_x + \beta u^2 u_x + u_{xxx} = 0 \tag{8}$$

where α and β are nonzero parameters. This equation describes the wave propagation of bounded particle,sound wave and thermal pulse [26–28].

The travelling wave transformation $u(x,t) = U(\xi)$, $\xi = x - st + \xi_0$, transforms Eq.(8) into to the following ODE

$$-\ sU' + \alpha U\ U' + \beta U^2 U' + U''' = 0 \tag{9}$$

where s is the velocity of the wave and the superscript of U shows the derivative of U with respect to ξ. Next, we integrate Eq.(9) and deduce the following equation

$$C - sU + \frac{1}{2}\alpha U^2 + \frac{1}{3}\beta U^3 + U^{''} = 0 \tag{10}$$

where C is an integration constant to be found later. We use homogeneous balance method, such as balancing the terms U'' and U^3 in Eq.(10) we get $n = 1$, so we can write Eq.(4) as

$$U(\xi) = \frac{a_0 + a_1(G^{'}/G)}{b_0 + b_1(G'/G)} \tag{11}$$

Next we substitute Eq.(11) into Eq.(10) and organize the equation in terms of the powers of (G'/G). Hence equating the coefficients of (G'/G) and its powers to zero in the resulting equation, gives a system of algebraic equations for a_0, b_0, a_1, b_1, s and C. Solving the set of equations by using the computer programme Maple, we get the following set of solutions.

Set 1

$$\begin{aligned} a_0 &= \mp\frac{1}{2}\frac{b_0(\pm\sqrt{-\frac{6}{\beta}}\alpha + 6\lambda)}{\beta\sqrt{-\frac{6}{\beta}}}\ ,\ a_1 = \pm\sqrt{-\frac{6}{\beta}}b_0\ ,\ b_1 = 0 \\ s &= -\frac{2\beta\lambda^2 + \alpha^2 - 8\beta\mu}{4\beta}\ ,\ \ C = \frac{\alpha\left(6\beta\lambda^2 + \alpha^2 - 24\beta\mu\right)}{24\beta^2} \end{aligned} \tag{12}$$

where $b_0, \lambda, \alpha, \beta$ and μ are all arbitrary constants. Substituting Eq.(12) into Eq.(11) we get the following solution

$$U(\xi) = \pm\sqrt{-\frac{6}{\beta}}(G^{'}/G) - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}} \tag{13}$$

where

$$\xi = x + \left(\frac{\alpha^2}{4\beta} + \frac{\lambda^2 - 4\mu}{2}\right) t + \xi_0 \tag{14}$$

and (G'/G) is given in Eq.(7). Substituting Eq.(7) into Eq.(13), we deduce the following travelling wave solutions.

Case 1: If $\lambda^2 - 4\mu > 0$, then we have

$$\begin{aligned} U(\xi) &= \pm\frac{1}{2}\sqrt{-\frac{6\,(\lambda^2-4\mu)}{\beta}}\left(\frac{c_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+c_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}{c_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)+c_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi)}\right) \\ &\quad \mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}. \end{aligned} \tag{15}$$

If we choose $c_1 = \sinh(\xi_0)$ and $c_2 = \cosh(\xi_0)$, we get the following hyperbolic solution for the Eq.(10)

$$U(\xi) = \pm\frac{1}{2}\sqrt{-\frac{6\,(\lambda^2-4\mu)}{\beta}}\tanh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}+\xi_0\right))\mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$$

The plot of the solution for the values ($\lambda = 5, \mu = 4, \alpha = 3, \beta = -4, \xi_0 = 2$) is given in Fig 1.

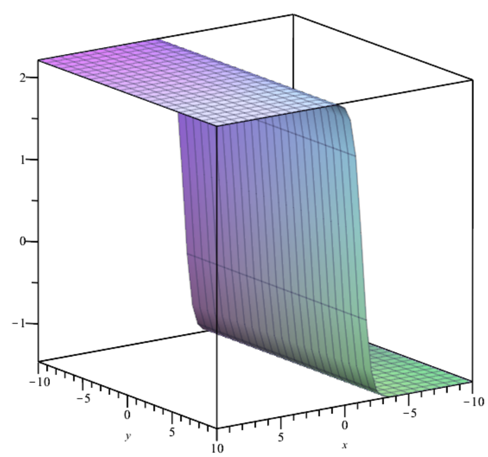

FIGURE 1. Hyperbolic solution for Eq.(8)

Case 2: If $\lambda^2 - 4\mu < 0$, then we have

$$\begin{aligned} U(\xi) &= \pm\frac{1}{2}\sqrt{-\frac{6\,(4\mu-\lambda^2)}{\beta}}\left(\frac{-c_1\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+c_2\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}{c_1\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+c_2\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}\right) \\ &\quad \mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}} \end{aligned}$$

If we choose $c_1 = \sin(\xi_0)$ and $c_2 = \cos(\xi_0)$, we get the following trigonometric solution for the Eq.(10)

$$U(\xi) = \pm\frac{1}{2}\sqrt{-\frac{6\,(4\mu-\lambda^2)}{\beta}}\tan\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}+\xi_0\right))\mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$$

The plot of the solution for the values ($\lambda = 4, \mu = 5, \alpha = 3, \beta = -6, \xi_0 = 2$) is given in Fig 2.

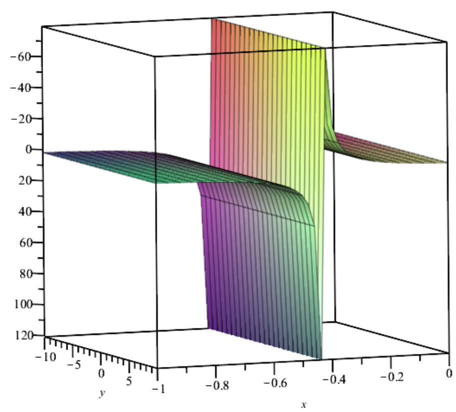

FIGURE 2. Trigonometric solution for Eq.(8)

Case 3: If $\lambda^2 - 4\mu = 0$, then we have

$$U(\xi) = \pm\sqrt{-\frac{6}{\beta}}\left(\frac{c_2}{c_1+c_2\xi}\right)\mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$$

The plot of the solution for the values ($\lambda = 4, \mu = 4, \alpha = 3, \beta = -6, \xi_0 = 2$) is given in Fig 3. In particular, if $c_1 = 0$ and $c_2 \neq 0$ and $\lambda > 0$ and $\mu = 0$, then

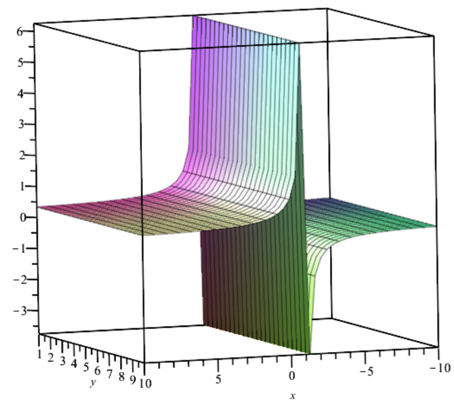

Figure 3. Rational solution for Eq.(8)

Eq.(15) becomes

$$U(\xi)=\pm\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}\tanh\left(\frac{\lambda}{2}\xi\right)\mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}} \tag{16}$$

or if $c_1\neq 0$ and $c_2=0$ and $\lambda>0$ and $\mu=0$, then Eq.(15) becomes

$$U(\xi)=\pm\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}\coth\left(\frac{\lambda}{2}\xi\right)\mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}} \tag{17}$$

where

$$\xi=x+\left(\frac{\alpha^2}{4\beta}+\frac{\lambda^2}{2}\right)t+\xi_0.$$

Note that Eq.(16) and Eq.(17) represents the solitary wave solutions of the combined KdV–mKdV equation Eq.(8)

Set 2

$$\begin{gathered} a_0=\frac{(-\lambda\alpha\mp\sqrt{-96\beta\mu^2-6\lambda^4\beta+48\mu\lambda^2\beta})b_1}{4\beta}\ ,\ a_1=\frac{-\alpha b_1}{2\beta}\ \ ,\ \ b_0=\frac{b_1\lambda}{2} \\ s=\frac{-2\lambda^2\beta+8\mu\beta-\alpha^2}{4\beta}\ ,\ C=\frac{\alpha\left(6\lambda^2\beta-24\mu\beta+\alpha^2\right)}{24\beta^2} \end{gathered} \tag{18}$$

where b_1, λ and μ are arbitrary constants. Substituting Eq.(18) into Eq.(11) we get the following solution

$$U(\xi) = \frac{-2\alpha(G^{'}/G) + (-\lambda\alpha \mp \sqrt{-96\beta\mu^2 - 6\lambda^4\beta + 48\mu\lambda^2\beta})}{4\beta(G^{'}/G) + 2\lambda\beta} \tag{19}$$

where

$$\xi = x + \left(2\mu - \frac{2\lambda^2\beta + \alpha^2}{4\beta}\right) t + \xi_0 \tag{20}$$

and (G'/G) is given in Eq.(7).

Set 3

$$\begin{aligned} a_0 &= \frac{-6b_0^2\lambda + 12b_1\mu b_0 + 3b_0b_1\lambda^2 - 6\lambda\mu b_1^2}{\mp\sqrt{-6b_1^2\lambda^2\beta + 24b_1\beta b_0\lambda - 24b_0^2\beta}} + \frac{\alpha b_0}{2\beta} \\ a_1 &= \frac{-\alpha b_1 \pm \sqrt{-6b_1^2\lambda^2\beta + 24b_1\beta b_0\lambda - 24b_0^2\beta}}{2\beta} \\ s &= \frac{-2\lambda^2\beta + 8\mu\beta - \alpha^2}{4\beta} \quad , \quad C = \frac{\alpha\left(6\lambda^2\beta - 24\mu\beta + \alpha^2\right)}{24\beta^2} \end{aligned} \tag{21}$$

where $b_0, b_1, \lambda, \alpha, \beta$ and μ are arbitrary constants. Substituting the values of constants from Eq.(21) into Eq.(11) gives

$$U(\xi) = \frac{\left(\frac{-\alpha b_1 \pm\sqrt{-6b_1^2\lambda^2\beta + 24b_1\beta b_0\lambda - 24b_0^2\beta}}{2\beta}\right)(G'/G) + (\frac{-6b_0^2\lambda + 12b_1\mu b_0 + 3b_0b_1\lambda^2 - 6\lambda\mu b_1^2}{\mp\sqrt{-6b_1^2\lambda^2\beta + 24b_1\beta b_0\lambda - 24b_0^2\beta}} + \frac{\alpha b_0}{2\beta})}{b_1(G'/G) + b_0}$$

where $\xi = x + (\frac{2\beta\lambda^2 + \alpha^2 - 8\beta\mu}{4\beta})t + \xi_0$.

Example 2. *The reaction-diffusion equation*

We have the reaction-diffusion equation [30]

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0 \tag{22}$$

where α , β and γ are nonzero constants. The traveling wave variable Eq.(2) reduces the Eq.(22) into an ODE

$$\left(\alpha + s^2\right) U'' + \beta U + U^3 = 0, \tag{23}$$

where s is the velocity of the wave. Next we express the solution of the Eq.(23) in terms of (G'/G) as it is written in Eq.(4), where $G = G(\xi)$ satisfies the second order linear ODE in Eq.(23). We use homogeneous balance method, such as balancing the terms U'' and U^3 in Eq.(23) we get $n = 1$, hence from Eq.(4), we have

$$U(\xi) = \frac{a_0 + a_1(G'/G)}{b_0 + b_1(G'/G)}. \tag{24}$$

Substituting Eq.(23) into Eq.(22) and write the left hand side in terms of (G'/G). Hence equating the coefficients of the resulting equation to zero, gives a system of algebraic equations for a_0, b_0, a_1, b_1 and s. Solving the set of equations by using the computer programme, we get the following set of solutions:

Set 1

$$\begin{gathered} a_0 = \pm\tfrac{1}{2}\sqrt{\tfrac{-\beta(\lambda^2-4\mu)}{\gamma}}b_1\ ,\ b_0 = \tfrac{1}{2}\lambda b_1\ , \\ a_1 = 0\ ,\ s = \pm\sqrt{\tfrac{2\beta}{\lambda^2-4\mu}-\alpha} \end{gathered} \tag{25}$$

where b_1, λ and μ are all arbitrary constants. Substituting Eq.(25) into Eq.(24) we get the following solution

$$U(\xi) = \frac{\pm\frac{1}{2}\sqrt{\frac{-\beta(\lambda^2-4\mu)}{\gamma}}}{(G'/G)+\lambda/2} \tag{26}$$

where

$$\xi = x \pm \left(\sqrt{\frac{2\beta}{\lambda^2-4\mu}-\alpha}\right)t + \xi_0 \tag{27}$$

and (G'/G) is given in Eq. (7). Substituting Eq.(7) into Eq.(26), we deduce the following travelling wave solutions.

Case 1: If $\lambda^2 - 4\mu > 0$, then we have

$$U(\xi) \quad = \quad \pm\sqrt{-\frac{\beta}{\gamma}}\left(\frac{c_1\sinh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})+c_2\cosh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})}{c_1\cosh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})+c_2\sinh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})}\right).$$

If we choose $c_1 = \cosh(\xi_0)$ and $c_2 = \sinh(\xi_0)$, we get the following hyperbolic solution for the Eq.(22)

$$U(\xi) = \pm\sqrt{-\frac{\beta}{\gamma}}\tanh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}+\xi_0\right).$$

Case 2: If $\lambda^2 - 4\mu < 0$, then we have

$$U(\xi) \quad = \quad \pm\sqrt{-\frac{\beta}{\gamma}}\left(\frac{c_1\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+c_2\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}{-c_1\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+c_2\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}\right).$$

If we choose $c_1 = \cos(\xi_0)$ and $c_2 = \sin(\xi_0)$, we get the following trigonometric solution for the Eq.(22)

$$U(\xi) = \mp\sqrt{-\frac{\beta}{\gamma}}\tan\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}+\xi_0\right).$$

Case 3: If $\lambda^2 - 4\mu = 0$, then we have trivial solution for the Eq.(22)

$$U(\xi) = 0.$$

Set 2

$$\begin{aligned} a_0 = \pm\frac{\lambda b_0\sqrt{\beta}}{\sqrt{\gamma(4\mu-\lambda^2)}}\ ,\ a_1 = \pm 2\sqrt{\frac{\beta}{\gamma(4\mu-\lambda^2)}}b_0\ \ , \\ b_1 = 0\ \ ,\ \ s = \pm\sqrt{\frac{2\beta}{\lambda^2-4\mu} - \alpha} \end{aligned} \tag{28}$$

where b_0, λ, β and μ are arbitrary constants. Substituting Eq.(28) into Eq.(24) we get the following solution

$$U(\xi) = \pm\frac{2\sqrt{\beta}}{\sqrt{\gamma\left(4\mu-\lambda^2\right)}}\left((G'/G) + \frac{\lambda}{2}\right), \tag{29}$$

where

$$\xi = x \pm \left(\sqrt{\frac{2\beta}{\lambda^2 - 4\mu} - \alpha}\right)t + \xi_0 \tag{30}$$

and (G'/G) is given in Eq.(7).

Set 3

$$\begin{aligned} a_0 = \pm\sqrt{\frac{\beta}{\gamma(4\mu-\lambda^2)}}\left(\lambda b_0 - 2\mu b_1\right)\ , \\ a_1 = \pm\sqrt{\frac{\beta}{\gamma(4\mu-\lambda^2)}}\left(\lambda b_1 - 2b_0\right)\ , s = \pm\sqrt{\frac{2\beta}{\lambda^2-4\mu} - \alpha} \end{aligned} \tag{31}$$

where b_0, b_1, λ and μ are arbitrary constants. Substituting Eq.(31) into Eq.(24) we get the following solution

$$U(\xi) = \pm\frac{\sqrt{\frac{\beta}{\gamma(4\mu-\lambda^2)}}\left(\lambda b_1 - 2b_0\right)(G'/G) + \sqrt{\frac{\beta}{\gamma(4\mu-\lambda^2)}}\left(\lambda b_0 - 2\mu b_1\right)}{b_1(G'/G) + b_0}$$

where

$$\xi = x \pm \left(\sqrt{\frac{2\beta}{\lambda^2 - 4\mu} - \alpha}\right)t + \xi_0$$

and (G'/G) is given in Eq.(7).

Set 4

$$\begin{aligned} a_0 = \pm\sqrt{\frac{\beta}{\gamma(4\mu-\lambda^2)}}\left(\left(\frac{\lambda}{2} \pm \frac{1}{6}\sqrt{3\lambda^2 - 12\mu}\right) - 2\mu\right)b_1\ ,\ a_1 = \pm\sqrt{\frac{-\beta}{3\gamma}}b_1\ , \\ b_0 = \left(\frac{\lambda}{2} \pm \frac{1}{6}\sqrt{3\lambda^2 - 12\mu}\right)b_1\ ,\ s = \pm\sqrt{\frac{2\beta}{\lambda^2-4\mu} - \alpha} \end{aligned} \tag{32}$$

where b_1, λ and μ are arbitrary constants. Substituting Eq.(32) into Eq.(24) we get the following solution

$$U(\xi) = \pm \frac{\sqrt{\frac{-\beta}{3\gamma}}(G'/G) + \sqrt{\frac{\beta}{\gamma(4\mu-\lambda^2)}}\left(\left(\frac{\lambda}{2} \pm \frac{1}{6}\sqrt{3\lambda^2 - 12\mu}\right) - 2\mu\right)}{(G'/G) + \frac{\lambda}{2} \pm \frac{1}{6}\sqrt{3\lambda^2 - 12\mu}}$$

where

$$\xi = x \pm \left(\sqrt{\frac{2\beta}{\lambda^2 - 4\mu}} - \alpha\right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Example 3. *The coupled Hirota-Satsuma KdV equations*

The coupled Hirota-Satsuma KdV equations (CHSK) describes an interaction of two long waves with different dispersion relations [31]. We will consider the CHSK equations in the following form

$$\begin{aligned} u_t &= \tfrac{1}{4}u_{xxx} + 3uu_x - 6vv_x \ , \\ v_t &= -\tfrac{1}{2}v_{xxx} - 3uv_x \ . \end{aligned} \tag{33}$$

Making the transformations $u(x,t) = U(\xi)$, $v(x,t) = V(\xi)$, $\xi = x - st + \xi_0$, where s is the velocity of the wave to be determined later. We get the CHSK equations in the following form

$$\begin{aligned} -sU' &= \frac{1}{4}U''' + 3UU' - 6VV', \\ -sV' &= -\frac{1}{2}V''' - 3UV'. \end{aligned} \tag{34}$$

By balancing the highest order derivatives and nonlinear terms in Eq.(34), we get $n = 2$ and from Eq.(4) we write the solutions of Eq.(33) as

$$\begin{aligned} U(\xi) &= \tfrac{a_0 + a_1(G'/G) + a_2(G'/G)^2}{b_0 + b_1(G'/G) + b_2(G'/G)^2}, \\ V(\xi) &= \tfrac{e_0 + e_1(G'/G) + e_2((G'/G)^2}{d_0 + d_1(G'/G) + d_2(G'/G)^2} \end{aligned} \tag{35}$$

Substituting Eq.(35) into Eq.(34), and we convert Eq.(34) into a polynomial in (G'/G). Equating the coefficients of the same power of (G'/G) to zero, yields a set of simultaneous algebraic equations. Solving the set of equations for $a_i, b_i, e_i, d_i (i = 0, 1, 2)$ and s by using the computer programme, we get the following set of solutions

Set 1

$$\begin{gathered} a_2 = -2b_0 \ , \ a_1 = -2\lambda b_0 \ , \ b_1 = b_2 = 0 \ , \ d_1 = d_2 = 0 \\ e_0 = -\tfrac{e_2\left(\lambda^2 b_0 + 8\mu b_0 + 8a_0\right)}{4b_0} \ , \ e_1 = \lambda e_2 \ , \ d_0 = e_2 \\ s = \tfrac{\lambda^2 b_0 + 8\mu b_0 + 6a_0}{2b_0} \end{gathered} \tag{36}$$

where a_0, b_0, e_2, λ and μ are constants. Substituting Eq.(36) into Eq.(35), hence we reach the following solutions

$$U(\xi) = -2\left[(G'/G)^2 + \lambda(G'/G)\right] + \frac{a_0}{b_0} \tag{37}$$
$$V(\xi) = (G'/G)^2 + \lambda(G'/G) - \frac{\lambda^2+8\mu}{4b_0} - 2\frac{a_0}{b_0}$$

where

$$\xi = x - \left(\frac{\lambda^2+8\mu}{2b_0} + 3\frac{a_0}{b_0}\right)t + \xi_0$$

and (G'/G) is given in Eq.(7). Substituting Eq.(7) into Eq.(37), we deduce the following travelling wave solutions.

Case 1: If $\lambda^2-4\mu>0$ and if we choose $c_1=\cosh(\xi_0)$, $c_2=\sinh(\xi_0)$, then we have the hyperbolic solutions for the Eq.(34)

$$U(\xi) = \tfrac{a_0}{b_0} + \tfrac{\lambda^2}{2} - \tfrac{\lambda^2-4\mu}{2}\coth^2\left(\tfrac{\xi}{2}\sqrt{\lambda^2-4\mu}+\xi_0\right),$$
$$V(\xi) = \tfrac{-2a_0}{b_0} - \tfrac{\lambda^2}{4} - \tfrac{\lambda^2+8\mu}{4b_0} + \tfrac{\lambda^2-4\mu}{4}\coth^2\left(\tfrac{\xi}{2}\sqrt{\lambda^2-4\mu}+\xi_0\right).$$

Case 2: If $\lambda^2-4\mu<0$ and if we choose $c_1=\cos(\xi_0)$, $c_2=\sin(\xi_0)$, then we have the trigonometric solutions for the Eq.(34)

$$U(\xi) = \tfrac{a_0}{b_0} + \tfrac{\lambda^2}{2} - \tfrac{\lambda^2-4\mu}{2}\cot^2\left(\tfrac{\xi}{2}\sqrt{4\mu-\lambda^2}+\xi_0\right),$$
$$V(\xi) = \tfrac{-2a_0}{b_0} - \tfrac{\lambda^2}{4} - \tfrac{\lambda^2+8\mu}{4b_0} + \tfrac{\lambda^2-4\mu}{4}\cot^2\left(\tfrac{\xi}{2}\sqrt{4\mu-\lambda^2}+\xi_0\right).$$

Case 3: If $\lambda^2-4\mu=0$, , then we have rational solutions for the Eq.(34)

$$U(\xi) = \tfrac{a_0}{b_0} + \tfrac{\lambda^2}{2} - \tfrac{2c_2^2}{(c_1+c_2\xi)^2},$$
$$V(\xi) = \tfrac{-2a_0}{b_0} - \tfrac{\lambda^2}{4} - \tfrac{\lambda^2+8\mu}{4b_0} + (\tfrac{c_2}{c_1+c_2\xi})^2.$$

Set 2

$$\begin{gathered} a_2=-b_0 \;\;,\;\; a_1=-\lambda b_0 \;\;,\;\; b_1=b_2=0 \;\;,\;\; d_2=0 \;\;,\;\; e_0=\tfrac{\lambda d_0 e_2}{2d_1} \\ a_0=-\tfrac{b_0\left(\lambda^2 d_1^2+8\mu d_1^2+4e_2^2\right)}{8d_1^2} \;\;,\;\; e_1=\tfrac{e_2(\lambda d_1+2d_0)}{2d_1} \\ s=\tfrac{\lambda^2 d_1^2-4\mu d_1^2-12e_2^2}{8d_1^2} \end{gathered} \tag{38}$$

where $b_0, d_0, d_1, e_2, \lambda$ and μ are constants. Substituting Eq.(38) into Eq.(35), hence we reach the following solutions

$$U(\xi) = -\left[(G'/G)^2 + \lambda(G'/G)\right] - \left(\frac{\lambda^2+8\mu}{8}\right) - \left(\frac{e_2^2}{2d_1^2}\right)$$
$$V(\xi) = e_2\frac{(G'/G)^2 + (\frac{\lambda d_1+2d_0}{2d_1})(G'/G) + \frac{\lambda d_0}{2d_1}}{d_1(G'/G)+d_0}$$

where

$$\xi = x - \left(\frac{\lambda^2 - 4\mu}{8} + \frac{4}{3}\frac{e_2^2}{d_1^2} \right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Set 3

$$\begin{array}{c} a_2 = -b_0 \quad , \quad a_1 = -\lambda b_0 \quad , \quad b_1 = b_2 = 0 \\ e_0 = \frac{e_2 d_0}{d_2} \quad , \quad e_1 = \frac{e_2 d_1}{d_2} \\ s = -\frac{\lambda^2 b_0 + 8\mu b_0 + 12 a_0}{4 b_0} \end{array} \tag{39}$$

where $a_0, b_0, d_0, d_1, d_2, e_2, \lambda$ and μ are constants. Substituting Eq.(39) into Eq.(35) , hence we reach the following solutions

$$U(\xi) = -\left[(G^{'}/G)^2 + \lambda (G^{'}/G) \right] + \frac{a_0}{b_0}$$

$$V(\xi) = e_2 \frac{(G'/G)^2 + \frac{d_1}{d_2}(G'/G) + \frac{d_0}{d_2}}{d_2 (G'/G)^2 + d_1 (G'/G) + d_0}$$

where

$$\xi = x + \left(\frac{\lambda^2 + 8\mu}{4} + 3\frac{a_0}{b_0} \right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Set 4

$$\begin{array}{c} a_0 = \frac{\lambda^4 b_0 - 8\lambda^2 \mu b_0 + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda^2} \\ a_1 = \frac{\lambda^4 b_0 - 8\lambda^2 \mu b_0 + 16\mu^2 a_2 + 16\mu^2 b_0}{4\mu\lambda} \quad , \quad d_2 = 0 \\ b_1 = \frac{\lambda b_0}{\mu} \quad , \quad b_2 = \frac{\lambda^2 b_0}{4\mu^2} \quad , \quad e_0 = e_1 = e_2 = 0 \\ s = -\frac{\lambda^4 b_0 - 16\lambda^2 b_0 \mu + 48\mu^2 a_2 + 48\mu^2 b_0}{4 b_0 \lambda^2} \end{array} \tag{40}$$

where a_2, b_0, λ and μ are constants. Substituting Eq.(40) into Eq.(35), hence we reach the following solutions

$$U(\xi) = \frac{a_2 (G'/G)^2 + \left(\frac{\lambda^4 b_0 - 8\lambda^2 \mu b_0 + 16\mu^2 a_2 + 16\mu^2 b_0}{4\mu\lambda} \right)(G'/G) + \left(\frac{\lambda^4 b_0 - 8\lambda^2 \mu b_0 + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda^2} \right)}{\frac{\lambda^2 b_0}{4\mu^2}(G'/G)^2 + \frac{\lambda b_0}{\mu}(G'/G) + b_0}$$

$$V(\xi) = 0$$

where

$$\xi = x + \left(\frac{\lambda^2 - 16\mu}{4} + \frac{12\mu^2 a_2 + 12\mu^2 b_0}{b_0 \lambda^2} \right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Set 5

$$\begin{array}{ll} a_0 = \frac{\lambda^4 b_0 - 8\lambda^2 b_0 \mu + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda^2} & , \quad e_1 = e_2 = 0 \\ a_1 = \frac{\lambda^4 b_0 - 8\lambda^2 b_0 \mu + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda\mu} & , \quad d_1 = d_2 = 0 \\ b_1 = \frac{\lambda b_0}{\mu} \quad , \quad b_2 = \frac{\lambda^2 b_0}{4\mu^2} \quad , & s = -\frac{\lambda^4 b_0 - 16\lambda^2 b_0 \mu + 48\mu^2 a_2 + 48\mu^2 b_0}{4 b_0 \lambda^2} \end{array} \tag{41}$$

where a_2, b_0, λ and μ are constants. Substituting Eq.(41) into Eq.(35) , hence we reach the following solutions

$$U(\xi) = \frac{a_2(G'/G)^2 + \left(\frac{\lambda^4 b_0 - 8\lambda^2 b_0 \mu + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda\mu}\right)(G'/G) + \left(\frac{\lambda^4 b_0 - 8\lambda^2 b_0 \mu + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda}\right)}{\frac{\lambda^2 b_0}{4\mu^2}(G'/G)^2 + \frac{\lambda b_0}{\mu}(G'/G) + b_0}$$

$$V(\xi) = \frac{e_0}{d_0}$$

where e_0, d_0 are constants and

$$\xi = x + \left(\frac{\lambda^2 - 16\mu}{4} + \frac{12\mu^2 a_2 + 12\mu^2 b_0}{b_0 \lambda^2}\right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

4. Conclusion

In this paper, we have obtained various types of travelling wave solutions for the combined KdV-mKdV equation, the reaction-diffusion equation, and the coupled Hirota-Satsuma KdV equations that are solved by using the rational (G'/G)-expansion method. The main idea of this method is to reduce the partial differential equation to an ODE by using the travelling wave transformation (Eq.(2)), after integrating the ODE in Eq.(3), once or twice, then express the ODE in a compact form. This ODE can be written by a n-th degree polynomial in terms of (G'/G), where $G = G(\xi)$ is the general solution of the second order LODE in Eq.(5). In order to find the positive integer, we use the homogeneous balance method, that is balancing between the highest order derivative term and nonlinear term. The coefficients of the polynomials can be obtained by solving a set of algebraic equations. Generally, the resulted algebraic equations can be solved by using Maple software program. It is mostly possible to find a solution of the algebraic equations, but it is generally unable to guarantee the existence of a solution. Despite of this, the rational (G'/G)-expansion method is still powerful method for finding travelling wave solutions of nonlinear evolution equations. The rational (G'/G)-expansion method is also direct, concise, elementary that the general solution of the second order ODE Eq.(5) is well known and effective that it can be used for many other nonlinear evolution equations, such as the generalized shallow water wave equation, the compound KdV-Burgers equations, the Klein-Gordon equation, the generalized KPP equation, the approximate long water wave equations, the coupled nonlinear

Klein-Gordon-Zakharov equations, and so on. Therefore, various explicit solutions of these nonlinear evolution equations can be obtained by this method.

Author Contribution Statements Authors contributed equally and they read and approved the final manuscript.

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ON THE MATRIX REPRESENTATION OF 5^{th} ORDER BÉZIER CURVE AND ITS DERIVATIVES IN E^3

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ABSTRACT. Using the matrix representation form, the first, second, third, fourth, and fifth derivatives of 5th order Bézier curves are examined based on the control points in E^3. In addition to this, each derivative of 5th order Bézier curves is given by their control points. Further, a simple way has been given to find the control points of a Bézier curves and its derivatives by using matrix notations. An example has also been provided and the corresponding figures which are drawn by Geogebra v5 have been presented in the end.

1. INTRODUCTION

French engineer Pierre Bézier, who used Bézier curves to design automobile bodies studied with them in 1962. But the study of these curves was first developed in 1959 by mathematician Paul de Casteljau using deCasteljau's algorithm, a numerically stable method to evaluate Bézier curves. A Bézier curve is frequently used in computer graphics and related fields, in vector graphics, and in animations as a tool to control motion. To guarantee smoothness, the control points at which two curves meet must be on the line between two control points on either side. In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For 3D animation Bézier curves are often used to define 3D paths as well as 2D curves for key frame interpolation. We have been motivated by the following studies. First Bézier-curves with curvature and torsion continuity has been examined in [6]. Also in [4], [7] and [10], Bézier curves and surfaces has been given. In [1] and [5], Bézier curves are designed for Computer-Aided Geometric Designs.

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Recently equivalence conditions of control points and application to planar Bézier curves have been examined in [8] and [9].The Serret-Frenet frame and curvatures of Bézier curves are examined those in E^4 in [3]. Frenet apparatus of the cubic Bézier curves and involute of the cubic Bezier curve by using matrix representation have been examined in E^3, in [11] and [12], respectively.

2. Preliminaries

A Bézier curve is defined by a set of control points P_0 through P_n, where n is called its order. If $n = 1$ for linear, if $n = 2$ for quadratic, if $n = 3$ for cubic Bézier curve, etc. The first and last control points are always the end points of the curve; however, the intermediate control points (if any) generally do not lie on the curve. Generally, Béziers curve can be defined by $n + 1$ control points $P_0, P_1, ..., P_n$ and has the following form:

$$\mathbf{B}(t) = \sum_{I=0}^{n} \binom{n}{I} t^I (1-t)^{n-I} (t) \quad [P_I] \ , \quad t \in [0,1] \, ,$$

where $\binom{n}{I} = \frac{n!}{I!(n-I)!}$ are the binomial coefficients [2]. The points P_I are called control points for the Bézier curve. The polygon formed by connecting the Bézier points with lines, starting with P_0 and finishing with P_n, is called the Bézier polygon (or control polygon). The convex hull of the Bézier polygon contains the Bézier curve.

The derivatives of the any Bézier curve $\mathbf{B}(t)$ is

$$\mathbf{B}'(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{n-i-1} Q_i$$

where $Q_0 = n(P_1 - P_0), Q_1 = n(P_2 - P_1), Q_2 = n(P_3 - P_2), ...$ [2].

Given points P_0 and P_1, a linear Bézier curve is simply a straight line between those two points. Linear Bézier curve is given by

$$\boldsymbol{\alpha}(t) = (1-t) P_0 + t P_1$$

and also it has the matrix form with control points P_0 and P_1

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}.$$

A quadratic Bézier curve is the path traced by the function $\boldsymbol{\alpha}(t)$, given points P_0, P_1 and P_2, which can be interpreted as the linear interpolant of corresponding points on the linear Bézier curves from P_0 to P_1 and from P_1 to P_2, respectively. A quadratic Bézier curve has also the matrix form with control points P_0 , P_1 and P_2

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}.$$

Four points P_0, P_1, P_2, P_3, and P_4 in the plane or in higher-dimensional space define a cubic Bézier curve with the following equation

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3.$$

We have already examined the cubic Bézier curves and involutes in [11] and [12], respectively. The matrix form of the cubic Bézier curve with control points P_0, P_1, P_2, and P_3 is

$$\alpha(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

The matrix form of the first derivative of a cubic Bézier curve based on the control points P_0, P_1, P_2, and P_3 is

$$\alpha'(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -3 & 9 & -9 & 3 \\ 6 & -12 & 6 & 0 \\ -3 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

The first derivative of a cubic Bézier curve is a quadratic Bézier curve with control points $Q_0 = 3(P_1 - P_0)$, $Q_1 = 3(P_2 - P_1)$, and $Q_2 = 3(P_3 - P_2)$,

$$\alpha'(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3(P_1 - P_0) \\ 3(P_2 - P_1) \\ 3(P_3 - P_2) \end{bmatrix}.$$

The matrix form of the second derivative of a cubic Bézier curve based on the control points P_0, P_1, P_2, and P_3 is

$$\alpha''(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -6 & 18 & -18 & 6 \\ 6 & -12 & 6 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

The second derivative of a cubic Bézier curve is a linear Bézier curve with control points $6(P_2 - 2P_1 + P_0)$, and $6(P_3 - 2P_2 + P_1)$,

$$\alpha''(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6(P_2 - 2P_1 + P_0) \\ 6(P_3 - 2P_2 + P_1) \end{bmatrix}.$$

Five points P_0, P_1, P_2, P_3, and P_4 in the plane or in higher-dimensional space define a $4th$ *order* Bézier curve with the following equation

$$\boldsymbol{\alpha}(t) = \sum_{I=0}^{4} \binom{4}{I} t^I (1-t)^{4-I} (t) [P_I] \ , \quad t \in [0,1].$$

The matrix form of the $4th$ *order* Bézier curve based on the control points is

$$\alpha(t)=\begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}.$$

3. 5^{th} Order Bézier Curve and Its Derivatives

Definition 1. *In the plane or in higher-dimensional space define a 5^{th} order Bézier curve with six points P_0, P_1, P_2, P_3, P_4, and P_5 and it has the following equation*

$$\boldsymbol{\alpha}(t)=\sum_{I=0}^{5}\binom{5}{I}t^I\,(1-t)^{5-I}\,(t)\,[P_I]\quad,\qquad t\in[0,1]\,.$$

Theorem 1. *The matrix representation of 5^{th} order Bézier curve with control points P_0, P_1, P_2, P_3, P_4, and P_5 is*

$$\alpha(t)=\begin{bmatrix} t^5 & t^4 & t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Proof. We have already found that

$$\alpha(t)=\begin{bmatrix} t^5 & t^4 & t^3 & t^2 & t & 1 \end{bmatrix}[5Bc]\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

where $[5Bc]_{6\times 6}$ is the coefficient matrix of 5^{th} *order* of Bézier curve. "$[5Bc]_{6\times 6}$ " is obtained by the initial letters of "5^{th} *order* Bézier curve" , and the coefficient matrix of 5^{th}*degree* Bézier curve is

$$\begin{bmatrix} -\binom{5}{0}\binom{5}{5} & \binom{5}{1}\binom{4}{4} & -\binom{5}{2}\binom{3}{3} & \binom{5}{3}\binom{2}{2} & -\binom{5}{4}\binom{1}{1} & \binom{5}{5}\binom{0}{0} \\ \binom{5}{0}\binom{5}{4} & -\binom{5}{1}\binom{4}{3} & \binom{5}{2}\binom{3}{2} & -\binom{5}{3}\binom{2}{1} & \binom{5}{4}\binom{1}{0} & 0 \\ -\binom{5}{0}\binom{5}{3} & \binom{5}{1}\binom{4}{2} & -\binom{5}{2}\binom{3}{1} & \binom{5}{3}\binom{2}{0} & 0 & 0 \\ \binom{5}{0}\binom{5}{2} & -\binom{5}{1}\binom{4}{1} & \binom{5}{2}\binom{3}{0} & 0 & 0 & 0 \\ -\binom{5}{0}\binom{5}{1} & \binom{5}{1}\binom{4}{0} & 0 & 0 & 0 & 0 \\ \binom{5}{0}\binom{5}{0} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Inverse matrix $[5Bc]$, *of* 5^{th} *order* of Bézier curve is

$$[5Bc]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{6\times 6}.$$

□

Theorem 2. *The matrix representation of the first derivative of* 5^{th} *order of a Bézier curve with control points* P_0, P_1, P_2, ..., *and* P_5 *is*

$$\alpha'(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Also as a 4^{th} *order Bézier curve, the matrix representation of the first derivative of* 5^{th} *order of a Bézier curve with control points* Q_0, Q_1, ..., Q_4 *is*

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},$$

where the control points, $(5P_1 - 5P_0)$, $(5P_2 - 5P_1)$, $(5P_3 - 5P_2)$, $(5P_4 - 5P_3)$, *and* $(5P_5 - 5P_4)$, *respectively.*

Proof. We have already found that

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} [5Bc]' \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

where $[5Bc]'$ is the coefficient matrix of the first derivative of 5^{th} *order* of a Bézier curve defined by following

$$[5Bc]' = \begin{bmatrix} -5\binom{5}{0}\binom{5}{5} & 5\binom{5}{1}\binom{4}{4} & -5\binom{5}{2}\binom{3}{3} & 5\binom{5}{3}\binom{2}{2} & -5\binom{5}{4}\binom{1}{1} & 5\binom{5}{5}\binom{0}{0} \\ 4\binom{5}{0}\binom{5}{4} & -4\binom{5}{1}\binom{4}{3} & 4\binom{5}{2}\binom{3}{2} & -4\binom{5}{3}\binom{2}{1} & 4\binom{5}{4}\binom{1}{0} & 0 \\ -3\binom{5}{0}\binom{5}{3} & 3\binom{5}{1}\binom{4}{2} & -3\binom{5}{2}\binom{3}{1} & 3\binom{5}{3}\binom{2}{0} & 0 & 0 \\ 2\binom{5}{0}\binom{5}{2} & -2\binom{5}{1}\binom{4}{1} & 2\binom{5}{2}\binom{3}{0} & 0 & 0 & 0 \\ -\binom{5}{0}\binom{5}{1} & \binom{5}{1}\binom{4}{0} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and thus,

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}. \quad (1)$$

Also the first derivative of 5^{th} *order* of a Bézier curve is a 4^{th} *order* Bézier curve. Hence, the matrix representation of 4^{th} *order* Bézier curve with control points Q_0, Q_1, ..., Q_4 is

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}, \quad (2)$$

where $Q_0 = 5P_1 - 5P_0, Q_1 = 5P_2 - 5P_1, Q_2 = 5P_3 - 5P_2, Q_3 = 5P_4 - 5P_3$ and $Q_4 = 5P_5 - 5P_4$ are the control points. From (1) and (2), we write

$$\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Since,

$$\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} 5P_1 - 5P_0 \\ 5P_2 - 5P_1 \\ 5P_3 - 5P_2 \\ 5P_4 - 5P_3 \\ 5P_5 - 5P_4 \end{bmatrix},$$

or equivalently we may write

$$\alpha'(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

$\square$

Theorem 3. *The matrix representation of the second derivative of 5^{th} order of a Bézier curve with control points P_0, P_1, P_2, ..., P_5 is*

$$\alpha''(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -20 & 100 & -200 & 200 & -100 & 20 \\ 60 & -240 & 360 & -240 & 60 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 20 & -40 & 20 & 0 & 0 & 0 \\ 0 & 20 & -40 & 20 & 0 & 0 \\ 0 & 0 & 20 & -40 & 20 & 0 \\ 0 & 0 & 0 & 20 & -40 & 20 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Also as a cubic Bézier curve, it has the following form

$$\alpha^{''}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix},$$

where the control points R_0, R_1,, ..., R_3 *are given by*

$$\begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 20P_0 - 40P_1 + 20P_2 \\ 20P_1 - 40P_2 + 20P_3 \\ 20P_2 - 40P_3 + 20P_4 \\ 20P_3 - 40P_4 + 20P_5 \end{bmatrix}.$$

Proof. We have already found $\alpha^{''}(t)$, therefore the coefficient matrix of the second derivative of $5^{th}order$ of a Bézier curve is

$$[5Bc]^{''} = \begin{bmatrix} -5.4\binom{5}{0}\binom{5}{5} & 5.4\binom{5}{1}\binom{4}{4} & -5.4\binom{5}{2}\binom{3}{3} & 5.4\binom{5}{3}\binom{2}{2} & -5.4\binom{5}{4}\binom{1}{1} & 5.4\binom{5}{5}\binom{0}{0} \\ 4.3\binom{5}{0}\binom{5}{4} & -4.3\binom{5}{1}\binom{4}{3} & 4.3\binom{5}{2}\binom{3}{2} & -4.3\binom{5}{3}\binom{2}{1} & 4.3\binom{5}{4}\binom{1}{0} & 0 \\ -3.2\binom{5}{0}\binom{5}{3} & 3.2\binom{5}{1}\binom{4}{2} & -3.2\binom{5}{2}\binom{3}{1} & 3.2\binom{5}{3}\binom{2}{0} & 0 & 0 \\ 2\binom{5}{0}\binom{5}{2} & -2\binom{5}{1}\binom{4}{1} & 2\binom{5}{2}\binom{3}{0} & 0 & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} -20 & 100 & -200 & 200 & -100 & 20 \\ 60 & -240 & 360 & -240 & 60 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \end{bmatrix}.$$

By the definition of a cubic Bézier curve that

$$\alpha^{''}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix},$$

and by using the equality of these, we get

$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} -20 & 100 & -200 & 200 & -100 & 20 \\ 60 & -240 & 360 & -240 & 60 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Since inverse is

$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -20 & 100 & -200 & 200 & -100 & 20 \\ 60 & -240 & 360 & -240 & 60 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} 20 & -40 & 20 & 0 & 0 & 0 \\ 0 & 20 & -40 & 20 & 0 & 0 \\ 0 & 0 & 20 & -40 & 20 & 0 \\ 0 & 0 & 0 & 20 & -40 & 20 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Here,

$$R_0 = 20P_0 - 40P_1 + 20P_2, \quad R_1 = 20P_1 - 40P_2 + 20P_3,$$
$$R_2 = 20P_2 - 40P_3 + 20P_4, \quad R_3 = 20P_3 - 40P_4 + 20P_5$$

are the control points. By combining the calculations above, we finally write

$$\alpha''(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 20 & -40 & 20 & 0 & 0 & 0 \\ 0 & 20 & -40 & 20 & 0 & 0 \\ 0 & 0 & 20 & -40 & 20 & 0 \\ 0 & 0 & 0 & 20 & -40 & 20 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

which completes the proof. $\square$

Theorem 4. *The matrix representation of the third derivative of a 5^{th} order Bézier curve with control points P_0, P_1, P_2, ..., P_5 is*

$$\alpha'''(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -120 & 360 & -360 & 120 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -120 & 360 & -360 & 120 & 0 & 0 \\ -60 & 120 & 0 & -120 & 60 & 0 \\ -60 & 180 & -240 & 240 & -180 & 60 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Also, since the third derivative of 5^{th} order of a Bézier curve is a quadratic Bézier curve, with control points S_0, S_1, and S_2 , $\alpha'''(t)$ has the following representation

$$\alpha^{'''}(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix},$$

where

$$\begin{aligned} S_0 &= -60P_0 + 180P_1 - 180P_2 + 60P_3, \\ S_1 &= -60P_1 + 180P_2 - 180P_3 + 60P_4, \\ S_2 &= -60P_2 + 180P_3 - 180P_4 + 60P_5. \end{aligned}$$

Proof. We have already found that

$$\alpha'''(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} [5Bc]''' \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

where the coefficient matrix of the third derivative of $5^{th} order$ of a Bézier curve is

$$[5Bc]''' = \begin{bmatrix} -5.4.3\binom{5}{0}\binom{5}{5} & 5.4.3\binom{5}{1}\binom{4}{4} & -5.4.3\binom{5}{2}\binom{3}{3} & 5.4.3\binom{5}{3}\binom{2}{2} & -5.4.3\binom{5}{4}\binom{1}{1} & 5.4.3\binom{5}{5}\binom{0}{0} \\ 4.3.2\binom{5}{0}\binom{5}{4} & -4.3.2\binom{5}{1}\binom{4}{3} & 4.3.2\binom{5}{2}\binom{3}{2} & -4.3.2\binom{5}{3}\binom{2}{1} & 4.3.2\binom{5}{4}\binom{1}{0} & 0 \\ -3.2\binom{5}{0}\binom{5}{3} & 3.2\binom{5}{1}\binom{4}{2} & -3.2\binom{5}{2}\binom{3}{1} & 3.2\binom{5}{3}\binom{2}{0} & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \end{bmatrix}.$$

Hence

$$\alpha'''(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Also Bézier curve is a quadratic curve with control points S_0, S_1 and S_2, it has the following form

$$\alpha^{'''}(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix}.$$

By using the equality of these, we get

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Since again the inverse is

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} -60 & 180 & -180 & 60 & 0 & 0 \\ 0 & -60 & 180 & -180 & 60 & 0 \\ 0 & 0 & -60 & 180 & -180 & 60 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

or correspondingly,

$$\alpha^{'''}(t)=\begin{bmatrix} t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix} -60 & 180 & -180 & 60 & 0 & 0\\ 0 & -60 & 180 & -180 & 60 & 0\\ 0 & 0 & -60 & 180 & -180 & 60\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix}.$$

□

Theorem 5. *The matrix representation of the fourth derivative of a 5^{th} order Bézier curve with control points P_0, P_1, P_2, ..., and P_5 is*

$$\begin{aligned}\alpha^{(4)}(t) &= \begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120\\ 120 & -480 & 720 & -480 & 120 & 0\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix},\\ &= \begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} 120P_0-480P_1+720P_2-480P_3+120P_4\\ 120P_1-480P_2+720P_3-480P_4+120P_5\end{bmatrix}.\end{aligned}$$

Also the fourth derivative of a 5^{th} order Bézier curve is a linear Bézier curve, with control points T_0, and T_1, and it has the following equation

$$\alpha^{(4)}(t)=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} T_0\\ T_1\end{bmatrix},$$

where

$$\begin{aligned} T_0 &= 120P_0-480P_1+720P_2-480P_3+120P_4,\\ T_1 &= 120P_1-480P_2+720P_3-480P_4+120P_5\end{aligned}$$

are the control points of the fourth derivative of a 5^{th}order Bézier curve based on the points P_0, P_1, P_2, ..., and P_5.

Proof. We have already found that

$$\alpha^{(4)}(t)=\begin{bmatrix} t & 1\end{bmatrix}[5Bc]^{(4)}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix},$$

where the coefficient matrix of the fourth derivative of $5^{th}order$ of a Bézier curve is

$$[5Bc]^{(4)}=\begin{bmatrix} -5.4.3.2\binom{5}{0}\binom{5}{5} & 5.4.3.2\binom{5}{1}\binom{4}{4} & -5.4.3.2\binom{5}{2}\binom{3}{3} & 5.4.3.2\binom{5}{3}\binom{2}{2} & -5.4.3.2\binom{5}{4}\binom{1}{1} & 5.4.3.2\binom{5}{5}\binom{0}{0} \\ 4.3.2\binom{5}{0}\binom{5}{4} & -4.3.2\binom{5}{1}\binom{4}{3} & 4.3.2\binom{3}{2} & -4.3.2\binom{5}{3}\binom{2}{1} & 4.3.2\binom{5}{4}\binom{1}{0} & 0 \end{bmatrix},$$

$$=\begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \\ 120 & -480 & 720 & -480 & 120 & 0 \end{bmatrix}.$$

Hence

$$\alpha^{(4)}(t)=\begin{bmatrix} t & 1 \end{bmatrix}\begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \\ 120 & -480 & 720 & -480 & 120 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

And also as a linear Bézier curve it has the matrix form with control points T_0 and T_1

$$\alpha^{(4)}(t)=\begin{bmatrix} t & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} T_0 \\ T_1 \end{bmatrix}.$$

By using the equality of these, we get

$$\begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \\ 120 & -480 & 720 & -480 & 120 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}=\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} T_0 \\ T_1 \end{bmatrix},$$

Since the inverse matrix is

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

we get

$$\begin{bmatrix} T_0 \\ T_1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \\ 120 & -480 & 720 & -480 & 120 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Therefore, the control points of the fourth derivative of a $5^{th}order$ Bézier curve based on the points P_0, P_1, P_2, ..., and P_5 are given by

$$T_0=120P_0-480P_1+720P_2-480P_3+120P_4,$$
$$T_1=120P_1-480P_2+720P_3-480P_4+120P_5,$$

and accordingly the matrix represented form of the curve is

$$\alpha^{(4)}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 120P_0 - 480P_1 + 720P_2 - 480P_3 + 120P_4 \\ 120P_1 - 480P_2 + 720P_3 - 480P_4 + 120P_5 \end{bmatrix}.$$

□

Theorem 6. *The matrix representation of the fifth derivative of a 5^{th} order Bézier curve with control points P_0, P_1, P_2, ..., and P_5 is*

$$\alpha^{(5)}(t) = 600P_1 - 120P_0 - 1200P_2 + 1200P_3 - 600P_4 + 120P_5.$$

Proof. It is clear that

$$\alpha^{(5)}(t) = [5Bc]^{(5)} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

where $[5Bc]^{(5)} = \begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \end{bmatrix}$. □

Now, we may consider an example of a curve given by its parametric form. Our first attempt is to find its control points with the help of matrix representation. Second we examine its derivatives and their control points. Finally, we represent each control point of every derivatives by the control points of initial curve, and draw their correspondence figures by using a free-ware program Geogebra v5.

Example 1. *Let us consider the 5th order Bézier curve parameterized as*

$$\begin{aligned} \alpha(t) = \big(&74t^5 - 210t^4 + 180t^3 - 50t^2 + 5t + 1, \\ &-79t^5 + 185t^4 - 130t^3 + 10t^2 + 10t + 1, \\ &-63t^5 + 95t^4 - 30t^3 - 5t + 2\big). \end{aligned}$$

To find the control points, we first write it as in the matrix product form by following:

$$\alpha(t) = \begin{bmatrix} t^5 & t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 74 & -79 & -63 \\ -210 & 185 & 95 \\ 180 & -130 & -30 \\ -50 & 10 & 0 \\ 5 & 10 & -5 \\ 1 & 1 & 2 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} 74 & -79 & -63 \\ -210 & 185 & 95 \\ 180 & -130 & -30 \\ -50 & 10 & 0 \\ 5 & 10 & -5 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 74 & -79 & -63 \\ -210 & 185 & 95 \\ 180 & -130 & -30 \\ -50 & 10 & 0 \\ 5 & 10 & -5 \\ 1 & 1 & 2 \end{bmatrix} = \mathbf{I} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ -2 & 6 & 0 \\ 7 & -3 & -4 \\ 5 & 0 & 5 \\ 0 & -3 & -1 \end{bmatrix},$$

where $\mathbf{I}$ *is a six by six identity matrix.*

Inversely, we find the parametric form of a $5th$ *order* Bézier curve, $\alpha(t)$ with control points $P_0 = (1,1,2)$, $P_1 = (2,3,1)$, $P_2 = (-2,6,0)$, $P_3 = (7,-3,-4)$, $P_4 = (5,0,5)$, $P_5 = (0,-3,-1)$ as follows:

$$\alpha(t) = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ -2 & 6 & 0 \\ 7 & -3 & -4 \\ 5 & 0 & 5 \\ 0 & -3 & -1 \end{bmatrix},$$

$$= \big(74t^5 - 210t^4 + 180t^3 - 50t^2 + 5t + 1, -79t^5 + 185t^4 - 130t^3 + 10t^2 + 10t + 1,$$
$$- 63t^5 + 95t^4 - 30t^3 - 5t + 2\big).$$

Let us find the control points of the first derivative $\alpha'(t)$

$$\alpha'(t) = \big(370t^4 - 840t^3 + 540t^2 - 100t + 5, -395t^4 + 740t^3 - 390t^2 + 20t + 10,$$
$$- 315t^4 + 380t^3 - 90t^2 - 5\big).$$

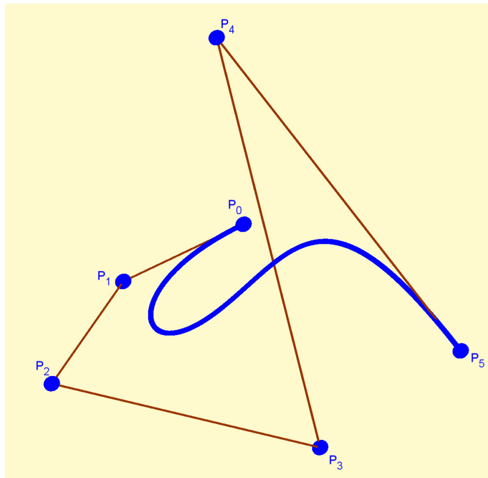

FIGURE 1. 5^{th} order Bézier curve with control points P_j $(j = 0, ..., 5)$

First we need to write its matrix product form as:

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 370 & -395 & -315 \\ -840 & 740 & 380 \\ 540 & -390 & -90 \\ -100 & 20 & 0 \\ 5 & 10 & -5 \end{bmatrix}.$$

Next, by equating the terms we have

$$\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 370 & -395 & -315 \\ -840 & 740 & 380 \\ 540 & -390 & -90 \\ -100 & 20 & 0 \\ 5 & 10 & -5 \end{bmatrix} = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},$$

$$\begin{bmatrix} 370 & -395 & -315 \\ -840 & 740 & 380 \\ 540 & -390 & -90 \\ -100 & 20 & 0 \\ 5 & 10 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},$$

$$\Longrightarrow \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 5 & 10 & -5 \\ -20 & 15 & -5 \\ 45 & -45 & -20 \\ -10 & 15 & 45 \\ -25 & -15 & -30 \end{bmatrix},$$

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$\begin{bmatrix} 5 & 10 & -5 \\ -20 & 15 & -5 \\ 45 & -45 & -20 \\ -10 & 15 & 45 \\ -25 & -15 & -30 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$\begin{bmatrix} 5 & 10 & -5 \\ -20 & 15 & -5 \\ 45 & -45 & -20 \\ -10 & 15 & 45 \\ -25 & -15 & -30 \end{bmatrix} = \begin{bmatrix} 5P_1 - 5P_0 \\ 5P_2 - 5P_1 \\ 5P_3 - 5P_2 \\ 5P_4 - 5P_3 \\ 5P_5 - 5P_4 \end{bmatrix}.$$

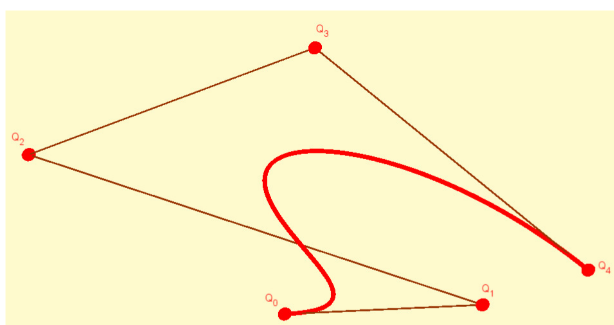

FIGURE 2. 1^{st} derivative of a 5^{th} order Bézier curve with control points Q_j $(j = 0, ..., 4)$

By following same steps given above, we may find the control points of the second and third derivative of the curve $\alpha(t)$ and draw them as in Fig. 3 and Fig. 4.

$$\alpha''(t) = \big(1080t - 2520t^2 + 1480t^3 - 100, -780t + 2220t^2 - 1580t^3 + 20$$

$$-180t + 1140t^2 - 1260t^3\big),$$

$$\alpha'''(t) = \left(-5040t + 4440t^2 + 1080, 4440t - 4740t^2 - 780, 2280t - 3780t^2 - 180\right).$$

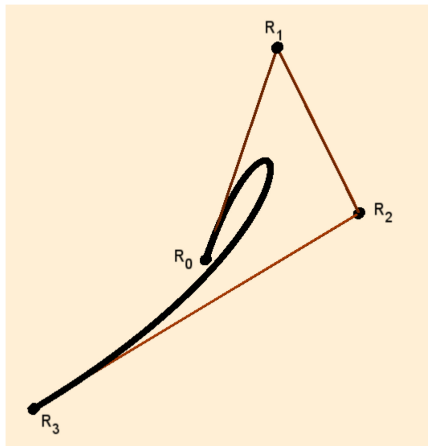

FIGURE 3. 2^{nd} derivative of a 5^{th} order Bézier curve with control points R_j $(j = 0, ..., 3)$

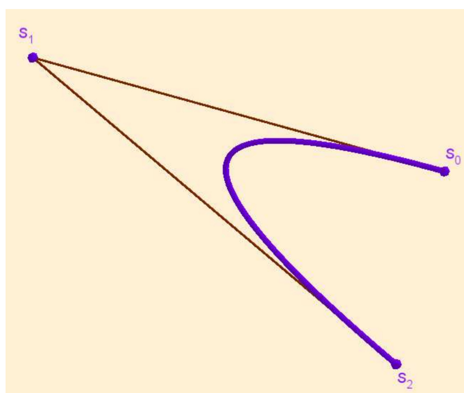

FIGURE 4. 3^{th} derivative of a 5^{th} order Bézier curve with control points S_j $(j = 0, ..., 2)$

The fourth derivative of the curve, $\alpha(t)$ is simply draws a line while the fifth derivative is a single point:

$$\alpha^{(4)}(t) = (8880t - 5040, -9480t + 4440, -7560t + 2280),$$

$$\alpha^{(5)}(t) = (8880, -9480, -7560).$$

4. Conclusion

We can write the parametric form of 5^{th} order Bézier curve using a simple matrix product. Further, we can find the control points using a simple matrix product, inversely. Also the second derivative of a 5^{th} order Bézier curve with the control points P_i, $(i = 0, ..., 4)$ can be considered another 4^{th} order Bézier curve having $(5+1)-2 = 4$ control points as $R_j = n(n-1)(P_j - 2P_{j+1} + P_{j+2})$, $j = 0, ..., 3$.The third derivative of a 5^{th} order Bézier curve with the control points P_i, $(i = 0, ..., 5)$ can be considered another cubic Bézier curve having $(5+1) - 3 = 3$ control points as $S_j = n(n-1)(n-2)(-P_j + 3P_{j+1} - 3P_{j+2} + P_{j+3})$, $j = 0, ..., 2$.The third derivative of an 5^{th} order Bézier curve with the control points P_i, $(i = 0, ..., 5)$, can be considered a quadratic Bézier curve having $(5+1) - 3 = 5 - 2 = 3$ control points as $N_j = n(n-1)(n-2)(-P_j + 3P_{j+1} - 3P_{j+2} + P_{j+3})$, $j = 0, ..., 2$.

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SPLIT COMPLEX BI-PERIODIC FIBONACCI AND LUCAS NUMBERS

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Abstract. The initial idea of this paper is to investigate the split complex bi-periodic Fibonacci and Lucas numbers by using SCFLN now on. We try to show some properties of SCFLN by taking into account the properties of the split complex numbers. Then, we present interesting relationships between SCFLN.

1. Introduction

The literature contains many articles that related to the special number sequences such as Fibonacci, Lucas, Pell ([2, 3, 6, 8, 14, 15, 17, 18]). One of these articles goes through to the bi-periodic Fibonacci (or, equivalently, generalized Fibonacci) and the bi-periodic Lucas (or, equivalently, generalized Lucas). In [3, 6], the authors introduced and studied bi-periodic Fibonacci $\{q_n\}_{n=0}^{\infty}$ and bi-periodic Lucas $\{l_n\}_{n=0}^{\infty}$ sequences that depend on two real parameters used in a non-linear (piecewise linear) recurrence relation as defined below.

$$q_n = a^{1-\xi(n)} b^{\xi(n)} q_{n-1} + q_{n-2}, \quad n \geq 2\,, \tag{1}$$

$$l_n = a^{\xi(n)} b^{1-\xi(n)} l_{n-1} + l_{n-2}, \quad n \geq 2\,, \tag{2}$$

where a and b are any two nonzero real numbers, $q_0 = 0, q_1 = 1$, $l_0 = 2, l_1 = a$ and the condition $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ can be read as

$$\xi(n) = \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}. \tag{3}$$

Furthermore, the authors in the references [3], [6] and [18] gave so many properties on the bi-periodic Fibonacci and bi-periodic Lucas sequences as in the following:

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- The Binet formulas are given by

$$q_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{4}$$

and

$$l_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha^n + \beta^n \right), \tag{5}$$

where the condition $\xi(n)$ is depend to definition in equation (3) and α, β are roots of the characteristic equation of $\lambda^2 - ab\lambda - ab = 0$.
- The generating functions for the bi-periodic Fibonacci and the bi-periodic Lucas sequences with odd and even subscripted are

$$\sum_{m=0}^{\infty} q_{2m+1} x^{2m+1} = \frac{x - x^3}{1 - (ab+2)x^2 + x^4}, \sum_{m=0}^{\infty} q_{2m} x^{2m} = \frac{ax^2}{1 - (ab+2)x^2 + x^4}, \tag{6}$$

and

$$\sum_{m=0}^{\infty} l_{2m+1} x^{2m+1} = \frac{ax + ax^3}{1 - (ab+2)x^2 + x^4}, \sum_{m=0}^{\infty} l_{2m} x^{2m} = \frac{2 - (ab+2)x^2}{1 - (ab+2)x^2 + x^4}. \tag{7}$$

- The bi-periodic Fibonacci and the bi-peridoic Lucas sequences provide the equations

$$l_n = q_{n-1} + q_{n+1} \tag{8}$$

$$(ab+4)q_n = l_{n-1} + l_{n+1}, \tag{9}$$

and

$$q_{m+n} = \frac{1}{2} \left[(\frac{b}{a})^{\xi(m+1)\xi(n)} q_m l_n + (\frac{b}{a})^{\xi(m)\xi(n+1)} q_n l_m \right], \tag{10}$$

$$l_{m+n} = \frac{1}{2} \left[(a^2b^2 + 4ab)(\frac{1}{a^2})^{\xi(m+1)\xi(n+1)} (\frac{1}{ab})^{1-\xi(m+1)\xi(n+1)} q_m q_n + (\frac{b}{a})^{\xi(m)\xi(n)} l_m l_n \right], \tag{11}$$

and

$$q_{-n} = (-1)^{n-1} q_n, \tag{12}$$

$$l_{-n} = (-1)^n l_n. \tag{13}$$

On the other hand, split complex numbers have applications in different areas of mathematics and theoretical physics. A split complex number (or hyperbolic number, also perplex number, double number) has two real number components a and b, and the set of split complex numbers is

$$\mathbb{H} = \left\{ x = a + hb : \quad h^2 = 1, a, b \in \mathbb{R} \right\}.$$

The split complex ring $\mathbb{H}$ is a bidimensional Clifford algebra, look at [10] for details. Also, split complex numbers are useful for measuring distances in the Lorentz space-time plane (you can examine [12]). The addition and multiplication of any two split complex numbers such as $x = a + hb, y = c + hd$ are defined by

$$x + y = a + c + h(b + d) \text{ and } xy = ac + bd + h(ad + bc)\,.$$

It is clear that this algebra of split complex number is commutative. The conjugate and norm of x are enounced by

$$\overline{x} = a - bh, \qquad x\overline{x} = a^2 - b^2\,. \tag{14}$$

For more information on split complex numbers, see for example, [4], [7], [9], [10], [11] and [12].

Many researchs activities can be seen in resent years studies on split complex(or hyperbolic) Fibonacci, Lucas, Jacobsthal and Tribonacci numbers (see [1], [5], [13], [16]). For example, in [1], it was investigated some properties of the split complex Fibonacci numbers are defined as $\tilde{F}_n = F_n + hF_{n+1}$.

2. The split complex bi-periodic Fibonacci and Lucas numbers

The objective of this paper is to define split complex bi-periodic Fibonacci and Lucas numbers(SCFLN) with a different aspect. In this part, we introduce the SCFLN that generalize split complex Fibonacci, split complex Lucas, split complex Pell and split complex Pell-Lucas numbers. We give some properties of the SCFLN such as the Binet formulas, the generating functions, sums, binomial sums of the SCFLN. We also present the Catalan, Cassini, D'Ocagne and other identities of the SCFLN.

Definition 1. *The split complex bi-periodic Fibonacci* ($\tilde{q}_n$) *and Lucas* ($\tilde{l}_n$) *numbers are defined by*

$$\tilde{q}_n = q_n + ha^{1-\xi(n)}b^{\xi(n)}q_{n+1}, \quad \tilde{q}_0 = ha, \ \tilde{q}_1 = hab + 1 \tag{15}$$

and

$$\tilde{l}_n = l_n + ha^{\xi(n)}b^{1-\xi(n)}l_{n+1}, \quad \tilde{l}_0 = hab + 2, \ \tilde{l}_1 = h(a^2b + 2a) + a \tag{16}$$

where $n \in \mathbb{N}, h^2 = 1$, $\xi(n)$ *is as defined the equation* (3) *and* q_n, l_n *are the bi-periodic Fibonacci and Lucas numbers, respectively.*

It can be easily shown that

$$\tilde{q}_n = \begin{cases} a\tilde{q}_{n-1} + \tilde{q}_{n-2}, & \text{if } n \text{ is even} \\ b\tilde{q}_{n-1} + \tilde{q}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \tag{17}$$

and

$$\tilde{l}_n = \begin{cases} b\tilde{l}_{n-1} + \tilde{l}_{n-2}, & \text{if } n \text{ is even} \\ a\tilde{l}_{n-1} + \tilde{l}_{n-2}, & \text{if } n \text{ is odd} \end{cases}\,. \tag{18}$$

From the equations (12), (13), (17) and (18), the SCFLN with negative subscripts are defined by

$$\tilde{q}_{-n} = \begin{cases} -a\tilde{q}_{-(n-1)} + \tilde{q}_{-(n-2)}, & \text{if } n \text{ is even} \\ -b\tilde{q}_{-(n-1)} + \tilde{q}_{-(n-2)}, & \text{if } n \text{ is odd} \end{cases}, \tag{19}$$

and

$$\tilde{l}_{-n} = \begin{cases} -b\tilde{l}_{-(n-1)} + \tilde{l}_{-(n-2)}, & \text{if } n \text{ is even} \\ -a\tilde{l}_{-(n-1)} + \tilde{l}_{-(n-2)}, & \text{if } n \text{ is odd} \end{cases}, \tag{20}$$

where $n \in \mathbb{N}$.

After all, we give the following Table 1. This table show that the first few SCFLN with positive and negative subscripts.

TABLE 1. The first few SCFLN with positive and negative subscripts.

n	$\tilde{q}_n$	$\tilde{l}_n$
-4	$h(a^2b+a)-a^2b-2a$	$h(a^2b+3a)-a^2b^2+4ab+2$
-3	$-hab+ab+1$	$h(a^2b+2a)-a^2b-3a$
-2	$ha-a$	$-hab+ab+2$
-1	1	$2ha-a$
0	ha	$hab+2$
1	$hab+1$	$h(a^2b+2a)+a$
2	$h(a^2b+a)+a$	$h(a^2b^2+3ab)+ab+2$
3	$h(a^2b^2+2ab)+ab+1$	$h(a^3b^2+4a^2b+2a)+a^2b+3a$
4	$h(a^3b^2+3a^2b+a)+a^2b+2a$	$h(a^3b^3+5a^2b^2+5ab)+a^2b^2+4ab+2$

Now, we give the Binet formulas for the SCFLN and so find some well-known mathematical properties.

Theorem 1. *For any integer n, the Binet formulas for the SCFLN are*

$$\tilde{q}_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\tilde{\alpha}\alpha^n - \tilde{\beta}\beta^n}{\alpha - \beta} \right) \tag{21}$$

and

$$\tilde{l}_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n \right), \tag{22}$$

where α, β are as the equation (4) *and $\tilde{\alpha} = 1 + h\alpha$, $\tilde{\beta} = 1 + h\beta$.*

Proof. It can easily established by using the Definition 1 and the Equations (4), (5). □

The generating functions of the SCFLN are given in the following theorem.

Theorem 2. *For the SCFLN, we have the generating functions*

$$i)\ \sum_{n=0}^{\infty} \tilde{q}_n x^n = \frac{h(abx+a-ax^2)+x+ax^2-x^3}{1-(ab+2)x^2+x^4},$$

$$ii)\ \sum_{n=0}^{\infty} \tilde{l}_n x^n = \frac{h(ab+a^2bx+2ax+abx^2-2ax^3)+2+ax-abx^2-2x^2+ax^3}{1-(ab+2)x^2+x^4}.$$

Proof. $i)$ Let $f(x)=\sum_{n=0}^{\infty} \tilde{q}_n x^n$. From the Definition 1, we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}(ha^{1-\xi(n)}b^{\xi(n)}q_{n+1}+q_n)x^n \\ &= hb\sum_{n=0}^{\infty} q_{2n+2}x^{2n+1}+ha\sum_{n=0}^{\infty} q_{2n+1}x^{2n}+\sum_{n=0}^{\infty} q_n x^n. \end{aligned}$$

By considering the Equation (6), we obtain

$$f(x)=\frac{habx}{1-(ab+2)x^2+x^4}+\frac{ha(1-x^2)}{1-(ab+2)x^2+x^4}+\frac{x+ax^2-x^3}{1-(ab+2)x^2+x^4},$$

as needed.

$ii)$ Similarly, we obtain equation in ii.

$\square$

Next, we give the formulas which give the summations and binomial sums of the SCFLN.

Theorem 3. *For $n \geq 0$, the following formulas are true:*

$$i)\ \sum_{i=0}^{n} a^{\xi(i)}(ab)^{\lfloor \frac{i}{2} \rfloor}\tilde{q}_i = \frac{a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor+1}\tilde{q}_n + a^{\xi(n+1)}(ab)^{\lfloor \frac{n+1}{2} \rfloor}\tilde{q}_{n+1}-a-ha}{2ab-1},$$

$$ii)\ \sum_{i=0}^{n} a^{-\xi(i)}(ab)^{\lfloor \frac{i+1}{2} \rfloor}\tilde{l}_i = \frac{a^{-\xi(n)}(ab)^{\lfloor \frac{n+1}{2} \rfloor+1}\tilde{l}_n + a^{-\xi(n+1)}(ab)^{\lfloor \frac{n+2}{2} \rfloor}\tilde{l}_{n+1}-2+ab-3hab}{2ab-1},$$

$$iii)\ \sum_{i=0}^{n} \binom{n}{i} a^{\xi(i)}(ab)^{\lfloor \frac{i}{2} \rfloor}\tilde{q}_i = \tilde{q}_{2n},$$
$$\sum_{i=0}^{n} \binom{n}{i} a^{\xi(i+1)}(ab)^{\lfloor \frac{i+1}{2} \rfloor}\tilde{q}_{i+1} = a\tilde{q}_{2n+1},$$

$$iv)\ \sum_{i=0}^{n} \binom{n}{i} a^{\xi(i+1)}(ab)^{\lfloor \frac{i+1}{2} \rfloor}\tilde{l}_i = a\tilde{l}_{2n},$$
$$\sum_{i=0}^{n} \binom{n}{i} a^{\xi(i)}(ab)^{\lfloor \frac{i+2}{2} \rfloor-1}\tilde{l}_{i+1} = \tilde{l}_{2n+1},$$

Proof. We will prove the parts i and iv, since the proof of the others can be done similarly with them.

i) The proof will be handled just the outcome of Theorem 1. Thus, we consider:

$$\begin{aligned}
\sum_{i=0}^{n} a^{\xi(i)}(ab)^{\lfloor \frac{i}{2} \rfloor}\tilde{q}_i &= \sum_{i=0}^{n} a\frac{\tilde{\alpha}\alpha^i - \tilde{\beta}\beta^i}{\alpha-\beta} \\
&= \frac{a\tilde{\alpha}}{\alpha-\beta}\left(\frac{\alpha^{n+1}-1}{\alpha-1}\right) + \frac{a\tilde{\beta}}{\alpha-\beta}\left(\frac{\beta^{n+1}-1}{\beta-1}\right) \\
&= \frac{a}{\alpha-\beta}\left[\frac{\tilde{\alpha}(\alpha^{n+1}\beta-\alpha^{n+1}-\beta+1)-\tilde{\beta}(\beta^{n+1}\alpha-\beta^{n+1}-\alpha+1)}{\alpha\beta-\alpha-\beta+1}\right].
\end{aligned}$$

At this point, by rearragement the last equality by using the equalities $\alpha\beta = -ab$ and $\alpha+\beta = ab$, we give

$$\begin{aligned}
\sum_{i=0}^{n} a^{\xi(i)}(ab)^{\lfloor \frac{i}{2} \rfloor}\tilde{q}_i &= \frac{-a^2b}{1-2ab}\tilde{q}_n\frac{(ab)^{\lfloor \frac{n}{2} \rfloor}}{a^{1-\xi(n)}} - \frac{a}{1-2ab}\tilde{q}_{n+1}\frac{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}{a^{1-\xi(n+1)}} \\
&\quad -\frac{a}{1-2ab}\frac{\beta(1+h\alpha)-\alpha(1+h\beta)}{\alpha-\beta} + \frac{a}{1-2ab}h \\
&= \frac{a^{\xi(n)}(ab)^{\lfloor \frac{n}{2} \rfloor+1}\tilde{q}_n + a^{\xi(n+1)}(ab)^{\lfloor \frac{n+1}{2} \rfloor}\tilde{q}_{n+1} - a - ha}{2ab-1}.
\end{aligned}$$

iv) From Theorem 1, we have

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} a^{\xi(i+1)}(ab)^{\lfloor \frac{i+1}{2} \rfloor}\tilde{l}_i &= \sum_{i=0}^{n} \binom{n}{i} a^{1-\xi(i)}(ab)^{\lfloor \frac{i+1}{2} \rfloor}\frac{a^{\xi(i)}}{(ab)^{\lfloor \frac{i+1}{2} \rfloor}}(\tilde{\alpha}\alpha^i + \tilde{\beta}\beta^i) \\
&= a\tilde{\alpha}(1+\alpha)^n + a\tilde{\beta}(1+\beta)^n.
\end{aligned}$$

By using the equalities $ab(1+\alpha) = \alpha^2$ and $ab(1+\beta) = \beta^2$, we get

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} a^{\xi(i+1)}(ab)^{\lfloor \frac{i+1}{2} \rfloor}\tilde{l}_i &= \frac{a}{(ab)^n}(\tilde{\alpha}\alpha^{2n} + \tilde{\beta}\beta^{2n}) \\
&= a\tilde{l}_{2n}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} a^{\xi(i)}(ab)^{\lfloor \frac{i+2}{2} \rfloor-1}\tilde{l}_{i+1} &= \sum_{i=0}^{n} \binom{n}{i} \frac{a}{ab}(\tilde{\alpha}\alpha^{i+1} + \tilde{\beta}\beta^{i+1}) \\
&= \frac{a\alpha}{ab}\tilde{\alpha}(1+\alpha)^n + \frac{a\beta}{ab}\tilde{\beta}(1+\beta)^n \\
&= \frac{a}{(ab)^{n+1}}(\tilde{\alpha}\alpha^{2n+1} + \tilde{\beta}\beta^{2n+1}) \\
&= \tilde{l}_{2n+1}.
\end{aligned}$$

$\square$

Theorem 4. *The relations of between the SCFLN are*

i)

$$\begin{aligned}\tilde{q}_{n+r} &= \left(\frac{b}{a}\right)^{\xi(n+1)\xi(r)} q_{r+1}\left(ha^{1-\xi(n)}b^{\xi(n)}q_{n+1}+q_n\right)\\ &+\left(\frac{b}{a}\right)^{\xi(r+1)\xi(n)} q_r\left(ha^{\xi(n)}b^{1-\xi(n)}q_n+q_{n-1}\right),\end{aligned}$$

ii)

$$\begin{aligned}\tilde{l}_{n+r} &= \left(\frac{b}{a}\right)^{\xi(n)\xi(r)} q_{r+1}\left(ha^{\xi(n)}b^{1-\xi(n)}l_{n+1}+l_n\right)\\ &+\left(\frac{b}{a}\right)^{\xi(n+1)\xi(r+1)} q_r\left(ha^{1-\xi(n)}b^{\xi(n)}l_n+l_{n-1}\right),\end{aligned}$$

iii) $\tilde{q}_{-n} = (-1)^n\tilde{q}_{n-2} + (-1)^{n+1}p_{n-1}$,

iv) $\tilde{l}_{-n} = (-1)^{n-1}\tilde{l}_{n-2} + (-1)^n(ab+4)q_{n-1}$,

v) $\tilde{q}_n = \tilde{l}_{n+1} + \tilde{l}_{n-1}$,

vi) $(ab+4)\tilde{l}_n = \tilde{q}_{n+1} + \tilde{q}_{n-1}$,

where $n, r \in \mathbb{N}$.

Proof. *i)* Using the Equation (1), Definition 1, Table 1 and iterative method, it was obtained

$$\begin{aligned}\tilde{q}_n &= ha^{1-\xi(n)}b^{\xi(n)}q_{n+1}+q_n,\\ \tilde{q}_{n+1} &= (hab+1)q_{n+1}+ha^{\xi(n)}b^{1-\xi(n)}q_n,\\ \tilde{q}_{n+2} &= a^{1-\xi(n)}b^{\xi(n)}(hab+1+h)q_{n+1}+(hab+1)q_n,\\ \tilde{q}_{n+3} &= (ha^2b^2+ab+2hab+1)q_{n+1}+ha^{\xi(n)}b^{1-\xi(n)}(hab+1+h)q_n,\\ &\vdots\\ \tilde{q}_{n+r} &= \left(\frac{b}{a}\right)^{\xi(n+1)\xi(r)} q_{r+1}\left(ha^{1-\xi(n)}b^{\xi(n)}q_{n+1}+q_n\right)\\ &+\left(\frac{b}{a}\right)^{\xi(r+1)\xi(n)} q_r\left(ha^{\xi(n)}b^{1-\xi(n)}q_n+q_{n-1}\right).\end{aligned}$$

ii) Using the Equation (2), Definition 1, Table 1 and iterative method, it was obtained

$$\begin{aligned}\tilde{l}_n &= ha^{\xi(n)}b^{1-\xi(n)}l_{n+1}+l_n,\\ \tilde{l}_{n+1} &= (hab+1)l_{n+1}+ha^{1-\xi(n)}b^{\xi(n)}l_n,\\ \tilde{l}_{n+2} &= a^{\xi(n)}b^{1-\xi(n)}(hab+1+h)l_{n+1}+(hab+1)l_n,\end{aligned}$$

$$\tilde{l}_{n+3} = (ha^2b^2 + ab + 2hab + 1)l_{n+1} + ha^{1-\xi(n)}b^{\xi(n)}(hab + 1 + h)l_n,$$

$$\vdots$$

$$\begin{aligned}\tilde{l}_{n+r} &= \left(\frac{b}{a}\right)^{\xi(n)\xi(r)} q_{r+1}\left(ha^{\xi(n)}b^{1-\xi(n)}l_{n+1} + l_n\right)\\ &\quad + \left(\frac{b}{a}\right)^{\xi(n+1)\xi(r+1)} q_r\left(ha^{1-\xi(n)}b^{\xi(n)}l_n + l_{n-1}\right).\end{aligned}$$

iii) By taking account of the Definition 1, Equations (4), (12) and (19), it was obtained

$$\begin{aligned}\tilde{q}_{-n} &= (-1)^{n-1}\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) + (-1)^n h\frac{ab^{\xi(n)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)\\ &= (-1)^{n-1}q_n + (-1)^n h\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor-1}}\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right) + (-1)^n q_{n-2} - (-1)^n q_{n-2}\\ &= (-1)^n\tilde{q}_{n-2} + (-1)^{n+1}l_{n-1}.\end{aligned}$$

iv) The proof can be done quite similarly as the part *iii* by using the Definition 1, Equations (5), (13) and (20).

v) The result is obtained by using Definition 1 and Equation (8). That is, we have

$$\begin{aligned}\tilde{l}_{n+1} + \tilde{l}_{n-1} &= ha^{\xi(n+1)}b^{1-\xi(n+1)}l_{n+2} + l_{n+1} + ha^{\xi(n-1)}b^{1-\xi(n-1)}l_n + l_{n-1}\\ &= ha^{\xi(n-1)}b^{1-\xi(n-1)}q_{n+1} + q_n\\ &= \tilde{q}_n\end{aligned}$$

as required.

vi) The proof can be done quite similarly as the part *v* by using Definition 1 and Equation (9).

□

Following Theorem gives Catalan's identities for the SCFLN;

Theorem 5. *For $n, r \in \mathbb{N}$ and $n \geq r$, we get*

i) $a^{\xi(n-r)}b^{1-\xi(n-r)}\tilde{q}_{n-r}\tilde{q}_{n+r} - a^{\xi(n)}b^{1-\xi(n)}\tilde{q}_n^2 = (-1)^{n+1-r}a^{\xi(r)}b^{1-\xi(r)}(1+hab-ab)q_r^2,$

ii) $\left(\frac{b}{a}\right)^{\xi(n+r)}\tilde{l}_{n-r}\tilde{l}_{n+r} - \left(\frac{b}{a}\right)^{\xi(n)}\tilde{l}_n^2 = (-1)^{n-r}\left(\frac{b}{a}\right)^{1-\xi(r)}(1+hab-ab)(ab+4)q_r^2.$

Proof. *i*) From Theorem 1, we have

$$a^{\xi(n-r)}b^{1-\xi(n-r)}\tilde{q}_{n-r}\tilde{q}_{n+r} = \frac{a}{(ab)^{n-1}}\left(\frac{\tilde{\alpha}^2\alpha^{2n} - \tilde{\alpha}\tilde{\beta}\alpha^{n-r}\beta^{n+r} - \tilde{\beta}\tilde{\alpha}\alpha^{n+r}\beta^{n-r} + \tilde{\beta}^2\beta^{2n}}{(\alpha-\beta)^2}\right),$$

$$a^{\xi(n)}b^{1-\xi(n)}\tilde{q}_n^2 = \frac{a}{(ab)^{n-1}}\left(\frac{\tilde{\alpha}^2\alpha^{2n} - \tilde{\alpha}\tilde{\beta}(\alpha\beta)^n - \tilde{\beta}\tilde{\alpha}(\alpha\beta)^n + \tilde{\beta}^2\beta^{2n}}{(\alpha-\beta)^2}\right).$$

From the properties of split complex numbers, we write $\tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha} = 1 + hab - ab$. Then, by using equation (4), we have

$$a^{\xi(n-r)}b^{1-\xi(n-r)}\tilde{q}_{n-r}\tilde{q}_{n+r} - a^{\xi(n)}b^{1-\xi(n)}\tilde{q}_n^2 = -\frac{a}{(ab)^{n-1}}(-ab)^{n-r}(1+hab-ab)\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2$$

$$= (-1)^{n+1-r}a^{\xi(r)}b^{1-\xi(r)}(1+hab-ab)q_r^2$$

ii) The proof can be done analogously to *i*.

□

Note that for $r = 1$ in Theorem 5, we obtain the following result which are Cassini's identities for the SCFLN.

Corollary 1. *For any positive integer n, we have*

i) $a^{\xi(n-1)}b^{1-\xi(n-1)}\tilde{q}_{n-1}\tilde{q}_{n+1} - a^{\xi(n)}b^{1-\xi(n)}\tilde{q}_n^2 = a(-1)^n(1+hab-ab)$,

ii) $\left(\frac{b}{a}\right)^{\xi(n+1)}\tilde{l}_{n-1}\tilde{l}_{n+1} - \left(\frac{b}{a}\right)^{\xi(n)}\tilde{l}_n^2 = (-1)^{n-1}(1+hab-ab)(ab+4)$.

Note that for $r = n$ in Theorem 5, we obtain the following result.

Corollary 2. *For any positive integer n, we have*

i) $hab\tilde{q}_{2n} - a^{\xi(n)}b^{1-\xi(n)}\tilde{q}_n^2 = -a^{\xi(n)}b^{1-\xi(n)}(1+hab-ab)q_n^2$,

ii) $(hab+2)\tilde{l}_{2n} - \left(\frac{b}{a}\right)^{\xi(n)}\tilde{l}_n^2 = \left(\frac{b}{a}\right)^{1-\xi(n)}(1+hab-ab)(ab+4)q_n^2$.

Following Theorem gives D'ocagne identities for the SCFLN;

Theorem 6. *For $m \geq n$ and $m, n \in \mathbb{N}$, we obtain*

i) $a^{\xi(mn+m)}b^{\xi(mn+n)}\tilde{q}_m\tilde{q}_{n+1} - a^{\xi(mn+n)}b^{\xi(mn+m)}\tilde{q}_{m+1}\tilde{q}_n = (-1)^n a^{\xi(m-n)}(1+hab-ab)q_{m-n}$,

ii) $a^{\xi(mn+n)}b^{\xi(mn+m)}\tilde{l}_m\tilde{l}_{n+1} - a^{\xi(mn+m)}b^{\xi(mn+n)}\tilde{l}_{m+1}\tilde{l}_n = (-1)^{n+1}a^{\xi(m-n)}(1+hab-ab)(ab+4)q_{m-n}$.

Proof. *i)* From Theorem 1, we have

$$a^{\xi(mn+m)}b^{\xi(mn+n)}\tilde{q}_m\tilde{q}_{n+1} = \frac{a(ab)^{-n}}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}\left(\frac{\tilde{\alpha}^2\alpha^{m+n+1} - \tilde{\alpha}\tilde{\beta}\alpha^m\beta^{n+1} - \tilde{\beta}\tilde{\alpha}\alpha^{n+1}\beta^m + \tilde{\beta}^2\beta^{m+n+1}}{(\alpha-\beta)^2}\right),$$

$$a^{\xi(mn+n)}b^{\xi(mn+m)}\tilde{q}_{m+1}\tilde{q}_n = \frac{a(ab)^{-n}}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}\left(\frac{\tilde{\alpha}^2\alpha^{m+n+1} - \tilde{\alpha}\tilde{\beta}\alpha^{m+1}\beta^n - \tilde{\beta}\tilde{\alpha}\alpha^n\beta^{m+1} + \tilde{\beta}^2\beta^{m+n+1}}{(\alpha-\beta)^2}\right).$$

Let us label the left-hand side of the equality in *i* as LHS. Then, from equation (4), we write

$$\begin{aligned} LHS &= \frac{a(ab)^{-n}}{(ab)^{\lfloor\frac{m-n}{2}\rfloor}}\tilde{\alpha}\tilde{\beta}(\alpha\beta)^n\left(\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}\right) \\ &= (-1)^n a^{\xi(m-n)}(1+hab-ab)q_{m-n}. \end{aligned}$$

ii) The proof can be done analogously to *i*.

□

We give summation formulas for the SCFLN in the following theorem.

Theorem 7. *For $m, n \in \mathbb{Z}$, we have*

i) $\left(\frac{b}{a}\right)^{\xi(m+1)\xi(n)} \tilde{q}_m \tilde{l}_n + \left(\frac{b}{a}\right)^{\xi(m)\xi(n+1)} \tilde{q}_n \tilde{l}_m = 2\tilde{q}_{m+n} + 2ha^{1-\xi(m+n)} b^{\xi(m+n)} \tilde{q}_{m+n+1}$,

ii) $(ab+4)\left(\frac{b}{a}\right)^{\xi(m+1)\xi(n+1)} \tilde{q}_m \tilde{q}_n + \left(\frac{b}{a}\right)^{\xi(m)\xi(n)} \tilde{l}_m \tilde{l}_n = 2\tilde{l}_{m+n} + 2ha^{\xi(m+n)} b^{1-\xi(m+n)} \tilde{l}_{m+n+1}$.

Proof. *i)* The proof can be done analogously to *ii*.

ii) We must express that the proof should be examined for both cases of m and n.

If both of m and n are even, from equations (15), (16) and (11), we find

$$\begin{aligned}(ab+4)\left(\frac{b}{a}\right)\tilde{q}_m\tilde{q}_n + \tilde{l}_m\tilde{l}_n &= (ab+4)\left(\frac{b}{a}\right)(q_m q_n + haq_m q_{n+1} + haq_n q_{m+1} + a^2 q_{m+1} q_{n+1}) \\ &\quad + l_m l_n + hbl_m l_{n+1} + hbl_n l_{m+1} + b^2 l_{m+1} l_{n+1} \\ &= 2l_{m+n} + 4hbl_{m+n+1} + 2abl_{m+n+2} \\ &= 2\tilde{l}_{m+n} + 2hb\tilde{l}_{m+n+1}.\end{aligned}$$

If both of m and n are odd, from equations (15), (16) and (11), we get

$$\begin{aligned}(ab+4)\tilde{q}_m\tilde{q}_n + \left(\frac{b}{a}\right)\tilde{l}_m\tilde{l}_n &= (ab+4)(q_m q_n + hbq_m q_{n+1} + hhq_n q_{m+1} + b^2 q_{m+1} q_{n+1}) \\ &\quad + \left(\frac{b}{a}\right)(l_m l_n + hal_m l_{n+1} + hal_n l_{m+1} + a^2 l_{m+1} l_{n+1}) \\ &= 2l_{m+n} + 4hbl_{m+n+1} + 2abl_{m+n+2} \\ &= 2\tilde{l}_{m+n} + 2hb\tilde{l}_{m+n+1}.\end{aligned}$$

If one of m and n is even and the other is odd, from equations (15), (16) and (11), we obtain

$$\begin{aligned}(ab+4)\tilde{q}_m\tilde{q}_n + \tilde{l}_m\tilde{l}_n &= 2l_{m+n} + 4hal_{m+n+1} + 2abl_{m+n+2} \\ &= 2\tilde{l}_{m+n} + 2ha\tilde{l}_{m+n+1}.\end{aligned}$$

If we put the all results together, we obtain the desired equation. □

If we take $m = 0$ in Theorem 7, it is easy to see the following:

Corollary 3. *For $n \in \mathbb{Z}$, we have*

i) $b\tilde{q}_n + \left(\frac{b}{a}\right)^{\xi(n)} \tilde{l}_n = 2\left(\frac{b}{a}\right)^{\xi(n)} \tilde{q}_{n+1}$,

ii) $(ab+4)\left(\frac{b}{a}\right)^{\xi(n+1)} \tilde{q}_n + b\tilde{l}_n = 2\left(\frac{b}{a}\right)^{\xi(n+1)} \tilde{l}_{n+1}$.

If we take $m = n$ in Theorem 7, it is easy to see the following:

Corollary 4. *For $n \in \mathbb{Z}$, we have*

i) $\tilde{q}_n \tilde{l}_n = \tilde{q}_{2n} + ha\tilde{q}_{2n+1}$,

ii) $(ab+4)\left(\frac{b}{a}\right)^{1-\xi(n)} \tilde{q}_n^2 + \left(\frac{b}{a}\right)^{\xi(n)} \tilde{l}_n^2 = 2\tilde{l}_{2n} + 2hb\tilde{l}_{2n+1}$.

If we take $m = 1$ in Theorem 7, it is easy to see the following:

Corollary 5. *For $n \in \mathbb{Z}$, we have*

i) $\tilde{q}_1 \tilde{l}_n + \left(\frac{b}{a}\right)^{\xi(n+1)} \tilde{q}_n \tilde{l}_1 = 2\tilde{q}_{n+1} + 2ha^{\xi(n)} b^{\xi(n+1)} \tilde{q}_{n+2}$,

ii) $(ab+4)\tilde{q}_1 \tilde{q}_n + \left(\frac{b}{a}\right)^{\xi(n)} \tilde{l}_1 \tilde{l}_n = 2\tilde{l}_{n+1} + 2ha^{\xi(n+1)} b^{\xi(n)} \tilde{l}_{n+2}$.

3. Conclusion

In this paper, we define split complex bi-periodic Fibonacci and Lucas numbers and give some properties of these new numbers. Thus, it is obtained a new genaralization for the split complex number sequences that have the similar recurrence relation. That is, in the all results of Section 2, we can express certain and immediate relationships as follows:

- If we replace $a = b = 1$ in $\tilde{q}_n$ and $\tilde{l}_n$, we get the same result in [1] for the split complex Fibonacci and Lucas numbers.
- If we replace $a = b = 2$ in $\tilde{q}_n$ and $\tilde{l}_n$, we find the split complex Pell and Pell-Lucas numberss.
- If we replace $a = b = k$ in $\tilde{q}_n$ and $\tilde{l}_n$, we obtain the split complex k-Fibonacci and k-Lucas numbers.

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STUDY STRONG SHEFFER STROKE NON-ASSOCIATIVE MV-ALGEBRAS BY FUZZY FILTERS

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Abstract. In this paper, some types of fuzzy filters of a strong Sheffer stroke non-associative MV-algebra (for short, strong Sheffer stroke NMV-algebra) are introduced. By presenting new properties of filters, we define a prime filter in this algebraic structure. Then (prime) fuzzy filters of a strong Sheffer stroke NMV-algebra are determined and some features are proved. Finally, we built quotient strong Sheffer stroke NMV-algebra by a fuzzy filter.

1. Introduction

Sheffer operation was introduced by H. M. Sheffer as a single binary operation on a Boolean algebra restated all Boolean operations or formulas [16]. Since it has all diods on the chip forming processor in a computer, producing a single diod for this operation is simpler and cheaper than to produce different diods for other Boolean operations. Therefore, it is applied to algebraic structures such as Boolean algebras ([9], [16]), ortholattices [3], orthoimplication algebras [1], Hilbert algebras [11], UP-algebras [14] and BL-algebras [13]. In recent times, Chajda et al. introduced and studied non-associative MV-algebras (briefly, NMV-algebras) ([4], [5], [6]) because associativity of the binary relation of a MV-algebra causes serious problems in expert systems in artificial intelligence ([2], [6]). Also, Oner et al. analyzed filters and neutrosophic structures on strong Sheffer stroke NMV-algebras ([10], [15]). On the other side, the notion of fuzzy logic was originally introduced by Lotfi Zadeh [18] and has been developing expeditiously. Since these

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concepts have an important position in classic or nonclassic logical algebras, it leads to interesting results ([7], [8], [12], [17]).

In this study, basic concepts and new properties of a strong Sheffer stroke NMV-algebra are presented. Then a (prime) filter of strong Sheffer stroke NMV-algebras is defined and some features examined. It is shown that a filter of a strong sheffer stroke NMV-algebra is prime if and only if it is not contained by another filter of this algebraic structure. Indeed, it is proved that a filter of a strong Sheffer stroke NMV-algebra is prime if and only if the quotient structure defined by the filter is totally ordered strong Sheffer stroke NMV-algebra and its cardinality is less than or equals to 2. By describing a (prime) fuzzy filter of strong Sheffer stroke NMV-algebras, related notions are stated. It is proved that α is a (prime) fuzzy filter of a strong Sheffer stroke NMV-algebra if and only if $\alpha_a = \{x \in A : a \leq \alpha(x)\}$ is empty or a (prime) filter of A, for all $a \in [0, 1]$. Besides, it is shown that F is a (prime) filter of a strong Sheffer stroke NMV-algebra if and only if a fuzzy subset α_F defined by F is a (prime) fuzzy filter of this algebraic structure. It is demonstrated that a strong Sheffer stroke NMV-algebra is totally ordered if and only if every fuzzy filter is prime if and only if the filter $\{1\}$ is prime. Also, we prove that a fuzzy filter α of a strong Sheffer stroke NMV-algebra is prime if and only if α_h is a prime fuzzy filter of this algebra, for a surjective endomorphism h on this algebra, and that $\alpha_h = \alpha$ if and only if $h(\alpha_a) = \alpha_a$, for an automorphism h on this algebra and $a \in Im(\alpha)$. Finally, a congruence relation on a strong Sheffer stroke NMV-algebra is defined by a fuzzy filter, and so, a quotient strong Sheffer stroke NMV-algebra is constructed by means of the congruence relation. In fact, a fuzzy filter α of a strong Sheffer stroke NMV-algebra is prime if and only if the quotient structure is a totally ordered strong Sheffer stroke NMV-algebra and its cardinality is less than or equals to 2. In addition, it is shown that $\alpha \circ h$ is a fuzzy filter of A and the quotient structures defined by the fuzzy filters $\alpha \circ h$ and α are isomorphic, for strong Sheffer stroke NMV-algebras A and B, an epimorphism h between these algebras and a fuzzy filter α of B. Consequently, it is stated that the class of all fuzzy filters of a strong Sheffer stroke NMV-algebra forms a complete lattice since the interval $[0, 1]$ is a complete lattice and has important properties.

2. Preliminaries

In this section, basic definitions and notions about strong Sheffer stroke NMV-algebras are presented.

Definition 1. *[3] Let $\mathcal{A} = (A, |)$ be a groupoid. The operation $|$ on A is said to be a Sheffer stroke operation if it satisfies the following conditions:*

($S1$) $x|y = y|x$,
($S2$) $(x|x)|(x|y) = x$,
($S3$) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
($S4$) $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2. *[4] A strong Sheffer stroke NMV-algebra is an algebra* $(A, |, 1)$ *of type* $(2, 0)$ *satisfying the identities for all* $x, y, z \in A$*:*

$(n1)$ $x|y \approx y|x$,
$(n2)$ $x|0 \approx 1$,
$(n3)$ $(x|1)|1 \approx x$,
$(n4)$ $((x|1)|y)|y \approx ((y|1)|x)|x$,
$(n5)$ $(x|1)|((x|y)|1) \approx 1$,
$(n6)$ $x|(((((x|y)|y)|z)|z)|1) \approx 1$

where 0 *denotes the algebraic constant* $1|1$.

Lemma 1. *[10] Let* $(A, |, 1)$ *be a strong Sheffer Stroke NMV-algebra. Then the binary relation* $\leq$ *defined by*

$$x \leq y \text{ if and only if } x|(y|1) \approx 1$$

is a partial order on A*. Hence,* $(A, \leq)$ *is a poset with the least element* 0 *and the greatest element* 1.

Lemma 2. *[10] In a strong Sheffer stroke NMV-algebra* A*, the following properties hold for all* $x, y, z \in A$*:*

(i) $x|(x|1) \approx 1$,
(ii) $x \leq y \Leftrightarrow y|1 \leq x|1$,
(iii) $y \leq x|(y|1)$,
(iv) $y|1 \leq x|y$,
(v) $x \leq (x|y)|y$,
(vi) $x \leq (((x|y)|y)|z)|z$,
(vii) $((x|y)|y)|y \approx x|y$,
$(viii)$ $x|1 \approx x|x$,
(ix) $x|(x|x) \approx 1$,
(x) $1|(x|x) \approx x$,
(xi) $x \leq y \Rightarrow y|z \leq x|z$,
(xii) $x|(y|1) \leq (y|(z|1))|((x|(z|1))|1)$,
$(xiii)$ $x|(y|1) \leq (z|(x|1))|((z|(y|1))|1)$.

Definition 3. *[10] A nonempty subset* $F \subseteq A$ *is called a filter of* A *if it satisfies the following properties:*
$(S_f - 1)$ $1 \in F$,
$(S_f - 2)$ *For all* $x, y \in A$*,* $x|(y|1) \in F$ *and* $x \in F$ *imply* $y \in F$.

Definition 4. *[10] Let* F *be a filter of a strong Sheffer stroke NMV-algebra* $(A, |, 1)$*. Define the binary relation* $\propto_F$ *on* A *as below: for all* $x, y \in A$

$$x \propto_F y \text{ if and only if } x|(y|1) \in F \text{ and } y|(x|1) \in F. \tag{1}$$

Definition 5. *[10] If* $x\xi y$ *implies* $x|k\xi y|k$*, for all* $x, y, k \in A$*, then the equivalence relation* ξ *is called a congruence relation on* A.

Lemma 3. *[10] An equivalence relation ξ is a congruence relation on A if and only if $x\xi y$ and $k_1\xi k_2$ imply $x|k_1\xi y|k_2$.*

Lemma 4. *[10] Let F be a filter of a strong Sheffer stroke NMV-algebra $(A,|,1)$ and the binary relation $\propto_F$ be defined as (1). Then $\propto_F$ is a congruence relation on A.*

Theorem 1. *[10] Let F be a filter of a strong Sheffer stroke NMV-algebra $(A,|,1)$ and $\propto$ be a congruence relation on A defined by F. Then $(A/\propto,|_\propto,[1]_\propto)$ is also a strong Sheffer stroke NMV-algebra where $A/F \equiv A/\propto = \{[x]_\propto : x \in A\}$, the strong Sheffer stroke $|_\propto$ on A/F is defined by $[x]_\propto|_\propto[y]_\propto \approx [x|y]_\propto$, for all $x,y \in A$ and $F \approx [1]_\propto$.*

Definition 6. *[10] Let $(A,|_A,1_A)$ and $(B,|_B,1_B)$ be strong Sheffer stroke NMV-algebras. A mapping $h : A \longrightarrow B$ is called a homomorphism if*

$$h(x|_Ay) = h(x)|_Bh(y),$$

for all $x,y \in A$.

3. Some Results in Strong Sheffer Stroke NMV-Algebras

In this section, new properties of strong Sheffer stroke NMV-algebras are given. Unless otherwise stated, A represents a strong Sheffer stroke NMV-algebra.

Lemma 5. *Let A be a strong Sheffer stroke NMV-algebra. Then $(A,\leq)$ is a bounded lattice with the least element 0 and the greatest element 1 of A, where $x \vee y \approx (x|(y|1))|(y|1)$ and $x \wedge y \approx (((x|1)|y)|y)|1$, for all $x,y \in A$.*

Proof. It is known from Lemma 1 that $(A,\leq)$ is a poset. Then $x \leq (x|(y|1))|(y|1)$ and $y \leq (x|(y|1))|(y|1)$ from Lemma 2 (v) and (iii), respectively. Thus, $(x|(y|1))|(y|1)$ is an upper bound of x and y. Let $x,y \leq z$. So, $x|(z|1) \approx 1$ and $y|(z|1) \approx 1$ from Lemma 1. Since

$$\begin{aligned}(x|(y|1))|(y|1) &\leq (z|(y|1))|(y|1)\\ &\approx (((z|1)|1)|(y|1))|(y|1)\\ &\approx (((y|1)|1)|(z|1))|(z|1)\\ &\approx (y|(z|1))|(z|1)\\ &\approx (z|1)|1\\ &\approx z\end{aligned}$$

from Lemma 2 (i), (xi), (n1), (n3) and (n4), it follows that $(x|(y|1))|(y|1)$ is the least upper bound of x and y. Hence,$x \vee y \approx (x|(y|1))|(y|1)$, and similarly, $x \wedge y \approx (((x|1)|y)|y)|1$, for all $x,y \in A$.

Since $0|(x|1) \approx (x|1)|0 \approx 1$ and $x|(1|1) \approx x|0 \approx 1$ from (n1) and (n2), it is obtained from Lemma 1 that $0 \leq x$ and $x \leq 1$, for all $x,y \in A$. Therefore, 0 is the least element and 1 is the greatest element of A □

Proposition 1. *Let A be a strong Sheffer stroke NMV-algebra. Then*

$$x|((y|(z|1))|1) \approx (x|(y|1))|((x|(z|1))|1),$$

for all $x, y, z \in A$.

Proof. Let A be a strong Sheffer stroke NMV-algebra.

$$\begin{aligned} x|((y|(z|1))|1) &\approx x|((y|(z|1))|(y|(z|1))) \\ &\approx y|((x|(z|1))|(x|(z|1))) \\ &\approx y|((x|(z|1))|1) \\ &\geq (x|(y|1))|((x|(z|1))|1) \end{aligned}$$

from Lemma 2 (viii), (iii), (xi), (S1) and (S3). Also,

$$\begin{aligned} x|((y|(z|1))|1) &\approx x|((y|(z|1))|(y|(z|1))) \\ &\approx y|((x|(z|1))|(x|(z|1))) \\ &\approx y|((x|(z|1))|1) \\ &\leq (x|(y|1))|((x|((x|(z|1))|1))|1) \\ &\approx (x|(y|1))|((x|((x|(z|1))|(x|(z|1))))|1) \\ &\approx (x|(y|1))|((((x|x)|(x|x))|(z|1))|1) \\ &\approx (x|(y|1))|((x|(z|1))|1) \end{aligned}$$

from Lemma 2 (viii), (xiii), (S1)-(S3).

Hence, $x|((y|(z|1))|1) \approx (x|(y|1))|((x|(z|1))|1)$, for all $x, y, z \in A$. □

Proposition 2. *Let A be a strong Sheffer stroke NMV-algebra. Then*

$$(x|y)|1 \leq x \text{ and } (x|y)|1 \leq y,$$

for all $x, y \in A$.

Proof. Let A be a strong Sheffer stroke NMV-algebra. Since $((x|y)|1)|(x|1) \approx (x|1)|((x|y)|1) \approx 1$ and $((x|y)|1)|(y|1) \approx (y|1)|((y|x)|1) \approx 1$ from (n1) and (n5), it is obtained from Lemma 1 that $(x|y)|1 \leq x$ and $(x|y)|1 \leq y$, for all $x, y \in A$. □

Lemma 6. *A nonempty subset F of A is a filter of A if and only if*
$(S_f - 3)$ *$x, y \in F$ imply $(x|y)|1 \in F$,*
$(S_f - 4)$ *$x \in F$ and $x \leq y$ imply $y \in F$,*
for all $x, y \in A$.

Proof. ($\Rightarrow$) Let F be a filter of A and $x, y \in A$. Since

$$\begin{aligned} x|(((x|y)|y)|1) &\approx x|(((x|y)|y)|((x|y)|y)) \\ &\approx (x|y)|((x|y)|(x|y)) \\ &\approx 1 \end{aligned}$$

from Lemma 2 (viii), (ix), (S1) and (S3), it follows from $(S_f - 2)$ that $(x|y)|y \in F$. Since $y|(((x|y)|1)|1) = (x|y)|y \in F$ from (n1) and (n3), respectively, it is obtained from $(S_f - 2)$ that $(x|y)|1 \in F$. Let $x \in F$ and $x \leq y$. Then $x|(y|1) \in F$ from Lemma 1 and $(S_f - 1)$. Thus, $y \in F$ from $(S_f - 2)$.

$(\Leftarrow)$ Let F be a nonempty subset of A satisfying $(S_f - 3)$ and $(S_f - 4)$. Assume that $x \in F$. Since $x \leq 1$ for all $x \in A$, it follows from $(S_f - 4)$ that $1 \in F$. Let $x|(y|1) \in F$ and $x \in F$. Then $(x|(x|(y|1)))|1 \in F$ from $(S_f - 3)$. Since

$$\begin{aligned}((x|(x|(y|1)))|1)|(y|1) &\approx ((((y|1)|x)|x)|1)|(y|1) \\ &\approx ((((x|1)|y)|y)|1)|(y|1) \\ &\approx (y|1)|((y|(y|(x|1)))|1) \\ &\approx 1\end{aligned}$$

from (n1), (n4) and (n5), it is obtained from Lemma 1 that $(x|(x|(y|1)))|1 \leq y$. Thus, $y \in F$ from $(S_f - 4)$. □

Lemma 7. *Let F be a filter of A. Then*

(*a*) *$z|(((y|(x|1))|(x|1))|1) \in F$ and $z \in F$ imply $(x|(y|1))|(y|1) \in F$,*
(*b*) *$z|((y|(x|1))|1) \in F$ and $z \in F$ imply $((x|(y|1))|(y|1))|(x|1) \in F$ and*
(*c*) *$x|((y|(z|1))|1) \in F$ and $x|(y|1) \in F$ imply $x|(z|1) \in F$,*

for all $x, y, z \in A$.

Proof. (*a*) Since

$$\begin{aligned}z|(((x|(y|1))|(y|1))|1) &\approx z|(((((x|1)|1)|(y|1))|(y|1))|1) \\ &\approx z|(((((y|1)|1)|(x|1))|(x|1))|1) \\ &\approx z|(((y|(x|1))|(x|1))|1) \in F\end{aligned}$$

from (n3) and (n4) and $z \in F$, it follows from $(S_f - 2)$ that $(x|(y|1))|(y|1) \in F$.

(*b*) Since

$$\begin{aligned}z|((((x|(y|1))|(y|1))|(x|1))|1) &\approx z|((((((x|1)|1)|(y|1))|(y|1))|(x|1))|1) \\ &\approx z|((((((y|1)|1)|(x|1))|(x|1))|(x|1))|1) \\ &\approx z|((((y|(x|1))|(x|1))|(x|1))|1) \\ &\approx z|((y|(x|1))|1) \in F\end{aligned}$$

from (n3), (n4) and Lemma 2 (vii) and $z \in F$, it is obtained from $(S_f - 2)$ that $((x|(y|1))|(y|1))|(x|1) \in F$.

(*c*) Since $(x|(y|1))|((x|(z|1))|1) \approx x|((y|(z|1))|1) \in F$ from Proposition 1 and $x|(y|1) \in F$, it follows from $(S_f - 2)$ that $x|(z|1) \in F$. □

Definition 7. *Let F be a filter of A. Then F is a prime filter of A if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for all $x, y \in A$.*

Example 1. *Consider a strong Shefeer stroke NMV-algebra $(A, |, 1)$ where a set $A = \{0, a, b, c, d, e, f, 1\}$ and the operation $|$ on A has the following Cayley table ([10]):*

TABLE 1. Cayley table of $|$

$\vert$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	0

Then $\{a,d,e,1\}$ *is a prime filter of* A *while* $\{e,1\}$ *is not since* $a \notin \{e,1\}$ *and* $c \notin \{e,1\}$ *when* $a \vee c \approx (a|(c|1))|(c|1) \approx (a|d)|d \approx f|d \approx e \in \{e,1\}$.

Lemma 8. *Let* F *be a filter of* A*. Then* F *is a prime filter of* A *if and only if* $x \in F$ *or* $x|1 \in F$*, for all* $x \in A$.

Proof. Let F be a prime filter of A. Since

$$\begin{aligned} x \vee (x|1) &\approx (x|((x|1)|1))|((x|1)|1) \\ &\approx x|(x|x) \\ &\approx 1 \in F \end{aligned}$$

from Lemma 5, (n1), (n3), Lemma 2 (ix) and (S_f-1), it is obtained that $x \in F$ or $x|1 \in F$, for all $x \in A$.

Conversely, let F be a filter of A such that $x \in F$ or $x|1 \in F$, for all $x \in A$. Assume that $x \vee y \in F$ such that $x \notin F$ and $y \notin F$, for some $x, y \in A$. Then $x|1 \in F$ and $y|1 \in F$. Since $x|1 \leq (y|1)|((x|1)|1) \approx x|(y|1)$ and $y|1 \leq (x|1)|((y|1)|1) \approx y|(x|1)$ from Lemma 2 (iii), (n1) and (n3), it follows from (S_f-4) that $x|(y|1) \in F$ and $y|(x|1) \in F$. Since $(x|(y|1))|(y|1) \approx x \vee y \in F$ and $(y|(x|1))|(x|1) \approx y \vee x \approx x \vee y \in F$ from Lemma 5, it is obtained from (S_f-2) that $x \in F$ and $y \in F$. This is a contradiction. Thus, $x \vee y \in F$ implies $x \in F$ or $y \in F$ which means that F is a prime filter of A. □

Lemma 9. *Let* F *be a filter of* A*. Then* F *is a prime filter of* A *if and only if* $x \notin F$ *and* $y \notin F$ *imply* $x|(y|1) \in F$ *and* $y|(x|1) \in F$*, for all* $x, y \in A$.

Proof. Let F be a prime filter of A, $x \notin F$ and $y \notin F$. Then $x|1 \in F$ and $y|1 \in F$. Since $x|1 \leq (y|1)|((x|1)|1) \approx x|(y|1)$ and $y|1 \leq (x|1)|((y|1)|1) \approx y|(x|1)$ from Lemma 2 (iii), (n1) and (n3), it follows from (S_f-4) that $x|(y|1) \in F$ and $y|(x|1) \in F$.

Conversely, let F be a filter of A such that $x \notin F$ and $y \notin F$ imply $x|(y|1) \in F$ and $y|(x|1) \in F$, for all $x, y \in A$. Assume that $x \notin F$ and $x|1 \notin F$, for some $x \in A$. Then $x|1 \approx x|x \approx x|((x|1)|1) \in F$ and $x \approx (x|x)|(x|x) \approx (1|((x|x)|(x|x)))|(1|((x|x)$

$|(x|x))) \approx (x|1)|(x|1) \in F$ from (n3), Lemma 2 (viii), (x) and (S1)-(S2). This is a contradiction. Thus, $x \in F$ or $x|1 \in F$, for all $x \in F$, i.e., F is a prime filter of A. □

Lemma 10. *Let F be a filter of A. Then*

(i) *$x \in F$ and $y \in F$ imply $x \wedge y \in F$,*
(ii) *F is a prime filter of A if and only if $x|(y|1) \in F$ or $y|(x|1) \in F$,*

for all $x, y \in A$.

Proof. (i) It is clear.

(ii) Let F be a prime filter of A. Since

$$\begin{aligned}(x|(y|1)) \vee (y|(x|1)) &\approx ((x|(y|1))|((y|(x|1))|1))|((y|(x|1))|1)\\ &\approx ((x|(y|1))|((y|(x|x))|(y|(x|x))))|((y|(x|x))|1)\\ &\approx ((((x|(y|1))|(x|x))|((x|(y|1))|(x|x)))|y)|((y|(x|x))|1)\\ &\approx (y|(x|x))|((y|(x|x))|1)\\ &\approx 1 \in F,\end{aligned}$$

from Lemma 5, Lemma 2 (i) and (viii), (S1)-(S3), it follows that $x|(y|1) \in F$ or $y|(x|1) \in F$

Conversely, let F be a filter of A such that $x|(y|1) \in F$ or $y|(x|1) \in F$, for all $x, y \in A$. Suppose that $x \vee y \in F$. If $x|(y|1) \in F$, then we have from $(S_f - 2)$ that $y \in F$ since $(x|(y|1))|(y|1) \approx x \vee y \in F$ from Lemma 5. Similarly, if $y|(x|1) \in F$, then we get from $(S_f - 2)$ that $x \in F$ since $(y|(x|1))|(x|1) \approx y \vee x \approx x \vee y \in F$ from Lemma 5. Hence, F is a prime filter of A.

□

Corollary 1. *Let F be a filter of A such that $F \neq A$. Then F is a prime filter of A if and only if $(x|(y|1)) \vee (y|(x|1)) \in F$, for all $x, y \in A$.*

Lemma 11. *Let F be a filter of A such that $F \neq A$. Then F is a prime filter of A if and only if there is no a filter G of A such that $F \subset G \subset A$.*

Proof. Let F be a prime filter of A. Assume that G is a filter of A such that $F \subset G \subset A$ and $y \in G$ such that $y \notin F$. Then $y|1 \in F$, and so, $y|1 \in G$. Since $y \in G$ and $y|1 \in G$, it follows from Lemma 2 (ix), (n1), Lemma 5 and Lemma 10 (i) that

$$\begin{aligned}0 &\approx 1|1\\ &\approx ((y|1)|((y|1)|(y|1)))|1\\ &\approx (((y|1)|(y|1))|(y|1))|1\\ &\approx y \wedge (y|1) \in G.\end{aligned}$$

Since $0 \in G$ and 0 is the least element of A, we have from $(S_f - 4)$ that $x \in G$, for all $x \in A$. Thus, $G = A$ which is a contradiction. Therefore, there is no a filter G of A such that $F \subset G \subset A$.

Conversely, let there be no a filter G of A such that $F \subset G \subset A$. Suppose that $x \vee y \in F$ such that $x, y \notin F$. Then there exists a filter G of A such that $x \in G$ or $y \in G$. Since $x, y \leq x \vee y$, we have from $(S_f - 4)$ that $x \vee y \in G$. Thus, $F \subset G$ which is a contradiction. Hence, $x \vee y \in F$ implies $x \in F$ or $y \in F$ which means that F is a prime filter of A. □

Lemma 12. *Let F be a filter of A and $\propto_F$ be a congruence relation on A defined by F. Define a relation $\subseteq$ on A/F by*

$$[x]_{\propto_F} \subseteq [y]_{\propto_F} \Leftrightarrow x|(y|1) \in F,$$

for all $x, y \in A$. Then the relation $\subseteq$ is a partial order on A/F.

Proof. Let F be a filter of A and $\propto_F$ be a congruence relation on A defined by F. Then $(A/F, |_{\propto_F}, F)$ is a strong Sheffer stroke NMV-algebra by Theorem 1.

• Since $x|(x|1) \approx 1 \in F$ from Lemma 2 (i) and $(S_f - 1)$, it follows that $[x]_{\propto_F} \subseteq [x]_{\propto_F}$, for all $x \in A$.

• Let $[x]_{\propto_F} \subseteq [y]_{\propto_F}$ and $[y]_{\propto_F} \subseteq [x]_{\propto_F}$. Then $x|(y|1) \in F$ and $y|(x|1) \in F$, and so, $x \propto_F y$. Thus, $[x]_{\propto_F} = [y]_{\propto_F}$.

• Let $[x]_{\propto_F} \subseteq [y]_{\propto_F}$ and $[y]_{\propto_F} \subseteq [z]_{\propto_F}$. Then $x|(y|1) \in F$ and $y|(z|1) \in F$. Since $x|(y|1) \leq (y|(z|1))|((x|(z|1))|1)$ from Lemma 2 (xii), it is obtained from $(S_f - 4)$ that $(y|(z|1))|((x|(z|1))|1) \in F$. Thus, it follows from $(S_f - 2)$ that $x|(z|1) \in F$ which implies that $[x]_{\propto_F} \subseteq [z]_{\propto_F}$.

Hence, the relation $\subseteq$ is a partial order on A/F. □

Theorem 2. *Let F be a filter of A and $\propto_F$ be a congruence relation on A defined by F. Then F is a prime filter of A if and only if $(A/F, |_{\propto_F}, F)$ is totally ordered and $|A/F| \leq 2$.*

Proof. Let F be a filter of A and $\propto_F$ be a congruence relation on A defined by F. Then $(A/F, |_{\propto_F}, F)$ is a strong Sheffer stroke NMV-algebra by Theorem 1. Let F be a prime filter of A. Then $x|(y|1) \in F$ or $y|(x|1) \in F$ by Lemma 10 (ii). Thus, $[x]_{\propto_F} \subseteq [y]_{\propto_F}$ or $[y]_{\propto_F} \subseteq [x]_{\propto_F}$ from Lemma 12. Hence, $(A/F, |_{\propto_F}, F)$ is totally ordered. Moreover, let $|A/F| > 2$. Then $[x]_{\propto_F} \in A/F$ such that $[0]_{\propto_F} \subset [x]_{\propto_F} \subset [1]_{\propto_F}$. Since F is a prime filter of A, it is known that $x \in F$ or $x|1 \in F$. Assume that $x|1 \in F$. Since $x|(0|1) \approx x|1 \in F$ and $0|(x|1) \approx 1 \in F$ from (n2), we get $[x]_{\propto_F} = [0]_{\propto_F}$ which is a contradiction. Therefore, $|A/F| \leq 2$.

Conversely, let $(A/F, |_{\propto_F}, F)$ be totally ordered. Then $[x]_{\propto_F} \subseteq [y]_{\propto_F}$ or $[y]_{\propto_F} \subseteq [x]_{\propto_F}$, for all $x, y \in A$. So, $x|(y|1) \in F$ or $y|(x|1) \in F$ by Lemma 12. Thus, F is a prime filter of A from Lemma 10 (ii). □

4. Fuzzy Filters of Strong Sheffer Stroke Nmv-Algebras

In this section, fuzzy filters strong Sheffer stroke NMV-algebras are introduced.

Definition 8. *A fuzzy subset α of A is called a fuzzy filter of A if*
$(FF1)$ $\alpha(x) \leq \alpha(1)$,
$(FF2)$ $\min\{\alpha(x), \alpha(x|(y|1))\} \leq \alpha(y)$,
for all $x, y \in A$.

Example 2. *Consider the strong Shefeer stroke NMV-algebra A in Example 1. Then a fuzzy subset α of A defined by*

$$\alpha(x) = \begin{cases} 0.19, & \text{if } x \approx 0, a, b, d \\ 0.81, & \text{otherwise} \end{cases}$$

is a fuzzy filter of A.

Lemma 13. *Let α be a fuzzy filter of A. Then*

(1) *if $x \leq y$, then $\alpha(x) \leq \alpha(y)$,*
(2) $\alpha(x|(y|1)) = \alpha(1)$ *implies* $\alpha(x) \leq \alpha(y)$,
(3) $\alpha((x|y)|1) = \alpha(x) \wedge \alpha(y)$,
(4) $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$,
(5) $\alpha(x) \wedge \alpha(x|1) = \alpha(0)$,
(6) $\alpha(x|(y|1)) \wedge \alpha(y|(z|1)) \leq \alpha(x|(z|1))$,
(7) $\alpha(x) \wedge \alpha(x|(y|1)) = \alpha(y) \wedge \alpha(y|(x|1)) = \alpha(x) \wedge \alpha(y)$ *and*
(8) $\alpha((((x|1)|y)|y)|1) = \alpha((((y|1)|x)|x)|1) = \alpha(x \wedge y)$,

for all $x, y, z \in A$.

Proof. (1) Let $x \leq y$. Then $x|(y|1) \approx 1$ from Lemma 1. Thus,

$$\begin{aligned} \alpha(x) &= \min\{\alpha(x), \alpha(1)\} \\ &= \min\{\alpha(x), \alpha(x|(y|1))\} \\ &\leq \alpha(y) \end{aligned}$$

from (FF1) and (FF2).

(2) Let $\alpha(x|(y|1)) = \alpha(1)$. Then

$$\begin{aligned} \alpha(x) &= \min\{\alpha(x), \alpha(1)\} \\ &= \min\{\alpha(x), \alpha(x|(y|1))\} \\ &\leq \alpha(y) \end{aligned}$$

from (FF1) and (FF2).

(3) Since $(x|y)|1 \leq x$ and $(x|y)|1 \leq y$ from Proposition 2, it follows from (1) that $\alpha((x|y)|1) \leq \alpha(x)$ and $\alpha((x|y)|1) \leq \alpha(y)$. Thus, $\alpha((x|y)|1) \leq \alpha(x) \wedge \alpha(y)$. Also,

$$\begin{aligned} \alpha(x) \wedge \alpha(y) &= \min\{\alpha(x), \alpha(y)\} \\ &\leq \min\{\alpha((x|y)|y), \alpha(y)\} \\ &= \min\{\alpha(y), \alpha(y|(((x|y)|1)|1))\} \\ &= \alpha((x|y)|1) \end{aligned}$$

from Lemma 2 (v), (1), (n1), (n3) and (FF2), respectively, Hence,

$$\alpha((x|y)|1) = \alpha(x) \wedge \alpha(y),$$

for all $x, y \in A$.

(4) Since $x \wedge y \leq x$ and $x \wedge y \leq y$, it is obtained from (1) that $\alpha(x \wedge y) \leq \alpha(x)$ and $\alpha(x \wedge y) \leq \alpha(y)$. So, $\alpha(x \wedge y) \leq \alpha(x) \wedge \alpha(y)$. Moreover, since $(x|y)|1 \leq x$ and $(x|y)|1 \leq y$ from Proposition 2, we have $(x|y)|1 \leq x \wedge y$. Thus, $\alpha(x) \wedge \alpha(y) = \alpha((x|y)|1) \leq \alpha(x \wedge y)$ from (3) and (1), respectively. Therefore, $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$, for all $x, y \in A$.

(5) $\alpha(x) \wedge \alpha(x|1) = \alpha((x|(x|1))|1) = \alpha(1|1) = \alpha(0)$ from (3) and Lemma 2 (i).

(6)

$$\begin{aligned}
\alpha(x|(y|1)) \wedge \alpha(y|(z|1)) &= \min\{\alpha(x|(y|1)), \alpha(y|(z|1))\} \\
&= \min\{\alpha(x|(y|1)), \alpha(x|((y|(z|1))|1))\} \\
&= \min\{\alpha(x|(y|1)), \alpha((x|(y|1))|((x|(z|1))|1))\} \\
&\leq \alpha(x|(z|1))
\end{aligned}$$

from Lemma 2 (iii), (1), Proposition 1 and (FF2).

(7)

$$\begin{aligned}
\alpha(y) \wedge \alpha(y|(x|1)) &= \alpha((y|(y|(x|1)))|1) \\
&= \alpha((((x|1)|y)|y)|1) \\
&= \alpha(x \wedge y) \\
&= \alpha(x) \wedge \alpha(y),
\end{aligned}$$

and similarly, $\alpha(x) \wedge \alpha(x|(y|1)) = \alpha(y) \wedge \alpha(x) = \alpha(x) \wedge \alpha(y)$ from (3), (n1), Lemma 5 and (4), respectively. Thus, $\alpha(x) \wedge \alpha(x|(y|1)) = \alpha(y) \wedge \alpha(y|(x|1)) = \alpha(x) \wedge \alpha(y)$, for all $x, y \in A$.

(8) It is proved Lemma 5.

□

Theorem 3. *Let α be a fuzzy subset of A. Then α is a fuzzy filter of A if and only if*

(*i*) *α is order-preserving,*
(*ii*) *$\alpha(x) \wedge \alpha(y) \leq \alpha((x|y)|1)$, for all $x, y \in A$.*

Proof. Let α be a fuzzy filter of A. Then it follows from Lemma 13 (1) and (3).

Conversely, let α be a fuzzy subset of A satisfying (i) and (ii). Since $x \leq 1$, it is obtained from (i) that $\alpha(x) \leq \alpha(1)$, for all $x \in A$.

$$\begin{aligned}
\min\{\alpha(x), \alpha(x|(y|1))\} &= \alpha(x) \wedge \alpha(x|(y|1)) \\
&\leq \alpha((x|(x|(y|1)))|1) \\
&= \alpha(y \wedge x) \\
&\leq \alpha(y)
\end{aligned}$$

from (ii), (n1), Lemma 5 and (i), respectively. Thus, α is a fuzzy filter of A. $\square$

Theorem 4. *Let α be a fuzzy subset of A. Then α is a fuzzy filter of A if and only if $x \leq y|(z|1)$ implies $\alpha(x) \wedge \alpha(y) \leq \alpha(z)$, for all $x, y, z \in A$.*

Proof. Let α be a fuzzy filter of A and $x \leq y|(z|1)$. Then $x|((y|(z|1))|1) \approx 1$ from Lemma 1. Since

$$\begin{aligned}((x|y)|1)|(z|1) &\approx ((x|y)|(x|y))|(z|1)\\ &\approx x|((y|(z|1))|(y|(z|1)))\\ &\approx x|((y|(z|1))|1)\\ &\approx 1\end{aligned}$$

from Lemma 2 (viii) and (S3), it follows from Lemma 1 that $(x|y)|1 \leq z$. So, $\alpha(x) \wedge \alpha(y) = \alpha((x|y)|1) \leq \alpha(z)$ from Lemma 13 (3) and (1), respectively.

Conversely, let α be a fuzzy subset of A such that $x \leq y|(z|1)$ implies $\alpha(x) \wedge \alpha(y) \leq \alpha(z)$, for all $x, y, z \in A$. Since $x \leq 1 \approx x|0 \approx x|(1|1)$, from (n2), it is obtained that $\alpha(x) = \alpha(x) \wedge \alpha(x) \leq \alpha(1)$, for all $x \in A$. Since $x \leq x \vee y \approx (x|(y|1))|(y|1)$ from Lemma 5, it follows that $\min\{\alpha(x), \alpha(x|(y|1))\} = \alpha(x) \wedge \alpha(x|(y|1)) \leq \alpha(y)$, for all $x, y \in A$. Hence, α is a fuzzy filter of A. $\square$

Theorem 5. *Let A be a strong Sheffer stroke NMV-algebra. Then α is a fuzzy filter of A if and only if $\alpha_a = \{x \in A : a \leq \alpha(x)\}$ is empty or a filter of A, for all $a \in [0, 1]$.*

Proof. Let α be a fuzzy filter of A and $\alpha_a = \{x \in A : a \leq \alpha(x)\} \neq \emptyset$. Suppose that $x \in \alpha_a$. Since $a \leq \alpha(x) \leq \alpha(1)$, we have $1 \in \alpha_a$. Let $x, x|(y|1) \in \alpha_a$. So, $a \leq \alpha(x)$ and $a \leq \alpha(x|(y|1))$. Since $a \leq \min\{\alpha(x), \alpha(x|(y|1))\} \leq \alpha(y)$, it is obtained that $y \in \alpha_a$. Hence, α_a is a filter of A.

Conversely, let $\alpha_a \neq \emptyset$ be a filter of A. Assume that $x \in \alpha_a$ such that $\alpha(1) < \alpha(x)$. If $a = 1/2(\alpha(1) + \alpha(x))$, then $\alpha(1) < a < \alpha(x)$. Thus, $1 \notin \alpha_a$ which is a contradiction with $(S_f - 1)$. Hence, $\alpha(x) \leq \alpha(1)$, for all $x \in A$. Suppose that $x, x|(y|1) \in \alpha_a$ such that $\alpha(y) < \min\{\alpha(x), \alpha(x|(y|1))\}$. If $a = 1/2(\alpha(y) + \min\{\alpha(x), \alpha(x|(y|1))\})$, then $\alpha(y) < a < \min\{\alpha(x), \alpha(x|(y|1))\} \leq \alpha(x)$ and $\alpha(y) < a < \min\{\alpha(x),\ \alpha(x|(y|1))\} \leq \alpha(x|(y|1))$. Thus, $y \notin \alpha_a$ which is a contradiction with $(S_f - 2)$. So, $\min\{\alpha(x), \alpha(x|(y|1))\} \leq \alpha(y)$, for all $x, y \in A$. Therefore, α is a fuzzy filter of A. $\square$

Lemma 14. *Let α_a and α_b be two filter of A such that $a < b$. Then $\alpha_a = \alpha_b$ if and only if there exist no $x_0 \in A$ such that $a \leq \alpha(x_0) < b$.*

Proof. Let $\alpha_a = \alpha_b$ be such that $a < b$. Then $\alpha_a = \{x \in A : a \leq \alpha(x)\} = \{x \in A : b \leq \alpha(x)\} = \alpha_b$. If there exists $x_0 \in A$ such that $a \leq \alpha(x_0) < b$, then $x_0 \notin \alpha_b = \alpha_a$ which is a contradiction with $x_0 \in \alpha_a$. Thus, there exist no $x_0 \in A$ such that $a \leq \alpha(x_0) < b$.

Conversely, suppose that there exist no $x_0 \in A$ such that $a \leq \alpha(x_0) < b$. Let $\alpha_a \neq \alpha_b$ be such that $a < b$. Then there exist $x_0 \in A$ such that $a \leq c = \alpha(x_0) < b$ which is a contradiction. Hence, $\alpha_a = \alpha_b$. □

Corollary 2. *Let α be a fuzzy filter of A. Then $\alpha_a = \alpha_b$, for any $a, b \in Im(\alpha)$ if and only if $a = b$.*

Proof. It is obvious that $\alpha_a = \alpha_b$, for any $a, b \in Im(\alpha)$ if $a = b$.

Conversely, let $\alpha_a = \alpha_b$, for any $a, b \in Im(\alpha)$. Then there exist $x_0, x_1 \in A$ such that $\alpha(x_0) = a$ and $\alpha(x_1) = b$. So, $x_0 \in \alpha_a = \alpha_b$ and $x_1 \in \alpha_b = \alpha_a$. Thus, $b \leq \alpha(x_0) = a$ and $a \leq \alpha(x_1) = b$ which imply $a = b$. □

Lemma 15. *Let α be a fuzzy filter of A and $x_0 \in A$. Then $\alpha(x_0) = a$ if and only if $x_0 \in \alpha_a$ and $x_0 \notin \alpha_b$, for all $a < b$.*

Proof. Let $\alpha(x_0) = a$. Since $\alpha(x_0) = a < b$, we get $x_0 \in \alpha_a$ and $x_0 \notin \alpha_b$, for all $a < b$.

Conversely, let $x_0 \in \alpha_a$ and $x_0 \notin \alpha_b$, for all $a < b$. Then $a \leq \alpha(x_0) < b$. If $a \leq \alpha(x_0) = b_0$, then $x_0 \notin \alpha_{b_0}$ which is a contradiction. Hence, $\alpha(x_0) = a$. □

Let α be a fuzzy subset of A. Define a subset

$$A_\alpha = \{x \in A : \alpha(x) = \alpha(1)\}$$

of A.

Lemma 16. *Let F be a nonempty subset of A and α_F be a fuzzy subset of A by*

$$\alpha_F(x) = \begin{cases} a_1, & \text{if } x \in F \\ a_2, & \text{otherwise} \end{cases}$$

where $a_1, a_2 \in [0, 1]$ such that $a_1 > a_2$. Then α_F is a fuzzy filter of A if and only if F is a filter of A. Also, $A_{\alpha_F} = F$.

Proof. Let α_F be a fuzzy filter of A. Since $\alpha_F(1) = a_1$ by (FF1), we get $1 \in F$. Let $x, x|(y|1) \in F$. Then $\alpha_F(x) = a_1$ and $\alpha_F(x|(y|1)) = a_1$. Since $a_1 = \min\{\alpha_F(x), \alpha_F(x|(y|1))\} \leq \alpha(y)$, we have $\alpha_F(y) = a_1$, i.e., $y \in F$.

Conversely, let F be a filter of A. Since $1 \in F$, $\alpha_F(x) \leq \alpha_F(1) = a_1$, for all $x \in A$. Let $\min\{\alpha_F(x), \alpha_F(x|(y|1))\} = a_1$. Then $\alpha_F(x) = a_1 = \alpha_F(x|(y|1))$ which means that $x \in F$ and $x|(y|1) \in F$. So, $y \in F$ which implies $\alpha_F(y) = a_1$. Thus, $\min\{\alpha_F(x), \alpha_F(x|(y|1))\} \leq \alpha(y)$. Moreover, if $\min\{\alpha_F(x), \alpha_F(x|(y|1))\} = a_2$, then $\min\{\alpha_F(x), \alpha_F(x|(y|1))\} \leq \alpha(y)$, for all $x, y \in A$. Hence, α_F is a fuzzy filter of A.

Since F is a filter of A,

$$\begin{aligned} A_{\alpha_F} &= \{x \in A : \alpha_F(x) = \alpha_F(1)\} \\ &= \{x \in A : \alpha_F(x) = a_1\} \\ &= \{x \in A : x \in F\} \\ &= A \cap F = F. \end{aligned}$$

□

Definition 9. *Let α be a fuzzy filter of A. Then α is called a prime fuzzy filter of A if $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$, for all $x, y \in A$.*

Example 3. *Consider the strong Sheffer stroke NMV-algebra A in Example 1. Then a fuzzy subset α_1 of A defined by*

$$\alpha_1(x) = \begin{cases} 0.007, & \text{if } x \approx 0, a, c, e \\ 0.993, & \text{otherwise} \end{cases}$$

is a prime fuzzy filter of A.

However, a fuzzy subset α_2 of A defined by

$$\alpha_2(x) = \begin{cases} 0.92, & \text{if } x \approx 1 \\ 0.9, & \text{otherwise} \end{cases}$$

is not a prime fuzzy filter of A since $\alpha_2(b \vee e) = \alpha_2((b|(e|1))|(e|1)) = \alpha_2(b|(b|b)) = \alpha_2(b|e) = \alpha_2(1) \neq \alpha_2(b) = \alpha_2(b) \vee \alpha_2(e)$.

Theorem 6. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$, for all $x \in A$.*

Proof. Let α be a prime fuzzy filter of A. Since

$$\begin{aligned} \alpha(x) \vee \alpha(x|1) &= \alpha(x \vee (x|1)) \\ &= \alpha((x|((x|1)|1))|((x|1)|1)) \\ &= \alpha(x|(x|x)) \\ &= \alpha(1) \end{aligned}$$

from Lemma 5, (n1), (n3) and Lemma 2 (ix), it follows that $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$, for all $x \in A$.

Conversely, let α be a fuzzy filter of A such that $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$, for all $x \in A$. Since $x \leq x \vee y$ and $y \leq x \vee y$, it follows from Lemma 13 (1) that $\alpha(x) \leq \alpha(x \vee y)$ and $\alpha(y) \leq \alpha(x \vee y)$, and so, $\alpha(x) \vee \alpha(y) \leq \alpha(x \vee y)$, for all $x, y \in A$. If $\alpha(x) = \alpha(1)$ or $\alpha(y) = \alpha(1)$, then $\alpha(x \vee y) \leq \alpha(x) \vee \alpha(y)$ from (FF1). If $\alpha(x) \neq \alpha(1)$ and $\alpha(y) \neq \alpha(1)$, then $\alpha(x|1) = \alpha(1)$ and $\alpha(y|1) = \alpha(1)$. Since

$$\begin{aligned} \alpha(x \vee y) &= \alpha(y \vee x) \\ &= \alpha(1) \wedge \alpha(y \vee x) \\ &= \alpha(x|1) \wedge \alpha(y \vee x) \\ &= \alpha(((x|1)|(y \vee x))|1) \\ &= \alpha(((x|1)|((y|(x|1))|(x|1)))|1) \\ &= \alpha((y|(x|1))|1) \\ &\leq \alpha(y), \end{aligned}$$

and similarly, $\alpha(x \vee y) \leq \alpha(x)$ from Lemma 13 (1) and (3), Lemma 5, (n1), (n3), Lemma 2 (iv), (vii) and (ix), it is obtained that $\alpha(x \vee y) \leq \alpha(x) \vee \alpha(y)$. Hence, $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$, for all $x, y \in A$,, i.e., F is a prime fuzzy filter of A. $\square$

Theorem 7. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x) \neq \alpha(1)$ and $\alpha(y) \neq \alpha(1)$ imply $\alpha(x|(y|1)) = \alpha(1)$ and $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$.*

Proof. Let α be a prime fuzzy filter of A and $\alpha(x) \neq \alpha(1)$ and $\alpha(y) \neq \alpha(1)$. Then $\alpha(x|1) = \alpha(1)$ and $\alpha(y|1) = \alpha(1)$ from Theorem 6. Since $(x|1)|((x|(y|1))|1) \approx 1$ and $(y|1)|((y|(x|1))|1) \approx 1$ from (n5), it follows from (FF2) that

$$\alpha(1) = \min\{\alpha(1), \alpha(1)\} = \min\{\alpha(x|1), \alpha((x|1)|((x|(y|1))|1))\} \leq \alpha(x|(y|1))$$

and

$$\alpha(1) = \min\{\alpha(1), \alpha(1)\} = \min\{\alpha(y|1), \alpha((y|1)|((y|(x|1))|1))\} \leq \alpha(y|(x|1)),$$

respectively. Thus, $\alpha(x|(y|1)) = \alpha(1)$ and $\alpha(y|(x|1)) = \alpha(1)$ from (FF1).

Conversely, let α be a fuzzy filter of A such that $\alpha(x) \neq \alpha(1)$ and $\alpha(y) \neq \alpha(1)$ imply $\alpha(x|(y|1)) = \alpha(1)$ and $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$. If $\alpha(x) \neq \alpha(1)$ and $\alpha(1|1) = \alpha(0) \neq \alpha(1)$ for any $x \in A$, then $\alpha(x|1) = \alpha(x|(0|1)) = \alpha(1)$ and $\alpha(0|(x|1)) = \alpha(1)$ from (n1) and (n2). Also, if $\alpha(x|1) \neq \alpha(1)$ and $\alpha(1|1) = \alpha(0) \neq \alpha(1)$ for any $x \in A$, then $\alpha(x) = \alpha((x|1)|1) = \alpha((x|1)|(0|1)) = \alpha(1)$ and $\alpha(0|((x|1)|1)) = \alpha(1)$ from (n1)-(n3). Therefore, $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$, for all $x \in A$. Hence, α is a prime fuzzy filter of A by Theorem 6. $\square$

Corollary 3. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x \vee (x|1)) = \alpha(1)$, for all $x, y \in A$.*

Theorem 8. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$.*

Proof. Let α be a prime fuzzy filter of A. Since

$$\begin{aligned}
\alpha(x|(y|1)) \vee \alpha(y|(x|1)) &= \alpha((x|(y|1)) \vee (y|(x|1))) \\
&= \alpha(((x|(y|1))|((y|(x|1))|1))|((y|(x|1))|1)) \\
&= \alpha(((x|(y|y))|((y|(x|x))|(y|(x|x))))|((y|(x|x))|(y|(x|x)))) \\
&= \alpha(((((x|(y|y))|(x|x))|((x|(y|y))| \\
&\quad (x|x)))|y)|((y|(x|x))|(y|(x|x)))) \\
&= \alpha((y|(x|x))|((y|(x|x))|(y|(x|x)))) \\
&= \alpha(1)
\end{aligned}$$

from Lemma 5, Lemma 2 (viii), (ix) and (S1)-(S3), it follows that $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$.

Conversely, let α be a fuzzy filter of A such that $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$. By substituting $[y := x|1]$ in the hypothesis, we have $\alpha(1) = \alpha(x|((x|1)|1)) = \alpha(x|x) = \alpha(x|1)$ and $\alpha(1) = \alpha((x|1)|(x|1)) = \alpha((x|x)|(x|x)) = \alpha(x)$ from (n3), Lemma 2 (viii) and (S2). Thus, α is a prime fuzzy filter of A. $\square$

Corollary 4. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x|(y|1)) \vee \alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$.*

Theorem 9. *Let A be a strong Sheffer stroke NMV-algebra. Then α is a prime fuzzy filter of A if and only if α_a is empty or a prime filter of A, for all $a \in [0, 1]$.*

Proof. Let α be a prime fuzzy filter of A and $\alpha_a \neq \emptyset$. Assume that $x \vee y \in \alpha_a$. Since $a \leq \alpha(x \vee y) = \alpha(x) \vee \alpha(y)$, it follows that $a \leq \alpha(x)$ or $a \leq \alpha(y)$. Thus, $x \in \alpha_a$ or $y \in \alpha_a$ which imply that α_a is a prime filter of A.

Conversely, $\alpha_a \neq \emptyset$ be a prime filter of A and $a = \alpha(x \vee y)$. Since $x \vee y \in \alpha_a$, it is obtained that $x \in \alpha_a$ or $y \in \alpha_a$. Hence, $a \leq \alpha(x)$ or $a \leq \alpha(y)$, and so, $\alpha(x \vee y) = a \leq \alpha(x) \vee \alpha(y)$. Since $x \leq x \vee y$ and $y \leq x \vee y$, we get from Lemma 13 (1) that $\alpha(x) \leq \alpha(x \vee y)$ and $\alpha(y) \leq \alpha(x \vee y)$. So, $\alpha(x) \vee \alpha(y) \leq \alpha(x \vee y)$. Therefore, $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$ which means that α is a prime fuzzy filter of A. □

Corollary 5. *Let A be a strong Sheffer stroke NMV-algebra. Then α is a (prime) fuzzy filter of A if and only if $\alpha_{\alpha_{(1)}}$ is a (prime) filter of A.*

Corollary 6. *Let F be a nonempty subset of A. Then F is a (prime) filter of A if and only if the characteristic function χ_F of F is a (prime) fuzzy filter of A.*

Corollary 7. *Let F be a nonempty subset of A and α_F be a fuzzy subset of A by*

$$\alpha_F(x) = \begin{cases} a_1, & \text{if } x \in F \\ a_2, & \text{otherwise} \end{cases}$$

where $a_1, a_2 \in [0, 1]$ such that $a_1 > a_2$. Then α_F is a prime fuzzy filter of A if and only if F is a prime filter of A.

Proof. Let α_F be a prime fuzzy filter of A. It is obvious that F is a filter of A by Lemma 16. Since $\alpha_F(x) = \alpha_F(1) = a_1$ or $\alpha_F(x|1) = \alpha_F(1) = a_1$ from $(S_f - 1)$, it follows that $x \in F$ or $x|1 \in F$ which means that F is a prime filter of A by Lemma 8.

Let F be a prime filter of A. It is clear that α_F is a fuzzy filter of A by Lemma 16. Since $x \in F$ or $x|1 \in F$, for all $x \in A$, it is obtained from $(S_f - 1)$ that $\alpha_F(x) = a_1 = \alpha_F(1)$ or $\alpha_F(x|1) = a_1 = \alpha_F(1)$ which means that α_F is a prime fuzzy filter of A by Theorem 6. □

Theorem 10. *Let A be a strong Sheffer stroke NMV-algebra. Then the following conditions are equivalent:*
(1) A is totally ordered.
(2) Every fuzzy filter of A is prime.
(3) $\{1\}$ is a prime filter of A.

Proof. Let A be a strong Sheffer stroke NMV-algebra.

(1)⇒(2) Let A be totally ordered and α be a fuzzy filter of A. Then $x \leq y$ or $y \leq x$, for all $x, y \in A$. Since $x|(y|1) \approx 1$ or $y|(x|1) \approx 1$ from Lemma 1, it follows that $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$ for all $x, y \in A$ which means that α is a prime fuzzy filter of A from Theorem 8.

(2)⇒(3) Let every fuzzy filter of A be prime. Then $\chi_{\{1\}}$ is a prime fuzzy filter of A. Thus, $\{1\}$ is a prime filter of A by Corollary 6.

(3)⇒(1) Let the filter $\{1\}$ of A be prime. Then $\chi_{\{1\}}$ is a prime fuzzy filter of A by Corollary 6. Since $\chi_{\{1\}}(x|(y|1)) \vee \chi_{\{1\}}(y|(x|1)) = \chi_{\{1\}}(1) = 1$ from Corollary 4, it follows that $\chi_{\{1\}}(x|(y|1)) = 1$ or $\chi_{\{1\}}(y|(x|1)) = 1$, for all $x, y \in A$. Thus, $x|(y|1) \approx 1$ or $y|(x|1) \approx 1$ which implies that $x \leq y$ or $y \leq x$ from Lemma 1. Hence, A is totally ordered. □

Let h be an endomorphism on A and α be a fuzzy subset of A. Define a new fuzzy subset of A by

$$\alpha_h(x) = \alpha(h(x)),$$

for all $x \in A$.

Theorem 11. *Let h be a surjective endomorphism on A. Then α is a (prime) fuzzy filter of A if and only if α_h is a (prime) fuzzy filter of A.*

Proof. (⇒) Let h be a surjective endomorphism on A and α be a fuzzy filter of A. Then $\alpha_h(x) = \alpha(h(x)) \leq \alpha(1) = \alpha(h(1)) = \alpha_h(1)$, for all $x \in A$. Also,

$$\begin{aligned}
\min\{\alpha_h(x), \alpha_h(x|(y|1))\} &= \min\{\alpha(h(x)), \alpha(h(x|(y|1)))\} \\
&= \min\{\alpha(h(x)), \alpha(h(x)|(h(y)|h(1)))\} \\
&\leq \alpha(h(y)) \\
&= \alpha_h(y),
\end{aligned}$$

for all $x, y \in A$. Thus, α_h is a fuzzy filter of A. If α is prime, then $\alpha_h(x) = \alpha(h(x)) = \alpha(1) = \alpha(h(1)) = \alpha_h(1)$ or $\alpha_h(x|1) = \alpha(h(x|1)) = \alpha(h(x)|h(1)) = \alpha(h(x)|1) = \alpha(1) = \alpha(h(1)) = \alpha_h(1)$, for all $x \in A$, for all $x \in A$ so that α_h is prime.

(⇐) Let h be a surjective endomorphism on A and α_h be a fuzzy filter of A. Then $\alpha(x) = \alpha(h(a)) = \alpha_h(a) \leq \alpha_h(1) = \alpha(h(1)) = \alpha(1)$ and

$$\begin{aligned}
\min\{\alpha(x), \alpha(x|(y|1))\} &= \min\{\alpha(h(a)), \alpha(h(a)|(h(b)|h(1)))\} \\
&= \min\{\alpha(h(a)), \alpha(h(a|(b|1)))\} \\
&= \min\{\alpha_h(a), \alpha_h(a|(b|1))\} \\
&\leq \alpha_h(b) \\
&= \alpha(h(b)) \\
&= \alpha(y)
\end{aligned}$$

where $x = h(a)$ and $y = h(b)$, for all $x, y, a, b \in A$. If α_h is prime, then $\alpha(x) = \alpha(h(a)) = \alpha_h(a) = \alpha_h(1) = \alpha(h(1)) = \alpha(1)$ or $\alpha(x|1) = \alpha(h(a)|h(1)) = \alpha(h(a|1)) = \alpha_h(a|1) = \alpha_h(1) = \alpha(h(1)) = \alpha(1)$, for all $x, a \in A$, for all $x \in A$. Hence, α is prime. □

Theorem 12. *Let h be an automorphism on A and α be a fuzzy filter of A. Then $\alpha_h = \alpha$ if and only if $h(\alpha_a) = \alpha_a$, for any $a \in Im(\alpha)$.*

Proof. Let $\alpha_h = \alpha$, $a \in Im(\alpha)$ and $x \in \alpha_a$. Then $h(x) \in h(\alpha_a)$. Since $a \leq \alpha(x) = \alpha_h(x) = \alpha(h(x))$, it follows that $h(x) \in \alpha_a$, i.e., $h(\alpha_a) \subseteq \alpha_a$. Let $x \in \alpha_a$ and $y \in A$ such that $h(y) = x$. Since $a \leq \alpha(x) = \alpha(h(y)) = \alpha_h(y) = \alpha(y)$, it is obtained that $y \in \alpha_a$. Then $x = h(y) \in h(\alpha_a)$ which implies that $\alpha_a \subseteq h(\alpha_a)$. Thus, $h(\alpha_a) = \alpha_a$, for any $a \in Im(\alpha)$.

Conversely, let $h(\alpha_a) = \alpha_a$, for any $a \in Im(\alpha)$ and $\alpha(x) = a$. By Lemma 15, $x \in \alpha_a$ and $x \notin \alpha_b$, for all $a \leq b$. Since $h(x) \in h(\alpha_a) = \alpha_a$, we have $a \leq \alpha(h(x)) = \alpha_h(x)$. Suppose that $\alpha_h(x) = b$. Then $\alpha(h(x)) = \alpha_h(x) = b$, and so, $h(x) \in \alpha_b = h(\alpha_b)$. Since h is an automorphism, we get $x \in \alpha_b$ which is a contradiction. Thus, $\alpha_h(x) = \alpha(h(x)) = a = \alpha(x)$, for all $x \in A$, i.e., $\alpha_h = \alpha$. □

Definition 10. *Let α be a fuzzy filter of A. Define the binary relation $\sim_\alpha$ on A by for all $x, y \in A$*

$$x \sim_\alpha y \ if \ and \ only \ if \ \alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1)). \tag{2}$$

Example 4. *Consider the strong Sheffer stroke NMV-algebra A in Example 1. For a fuzzy filter α of A by*

$$\alpha(x) = \begin{cases} 0.87, & \textit{if } x \approx d, 1 \\ 0.03, & \textit{otherwise,} \end{cases}$$

$\sim_\alpha = \{(0,0), (a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (1,1), (d,1), (1,d), (c,0), (0,c), (a,e), (e,a), (b,f), (f,b)\}$ *is a binary relation on A.*

Lemma 17. *Let α be a fuzzy filter of A and the binary relation $\sim_\alpha$ be defined as (2). Then $\sim_\alpha$ is a congruence relation on A.*

Proof. • Reflexive: Since $\alpha(x|(x|1)) = \alpha(1)$ from Lemma 2 (i), it follows that $x \sim_\alpha x$, for all $x \in A$.

• Let $x \sim_\alpha y$. Then $\alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1))$. Since $\alpha(y|(x|1)) = \alpha(1) = \alpha(x|(y|1))$, we get $y \sim_\alpha x$.

• Let $x \sim_\alpha y$ and $y \sim_\alpha z$. Then $\alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1))$ and $\alpha(y|(z|1)) = \alpha(1) = \alpha(z|(y|1))$. Since $\alpha(1) = \alpha(1) \wedge (1) = \alpha(x|(y|1)) \wedge \alpha(y|(z|1)) \leq \alpha(x|(z|1))$ and $\alpha(1) = \alpha(1) \wedge (1) = \alpha(z|(y|1)) \wedge \alpha(y|(x|1)) \leq \alpha(z|(x|1))$ from Lemma 13 (6), it is obtained that $\alpha(x|(z|1)) = \alpha(1) = \alpha(z|(x|1))$. Thus, $x \sim_\alpha z$.

Hence, $\sim_\alpha$ is an equivalence relation on A.

Let $x \sim_\alpha y$ and $z \sim_\alpha t$. Then $\alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1))$ and $\alpha(z|(t|1)) = \alpha(1) = \alpha(t|(z|1))$.

(a) It follows from (n1), (n3) and Lemma 2 (xiii) that $x|(y|1) \approx (y|1)|((x|1)|1) \leq (z|((y|1)|1))|((z|((x|1)|1))|1) \approx (y|z)|((x|z)|1)$, and similarly, $y|(x|1) \leq (x|z)|((y|z)|1)$. Since $\alpha((x|z)|((y|z)|1)) = \alpha(1) = \alpha((y|z)|((x|z)|1))$ from Lemma 13 (1) and (FF1), it is obtained $x|z \sim_\alpha y|z$.

(b) By substituting $[x := z]$, $[y := t]$ and $[z := y]$ in (a), simultaneously, it follows from (n1) that $y|z \sim_\alpha y|t$.

Therefore, $x|z \sim_\alpha y|t$ from the transitivity of $\sim_\alpha$, and so, $\sim_\alpha$ is a congruence relation on A.

□

Theorem 13. *Let α be a fuzzy filter of A and $\sim$ be a congruence relation on A defined by α. Then $(A/\sim, |_\sim, [1]_\sim)$ is also a strong Sheffer stroke NMV-algebra where $A/\sim = \{[x]_\sim : x \in A\}$, the strong Sheffer stroke $|_\sim$ on $A/\sim$ is defined by $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim$, for all $x, y \in A$. Also, a relation $\preceq$ defined by $[x]_\sim \preceq [y]_\sim \Leftrightarrow \alpha(x|(y|1)) = \alpha(1)$, for all $x, y \in A$, is a partial order on $A/\sim$ and $[1]_\sim$ is the greatest element and $[0]_\sim$ is the least element of $A/\sim$.*

Proof. Let α be a fuzzy filter of A, $\sim$ be a congruence relation on A defined by α and the binary operation $|_\sim$ be defined by $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim$, for all $x, y \in A$. Since

(n1)(and (S1)): $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim = [y|x]_\sim = [y]_\sim|_\sim[x]_\sim$,

(n2): $[x]_\sim|_\sim[0]_\sim = [x|0]_\sim = [1]_\sim$,

(n3): $([x]_\sim|_\sim[1]_\sim)|_\sim[1]_\sim = [(x|1)|1]_\sim = [x]_\sim$,

(n4):

$$\begin{aligned}(([x]_\sim|_\sim[1]_\sim)|_\sim[y]_\sim)|_\sim[y]_\sim &= [((x|1)|y)|y]_\sim \\ &= [((y|1)|x)|x]_\sim \\ &= (([y]_\sim|_\sim[1]_\sim)|_\sim[x]_\sim)|_\sim[x]_\sim,\end{aligned}$$

(n5): $([x]_\sim|_\sim[1]_\sim)|_\sim(([x]_\sim|_\sim[y]_\sim)|_\sim[1]_\sim) = [(x|1)|((x|y)|1)]_\sim = [1]_\sim$,

(n6):

$$\begin{aligned}&[x]_\sim|_\sim(((((([x]_\sim|_\sim[y]_\sim)|_\sim|_\sim[y]_\sim)|_\sim[z]_\sim)|_\sim[z]_\sim)|_\sim[1]_\sim) \\ &= [x|(((((x|y)|y)|z)|z)|1)]_\sim \\ &= [1]_\sim,\end{aligned}$$

(S2): $([x]_\sim|_\sim[x]_\sim)|_\sim([x]_\sim|_\sim[y]_\sim) = [(x|x)|(x|y)]_\sim = [x]_\sim$,

(S3):

$$\begin{aligned}[x]_\sim|_\sim(([y]_\sim|_\sim[z]_\sim)|_\sim([y]_\sim|_\sim[z]_\sim)) &= [x|((y|z)|(y|z))]_\sim \\ &= [((x|y)|(x|y))|z]_\sim \\ &= (([x]_\sim|_\sim[y]_\sim)|_\sim([x]_\sim|_\sim[y]_\sim))|_\sim[z]_\sim\end{aligned}$$

and

(S4):

$$\begin{aligned}&([x]_\sim|_\sim(([x]_\sim|_\sim[x]_\sim)|_\sim([y]_\sim|_\sim[y]_\sim)))|_\sim([x]_\sim|_\sim(([x]_\sim|_\sim[x]_\sim)|_\sim([y]_\sim|_\sim[y]_\sim))) \\ &= [(x|((x|x)|(y|y)))|(x|((x|x)|(y|y)))]_\sim \\ &= [x]_\sim,\end{aligned}$$

for all $x, y, z \in A$, the binary operation $|_\sim$ is a strong Sheffer stroke.

• Reflexive: $[x]_\sim \preceq [x]_\sim$ since $\alpha(x|(x|1)) = \alpha(1)$, from Lemma 2 (i).

• Antisymmetric: let $[x]_\sim \preceq [y]_\sim$ and $[y]_\sim \preceq [x]_\sim$. Since $\alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1))$, we have $x \sim y$ which implies $[x]_\sim = [y]_\sim$.

• Transitive: let $[x]_\sim \preceq [y]_\sim$ and $[y]_\sim \preceq [z]_\sim$. Then $\alpha(x|(y|1)) = \alpha(1)$ and $\alpha(y|(z|1)) = \alpha(1)$. Since $\alpha(1) = \alpha(1) \wedge \alpha(1) = \alpha(x|(y|1)) \wedge \alpha(y|(z|1)) \leq \alpha(x|(z|1))$ from Lemma 13 (6), it follows from (FF1) that $\alpha(x|(z|1)) = \alpha(1)$, i.e., $[x]_\sim \preceq [z]_\sim$.

Thus, $\preceq$ is a partial order on $A/\sim$.

Since $\alpha(x|(1|1)) = \alpha(x|0) = \alpha(1)$ from (n2), it is ontained that $[x]_\sim \preceq [1]_\sim$, for all $x \in A$. Thus, $[1]_\sim$ is the greatest element, and so, $[0]_\sim = [1|1]_\sim = [1]_\sim|_\sim[1]_\sim$ is the least element of $A/\sim$. □

Example 5. *Consider the strong Shefeer stroke NMV-algebra A in Example 1. For a fuzzy filter α of A defined by*

$$\alpha(x) = \begin{cases} 1, & \text{if } x \approx f, 1 \\ 0.001, & \text{otherwise} \end{cases}$$

$\sim_\alpha = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(1,1),(f,1),(1,f),(a,0),(0,a),$ $(c,e),(e,c),(b,d),(d,b)\}$ *is a congruence relation on A. Then* $(A/\sim_\alpha, |_{\sim_\alpha}, [1]_\sim)$ *is also a strong Sheffer stroke NMV-algebra with the following Cayley table where* $A/\sim_\alpha = \{[0]_{\sim_\alpha}, [d]_{\sim_\alpha}, [e]_{\sim_\alpha}, [1]_{\sim_\alpha}\}$*:*

TABLE 2. Cayley table of $|_{\sim_\alpha}$

$\vert_{\sim_\alpha}$	$[0]_{\sim_\alpha}$	$[d]_{\sim_\alpha}$	$[e]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$
$[0]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$
$[d]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[e]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[e]_{\sim_\alpha}$
$[e]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[d]_{\sim_\alpha}$	$[d]_{\sim_\alpha}$
$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[e]_{\sim_\alpha}$	$[d]_{\sim_\alpha}$	$[0]_{\sim_\alpha}$

Theorem 14. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $A/\sim_\alpha$ is totally ordered and $|A/\sim_\alpha| \leq 2$.*

Proof. Let α be a prime fuzzy filter of A. By Theorem 8, $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$. Then $[x]_\sim \preceq [y]_\sim$ or $[y]_\sim \preceq [x]_\sim$ which means that $A/\sim_\alpha$ is totally ordered. Also, let $|A/\sim_\alpha| > 2$. Then $[x]_{\sim_\alpha} \in A/\sim_\alpha$ such that $[0]_{\sim_\alpha} < [x]_{\sim_\alpha} < [1]_{\sim_\alpha}$. Since α is a prime fuzzy filter of A, we have $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$. Assume that $\alpha(x|1) = \alpha(1)$. Since $\alpha(x|(0|1)) = \alpha(x|1) = \alpha(1)$ and $\alpha(0|(x|1)) = \alpha(1)$ from (n2), it follows that $[x]_{\sim_\alpha} = [0]_{\sim_\alpha}$ which is a contradiction. So, $|A/\sim_\alpha| \leq 2$.

Conversely, let $A/\sim_\alpha$ be totally ordered. Then $[x]_\sim \preceq [y]_\sim$ or $[y]_\sim \preceq [x]_\sim$, for all $x, y \in A$. Since $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$, it is obtained from Theorem 8 that α is a prime fuzzy filter of A. □

Theorem 15. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras, $h : A \longrightarrow B$ be an epimorphism and α be a fuzzy filter of B. Then $\alpha \circ h$ is a fuzzy filter of A and $A/\sim_{\alpha\circ h} \cong B/\sim_\alpha$.*

Proof. Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras, $h : A \longrightarrow B$ be an epimorphism and α be a fuzzy filter of B. It is first shown that $\alpha \circ h$ is a fuzzy filter of A.

- $\alpha \circ h(x) = \alpha(h(x)) \leq \alpha(1_B) = \alpha(h(1_A)) = \alpha \circ h(1_A)$ and
- $$\begin{aligned} \min\{\alpha \circ h(x), \alpha \circ h(x|_A(y|_A 1_A))\} &= \min\{\alpha(h(x)), \alpha(h(x|_A(y|_A 1_A)))\} \\ &= \min\{\alpha(h(x)), \alpha(h(x)|_B(h(y)|_B h(1_A)))\} \\ &= \min\{\alpha(h(x)), \alpha(h(x)|_B(h(y)|_B 1_B))\} \\ &\leq \alpha(h(y)) \\ &= \alpha \circ h(y), \end{aligned}$$

for all $x, y \in A$.

$A/\sim_{\alpha\circ h}$ and $B/\sim_\alpha$ are strong Sheffer stroke NMV-algebras by Theorem 13. Let $f : A/\sim_{\alpha\circ h} \longrightarrow B/\sim_\alpha$ be defined by $f([x]_{\sim_{\alpha\circ h}}) = [h(x)]_{\sim_\alpha}$, for all $x \in A$.

- f is well-defined and one-to-one: Let $[x]_{\sim_{\alpha\circ h}}, [y]_{\sim_{\alpha\circ h}} \in A/\sim_{\alpha\circ h}$. Then

$$\begin{aligned} [x]_{\sim_{\alpha\circ h}} = [y]_{\sim_{\alpha\circ h}} &\Leftrightarrow x \sim_{\alpha\circ h} y \\ &\Leftrightarrow \alpha \circ h(x|_A(y|_A 1_A)) = \alpha \circ h(1_A) = \alpha \circ h(y|_A(x|_A 1_A)) \\ &\Leftrightarrow \alpha(h(x)|_B(h(y)|_b h(1_A))) = \alpha(h(1_A)) \\ &\quad = \alpha(h(y)|_B(h(x)|_b h(1_A))) \\ &\Leftrightarrow \alpha(h(x)|_B(h(y)|_b 1_B)) = \alpha(1_B) = \alpha(h(y)|_B(h(x)|_b 1_B)) \\ &\Leftrightarrow h(x) \sim_\alpha h(y) \\ &\Leftrightarrow [h(x)]_{\sim_\alpha} = [h(y)]_{\sim_\alpha} \\ &\Leftrightarrow f([x]_{\sim_{\alpha\circ h}}) = f([y]_{\sim_{\alpha\circ h}}). \end{aligned}$$

- f is a homomorphism: Let $[x]_{\sim_{\alpha\circ h}}, [y]_{\sim_{\alpha\circ h}} \in A/\sim_{\alpha\circ h}$. Then

$$\begin{aligned} f([x]_{\sim_{\alpha\circ h}}|_{\sim_{\alpha\circ h}}[y]_{\sim_{\alpha\circ h}}) &= f([x|_A y]_{\sim_{\alpha\circ h}}) \\ &= [h(x|_A y)]_{\sim_\alpha} \\ &= [h(x)|_B h(y)]_{\sim_\alpha} \\ &= [h(x)]_{\sim_\alpha}|_{\sim_\alpha}[h(y)]_{\sim_\alpha} \\ &= f([x]_{\sim_{\alpha\circ h}})|_{\sim_\alpha} f([y]_{\sim_{\alpha\circ h}}). \end{aligned}$$

- f is onto: Let $[y]_{\sim_\alpha} \in B/\sim_\alpha$. Since h is an epimorphism, there exists $x \in A$ such that $h(x) = y$. Thus, there exists $[x]_{\sim_{\alpha\circ h}} \in A/\sim_{\alpha\circ h}$ such that $f([x]_{\sim_{\alpha\circ h}}) = [h(x)]_{\sim_\alpha} = [y]_{\sim_\alpha}$. □

Theorem 16. *The class $\mathcal{F}_A$ of all fuzzy filters of A forms a complete lattice.*

Proof. Since every fuzzy filter of A is a mapping from A to the interval $[0, 1]$ and $[0, 1]$ is a complete lattice where $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$, for all $a, b \in [0, 1]$, $\mathcal{F}_A$ forms a complete lattice. □

5. Conclusion

In present study, basic definitions and notions of a strong Sheffer stroke NMV-algebra are given. Then new properties, various filters, fuzzy filters of a strong Sheffer stroke NMV-algebra and the relationships between them are investigated. We prove that a filter of a strong Sheffer stroke NMV-algebra is prime if and only if it is not contained by another filter of this algebraic structure, and examine some features of a prime filter. Also, it is shown that the quotient structure of a strong Sheffer stroke NMV-algebra defined by a prime filter is totally ordered and it has at most 2 elements. Besides, we define a (prime) fuzzy filter of strong Sheffer stroke NMV-algebras and show that α is a (prime) fuzzy filter of a strong Sheffer stroke NMV-algebra if and only if $\alpha_a = \{x \in A : a \leq \alpha(x)\}$ is empty or a (prime) filter of A, for all $a \in [0, 1]$. It is demonstrated that a fuzzy subset α_F is a (prime) fuzzy filter of a strong Sheffer stroke NMV-algebra if and only if F is a (prime) filter of the algebra. Thus, the relationships between filters and fuzzy filters of a strong Sheffer stroke NMV-algebra are stated. We prove that a strong Sheffer stroke NMV-algebra is totally ordered if and only if every fuzzy filter is prime if and only if the filter $\{1\}$ is prime. It is shown that a fuzzy subset α_h of a strong Sheffer stroke NMV-algebra is a (prime) fuzzy filter defined by means of a (prime) fuzzy filter α and a surjective endomomorphism h on this algebra, and that $\alpha_h = \alpha$ if and only if $h(\alpha_a) = \alpha_a$ whenever h is an automorphism on this algebra and $a \in Im(\alpha)$. By describing a congruence relation on a strong Sheffer stroke NMV-algebra by a fuzzy filter, a quotient structure of a strong Sheffer stroke NMV-algebra is built via the congruence relation. Hence, it is shown that the structure forms a strong Sheffer stroke NMV-algebra. Indeed, we prove that the quotient structure defined by a prime fuzzy filter is totally ordered strong Sheffer stroke NMV-algebra and it has at most 2 elements. Moreover, we present that $\alpha \circ h$ is a fuzzy filter of A and the quotient structures defined by the fuzzy filters $\alpha \circ h$ and α are isomorphic when an epimorphism h between strong Sheffer stroke NMV-algebras A and B and a fuzzy filter α of B. Finally, it is easy to see that the class of all fuzzy filters of a strong Sheffer stroke NMV-algebra forms a complete lattice.

In the future works, we wish to investigate annihilators ans stabilizers on strong Sheffer stroke NMV-algebras.

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NONOSCILLATION AND OSCILLATION CRITERIA FOR CLASS OF HIGHER-ORDER DIFFERENCE EQUATIONS INVOLVING GENERALIZED DIFFERENCE OPERATOR

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Abstract. In this paper, sufficient conditions are obtained for nonoscillation/oscillation of all solutions of a class of higher-order difference equations involving the generalized difference operator of the form

$$\Delta_a^k(p_n\Delta_a^2 y_n) = f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1} y_n),$$

where Δ_a is generalized difference operator which is defined as $\Delta_a y_n = y_{n+1} - ay_n$, $a \neq 0$.

1. Introduction

In this paper, we study nonoscillation and oscillation of solutions of a class of higher-order difference equations of the form

$$\Delta_a^k(p_n\Delta_a^2 y_n) = f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1} y_n), \quad n \in \mathbb{N}, \tag{1}$$

where $\mathbb{N}$ is the set of natural numbers, $a \in \mathbb{R}\backslash\{0\}$, $\mathbb{R}$ is the set of real numbers, $\{p_n\}$ is a real sequence with $p_n \neq 0$ for $n \in \mathbb{N}$ and $f : \mathbb{N} \times \mathbb{R}^{k+2} \longrightarrow \mathbb{R}$. The generalized difference operator Δ_a is defined as $\Delta_a y_n = y_{n+1} - ay_n$. For $a = 1$, we write $\Delta_1 = \Delta$ where Δ is known forward difference operator. We define inductively $\Delta_a^k y_n = \Delta_a(\Delta_a^{k-1} y_n)$ for $k \geq 2$. By a solution of Eq. (1) we mean a sequence $\{y_n\}$ of real numbers which satisfies Eq. (1) identically. We consider only nontrivial solutions, i.e., such for which $\sup\{|y_n| : n \geq i\} > 0$ for every $i \in \mathbb{N}$. A solution of Eq. (1) is called non-oscillatory if it is eventually of constant sign (positive or negative) otherwise it is called oscillatory. For $a \in \mathbb{R}\backslash\{0\}$, Eq. (1) always admits a solution on $\mathbb{N}$. The oscillation and nonoscillation of solutions of difference

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equations are very popular for researchers in the last two decades. For this we refer the monograps [1,2,3]. The oscillation and nonoscillation of solutions of higher order difference equations has been studied by many authors. For example in [9], oscillation criteria are obtained for higher-order neutral-type nonlinear delay difference equations of the form

$$\Delta\left(r_n\left(\Delta^{k-1}\left(y_n+p_n y_{\tau_n}\right)\right)\right)+q_n f\left(y_{\sigma_n}\right)=0, n \geq n_0,$$

where $r_n, p_n, q_n \in [n_0, \infty)$, $r_n > 0$, $q_n > 0; 0 \leq p_n \leq p_0 < \infty$, $\lim_{n\to\infty} \tau_n = \infty$, $\lim_{n\to\infty} \sigma_n = \infty$, $\sigma_n \leq n$, σ_n is nondecreasing, $\Delta\tau_n \geq \tau_0 > 0$, $\tau_\sigma = \sigma_\tau$; $\frac{f(u)}{u} \geq m > 0$ for $u \neq 0$. In [5], Agarval et al. established some new criteria for the oscillation of higher order difference equations of the form

$$\Delta\left(\Delta^{m-1}\left(x_n\right)\right)^\alpha + q_n x^\alpha[n-\tau] = 0,$$

where $m \geq 2$, $\tau \geq 1$ and α is the ratio of positive odd integers. In [4], Agarval et al. established sufficient conditions for the oscillation of all solutions of the even order difference equations of the form

$$\Delta^m x_n + p_n \Delta^{m-1} x_n + F\left(x_{n-g}, \Delta x_{n-h}\right) = 0,\ m \text{ is even},$$

by comparing it with certain difference equations of lower order whose oscillatory character is known. In [6], some oscillation criteria for solutions of nonlinear higher-order forced difference equations are established. The investigations are carried out without assuming that the coefficients of the equations are of a definite sign and by showing that the forcing term needs not be the mth difference of an oscillatory function. In [13], Saker et al. established some new oscillation criteria for a certain class of third order nonlinear delay difference equations by employing the generalized Riccati transformation technique. In [7], sufficient conditions are established for the oscillatory and asymptotic behavior of higher–order half–linear delay difference equation of the form

$$\Delta\left(p_n\left(\Delta^{m-1}\left(x_n+q_n x_{\tau_n}\right)\right)^\alpha\right) + r_n x_{\sigma_n}^\beta = 0,\ n \geq n_0,$$

where it is assumed that $\sum_{s=n_0}^{\infty} \frac{1}{p_s{}^{\frac{1}{\alpha}}} < \infty$. In [8] Bolat et al. investigated the oscillatory behavior of solutions of the th order half-linear functional difference equations with damping term of the form

$$\Delta\left(p_n Q\left(\Delta^{m-1} y_n\right)\right) + q_n Q\left(\Delta^{m-1} y_n\right) + r_n Q\left(y_{\tau_n}\right) = 0, n \geq n_0,$$

where m is even and $Q(s) = |s|^{\alpha-2} s$, $\alpha > 1$ is a fixed real number.

The generalized difference operator Δ_a is a generalization of the difference operator Δ. Due to the relation between the ordinary difference operator Δ and generalized difference operator Δ_a, most difference equation can be considered more effectively by using generalized difference operator Δ_a. In the literature there are number of papers on the behavior of the difference equations involving operator

Δ_a. In [12], Popenda obtained sufficient conditions for nonoscillation/oscillation of solutions of a class of nonlinear nonhomogeneous second order difference equations involving generalized difference of the form

$$\Delta_a^2 x_n = F(n, x_n, \Delta_b x_n). \tag{2}$$

For some results of this type we refer the reader to the recent papers [11,14,15]. In [16], Tan and Yang generalized and improved the result of Popenda by considering the equation

$$\Delta_a(p_n \Delta_a x_n) + q_n \Delta_a x_n = F(n, x_n, \Delta_b x_n). \tag{3}$$

In [10], Parhi and Panda obtained sufficient conditions for nonoscillation /oscillation of all solutions of a class of nonlinear third order difference equations of the form

$$\Delta_a(p_n \Delta_a^2 y_n) + q_n \Delta_a^2 y_n = f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n). \tag{4}$$

Our purpose is to establish oscillation and nonoscillation criteria for a class of higher-order difference equations involving generalized difference operator of the form Eq. (1).

2. Auxiliary Lemmas

Lemma 1. *[10] Let $\{y_n\}$be a real sequence. If $\{\Delta_b y_n\}$, $b > 0$, is eventually of one sign, then $\{y_n\}$ is non-oscillatory.*

Lemma 2. *[10] For $b > 0$, a real sequence $\{y_n\}$ is oscillatory if and only if $\{\Delta_b^l y_n\}$ is oscillatory for all integers $l \geq 0$, where $\Delta_b^0 y_n \equiv y_n$*

Lemma 3. *For $n \in \mathbb{Z}$, $\Delta_a y_{n+1} = \Delta_a^2 y_n + a\Delta_a y_n$.*

Proof. By the definition of generalized difference operator, we write $\Delta_a y_n = y_{n+1} - ay_n$. Thus, If we apply the generalized difference to the both sides of this equality, we obtain that $\Delta_a^2 y_n = \Delta_a y_{n+1} - a\Delta_a y_n$. □

Lemma 4. *[10] Let $b < 0$ and $k \in \mathbb{N}$. Then $\Delta_b^k y_l = b^{l+k}\Delta^k(\frac{y_l}{b^l})$, $l \in \mathbb{N}$, for any sequence $\{y_n\}$ of real numbers.*

Lemma 5. *For $m \geq 1$, $\Delta_a^m(p_n \Delta_a^2 y_n) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} a^i p_{n+m-i} \Delta_a^2 y_{n+m-i}$.*

Proof. One can easily show it using the definition of the generalized difference operator. □

Lemma 6. *For $k \geq 1$, $\Delta_a y_l = \sum_{i=0}^{k} a^i \binom{k}{i} \Delta_a^{k+1-i} y_{l-k}$.*

Proof. From Lemma 3, we can write $\Delta_a y_{l-1} = \Delta_a^2 y_{l-2} + a\Delta_a y_{l-2}$. If we apply the generalized difference operator to the both sides of this equality, we obtain $\Delta_a^2 y_{l-1} = \Delta_a^3 y_{l-2} + a\Delta_a^2 y_{l-2}$. Also from Lemma 3, we can write $\Delta_a y_l = \Delta_a^2 y_{l-1} + a\Delta_a y_{l-1}$. Then we have

$$\Delta_a y_l = \Delta_a^3 y_{l-2} + 2a\Delta_a^2 y_{l-2} + a^2 \Delta_a y_{l-2}.$$

Thus we obtain

$$\Delta_a y_{l-1} = \Delta_a^3 y_{l-3} + 2a\Delta_a^2 y_{l-3} + a^2 \Delta_a y_{l-3}.$$

Similarly, by applying the generalized difference operator to the both sides of last equality , we obtain that

$$\Delta_a^2 y_{l-1} = \Delta_a^4 y_{l-3} + 2a\Delta_a^3 y_{l-3} + a^2 \Delta_a^2 y_{l-3}.$$

By writing $\Delta_a y_{l-1}$ and $\Delta_a^2 y_{l-1}$ in the last equation, we obtain

$$\Delta_a y_l = \Delta_a^4 y_{l-3} + 3a\Delta_a^3 y_{l-3} + 3a^2 \Delta_a^2 y_{l-3} + a^3 \Delta_a y_{l-3},$$

and so on , we reach

$$\Delta_a y_l = \sum_{i=0}^{k} a^i \binom{k}{i} \Delta_a^{k+1-i} y_{l-k}.$$

□

Lemma 7. $\Delta_a^2 y_{n+k-1} = \sum_{i=0}^{k-1} \binom{k-1}{i} a^i \Delta_a^{k+1-i} y_n$, *for* $k \geq 1$, $n \in \mathbb{N}$.

Proof. By the Lemma 3, we have

$$\Delta_a^2 y_{n+1} = \Delta_a^3 y_n + a\Delta_a^2 y_n. \tag{5}$$

From (5) we can write

$$\Delta_a^2 y_{n+2} = \Delta_a^3 y_{n+1} + a\Delta_a^2 y_{n+1}. \tag{6}$$

Applying generalized difference operator to the Equation (5), we obtain $\Delta_a^3 y_{n+1} = \Delta_a^4 y_n + a\Delta_a^3 y_n$. Hence from (5) and (6) we have

$$\Delta_a^2 y_{n+2} = \Delta_a^4 y_n + 2a\Delta_a^3 y_n + a^2 \Delta_a^2 y_n.$$

Similarly, we obtain

$$\Delta_a^2 y_{n+3} = \Delta_a^5 y_n + 3a\Delta_a^4 y_n + 3a^2 \Delta_a^3 y_n + a^3 \Delta_a^2 y_n$$

and so on we reach

$$\Delta_a^2 y_{n+k-1} = \sum_{i=0}^{k-1} \binom{k-1}{i} a^i \Delta_a^{k+1-i} y_n, \ \text{ for } k \geq 1, \ n \in \mathbb{N}. \tag{7}$$

From (7) we can write

$$\Delta_a^2 y_n = \sum_{j=0}^{k-1} \binom{k-1}{j} a^j \Delta_a^{k+1-j} y_{n-k+1}, \quad \text{for } k \geq 1, \ n \in \mathbb{N}. \tag{8}$$

The proof is completed. □

3. Nonoscillation of Solutions

In this section non-oscillatory behaviour of solutions of Eq. (1) is studied.

Theorem 1. *Let $a > 0$. Assume that*

$$\frac{\sum_{i=0}^{k} a^i \binom{k}{i} \Delta_a^{k+1-i} y_n}{p_{n+k}} \Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1} y_n) + \sum_{j=1}^{k} \binom{k}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n+k-m} \right) \Delta_a^{k+2-m} y_n \Big] \geq 0, \tag{9}$$

is satisfied. Then all solutions of Eq. (1) are non-oscillatory.

Proof. Assume that $\{y_n\}$ is a possible oscillatory solution of Eq. (1). Hence, for every $s \in \mathbb{N}$, there exists $l > s$ such that $y_l \geq 0$ and $y_{l+1} < 0$ or $y_l > 0$ and $y_{l+1} \leq 0$. Therefore, $\Delta_a y_l = y_{l+1} - a y_l < 0$. By the Lemma 5 and Lemma 7, for $n \geq l$, Eq. (1) can be written as

$$\Delta_a y_{n+1} = a \Delta_a y_n + \frac{1}{p_n} \Big[f(n-k, y_{n-k}, ..., \Delta_a^{k+1} y_{n-k}) + \sum_{j=1}^{k} \binom{k}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n-m} \right) \Delta_a^{k+2-m} y_{n-k} \Big]. \tag{10}$$

Multiplying (10) by $\Delta_a y_l$ and considering (9) we have

$$\Delta_a y_l \Delta_a y_{l+1} = a(\Delta_a y_l)^2 + \frac{\Delta_a y_l}{p_l} \Big[f(l-k, y_{l-k}, ..., \Delta_a^{k+1} y_{l-k}) + \sum_{j=1}^{k} \binom{k}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{l-m} \right) \Delta_a^{k+2-m} y_{l-k} \Big] > 0$$

Hence $\Delta_a y_{l+1} < 0$, since $\Delta_a y_l = \sum_{i=0}^{k} a^i \binom{k}{i} \Delta_a^{k+1-i} y_{l-k} < 0$. Putting $n = l+1$ in (10) and proceeding as above, we obtain $\Delta_a y_{l+1} \Delta_a y_{l+2} > 0$. Hence $\Delta_a y_{l+2} < 0$. Generally, we see that $\Delta_a y_{l+t} < 0$ for $t \in \mathbb{N}$. That is, $\Delta_a y_{l+t}$ is eventually of one sign. From Lemma 1 it follows that $\{y_n\}$ is eventually non-oscillatory. This is a contradiction to our assumption. Thus the theorem is proved.

□

Theorem 2. *Let $a > 0$. Assume that*

$$\frac{1}{p_{n+k}} \Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1} y_n)$$

$$+\sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\Bigg]\le 0, \tag{11}$$

is satisfied. Then all solutions of Eq. (1) are non-oscillatory.

Proof. Let $\{y_n\}$ be a solution of Eq. (1). We claim that it is non-oscillatory. If not, then $\{y_n\}$ is oscillatory. Hence, for every $s \in \mathbb{N}$, there exists $l > s$ such that $y_l \ge 0$ and $y_{l+1} < 0$ or $y_l > 0$ and $y_{l+1} \le 0$. Therefore, $\Delta_a y_l = y_{l+1} - ay_l < 0$. For $n \ge l$, we can write Eq. (1) as in (10). Considering (10) and $\Delta_a y_l < 0$, we have

$$\begin{aligned}\Delta_a y_{l+1} &= a\Delta_a y_l + \frac{1}{p_l}\Big[f(l-k, y_{l-k}, ..., \Delta_a^{k+1}y_{l-k}) \\ &\quad +\sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{l-m}\right)\Delta_a^{k+2-m}y_{l-k}\Bigg] \\ &< 0.\end{aligned}$$

Putting $n = l+1$ in (10) and by (11) we obtain $\Delta_a y_{l+2} < 0$. By similar processes, we reach that $\Delta_a y_{l+s} < 0$ for $s \in \mathbb{N}$. Hence $\{y_n\}$ is eventually non-oscillatory by Lemma 1. This contradiction completes the proof. □

Theorem 3. *Let $a > 0$. Assume that*

$$\begin{aligned}&\frac{1}{p_{n+k}}\Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1}y_n) \\ &+ \sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n \\ &\ge 0,\end{aligned}$$

is satisfied. Then all solutions of Eq. (1) are non-oscillatory.

Proof. Assume that $\{y_n\}$ is an oscillatory solution of Eq. (1). So we choose $n > n_0$, where $n_0 \in \mathbb{N}$, such that $y_n \le 0$ and $y_{n+1} > 0$ or $y_n < 0$ and $y_{n+1} \ge 0$. Thus $\Delta_a y_n = y_{n+1} - ay_n > 0$. The rest of proof can be made. □

Theorem 4. *Let $a > 0$. Assume that*

$$\begin{cases} f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1}y_n) = 0, & \text{if } \Delta_a^2 y_n = 0 \\ \frac{\sum_{j=0}^{k-1}\binom{k-1}{j}a^j\Delta_a^{k+1-j}y_n}{p_{n+k}}\Big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1}y_n) + \sum_{j=1}^{k}\binom{k}{j}a^j & \\ \times\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\Big] > 0, & \text{if } \Delta_a^2 y_n \ne 0 \end{cases} \tag{12}$$

is satisfied. Then all solutions of Eq. (1) are non-oscillatory.

Proof. Let X is the set of all solutions $y = \{y_n\}$ of Eq. (1). Assume that $X_1 = \{y \in X : \Delta_a^2 y_n = 0 \text{ for some } n \in \mathbb{N}\}$ and $X_2 = X - X_1$. Suppose that $y = \{y_n\}$ is a solution of Eq. (1). If $y \in X_1$, then there exists $t \in \mathbb{N}$ such that $\Delta_a^2 y_t = 0$. From the first part of assumption (12) it follows that $f(t, y_t, \Delta_a y_t, \Delta_a^2 y_t, ..., \Delta_a^{k+1} y_t) = 0$. Thus from Eq. (1) we obtain

$$\Delta_a^k (p_t \Delta_a^2 y_t) = 0.$$

Hence, by from Lemma 5, we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i p_{t+k-i} \Delta_a^2 y_{t+k-i} = 0.$$

In here we know that $p_{t+k}\Delta_a^2 y_{t+k} = \binom{k}{1} a p_{t+k-1} \Delta_a^2 y_{t+k-1} - \binom{k}{2} a^2 p_{t+k-2} \Delta_a^2 y_{t+k-2} + ... - (-1)^k a^k p_t \Delta_a^2 y_t$. If $\Delta_a^2 y_t = 0$, $\Delta_a y_{t+1} = a\Delta_a y_t$. Thus, If we apply the generalized difference operator to both sides of the last equality, we obtain that $\Delta_a^2 y_{t+1} = a\Delta_a^2 y_t$. Since $\Delta_a^2 y_t = 0$, $\Delta_a^2 y_{t+1} = 0$. Since $\Delta_a^2 y_{t+1} = 0$, $\Delta_a y_{t+2} = a\Delta_a y_{t+1}$. Likewise if we apply the generalized difference operator to both sides of the last equality , we obtain that $\Delta_a^2 y_{t+2} = a\Delta_a^2 y_{t+1}$. Since $\Delta_a^2 y_{t+1} = 0$, $\Delta_a^2 y_{t+2} = 0$. Continuing the progress in the same way yields $p_{t+k}\Delta_a^2 y_{t+k} = 0$, that is, $\Delta_a^2 y_{t+k} = 0$. Writing $t + k$ instead of n in Eq. (1) and using the first part of (15), we obtain $\Delta_a^k (p_{t+k} \Delta_a^2 y_{t+k}) = 0$, that is, $\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i p_{t+2k-i} \Delta_a^2 y_{t+2k-i} = 0$. If $\Delta_a^2 y_{t+k} = 0$, $\Delta_a y_{t+k+1} = a\Delta_a y_{t+k}$. If we apply the generalized difference operator to both sides of the last equality, we obtain that $\Delta_a^2 y_{t+k+1} = a\Delta_a^2 y_{t+k}$. Thus $\Delta_a^2 y_{t+k+1} = 0$. Continuing the progress in the same way for the first part of (12) yields $\Delta_a^2 y_{t+s} = 0$ for $s \in \mathbb{N}$. We may observe that $\Delta_a^2 y_{t+1} = 0$ implies $\Delta_a y_{t+2} = a\Delta_a y_{t+1}$ and $\Delta_a^2 y_{t+2} = 0$ implies $\Delta_a y_{t+3} = a\Delta_a y_{t+2} = a^2 \Delta_a y_{t+1}$. In general case, we obtain

$$\Delta_a y_{t+l} = a^{l-1} \Delta_a y_{t+1},\ l \in \mathbb{N}. \tag{13}$$

If $\Delta_a y_{t+1} = 0$, then $\Delta_a y_{t+l} = 0$ for $l \in \mathbb{N}$. Hence

$$y_{t+l+1} = a y_{t+l},\ l \in \mathbb{N}. \tag{14}$$

Since the solution $\{y_n\}$ of Eq. (1) is nontrivial, we can find $n_0 \in \mathbb{N}$, $n_0 \geq t+1$, such that $y_{n_0} \neq 0$. Putting $l = n_0 - t,\ n_0 - t + 1, ...$ in (13) we get $y_{n_0+1} = a y_{n_0}$, $y_{n_0+2} = a y_{n_0+1} = a^2 y_{n_0}$, etc. In general, $y_{n_0+s} = a^s y_{n_0}$, $s \in \mathbb{N}$. Hence $\{y_n\}$ is eventually of one sign, that is, $\{y_n\}$ is non-oscillatory. From (13) it follows that since $\Delta_a y_{t+1} > 0$ or < 0, $\Delta_a y_{t+l} > 0$ or < 0 for $l \in \mathbb{N}$. Hence $\{\Delta_a y_n\}$ is eventually of one sign. Thus $\{y_n\}$ is eventually of one sign by Lemma 1. Consequently, $\{y_n\}$ is non-oscillatory.

Now let $y \in X_2$. Then $\Delta_a^2 y_n \neq 0$ for all $n \in \mathbb{N}$. Eq. (1) can be written in the form

$$\Delta_a^2 y_{n+k} = \frac{1}{p_{n+k}} \left[f(n, y_n, ..., \Delta_a^{k+1} y_n) \right.$$

$$+\sum_{j=1}^{k}\binom{k}{j}a^{j}\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\Bigg]. \tag{15}$$

Putting $n=l-k+1$ in (15) for a fixed l and multiplying (15) by $\Delta_a^2 y_l$, we obtain

$$\begin{aligned}\Delta_a^2 y_l\Delta_a^2 y_{l+1} &= \frac{\Delta_a^2 y_l}{p_{l+1}}\Big[f(l-k+1,y_{l-k+1},...,\Delta_a^{k+1}y_{l-k+1}) \\ &\quad +\sum_{j=1}^{k}\binom{k}{j}a^{j}\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{l-m+1}\right)\Delta_a^{k+2-m}y_{l-k+1}\Bigg] \\ &> 0\end{aligned}$$

by the second part of the assumption (12). Since $\Delta_a^2 y_l\neq 0$, $\Delta_a^2 y_l>0$ or <0, also $\Delta_a^2 y_{l+1}>0$ or <0. Putting $n=l-k+2$ in (15) and considering the second part of (12), we have $\Delta_a^2 y_{l+2}\Delta_a^2 y_{l+1}>0$. Therefore since $\Delta_a^2 y_l>0$ or <0, $\Delta_a^2 y_{l+2}>0$ or <0. The repeated considering of the second part of (12), we yield $\Delta_a^2 y_{l+k}>0$ or <0 for $k\in\mathbb{N}$. Hence from (8) we have $\Delta_a^2 y_l=\sum_{j=0}^{k-1}\binom{k-1}{j}a^j\Delta_a^{k+1-j}y_{l-k+1}>0$ or <0.Thus $\{\Delta_a^2 y_n\}$ is non-oscillatory. From Lemma 2 it follows that $\{y_n\}$ is non-oscillatory. Thus the theorem is proved. □

4. Oscillation of Solutions

In this section, we study oscillatory behavior of all solutions of Eq. (1).

Theorem 5. *Let $a<0$. Assume that*

$$\frac{\sum_{i=0}^{k}a^i\binom{k}{i}\Delta_a^{k+1-i}y_n}{p_{n+k}}\Big[f(n,y_n,...,\Delta_a^{k+1}y_n)$$
$$+\sum_{j=1}^{k}\binom{k}{j}a^{j}\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\Bigg]\leq 0, \tag{16}$$

is satisfied. Then all solutions of Eq. (1) are oscillatory.

Proof. Let $\{y_n\}$ be a solution of Eq. (1). If $\Delta_a y_n=0$, then $y_{n+1}=ay_n$. Hence $\{y_n\}$ is oscillatory because of $a<0$. Suppose that $\Delta_a y_n\neq 0$. If we write Eq. (1) as in (10) and multiply both of this equality $\Delta_a y_n=\sum_{i=0}^{k}a^i\binom{k}{i}\Delta_a^{k+1-i}y_{n-k}$ for $\Delta_a y_n\neq 0$, we have

$$\begin{aligned}\Delta_a y_n\Delta_a y_{n+1} &= a(\Delta_a y_n)^2+\frac{\Delta_a y_n}{p_n}\Big[f(n-k,y_{n-k},...,\Delta_a^{k+1}y_{n-k}) \\ &\quad +\sum_{j=1}^{k}\binom{k}{j}a^{j}\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n-m}\right)\Delta_a^{k+2-m}y_{n-k}\Bigg] \\ &< 0.\end{aligned}$$

Hence (16) holds. By the Lemma 4 we have $a^{n+1}\Delta\left(\frac{y_n}{a^n}\right)a^{n+2}\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right) < 0$, that is, $a^{2n+3}\Delta\left(\frac{y_n}{a^n}\right)\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right) < 0$. Since $a < 0$, then

$$\Delta\left(\frac{y_n}{a^n}\right)\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right) > 0, n \in \mathbb{N}. \tag{17}$$

If $\Delta\left(\frac{y_n}{a^n}\right) > 0$, then $\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right) > 0$. As (17) holds for every $n \in \mathbb{N}$, then $\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right) > 0$ implies that $\Delta\left(\frac{y_{n+2}}{a^{n+2}}\right) > 0$ and so on. Hence $\{\Delta\left(\frac{y_n}{a^n}\right)\}$ is eventually of one sign. Consequently, $\{\frac{y_n}{a^n}\}$ is eventually of one sign by Lemma 1 for $b = 1$. This implies that $\{y_n\}$ is oscillatory because $a < 0$. Similarly, if $\Delta\left(\frac{y_n}{a^n}\right) < 0$, then $\{y_n\}$ is oscillatory. Thus the theorem is proved. □

Remark 1. *If*

$$\frac{1}{p_{n+k}}\left[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1} y_n)\right.$$
$$\left. + \sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\right] = 0, \tag{18}$$

then all solutions of Eq. (1) are oscillatory. Indeed, if $\{y_n\}$ is a non-oscillatory solution of Eq. (1), then there exists $k_o \in \mathbb{N}$ such that $y_n > 0$ or < 0 for $n \geq k_0$. Eq. (1) can be written in the form

$$\begin{aligned}\Delta_a^2 y_{n+k} &= \frac{1}{p_{n+k}}\left[f(n, y_n, ..., \Delta_a^{k+1} y_n)\right.\\ &\left.+ \sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\right].\end{aligned}$$

Then considering (18), we have $\Delta_a^2 y_{n+k} = 0$, $n \in \mathbb{N}$. Then for $k \geq 1$, $\Delta_a^2 y_{n+1} = 0$ implies that $\Delta_a y_{n+2} = a\Delta_a y_{n+1}$. Similarly, $\Delta_a^2 y_{n+2} = 0$ implies that $\Delta_a y_{n+3} = a\Delta_a y_{n+2} = a^2\Delta_a y_{n+1}$. In general case, $\Delta_a^2 y_{n+k} = 0$ implies that $\Delta_a y_{n+k+1} = a^k\Delta_a y_{n+1}$, $k \in \mathbb{N}$. In particular, $\Delta_a y_{k_0+k+1} = a^k\Delta_a y_{k_0+1}$ for $n \geq k_0$. Let $y_n > 0$ for $n \geq k_0$. We consider three possibilities for $\Delta_a y_{k_0+1}$, viz., $\Delta_a y_{k_0+1} = 0$, > 0 and < 0 and obtain a contradiction in each case. If $\Delta_a y_{k_0+1} = 0$, then $\Delta_a y_{k_0+k+1} = 0$, that is, $y_{k_0+k+2} = ay_{k_0+k+1} < 0$ for $k \in \mathbb{N}$, a contradiction to the fact that $y_n > 0$ for $n \geq k_0$. Let $\Delta_a y_{k_0+1} > 0$. Then $\Delta_a y_{k_0+2k+2} = a^{2k+1}\Delta_a y_{k_0+1} < 0$ implies that $y_{k_0+2k+3} = ay_{k_0+2k+2} < 0$, a contradiction. If $\Delta_a y_{k_0+1} < 0$, then $\Delta_a y_{k_0+2k+1} = a^{2k}\ \Delta_a y_{k_0+1} < 0$ implies that $y_{k_0+2k+2} < ay_{k_0+2k+1} < 0$, a contradiction. Thus $y_n > 0$ for $n \geq k_0$ is not possible. Let $y_n < 0$ for $n \geq k_0$. Proceeding as above we arrive at a contradiction in each of the three cases, viz., $\Delta_a y_{k_0+1} = 0$, > 0 and < 0. Hence $y_n < 0$ for $n \geq k_0$ is not possible. Thus $\{y_n\}$ is oscillatory.

Theorem 6. *Let $a < 0$. Assume that*

$$\begin{cases} f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1} y_n) = 0, & \text{if } \Delta_a^2 y_n = 0 \\ \frac{\sum_{j=0}^{k-1} \binom{k-1}{j} a^j \Delta_a^{k+1-j} y_n}{p_{n+k}} \Big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1} y_n) + \sum_{j=1}^{k} \binom{k}{j} a^j & \\ \quad \times \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n+k-m} \right) \Delta_a^{k+2-m} y_n \Big] < 0, & \text{if } \Delta_a^2 y_n \neq 0 \end{cases} \tag{19}$$

is satisfied. Then all solutions of Eq. (1) are oscillatory.

Proof. Let X be the set of all solutions $y = \{y_n\}$ of Eq. (1). Assume that $X_1 = \{y \in X : \Delta_a^2 y_n = 0 \text{ for some } n \in \mathbb{N}\}$ and $X_2 = X - X_1$. Suppose that $y = \{y_n\}$ be a non-oscillatory solution of Eq. (1). Hence $\{y_n\}$ is eventually of one sign. If $y \in X_1$, then there exists $t \in \mathbb{N}$ such that $\Delta_a^2 y_t = 0$. Thus from Eq. (1) and (19) it follows that $\Delta_a^k (p_t \Delta_a^2 y_t) = 0$, that is, $\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i p_{t+k-i} \Delta_a^2 y_{t+k-i} = 0$. Then

$$p_{t+k} \Delta_a^2 y_{t+k} = \binom{k}{1} a p_{t+k-1} \Delta_a^2 y_{t+k-1} - \binom{k}{2} a^2 p_{t+k-2} \Delta_a^2 y_{t+k-2} + ... - (-1)^k a^k p_t \Delta_a^2 y_t.$$

If $\Delta_a^2 y_t = 0$, then $\Delta_a y_{t+1} = a \Delta_a y_t$. If we apply the generalized difference operator to both sides of the last equality, we obtain that $\Delta_a^2 y_{t+1} = a \Delta_a^2 y_t$. Since $\Delta_a^2 y_t = 0$, $\Delta_a^2 y_{t+1} = 0$. Since $\Delta_a^2 y_{t+1} = 0$, $\Delta_a y_{t+2} = a \Delta_a y_{t+1}$. Likewise , if we apply the generalized difference operator to both sides of the last equality, we obtain that $\Delta_a^2 y_{t+2} = a \Delta_a^2 y_{t+1}$. Since $\Delta_a^2 y_{t+1} = 0$, $\Delta_a^2 y_{t+2} = 0$. By recurrence of the processes ,we obtain that $p_{t+k} \Delta_a^2 y_{t+k} = 0$, that is, $\Delta_a^2 y_{t+k} = 0$. If $\Delta_a^2 y_{t+1} = 0$, for $k \geq 1$, $\Delta_a y_{t+2} = a \Delta_a y_{t+1}$. Since $\Delta_a^2 y_{t+2} = 0$, $\Delta_a y_{t+3} = a \Delta_a y_{t+2} = a^2 \Delta_a y_{t+1}$ and so on. Generally, we have $\Delta_a y_{t+k} = a^{k-1} \Delta_a y_{t+1}$. We can choose $k_0 \in \mathbb{N}$ such that $y_k > 0$ or < 0 for $k \geq k_0$. Let $y_k > 0$ for $k \geq k_0$. If $\Delta_a y_{t+1} = 0$, then $\Delta_a y_{t+k_0} = 0$ and hence $y_{t+k_0+1} = a y_{t+k_0} < 0$, a contradiction. If $\Delta_a y_{t+1} > 0$, then $\Delta_a y_{t+2k_0} = a^{2k_0-1} \Delta_a y_{t+1} < 0$ and hence $y_{t+2k_0+1} = a y_{t+2k_0} < 0$, a contradiction. If $\Delta_a y_{t+1} < 0$, then $\Delta_a y_{t+2k_0+1} = a^{2k_0} \Delta_a y_{t+1} < 0$ implies that $y_{t+2k_0+2} = a y_{t+2k_0+1} < 0$, a contradiction. Similar contradiction is obtained if $y_k < 0$ for $k \geq k_0$. Thus $y \notin X_1$. Now let $y \in X_2$. Hence $\Delta_a^2 y_n \neq 0$ for all $n \in \mathbb{N}$. Writing Eq. (1) as we obtain

$$\begin{aligned} \Delta_a^2 y_n \Delta_a^2 y_{n+1} &= \frac{\Delta_a^2 y_n}{p_{n+1}} \Big[f(n-k+1, y_{n-k+1}, \Delta_a y_{n-k+1}, ..., \Delta_a^{k+1} y_{n-k+1}) \\ &\quad + \sum_{j=1}^{k} \binom{k}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n-m+1} \right) \Delta_a^{k+2-m} y_{n-k+1} \Big] \\ &< 0, \end{aligned}$$

by the second of (19). In here $\Delta_a^2 y_n = \sum_{j=0}^{k-1} \binom{k-1}{j} a^j \Delta_a^{k+1-j} y_{n-k+1}$. Applying Lemma 4 we get $a^{2n+5} \Delta^2 \left(\frac{y_n}{a^n}\right) \Delta^2 \left(\frac{y_{n+1}}{a^{n+1}}\right) < 0$. Hence $\Delta^2 \left(\frac{y_n}{a^n}\right) \Delta^2 \left(\frac{y_{n+1}}{a^{n+1}}\right) > 0$, $n \in \mathbb{N}$, since $a < 0$. If $\Delta^2 \left(\frac{y_n}{a^n}\right) > 0$, then $\Delta^2 \left(\frac{y_{n+1}}{a^{n+1}}\right) > 0$. This in turn implies that $\Delta^2\left(\frac{y_{n+2}}{a^{n+2}}\right) > 0$ and so on. If $\Delta^2\left(\frac{y_n}{a^n}\right) < 0$, then $\Delta^2\left(\frac{y_{n+1}}{a^{n+1}}\right) < 0$ which in turn implies that $\Delta^2\left(\frac{y_{n+2}}{a^{n+2}}\right) < 0$ and so on. Therefore $\{\Delta^2\left(\frac{y_n}{a^n}\right)\}$ is of one sign. By Lemma 1, $\{\Delta\left(\frac{y_n}{a^n}\right)\}$ is eventually of one sign and hence $\{\frac{y_n}{a^n}\}$ is eventually of one sign . Consequently $\{y_n\}$ is oscillatory. This contradicts our assumption $y = \{y_n\}$ be a non-oscillatory solution of Eq. (1) . Thus $y \notin X_2$. Consequently, all solutions of Eq. (1) are oscillatory and this completes the proof of the theorem. □

5. Examples

Example 1. *Consider*

$$4\Delta_a^4 y_n = (1-8a)\Delta_a^3 y_n + 2a(1-2a)\Delta_a^2 y_n + a^2\Delta_a y_n, \tag{20}$$

where $a > 0$, $p_n = 4$, $k = 2$ *and* $f(n, y_n, \Delta_a y_n, \Delta^2{}_a y_n, \Delta_a^3 y_n) = (1-8a)\Delta_a^3 y_n + 2a(1-2a)\Delta_a^2 y_n + a^2\Delta_a y_n$. *Since*

$$\begin{aligned}
&\frac{\sum_{i=0}^{2} a^i\binom{2}{i}\Delta_a^{2+1-i} y_n}{p_{n+2}}\left[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{2+1} y_n)\right.\\
&\left.+\sum_{j=1}^{2}\binom{2}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+2-m}\right)\Delta_a^{2+2-m} y_n\right]\\
= &\frac{\Delta_a^3 y_n + 2a\Delta_a^2 y_n + a^2\Delta_a y_n}{p_{n+2}}\left[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n)\right.\\
&\left.+2ap_{n+1}\Delta_a^3 y_n + a^2(2p_{n+1} - p_n)\Delta_a^2 y_n\right]\\
= &\frac{\Delta_a^3 y_n + 2a\Delta_a^2 y_n + a^2\Delta_a y_n}{4}\left[(1-8a)\Delta_a^3 y_n + 2a(1-2a)\Delta_a^2 y_n + a^2\Delta_a y_n\right.\\
&\left.+4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n\right]\\
= &\frac{(\Delta_a^3 y_n + 2a\Delta_a^2 y_n + a^2\Delta_a y_n)^2}{4}\\
\geq &\ 0,
\end{aligned}$$

all solutions of (20) are non-oscillatory by Theorem 1. In other way, Equation (20) can be written as

$$4y_{n+4} + (-1-8a)y_{n+3} + (4a^2+a)y_{n+2} = 0.$$

The characteristic equation concerning with this equation is given by

$$4\lambda^4 + (-1-8a)\lambda^3 + (4a^2+a)\lambda^2 = 0,$$

that is,

$$(\lambda - a)(4\lambda^3 + (-1-4a)\lambda^2) = 0.$$

A fundamental set of all solutions of (20) equation is $\left\{\{a^n\}, \{(\frac{1+4a}{4})^n\}\right\}$. Thus we again see that all solutions of (20) are non-oscillatory.

Example 2. *Consider the equation*

$$-2\Delta^5 y_n = 6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n + (\Delta y_n)^2, \tag{21}$$

where $a = 1$, $p_n = -2$, $k = 3$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n, \Delta_a^4 y_n) = 6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n + (\Delta y_n)^2$. *Thus*

$$\begin{aligned}
&\frac{1}{p_{n+3}}\Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{3+1} y_n) \\
&+\sum_{j=1}^{3}\binom{3}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+3-m}\right)\Delta_a^{3+2-m}y_n\Bigg] \\
= \quad &\frac{1}{p_{n+3}}\big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n, \Delta_a^4 y_n) \\
&+3ap_{n+2}\Delta_a^4 y_n + 3a^2(2p_{n+2} - p_{n+1})\Delta_a^3 y_n + a^3(3p_{n+2} - 3p_{n+1} + p_n)\Delta_a^2 y_n\big] \\
= \quad &\frac{1}{-2}\big[6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n + (\Delta y_n)^2 \\
&-6\Delta^4 y_n - 6\Delta^3 y_n - 2\Delta^2 y_n\big] \\
= \quad &-\frac{(\Delta y_n)^2}{2} \leq 0
\end{aligned}$$

and the condition of Theorem 2 is satisfied. Hence it follows that all solutions of (21) are non-oscillatory. In particular, $y_n \equiv c$, *where* $c \neq 0$ *is a constant, is a non-oscillatory solution of the equation.*

Example 3. *Consider*

$$-2\Delta^5 y_n = 6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n - (\Delta y_n)^2, \tag{22}$$

where $a = 1$, $p_n = -2$, $k = 3$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n, \Delta_a^4 y_n) = 6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n - (\Delta y_n)^2$. *Thus*

$$\begin{aligned}
&\frac{1}{p_{n+3}}\Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{3+1} y_n) \\
&+\sum_{j=1}^{3}\binom{3}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+3-m}\right)\Delta_a^{3+2-m}y_n\Bigg] \\
= \quad &\frac{1}{p_{n+3}}\big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n, \Delta_a^4 y_n) \\
&+3ap_{n+2}\Delta_a^4 y_n + 3a^2(2p_{n+2} - p_{n+1})\Delta_a^3 y_n + a^3(3p_{n+2} - 3p_{n+1} + p_n)\Delta_a^2 y_n\big] \\
= \quad &\frac{1}{-2}\big[6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n - (\Delta y_n)^2 \\
&-6\Delta^4 y_n + -6\Delta^3 y_n + -2\Delta^2 y_n\big] \\
= \quad &\frac{(\Delta y_n)^2}{2} \geq 0.
\end{aligned}$$

Then all solutions of the equation (22) are non-oscillatory due to Theorem 3.

Example 4. *Consider*

$$3\Delta_a^3 y_n = 2\Delta_a^2 y_n, \tag{23}$$

where $a > 0$, $p_n = 3, k = 1$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) = 2\Delta_a^2 y_n$. $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) = 0$ *if* $\Delta_a^2 y_n = 0$, *and if* $\Delta_a^2 y_n \neq 0$,

$$\begin{aligned}
&\frac{\sum_{j=0}^{1-1}\binom{1-1}{j} a^j \Delta_a^{1+1-j} y_n}{p_{n+1}} \left[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{1+1} y_n) + \sum_{j=1}^{1} \binom{1}{j} a^j \right.\\
&\left. \times \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n+1-m} \right) \Delta_a^{1+2-m} y_n \right]\\
= \;& \frac{\Delta_a^2 y_n}{p_{n+1}} \left[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) + a p_n \Delta_a^2 y_n \right]\\
= \;& \frac{\Delta_a^2 y_n}{3} \left[2\Delta_a^2 y_n + 3a\Delta_a^2 y_n \right]\\
= \;& \frac{(\Delta_a^2 y_n)^2 (2 + 3a)}{3} > 0,
\end{aligned}$$

Therefore all solution of (23) are non-oscillatory by Theorem 4. We can make the proof by the another way. For this, we can write the Eq. (23) as in the form

$$3y_{n+3} - (9a + 2)y_{n+2} + (9a^2 + 4a)y_{n+1} - (2a^2 + 3a^3)y_n = 0.$$

The characteristic equation concerning with this equation is

$$3\lambda^3 - (9a + 2)\lambda^2 + (9a^2 + 4a)\lambda - (2a^2 + 3a^3) = 0,$$

that is,

$$(\lambda - a)(3\lambda^2 - (6a + 2)\lambda + 3a^2 + 2a) = 0.$$

Hence a fundamental set of all solutions of Eq. (23) is $\left\{ \{a^n\}, \{na^n\}, \{\left(\frac{3a+2}{3}\right)^n\} \right\}$. *Thus all solutions of (23) are non-oscillatory.*

Example 5. *Consider*

$$\Delta_a^3 y_n = -(1 + a)\Delta_a^2 y_n - a\Delta_a y_n, \tag{24}$$

where $a < 0$, $p_n = 1$, $k = 1$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) = -(1 + a)\Delta_a^2 y_n - a\Delta_a y_n$. *Since*

$$\begin{aligned}
&\frac{\sum_{i=0}^{1} a^i \binom{1}{i} \Delta_a^{1+1-i} y_n}{p_{n+1}} \left[f(n, y_n, ..., \Delta_a^{1+1} y_n) \right.\\
&\left. + \sum_{j=1}^{1} \binom{1}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n+1-m} \right) \Delta_a^{1+2-m} y_n \right]\\
= \;& \frac{\Delta_a^2 y_n + a\Delta_a y_n}{p_{n+1}} \left[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) + a p_n \Delta_a^2 y_n \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\Delta_a^2 y_n + a\Delta_a y_n}{1}\left[-(1+a)\Delta_a^2 y_n - a\Delta_a y_n + a\Delta_a^2 y_n\right] \\
&= -(\Delta_a^2 y_n + a\Delta_a y_n)^2 \\
&\leq 0,
\end{aligned}$$

all solutions of the equation are oscillatory by Theorem 5. In particular, a fundamental set of all solutions of Eq. (24) is $\{\{(a^n)\},\{(a-1)^n\}\}$. *Thus all of solutions of (24) are oscillatory.*

Example 6. *Consider*

$$2\Delta_a^4 y_n = -(4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n), \tag{25}$$

where $a<0$, $p_n = 2$, $k=2$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n) = -(4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n)$. *Since*

$$\begin{aligned}
&\frac{1}{p_{n+2}}\Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{2+1} y_n) \\
&\quad + \sum_{j=1}^{2}\binom{2}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+2-m}\right)\Delta_a^{2+2-m}y_n\Big] \\
&= \frac{1}{p_{n+2}}\left[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n) + 2ap_{n+1}\Delta_a^3 y_n + a^2(2p_{n+1}-p_n)\Delta_a^2 y_n\right] \\
&= \frac{1}{2}\left[-(4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n) + 4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n\right] \\
&= 0,
\end{aligned}$$

all solutions of the equation (25) are oscillatory in view of Remark 1. In particular, $\{a^n\}$ *and* $\{na^n\}$ *are two oscillatory solutions of the equation.*

Example 7. *Consider*

$$3\Delta_a^3 y_n = -2\Delta_a^2 y_n, \tag{26}$$

where $a<0$, $k=1$, $p_n=3$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) = -2\Delta_a^2 y_n$. *Hence* $\Delta_a^2 y_n = 0$ *implies that* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) = 0$. *If* $\Delta_a^2 y_n \neq 0$, *then*

$$\begin{aligned}
&\frac{\sum_{j=0}^{1-1}\binom{1-1}{j}a^j\Delta_a^{1+1-j}y_n}{p_{n+k}}\Bigg[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{1+1} y_n) + \sum_{j=1}^{1}\binom{1}{j}a^j \\
&\quad \times\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+1-m}\right)\Delta_a^{1+2-m}y_n\Bigg] \\
&= \frac{\Delta_a^2 y_n}{p_{n+1}}\left[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) + ap_n\Delta_a^2 y_n\right] \\
&= \frac{\Delta_a^2 y_n}{3}\left[-2\Delta_a^2 y_n + 3a\Delta_a^2 y_n\right]
\end{aligned}$$

$$\begin{aligned} &= (\Delta_a^2 y_n)^2 \left(\frac{-2+3a}{3}\right) \\ &< 0. \end{aligned}$$

Hence by Theorem 6 all solution of (26) are oscillatory . On the other hand, the characteristic equation of (26) is

$$(\lambda - a)^2(3\lambda^2 + (2-6a)\lambda + 3a^2 - 2a) = 0.$$

Hence a fundamental set of all solutions of Eq. (26) is $\{\{a^n\}, \{na^n\}, \{\left(\frac{3a-2}{3}\right)^n\}\}$ *which consists of all oscillatory solutions.*

6. Conclusion

In this paper we investigated the sufficient conditions of the oscillation and non-oscillation of higher -order difference equations (1). In this study, we used definitions of generalized difference operator and oscillation/non-oscillation for the proof of the results. Also, we have considered both cases of $a < 0$ and $a > 0$. We have obtained non-oscillatory behaviour of solution of Eq. (1) in Section 3, we have studied oscillatory behaviour of solution of Eq. (1) in Section 4, respectively. Finally, we have discussed some examples related to our main results.

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OPERATOR INEQUALITIES IN REPRODUCING KERNEL HILBERT SPACES

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Abstract. In this paper, by using some classical Mulholland type inequality, Berezin symbols and reproducing kernel technique, we prove the power inequalities for the Berezin number $ber(A)$ for some self-adjoint operators A on $\mathcal{H}(\Omega)$. Namely, some Mulholland type inequality for reproducing kernel Hilbert space operators are established. By applying this inequality, we prove that $(ber(A))^n \leq C_1 ber(A^n)$ for any positive operator A on $\mathcal{H}(\Omega)$.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} \frac{1}{m} a_m^p < +\infty$ and $0 < \sum_{n=2}^{\infty} \frac{1}{n} b_n^q$, then the Mulholland's inequality [13, 20] is given by

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{mn \ln mn} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\sum_{n=2}^{\infty} \frac{1}{n} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=2}^{\infty} \frac{1}{n} b_n^q\right\}^{\frac{1}{q}} \tag{1}$$

and an equivalent form is

$$\sum_{n=2}^{\infty} \frac{1}{n} \left(\sum_{m=2}^{\infty} \frac{a_m}{m \ln mn}\right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \sum_{n=2}^{\infty} \frac{1}{n} a_n^p, \tag{2}$$

where the constants $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ and $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p$ are the best possible.

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The integral analogues of (1) and (2) are as follows:

$$\int_1^\infty \int_1^\infty \frac{f(x)\,g(y)}{xy\ln xy}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_1^\infty \frac{f^p(x)}{x}dx \right)^{1/p} \left(\int_1^\infty \frac{g^q(y)}{y}dy \right)^{1/q}, \tag{3}$$

$$\int_1^\infty \left(\int_1^\infty \frac{f(x)}{xy\ln xy}dx \right) dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_1^\infty \frac{f^p(x)}{x}dx.$$

Inequalities (1) and (3) are called the Mulholland's inequality and Mulholland's integral inequality, respectively (see [13, 20]). Some generalizations of these type inequalities are given in [5, 8, 11, 12, 14, 29].

Denote by $\mathcal{F}(\Omega)$ the set of all complex valued functions on some set Ω. A reproducing kernel Hilbert space (RKHS for short) on the set Ω is a Hilbert space $\mathcal{H} \subset \mathcal{F}(\Omega)$ with a function $k_\lambda : \Omega \times \Omega \to \mathcal{H}$, which is called the reproducing kernel enjoying the reproducing property $k_\lambda := k(.,\lambda) \in \mathcal{H}$ for all $\lambda \in \Omega$ and

$$f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$$

holds for all $\lambda \in \Omega$ and all $f \in \mathcal{H}$ (see [1, 23]).

Let $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For any bounded linear operator A on $\mathcal{H}$, the Berezin symbol of A is the function $\widetilde{A}$ defined by (see [4])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}} \quad (\lambda \in \Omega).$$

Recall that the Berezin set and the Berezin number for an operator $A \in \mathcal{B}(\mathcal{H}(\Omega))$ were introduced in [15, 16] as follows:

$$Ber(A) := Range(\widetilde{A}) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\} \quad \text{(Berezin set)}.$$

$$ber(A) := \sup \left\{ \left| \widetilde{A}(\lambda) \right| : \lambda \in \Omega \right\} \quad \text{(Berezin number)}.$$

Clearly, $Ber(A) \subset W(A) := \{\langle Ax, x\rangle : \|x\|_{\mathcal{H}} = 1\}$ (numerical range) and $ber(A) \leq w(A) := \sup\{|\langle Ax, x\rangle| : \|x\|_{\mathcal{H}} = 1\}$ (numerical radius). More information about numerical range and numerical radius can be found in [6, 7, 9, 18, 19, 21].

Using the Hardy-Hilbert type inequalities and some well-known inequalities, some important results about the Berezin number inequalities were obtained in [2, 3, 10, 22, 24–28].

In the present paper, by using inequalities (1), (2) and some ideas of paper [17], we will estimate Berezin number (which is a new numerical value of the bounded linear operators on RKHS) of operators acting in the reproducing kernel Hilbert spaces.

2. Mulholland Type Inequalities and Berezin Number of Some Operators

In the following result, we prove an analog of inequality (1) for some self-adjoint operators on a RKHS $\mathcal{H}=\mathcal{H}(\Omega)$.

Theorem 1. *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let f,g be two continuous functions defined on an interval $\Delta\subset(0,+\infty)$ and $f,g\geq 0$. Then the following is true:*

$$\begin{aligned}
&\frac{8\widetilde{f(A)g(A)}(\lambda)+\widetilde{f(C)g(C)}(\xi)}{32\ln 4}+\frac{\widetilde{f(A)}(\lambda)\widetilde{g(B)}(\mu)+\widetilde{f(B)}(\mu)\widetilde{g(A)}(\lambda)}{6\ln 6}\\
&+\frac{\widetilde{f(A)}(\lambda)\widetilde{g(C)}(\xi)+\widetilde{f(C)}(\xi)\widetilde{g(A)}(\lambda)}{8\ln 8}+\frac{\frac{1}{2}\widetilde{f(B)g(B)}(\mu)}{9\ln 9}\\
&+\frac{\widetilde{f(B)}(\mu)\widetilde{g(C)}(\xi)+\widetilde{f(C)}(\xi)\widetilde{g(B)}(\mu)}{12\ln 12}\\
&<\frac{\pi}{\sin(\pi/p)}\left\langle\left(\frac{f^p(A)}{2}+\frac{f^p(B)}{3}+\frac{f^p(C)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(B)}{3}+\frac{g^q(C)}{4}\right)^{1/q}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,
\end{aligned}\tag{4}$$

for all self-adjoint operators $A,B,C\in\mathcal{B}(\mathcal{H}(\Omega))$ with spectra contained in Δ and for all μ, λ, $\xi\in\Omega$.

Proof. Let a_2,a_3,a_4,b_2,b_3,b_4 be positive scalars. Then using inequality (1), we have

$$\begin{aligned}
&\frac{8a_2b_2+a_4b_4}{32\ln 4}+\frac{a_2b_3+a_3b_2}{6\ln 6}+\frac{a_2b_4+a_4b_2}{8\ln 8}+\frac{a_3b_3}{9\ln 9}+\frac{a_3b_4+a_4b_3}{12\ln 12}\\
&<\frac{\pi}{\sin(\pi/p)}\left(\frac{a_2^p}{2}+\frac{a_3^p}{3}+\frac{a_4^p}{4}\right)^{1/p}\left(\frac{b_2^q}{2}+\frac{b_3^q}{3}+\frac{b_4^q}{4}\right)^{1/q}.
\end{aligned}\tag{5}$$

Let $x,y,z\in\Delta$. By the hypothyses of the theorem $f(x)\geq 0$, $g(x)\geq 0$ for all $x\in\Delta$. If we put $a_2=f(x)$, $a_3=f(y)$, $a_4=f(z)$, $b_2=g(x)$, $b_3=g(y)$, $b_4=g(z)$ in (5), then we have

$$\begin{aligned}
&\frac{8f(x)g(x)+f(z)g(z)}{32\ln 4}+\frac{f(x)g(y)+f(y)g(x)}{6\ln 6}\\
&+\frac{f(x)g(z)+f(z)g(x)}{8\ln 8}+\frac{f(y)g(y)}{9\ln 9}+\frac{f(y)g(z)+f(z)g(y)}{12\ln 12}\\
&<\frac{\pi}{\sin(\pi/p)}\left(\frac{f^p(x)}{2}+\frac{f^p(y)}{3}+\frac{f^p(z)}{4}\right)^{1/p}\left(\frac{g^q(x)}{2}+\frac{g^q(y)}{3}+\frac{g^q(z)}{4}\right)^{1/q}
\end{aligned}\tag{6}$$

for all $x,y,z\in\Delta$. Let A be a self-adjoint operator. Then, by using functional calculus and inequality (6), we get

$$\frac{8f(A)g(A)+f(z)g(z)}{32\ln 4}+\frac{f(A)g(y)+f(y)g(A)}{6\ln 6}$$

$$+\frac{f(A)g(z)+f(z)g(A)}{8\ln 8}+\frac{f(y)g(y)}{9\ln 9}+\frac{f(y)g(z)+f(z)g(y)}{12\ln 12}$$
$$<\frac{\pi}{\sin(\pi/p)}\left(\frac{f^p(A)}{2}+\frac{f^p(y)}{3}+\frac{f^p(z)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(y)}{3}+\frac{g^q(z)}{4}\right)^{1/q},$$

and therefore, we have that

$$\frac{8\left\langle f(A)g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+f(z)g(z)}{32\ln 4}+\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle g(y)+f(y)\left\langle g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{6\ln 6}$$
$$+\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle g(z)+f(z)\left\langle g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{8\ln 8}+\frac{f(y)g(y)}{9\ln 9}+\frac{f(y)g(z)+f(z)g(y)}{12\ln 12}$$
$$<\frac{\pi}{\sin(\pi/p)}\left\langle\left(\frac{f^p(A)}{2}+\frac{f^p(y)}{3}+\frac{f^p(z)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(y)}{3}+\frac{g^q(z)}{4}\right)^{1/q}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,$$

for all $\lambda\in\Omega$ and any $y,z\in\Delta$.

Using the functional calculus once more to the self-adjoint operators B and C, we get

$$\frac{8\left\langle f(A)g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+f(C)g(C)}{32\ln 4}+\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle g(B)+f(B)\left\langle g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{6\ln 6}$$
$$(7)$$
$$+\frac{\left\langle f(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle g(C)+f(C)\left\langle g(A)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{8\ln 8}+\frac{f(B)g(B)}{9\ln 9}+\frac{f(B)g(C)+f(C)g(B)}{12\ln 12}$$
$$<\frac{\pi}{\sin(\pi/p)}\widetilde{\left[\left(\frac{f^p(A)}{2}+\frac{f^p(B)}{3}+\frac{f^p(C)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(B)}{3}+\frac{g^q(C)}{4}\right)^{1/q}\right]}(\lambda).$$

Hence, we have from (7) that

$$\frac{8\widetilde{f(A)g(A)}(\lambda)+\widetilde{f(C)g(C)}(\xi)}{32\ln 4}+\frac{\widetilde{f(A)}(\lambda)\widetilde{g(B)}(\mu)+\widetilde{f(B)}(\mu)\widetilde{g(A)}(\lambda)}{6\ln 6}$$
$$+\frac{\widetilde{f(A)}(\lambda)\widetilde{g(C)}(\xi)+\widetilde{f(C)}(\xi)\widetilde{g(A)}(\lambda)}{8\ln 8}+\frac{\widetilde{f(B)g(B)}(\mu)}{9\ln 9}$$
$$+\frac{\widetilde{f(B)}(\mu)\widetilde{g(C)}(\xi)+\widetilde{f(C)}(\xi)\widetilde{g(B)}(\mu)}{12\ln 12}$$
$$<\frac{\pi}{\sin(\pi/p)}\left\langle\left(\frac{f^p(A)}{2}+\frac{f^p(B)}{3}+\frac{f^p(C)}{4}\right)^{1/p}\left(\frac{g^q(A)}{2}+\frac{g^q(B)}{3}+\frac{g^q(C)}{4}\right)^{1/q}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,$$

for all self-adjoint operators $A,B,C\in\mathcal{B}(\mathcal{H}(\Omega))$ and for all $\lambda,\mu,\xi\in\Omega$. This proves the theorem. $\square$

Corollary 1. $(ber\,(f\,(A)))^2 < C_1 ber\left(f\,(A)^2\right)$ *for any self-adjoint operator* $A \in \mathcal{B}(H(\Omega))$ *with spectrum contained in* Δ; *in particular,*

$$(ber\,(A))^2 < C_1 ber\left(A^2\right),$$

where $C_1 = \left(\dfrac{2.904\pi}{\sin(\pi/p)} - 0.678\right)$.

Proof. Indeed, for $C = B = A$, $g = f$ and $\xi = \mu = \lambda$, we have from inequality (4) that

$$\frac{9\widetilde{f^2(A)}(\lambda)}{32\ln 4} + \frac{2\left[\widetilde{f(A)}(\lambda)\right]^2}{6\ln 6} + \frac{2\left[\widetilde{f(A)}(\lambda)\right]^2}{8\ln 8} + \frac{\widetilde{f^2(A)}(\lambda)}{18\ln 3} + \frac{2\left[\widetilde{f(A)}(\lambda)\right]^2}{12\ln 12}$$
$$< \frac{13\pi}{12\sin(\pi/p)}\widetilde{f^2(A)}(\lambda),$$

or equivalently

$$\left(\frac{4\log_6 e + 3\log_8 e + 2\log_{12} e}{12}\right)\left[\widetilde{f(A)}(\lambda)\right]^2$$
$$< \left(\frac{13\pi}{12\sin(\pi/p)} - \frac{81\log_4 e + 16\log_3 e}{288}\right)\widetilde{f^2(A)}(\lambda)$$

for all $\lambda \in \Omega$. Since $\left[\widetilde{f(A)}(\lambda)\right]^2 \geq 0$ and $\widetilde{f(A)^2}(\lambda) \geq 0$, we have that

$$\sup_{\lambda\in\Omega}\left[\widetilde{f(A)}(\lambda)\right]^2 < \left(\frac{2.904\pi}{\sin(\pi/p)} - 0.678\right)\sup_{\lambda\in\Omega}\widetilde{f^2(A)}(\lambda)$$

for all $\lambda \in \Omega$. This obviously implies that

$$ber(f(A))^2 < \left(\frac{2.904\pi}{\sin(\pi/p)} - 0.678\right)ber(f^2(A));$$

in particular, for $f(x) = x$, we have that

$$ber(A)^2 < \left(\frac{2.904\pi}{\sin(\pi/p)} - 0.678\right)ber(A^2).$$

□

Our more general result is the following theorem which gives a sharper estimate than Corollary 1.

Theorem 2. *Let* $p > 1$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Let* f *be a continuous function defined on an interval* $\triangle \subset (0, +\infty)$ *and* $f \geq 0$. *Let* $A : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ *be a positive operator on a RKHS* $\mathcal{H}(\Omega)$ *with spectrum contained in* $\triangle$. *Then there exists a constant* $C_1 > 1$ *such that*

$$[ber(f(A))]^p < C_1 ber(f^p(A));$$

in particular, $ber(A)^p < C_1 ber(A^p)$, *where* $C_1 = 1.73\left[\frac{\pi}{\sin(\pi/p)}\right]^p$.

Proof. Let a_2, a_3, a_4 be positive numbers. Then using (2), we have that

$$\begin{aligned} &\frac{1}{2}\left(\frac{a_2}{2\ln 4}+\frac{a_3}{3\ln 6}+\frac{a_4}{4\ln 8}\right)^p+\frac{1}{3}\left(\frac{a_2}{2\ln 6}+\frac{a_3}{3\ln 9}+\frac{a_4}{4\ln 12}\right)^p \\ &+\frac{1}{4}\left(\frac{a_2}{2\ln 8}+\frac{a_3}{3\ln 12}+\frac{a_4}{4\ln 16}\right)^p \\ &<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\frac{a_2^p}{2}+\frac{a_3^p}{3}+\frac{a_4^p}{4}\right). \end{aligned} \tag{8}$$

Let $x, y, z \in \Delta$. Since $f(x) \geq 0$ for all $x \in \Delta$, by putting $a_2 = f(x), a_3 = f(y)$ and $a_4 = f(z)$ in (8), we have

$$\begin{aligned} &\frac{1}{2}\left(\frac{f(x)}{2\ln 4}+\frac{f(y)}{3\ln 6}+\frac{f(z)}{4\ln 8}\right)^p+\frac{1}{3}\left(\frac{f(x)}{2\ln 6}+\frac{f(y)}{3\ln 9}+\frac{f(z)}{4\ln 12}\right)^p \\ &+\frac{1}{4}\left(\frac{f(x)}{2\ln 8}+\frac{f(y)}{3\ln 12}+\frac{f(z)}{4\ln 16}\right)^p \\ &<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\frac{f^p(x)}{2}+\frac{f^p(y)}{3}+\frac{f^p(z)}{4}\right). \end{aligned} \tag{9}$$

So, by using the same functional calculus arguments as in the proof of Theorem 1, finally we get from (9) that

$$\begin{aligned} &\frac{1}{2}\left(\frac{\left\langle f(A)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}{2\ln 4}+\frac{\left\langle f(B)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle}{3\ln 6}+\frac{\left\langle f(C)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle}{4\ln 8}\right)^p \\ &+\frac{1}{3}\left(\frac{\left\langle f(A)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}{2\ln 6}+\frac{\left\langle f(B)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle}{3\ln 9}+\frac{\left\langle f(C)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle}{4\ln 12}\right)^p \\ &+\frac{1}{4}\left(\frac{\left\langle f(A)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}{2\ln 8}+\frac{\left\langle f(B)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle}{3\ln 12}+\frac{\left\langle f(C)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle}{4\ln 16}\right)^p \\ &<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\frac{\left\langle f^p(A)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}{2}+\frac{\left\langle f^p(B)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle}{3}+\frac{\left\langle f^p(C)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle}{4}\right). \end{aligned}$$

and hence

$$\begin{aligned} &\frac{1}{2}\left(\frac{\widetilde{f(A)}(\lambda)}{2\ln 4}+\frac{\widetilde{f(B)}(\mu)}{3\ln 6}+\frac{\widetilde{f(C)}(\xi)}{4\ln 8}\right)^p \\ &+\frac{1}{3}\left(\frac{\widetilde{f(A)}(\lambda)}{2\ln 6}+\frac{\widetilde{f(B)}(\mu)}{3\ln 9}+\frac{\widetilde{f(C)}(\xi)}{4\ln 12}\right)^p \end{aligned}$$

$$+\frac{1}{4}\left(\frac{\widetilde{f(A)}(\lambda)}{2\ln 8}+\frac{\widetilde{f(B)}(\mu)}{3\ln 12}+\frac{\widetilde{f(C)}(\xi)}{4\ln 16}\right)^{p}$$
$$<\left(\frac{\pi}{\sin(\pi/p)}\right)^{p}\left(\frac{\widetilde{f^{p}(A)}(\lambda)}{2}+\frac{\widetilde{f^{p}(B)}(\mu)}{3}+\frac{\widetilde{f^{p}(C)}(\xi)}{4}\right).$$

for all positive operators A, B, C which spectrum contained in $\triangle$ and all $\lambda, \mu, \xi \in \Omega$. Now by replacing $C = B = A$ and $\xi = \mu = \lambda$, we have from the latter equality that

$$\left[\frac{129\log_2 e+32\log_3 e+192\log_6 e+96\log_{12} e}{576}\right]\left[\widetilde{f(A)}(\lambda)\right]^{p}<\frac{13}{12}\left[\frac{\pi}{\sin(\pi/p)}\right]^{p}\left[\widetilde{f^{p}(A)}(\lambda)\right],$$

for all $\lambda \in \Omega$. Since $\widetilde{f^{p}(A)}(\lambda) \geq 0$ for all $\lambda \in \Omega$ and for all $p > 1$, the last inequality shows that

$$\sup_{\lambda\in\Omega}\left[\widetilde{f(A)}(\lambda)\right]^{p}<1.73\left[\frac{\pi}{\sin(\pi/p)}\right]^{p}\sup_{\lambda\in\Omega}\left[\widetilde{f^{p}(A)}(\lambda)\right]$$

for all $\lambda \in \Omega$ and $p > 1$. This implies that

$$[ber(f(A))]^{p}<1.73\left[\frac{\pi}{\sin(\pi/p)}\right]^{p}ber(f^{p}(A)),$$

in particular,

$$[ber(A)]^{p}<1.73\left[\frac{\pi}{\sin(\pi/p)}\right]^{p}ber(A^{p}).$$

This proves the theorem. □

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GENERALIZED OSCULATING CURVES OF TYPE (n-3) IN THE n-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. In this paper, we give a generalization of the osculating curves to the n-dimensional Euclidean space. Based on the definition of an osculating curve in the 3 and 4 dimensional Euclidean spaces, a new type of osculating curve has been defined such that the curve is independent of the $(n-3)$th binormal vector in the n-dimensional Euclidean space, which has been called "a generalized osculating curve of type $(n-3)$". We find the relationship between the curvatures for any unit speed curve to be congruent to this osculating curve in E^n. In particular, we characterize the osculating curves in E^n in terms of their curvature functions. Finally, we show that the ratio of the $(n-1)$th and $(n-2)$th curvatures of the osculating curve is the solution of an $(n-2)$th order linear nonhomogeneous differential equation.

1. Introduction

Curve theory is a popular research interest in classical differential geometry and osculating curves are a known example in this field. There are many studies on osculating curves in the Euclidean 3-space E^3. The significant property of these curves is that the position vector of osculating curves always lie in their osculating planes. The osculating curve $\alpha : I \to E^3$ is defined by

$$\alpha(s) = \lambda(s) T(s) + \mu(s) N(s),$$

for some differentiable functions λ and μ of $s \in I \subset \mathbb{R}$, where $T(s)$ is the tangent vector field and $N(s)$ is the normal vector field. Similar curves are present in curve theory such as normal curves, where the position vector always lies in the normal plane, and the rectifying curves, where the position vector always lies in the

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rectifying plane [1]. B. Y. Chen has studied rectifying curves in his paper "When does the position vector of a space curve always lie in its rectifying plane?". Since this study, normal, rectifying and osculating curves have been studied from different perspectives. Some of the studies in the literature have been listed below.

Chen and Dillen have studied rectifying curves in [2], where they established a relationship between rectifying curves and centrodes in mechanics. They also show that rectifying curves satisfy the equality case of a general inequality in their study. The characterization of the rectifying curve in Euclidean $4-$space and Minkowski $3-$space are given in [3], [4] and [5]. Cambie et al. investigated rectifying curves in an arbitrary dimensional Euclidean space using conditions on their curvature [6]. Additionally, there are some papers on spacelike, timelike and null normal curves in Minkowski space [7], [8]. Characterizations of an osculating curve in the 3-dimensional Euclidean space has been given in [9] and a specific osculating curve has been defined in the Euclidean 4-space. Normal, osculating and rectifying curves have been defined in the Euclidean and semi Euclidean space by using quaternion algebra in [10], [11], [16] and [25]. Bi-null curves of these types have also been analyzed in $\mathbb{R}^6_3$ and $\mathbb{R}^5_2$ in [12, 13]. Several studies in the literature on the topic of interest of this study can be found in [14-18, 26, 27].

In this paper, using similar methods to those used in [6] and the definition of an osculating curve as stated in [9], we investigate the properties of a generalized form of osculating curves in the $n-$dimensional Euclidean space which are independent of the $(n-3)$th binormal vector. We call this osculating curve "a generalized osculating curve of type $(n-3)$". Firstly, basic concepts of curve theory in E^n are given as preliminaries. Then, the characterization of the osculating curves is given in E^n. The necessary and sufficient condition for a curve to be an osculating curve in the **n**-dimensional Euclidean space is also obtained. Additionally, using this necessary and sufficient condition, we show that if a curve is an osculating curve in the **n**-dimensional Euclidean space, its curvatures define a differential equation. Finally, we state the existence and uniqueness of the solution of this differential equation and propose a general form for the general solution of the equation.

2. Preliminaries

Basic concepts of curve theory in the n-dimensional Euclidean space E^n are given in this section. Let $\alpha : I \subset \mathbb{R} \to E^n$, $s \in I \to \alpha(s)$ be an arclength parameterized, n times continuously differentiable curve. The curve α is called a unit speed curve if $\langle \alpha, \alpha \rangle = 1$, where $\langle\ ,\ \rangle$ is the function that shows the standart inner product in the n-dimensional Euclidean space E^n given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$$

for each $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n) \in E^n$. The norm of X is given by $\|X\| = \sqrt{\langle X, X \rangle}$. On the other hand, if $\|X\| = 1$, then X is an unit vector.

Also, if the curve α in E^n is an arclength parameterized curve, then $\left\|\frac{d\alpha}{ds}\right\| = 1$. The Serret Frenet formulas for E^n are given as the following equations (see [19]):

$$\begin{aligned}
&T'(s) = \kappa_1(s)N(s)\\
&N'(s) = -\kappa_1(s)T(s) + \kappa_2(s)B_1(s)\\
&B_1'(s) = -\kappa_2(s)N(s) + \kappa_3(s)B_2(s)\\
&B_i'(s) = -\kappa_{i+1}(s)B_{i-1}(s) + \kappa_{i+2}(s)B_{i+1}(s),\ 2 \le i \le n-3\\
&B_{n-2}'(s) = -\kappa_{n-1}(s)B_{n-3}(s),
\end{aligned} \tag{1}$$

where κ_1, κ_2, κ_3,...,κ_{n-1} are the curvature functions of the curve and are positive. For more information on curve theory, the reader is advised to see the liteature [20-22].

3. Osculating curves of type (n-3) in the n−dimensional Euclidean space

In this section, generalizations of several fundamental definitions, theorems, and results to generalized osculating curves of type (n-3) in the n-dimensional Euclidean space are given. All of the mentions to osculating curves in our study refer to the generalized osculating curves of type (n-3) from this point.

Definition 1. *Let $\alpha : I \subset \mathbb{R} \to E^n$, $s \in I \to \alpha(s)$ be an arclength parameterized, n times continuously differentiable curve. In E^n, a curve for which the position vector always lies in the orthogonal complement $B_{n-3}^{\perp}(s)$ of its $(n-3)$th binormal vector field $B_{n-3}(s)$ is called the osculating curve. $B_{n-3}^{\perp}(s)$ is defined as*

$$B_{n-3}^{\perp}(s) = \{W \in E^n \,|\langle W, B_{n-3}(s)\rangle = 0\}$$

where $\langle\ ,\ \rangle$ denotes the standard scalar product in E^n. Thus $B_{n-3}^{\perp}(s)$ is a $(n-1)$-dimensional subspace of E^n, spanned by the tangent, the principal normal, the first binormal, second binormal,...,$(n-4)$th binormal and, $(n-2)$th binormal vector fields T,N,$B_1(s)$,...,$B_{n-4}(s)$,$B_{n-2}(s)$ respectively. Therefore, the position vector of an osculating curve with respect to a specific origin is given as

$$\alpha(s) = \mu_1(s)T(s) + \mu_2(s)N(s) + \sum_{i=1}^{n-4}\mu_{i+2}(s)B_i(s) + \mu_{n-1}(s)B_{n-2}(s) \tag{2}$$

for some differentiable functions μ_i $(1 \le i \le n-3)$ of $s \in I \subset \mathbb{R}$.

Theorem 1. *Let $\alpha(s)$ be a unit speed curve in E^n with nonzero curvatures. Then $\alpha(s)$ is congruent to a osculating curve in E^n if and only if*

$$\sum_{z=0}^{n-3}\left(\Gamma_{i,z}(s)\frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)\right)' - \kappa_1(s)\sum_{z=0}^{n-4}\Gamma_{i,z}(s)\frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) = \frac{1}{c},$$

$c \in \mathbb{R} - \{0\}$, where $1 \le i \le n-1$.

Proof. Let α be an arclength parameterized osculating curve in the n-dimensional Euclidean space. The derivative of (2) with respect to s for both sides of the equation is

$$\begin{aligned}\alpha^{'}(s) &= \mu_1^{'}(s)\,T(s)+\mu_1(s)\,T^{'}(s)+\mu_2^{'}(s)\,N(s)+\mu_2(s)\,N^{'}(s)\\ &\quad+\sum_{i=2}^{n-4}\left(\mu_{i+2}^{'}(s)\,B_i(s)+\mu_{i+2}(s)\,B_i^{'}(s)\right)\\ &\quad+\mu_{n-1}^{'}(s)\,B_{n-2}(s)+\mu_{n-1}(s)\,B_{n-2}(s)\,.\end{aligned}$$

Implementing the Serret Frenet formulas for the n-dimensional Euclidean space and rearranging the terms of the right hand side, we get

$$\begin{aligned}T(s) &= \left(\mu_1^{'}(s)-\mu_2(s)\,\kappa_1(s)\right)T(s)\\ &\quad+\left(\mu_1(s)\,\kappa_1(s)+\mu_2^{'}(s)-\mu_3(s)\,\kappa_2(s)\right)N(s)\\ &\quad+\left(\mu_2(s)\,\kappa_2(s)+\mu_3^{'}(s)-\mu_4(s)\,\kappa_3(s)\right)B_1(s)\\ &\quad+\sum_{i=2}^{n-5}\left(\mu_{i+1}(s)\,\kappa_{i+1}(s)+\mu_{i+2}^{'}(s)-\mu_{i+3}(s)\,\kappa_{i+2}(s)\right)B_i(s)\\ &\quad+\left(\mu_{n-3}(s)\,\kappa_{n-3}(s)+\mu_{n-2}^{'}(s)\right)B_{n-4}(s)\\ &\quad+\left(\mu_{n-2}(s)\,\kappa_{n-2}(s)-\mu_{n-1}(s)\,\kappa_{n-1}(s)\right)B_{n-3}(s)\\ &\quad+\mu_{n-1}^{'}(s)\,B_{n-2}(s)\end{aligned}$$

Using the equality of both sides, we get the following expressions for the coefficients of $T(s), N(s), B_i(s)$ for $i=2,3,\ldots,n-2$:

$$\mu_1^{'}(s)-\mu_2(s)\,\kappa_1(s)=1 \tag{3}$$

$$\mu_1(s)\,\kappa_1(s)+\mu_2^{'}(s)-\mu_3(s)\,\kappa_2(s)=0 \tag{4}$$

$$\mu_2(s)\,\kappa_2(s)+\mu_3^{'}(s)-\mu_4(s)\,\kappa_3(s)=0 \tag{5}$$

$$\mu_{i+1}(s)\,\kappa_{i+1}(s)+\mu_{i+2}^{'}(s)-\mu_{i+3}(s)\,\kappa_{i+2}(s)=0,\ 2\le i\le n-3 \tag{6}$$

$$\mu_{n-3}(s)\,\kappa_{n-3}(s)+\mu_{n-2}^{'}(s)=0 \tag{7}$$

$$\mu_{n-2}(s)\,\kappa_{n-2}(s)-\mu_{n-1}(s)\,\kappa_{n-1}(s)=0 \tag{8}$$

$$\mu_{n-1}^{'}(s)=0. \tag{9}$$

Starting from (9), we integrate these expressions with respect to s to obtain the coefficient functions

$$\mu_{n-1}(s)=c,\ c\in\mathbb{R}. \tag{10}$$

Similarly, the integrations of (7) and (8) yield

$$\mu_{n-2}(s) = -\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \tag{11}$$

and

$$\mu_{n-3}(s) = -\frac{c}{\kappa_{n-3}(s)}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'. \tag{12}$$

On the other hand, for $i = n-4$, and $n-5$, we get the following equations:

$$\begin{aligned} \mu_{n-4}(s) &= -c\frac{\kappa_{n-3}(s)}{\kappa_{n-4}(s)}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) \\ &\quad +\frac{c}{\kappa_{n-4}(s)}\left(\frac{1}{\kappa_{n-3}(s)}\right)'\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)' \\ &\quad +\frac{c}{\kappa_{n-4}(s)}\left(\frac{1}{\kappa_{n-3}(s)}\right)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'' \end{aligned} \tag{13}$$

and

$$\begin{aligned} \mu_{n-5}(s) &= -\frac{c}{\kappa_{n-5}(s)}\left(\frac{\kappa_{n-3}(s)}{\kappa_{n-4}(s)}\right)'\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) \\ &\quad -\left[\frac{c}{\kappa_{n-5}(s)}\left(\left(\frac{\kappa_{n-3}(s)}{\kappa_{n-4}(s)}\right)-\frac{\kappa_{n-4}(s)}{\kappa_{n-3}(s)}\right)\right]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)' \\ &\quad +\frac{c}{\kappa_{n-5}(s)}\left(\frac{1}{\kappa_{n-4}(s)}\left(\frac{1}{\kappa_{n-3}(s)}\right)'\right)'\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)' \\ &\quad +\frac{2c}{\kappa_{n-5}(s)}\frac{1}{\kappa_{n-4}(s)}\frac{1}{\kappa_{n-3}(s)}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'' \\ &\quad -\frac{c}{\kappa_{n-5}(s)}\frac{1}{\kappa_{n-4}(s)}\frac{1}{\kappa_{n-3}(s)}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'''. \end{aligned} \tag{14}$$

The other curvature functions have expressions of a complicated structure. Hence, we define the following functions to express these curvatures: The function $\Gamma_{n-4,0}(s)$ is defined as

$$\begin{aligned} \Gamma_{n-4,0}(s) &= -\frac{\kappa_{n-3}(s)}{\kappa_{n-4}(s)}, \Gamma_{n-4,1}(s) = \frac{1}{\kappa_{n-4}(s)}\left(\frac{1}{\kappa_{n-3}(s)}\right)', \\ \Gamma_{n-4,2}(s) &= \frac{1}{\kappa_{n-4}(s)}\left(\frac{1}{\kappa_{n-3}(s)}\right), \end{aligned}$$

then we get

$$\begin{aligned}\mu_{n-4}(s) &= c\Gamma_{n-4,0}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)+c\Gamma_{n-4,1}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)' \\ &\quad +c\Gamma_{n-4,2}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)''.\end{aligned}$$

Similarly, $\Gamma_{n-5,0}(s)$, $\Gamma_{n-5,1}(s)$, $\Gamma_{n-5,2}(s)$, and $\Gamma_{n-5,3}(s)$ are defined as

$$\begin{aligned}\Gamma_{n-5,0}(s) &= -\frac{1}{\kappa_{n-5}(s)}\left(\frac{\kappa_{n-3}(s)}{\kappa_{n-4}(s)}\right)', \\ \Gamma_{n-5,1}(s) &= \left[\frac{1}{\kappa_{n-5}(s)}\left(\left(\frac{\kappa_{n-3}(s)}{\kappa_{n-4}(s)}\right)-\frac{\kappa_{n-4}(s)}{\kappa_{n-3}(s)}\right)\right] \\ &\quad +\frac{1}{\kappa_{n-5}(s)}\left(\frac{1}{\kappa_{n-4}(s)}\left(\frac{1}{\kappa_{n-3}(s)}\right)'\right)', \\ \Gamma_{n-5,2}(s) &= \frac{2}{\kappa_{n-5}(s)}\frac{1}{\kappa_{n-4}(s)}\frac{1}{\kappa_{n-3}(s)}, \\ \Gamma_{n-5,3}(s) &= -\frac{1}{\kappa_{n-5}(s)}\frac{1}{\kappa_{n-4}(s)}\frac{1}{\kappa_{n-3}(s)},\end{aligned}$$

then we get

$$\begin{aligned}\mu_{n-5}(s) &= c\Gamma_{n-5,0}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)+c\Gamma_{n-5,1}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)' \\ &\quad +c\Gamma_{n-5,2}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)''+c\Gamma_{n-5,3}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'''.\end{aligned}$$

Altogether, the following expression can be defined for the functions defined above:

$$\mu_i(s)=\sum_{z=0}^{n-i-2}c\Gamma_{i,z}(s)\frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right),\ 1\leq i\leq n-1. \tag{15}$$

Thus we get the following coefficient functions for $i=1$, and $i=2$

$$\mu_1(s)=\sum_{z=0}^{n-3}c\Gamma_{i,z}(s)\frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right), \tag{16}$$

and

$$\mu_2(s)=\sum_{z=0}^{n-4}c\Gamma_{i,z}(s)\frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right). \tag{17}$$

Substituting (16) and (17) into (3), we obtain the relations below

$$\sum_{z=0}^{n-3}\left(\Gamma_{i,z}\left(s\right)\frac{d^{z}}{ds^{z}}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right)\right)^{'}-\kappa_{1}\left(s\right)\sum_{l=0}^{n-4}\Gamma_{i,z}\left(s\right)\frac{d^{z}}{ds^{z}}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right)=\frac{1}{c},\quad(18)$$

for $c\in\mathbb{R}-0$.

Conversely, consider an arbitrary unit speed curve in E^n for which the curvature functions satisfy the relation (18). Then, we consider the the vector $X\in E^n$ defined by

$$\begin{aligned}X\left(s\right) &= \alpha\left(s\right)+\sum_{z=0}^{n-3}c\Gamma_{i,z}\left(s\right)\frac{d^{z}}{ds^{z}}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right)T\left(s\right)\\ &+\sum_{z=0}^{n-4}c\Gamma_{i,z}\left(s\right)\frac{d^{z}}{ds^{z}}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right)N\left(s\right)\\ &+...+\sum_{z=0}^{n-i-4}c\Gamma_{i,z}\left(s\right)\frac{d^{z}}{ds^{z}}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right)B_{i}\left(s\right)\\ &+...-\left(\frac{c}{\kappa_{n-3}\left(s\right)}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right)^{'}\right)B_{n-5}\left(s\right)\\ &-\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right)B_{n-4}\left(s\right)\\ &+cB_{n-2}\left(s\right)\end{aligned}$$

It can be seen that $X^{'}(s)=0$ through the relations (1) and (18). Thus, X is a constant vector. This implies that α is congruent to an osculating curve. Hence, the proof is complete. □

Theorem 2. *Let $\alpha(s)$ be a unit speed osculating curve in E^n with nonzero curvatures. Then the following hold:*

i) *The tangential, the principal normal, the first, the second, ..., the i-th,..., the $(n-5)$th, and $(n-4)$th binormal components of the position vector of the curve are respectively given by*

$$\left\langle\alpha\left(s\right),T\left(s\right)\right\rangle=\sum_{z=0}^{n-3}c\Gamma_{i,z}\left(s\right)\frac{d^{z}}{ds^{z}}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right),\quad(19)$$

$$\left\langle\alpha\left(s\right),N\left(s\right)\right\rangle=\sum_{z=0}^{n-4}c\Gamma_{i,z}\left(s\right)\frac{d^{z}}{ds^{z}}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right).\quad(20)$$

$$\left\langle\alpha\left(s\right),B_{1}\left(s\right)\right\rangle=\sum_{z=0}^{n-5}c\Gamma_{i,z}\left(s\right)\frac{d^{z}}{ds^{z}}\left(\frac{\kappa_{n-1}\left(s\right)}{\kappa_{n-2}\left(s\right)}\right),\quad(21)$$

$$\langle \alpha(s), B_2(s) \rangle = \sum_{z=0}^{n-6} c\Gamma_{i,z}(s) \frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right), \tag{22}$$

$$\vdots$$

$$\langle \alpha(s), B_i(s) \rangle = \sum_{z=0}^{n-i-4} c\Gamma_{i,z}(s) \frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) \tag{23}$$

$$\vdots$$

$$\langle \alpha(s), B_{n-5}(s) \rangle = -\frac{c}{\kappa_{n-3}(s)}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)', \tag{24}$$

$$\langle \alpha(s), B_{n-4}(s) \rangle = -\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}. \tag{25}$$

ii) *The $(n-2)$th binormal component of the position vector of the curve is a non-zero constant.*

Conversely, if $\alpha(s)$ is a unit speed curve in E^n with non-zero curvatures and one of the statements (i), (ii) holds, then $\alpha(s)$ is an osculating curve or is congruent to an osculating curve in E^n.

Proof. By using the relations (2) and (3)-(9), the position vector of the curve can be written as follows:

$$\begin{aligned}
\alpha(s) \quad = \quad & \sum_{z=0}^{n-3} c\Gamma_{i,z}(s) \frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) T(s) \\
& + \sum_{z=0}^{n-4} c\Gamma_{i,z}(s) \frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) N(s) \\
& + ... + \sum_{z=0}^{n-i-4} c\Gamma_{i,z}(s) \frac{d^z}{ds^z}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) B_i(s) \\
& + ... - \left(\frac{c}{\kappa_{n-3}(s)}\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'\right) B_{n-5}(s) \\
& - \left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) B_{n-4}(s) + cB_{n-2}(s)
\end{aligned} \tag{26}$$

From (19), we get (19)- (25). Thus, (i) and (ii) have been proved.

Conversely, assume that statements (i) and (ii) hold. By taking the derivative of $\langle \alpha(s), B_{n-4}(s) \rangle = -\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}$ with respect to s and using (1) we get,

$$-\kappa_{n-3}(s) \langle \alpha(s), B_{n-5}(s) \rangle + \kappa_{n-2}(s) \langle \alpha(s), B_{n-3}(s) \rangle = -\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}.$$

By using $\langle \alpha(s), B_{n-5}(s) \rangle = -\frac{c}{\kappa_{n-3}(s)} \left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \right)'$ and $\kappa_{n-2}(s) \neq 0$, we get $\langle \alpha(s), B_{n-3}(s) \rangle = 0$, which means that this is an osculating curve.

If statement (ii) holds, then we have $\langle \alpha(s), B_{n-2}(s) \rangle = c$, $c \in \mathbb{R} - \{0\}$. Differentiating the previous equation with respect to s and using (1), we find $-\kappa_{n-1}(s) \langle \alpha(s), B_{n-3}(s) \rangle = 0$. It follows that $\langle \alpha(s), B_{n-3}(s) \rangle = 0$ and hence the curve α is an osculating curve. □

Theorem 3. *Let $\alpha(s)$ be a unit speed osculating curve in E^n with nonzero curvatures. The differential equation*

$$\sum_{z=0}^{n-3} \left(\Gamma_{i,z}(s) \frac{d^z}{ds^z} \left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \right) \right)' - \kappa_1(s) \sum_{z=0}^{n-4} \Gamma_{i,z}(s) \frac{d^z}{ds^z} \left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \right) = \frac{1}{c}, \, ,$$

where $c \in \mathbb{R} - \{0\}$., $n > 4, i = 1, 2, \ldots, n-1$ with the initial conditions

$$\frac{\kappa_{n-1}(s_0)}{\kappa_{n-2}(s_0)} = k_0, \; \left[\frac{\kappa_{n-1}(s_0)}{\kappa_{n-2}(s_0)} \right]' = k_1, \; \ldots, \; \left[\frac{\kappa_{n-1}(s_0)}{\kappa_{n-2}(s_0)} \right]^{(n-3)} = k_{(n-3)}$$

for $s_0 \in I \subset \mathbb{R}$ has a unique solution on an open interval $I \subset \mathbb{R}$ if the functions

$$\begin{aligned} &[\Gamma'_{i,0}(s) - \kappa_1 \Gamma_{i,0}(s)], [\Gamma_{i,0}(s) + \Gamma'_{i,1}(s) - \kappa_1 \Gamma_{i,1}(s)], \ldots, \\ &[\Gamma_{i,m-1}(s) + \Gamma'_{i,m}(s) - \kappa_1 \Gamma_{i,m}(s)], \ldots, [\Gamma_{i,n-4}(s) + \Gamma'_{i,n-3}(s)], [\Gamma_{i,n-3}(s)], \frac{1}{c} \end{aligned}$$

are continuous on I and $[\Gamma'_{i,0}(s) - \kappa_1 \Gamma_{i,0}(s)] \neq 0, \ldots, [\Gamma_{i,n-3}(s)] \neq 0$ for every $s \in I$. This equation has a general solution of the form

$$\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \right) = c_1 \left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_1 (s) + \ldots + c_{n-2} \left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_{n-2} (s) + \left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_p$$

where $\left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_1 (s), \left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_2 (s), \ldots, \left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_{n-2} (s)$ form the fundamental set of solutions for the homogeneous equation

$$\begin{aligned} &\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \right)^{(n-2)} + \left(\frac{\Gamma_{i,n-4}(s) + \Gamma'_{i,n-3}(s)}{\Gamma_{i,n-3}(s)} \right) \left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \right)^{(n-3)} + \ldots \\ &+ \left(\frac{\Gamma_{i,m-1}(s) + \Gamma'_{i,m}(s) - \kappa_1(s) \Gamma_{i,m}(s)}{\Gamma_{i,n-3}(s)} \right) \left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \right)^{(m)} + \ldots \\ &+ \left(\frac{\Gamma_{i,0}(s) - \kappa_1(s) \Gamma_{i,0}(s)}{\Gamma_{i,n-3}(s)} \right) \left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)} \right) = 0. \end{aligned}$$

satisfying the condition

$$W \left(\left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_1, \left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_2, \ldots, \left(\frac{\kappa_{n-1}}{\kappa_{n-2}} \right)_{n-2} \right)$$

$$
= \begin{vmatrix}
\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_1 & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_2 & \dots & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_{n-2} \\
\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)'_1 & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)'_2 & \dots & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)'_{n-2} \\
\dots & \dots & & \dots \\
\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)^{(n-3)}_1 & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)^{(n-3)}_2 & \dots & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)^{(n-3)}_{n-2}
\end{vmatrix} \neq 0,
$$

$\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_p$ *is a particular solution of the initial value problem and* $c_1, c_2, \dots, c_{n-2}$ *are arbitrary constants.*

Proof. If the summation operators are expanded in the differential equation, we get

$$
\begin{aligned}
&\left(\Gamma_{i,0}(s)\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)' + \left(\Gamma_{i,1}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'\right)' + \left(\Gamma_{i,2}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)''\right)' \\
&+ \left(\Gamma_{i,3}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'''\right)' + \dots + \left(\Gamma_{i,n-5}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-5)}\right)' \\
&+ \left(\Gamma_{i,n-4}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-4)}\right)' + \left(\Gamma_{i,n-3}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-3)}\right)' \\
&-\kappa_1\left[\Gamma_{i,0}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) + \dots + \Gamma_{i,n-4}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-4)}\right] = \frac{1}{c}.
\end{aligned}
$$

Applying the derivations in the first summation and collecting the derivatives of same order yields

$$
\begin{aligned}
&[\Gamma'_{i,0}(s) - \kappa_1\Gamma_{i,0}(s)]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) + [\Gamma_{i,0}(s) + \Gamma'_{i,1}(s) - \kappa_1\Gamma_{i,1}(s)]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)' \\
&+[\Gamma_{i,1}(s) + \Gamma'_{i,2}(s) - \kappa_1\Gamma_{i,2}(s)]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)'' + \dots \\
&+[\Gamma_{i,n-5}(s) + \Gamma'_{i,n-4}(s) - \kappa_1\Gamma_{i,n-4}(s)]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-4)} \\
&+[\Gamma_{i,n-4}(s) + \Gamma'_{i,n-3}(s)]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-3)} + \Gamma_{i,n-3}(s)\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-2)} = \frac{1}{c}.
\end{aligned}
$$

This equation is a nonhomogeneous linear differential equation of the order $n-2$. Considered along with the initial conditions, $\frac{\kappa_{n-1}(s_0)}{\kappa_{n-2}(s_0)} = k_0$, $\left[\frac{\kappa_{n-1}(s_0)}{\kappa_{n-2}(s_0)}\right]' = k_1$, $\dots$, $\left[\frac{\kappa_{n-1}(s_0)}{\kappa_{n-2}(s_0)}\right]^{(n-3)} = k_{n-3}$, it defines an initial value problem. The continuity of the coefficients of the higher order linear differential equation $[\Gamma'_{i,0}(s) - \kappa_1\Gamma_{i,0}(s)], [\Gamma_{i,0}(s) + \Gamma'_{i,1}(s) - \kappa_1\Gamma_{i,1}(s)], \dots, [\Gamma_{i,m-1}(s) + \Gamma'_{i,m}(s) - \kappa_1\Gamma_{i,m}(s)], \dots,$

$[\Gamma_{i,n-4}(s)+\Gamma'_{i,n-3}(s)], [\Gamma_{i,n-3}(s)]$ and the function $\frac{1}{c}$ guarentees the existence and uniqueness of the solution for the initial value problem on $I \subset \mathbb{R}$ since $[\Gamma'_{i,0}(s) - \kappa_1 \Gamma_{i,0}(s)] \neq 0, \ldots, [\Gamma_{i,n-3}(s)] \neq 0$ for every $s \in I$ [23]. Division of the equation by the coefficient of the highest order derivative gives

$$\begin{aligned}
&\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-2)} + \left[\frac{\Gamma_{i,n-4}(s)+\Gamma'_{i,n-3}(s)}{\Gamma_{i,n-3}(s)}\right]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-3)} \\
&+\left[\frac{\Gamma_{i,n-5}(s)+\Gamma'_{i,n-4}(s)-\kappa_1\Gamma_{i,n-4}(s)}{\Gamma_{i,n-3}(s)}\right]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)^{(n-4)} \\
&+\left[\frac{\Gamma_{i,0}(s)+\Gamma'_{i,1}(s)-\kappa_1\Gamma_{i,1}(s)}{\Gamma_{i,n-3}(s)}\right]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right)' \\
&+\left[\frac{\Gamma'_{i,0}(s)-\kappa_1\Gamma_{i,0}(s)}{\Gamma_{i,n-3}(s)}\right]\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) = \frac{1}{c\Gamma_{i,n-3}(s)},
\end{aligned}$$

such that $\Gamma_{i,n-3} \neq 0$. The continuity of the new coefficients comes from the continuity assumption of the theorem and the fact that $[\Gamma_{i,n-3}(s)] \neq 0$. Hence, The homogeneous version of this linear differential equation has a fundamental set of solutions on $I \subset \mathbb{R}$ containing solutions of the form $\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_k$ for $k = 1, 2, ...$ [23]. The fundamental set of solutions is linearly independent if and only if

$$\begin{aligned}
&W\left(\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_1, \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_2, \ldots, \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_{n-2}\right) \\
&= \begin{vmatrix}
\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_1 & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_2 & \cdots & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_{n-2} \\
\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)'_1 & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)'_2 & \cdots & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)'_{n-2} \\
\cdots & \cdots & & \cdots \\
\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)^{(n-3)}_1 & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)^{(n-3)}_2 & \cdots & \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)^{(n-3)}_{n-2}
\end{vmatrix} \neq 0
\end{aligned}$$

for every $s \in I$ and the superposition principle suggests that the homogeneous linear differential equation has a general solution of the form

$$\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) = c_1\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_1(s) + c_2\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_2(s) + \ldots + c_{n-2}\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_{n-2}(s).$$

for arbitrary constants $c_i,\ i = 1, 2, ..., n-2$ [23]. Using the initial conditions, the particular solution can be found as

$$\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) = c_1\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_1(s) + \ldots + c_{n-2}\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_{n-2}(s) + \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_p.$$

□

There are several methods in the literature for analyzing the solutions of higher order linear differential equations. For instance, the variation of parameters method proposes a particular solution for the nonhomogeneous differential equation in the form of

$$\left(\frac{\kappa_{n-1}(s)}{\kappa_{n-2}(s)}\right) = \sum_{m=1}^{n-2} \left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_m (s) \int \frac{(c\Gamma_{i,n-3}(t))^{-1} W_m(t)}{W(t)} dt$$

where $\left(\frac{\kappa_{n-1}}{\kappa_{n-2}}\right)_m (s)$ form the fundamental set of solutions and $W_m(t)$ are obtained by replacing the m-th column of the Wronskian by $(0, 0, \ldots, (c\Gamma_{i,n-3}(t))^{-1})$ [24].

4. Conclusion

In this paper, we have investigated some concepts of osculating curves, defined on 3- and 4-dimensional Euclidean spaces, on the n-dimensional Euclidean space. This generalization of osculating curves to E^n has been called "generalized osculating curve of type $(n-3)$. A total of $n-2$ generalizations of osculating curves to E^n can be found by using the other binormal vectors. However, we have used the $(n-3)$th binormal vector for the generalization since the relations between the curvatures provide meaningful results. Since the differential equation that gives the relation between curvature functions of the osculating curve in the n-dimensional Euclidean space is a higher order differential equation, we have invesitaged the existence and uniquness of a general solution for the initial value problem of order $n-2$. The differential equation of order n-2 is a linear differential equation with variable coefficients. Several methods in the literature can be used for analyzing the particular solution of the higher order differential equation.

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GENERALIZED RELATIVE NEVANLINNA ORDER (α, β) AND GENERALIZED RELATIVE NEVANLINNA TYPE (α, β) BASED SOME GROWTH PROPERTIES OF COMPOSITE ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. Our aim in this paper is to introduce some idea about generalized relative Nevanlinna order (α, β) and generalized relative Nevanlinna type (α, β) of an analytic function with respect to another analytic function in the unit disc where α and β are continuous non-negative functions on $(-\infty, +\infty)$. So we discuss about some growth properties relating to the composition of two analytic functions in the unit disc on the basis of generalized relative Nevanlinna order (α, β) and generalized relative Nevanlinna type (α, β) as compared to the growth of their corresponding left and right factors.

1. Introduction

A function g which is analytic in the unit disc $U = \{z : |z| < 1\}$ is said to have finite Nevanlinna order [1] if there exists a number μ for which the Nevanlinna characteristic function $T_g(r)$ of g satisfies $T_g(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$ where $T_g(r)$ is defined as

$$T(r, g) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| g\left(re^{i\theta}\right) \right| d\theta$$

where $\log^+ r = \max(0, \log r)$.

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The infimum of all such numbers μ is called the Nevanlinna order of g. Hence the Nevanlinna order $\rho(g)$ of g is formulated as

$$\rho(g) = \limsup_{r\to 1} \frac{\log T_g(r)}{-\log(1-r)}.$$

Similarly, the Nevanlinna lower order $\lambda(g)$ of g is formulated as

$$\lambda(g) = \liminf_{r\to 1} \frac{\log T_g(r)}{-\log(1-r)}.$$

Now let L be a class of continuous non-negative functions α defined on $(-\infty,\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow \infty$ as $x \to \infty$. Also throughout the present paper we take $\alpha,\ \beta \in L$. Considering the above, Sheremeta [5] introduced the concept of generalized order (α,β) of an entire function. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different directions. For the purpose of further applications, Biswas et al. [2] have introduced the definitions of the generalized Nevanlinna order (α,β) and generalized Nevanlinna lower order (α,β) of an analytic function g in the unit disc U which are as follows:

Definition 1. *[2] The generalized Nevanlinna order (α,β) denoted by $\rho_{(\alpha,\beta)}[g]$ and generalized Nevanlinna lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[g]$ of an analytic function g in the unit disc U are defined as:*

$$\rho_{(\alpha,\beta)}[g] = \limsup_{r\to 1} \frac{\alpha(\exp(T_g(r)))}{\beta\left(\frac{1}{1-r}\right)} \text{ and } \lambda_{(\alpha,\beta)}[g] = \liminf_{r\to 1} \frac{\alpha(\exp(T_g(r)))}{\beta\left(\frac{1}{1-r}\right)}.$$

Clearly $\rho_{(\log\log r,\log r)}[g] = \rho(g)$ and $\lambda_{(\log\log r,\log r)}[g] = \lambda(g)$.

Now we can introduce the definitions of the generalized relative Nevanlinna order (α,β) and generalized relative Nevanlinna lower order (α,β) of an analytic function g with respect to another entire function w in the unit disc U which are as follows:

Definition 2. *The generalized relative Nevanlinna order (α,β) denoted by $\rho_{(\alpha,\beta)}[g]_w$ and generalized relative Nevanlinna lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[g]_w$ of an analytic function g with respect to another entire function w in the unit disc U are defined as:*

$$\rho_{(\alpha,\beta)}[g]_w = \limsup_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_g(r))\right)}{\beta\left(\frac{1}{1-r}\right)} \text{ and } \lambda_{(\alpha,\beta)}[g]_w = \liminf_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_g(r))\right)}{\beta\left(\frac{1}{1-r}\right)}.$$

The previous definitions are easily generated as particular cases, e.g. if $w = z$, then Definition 2 reduces to Definition 1, and if $\alpha(r) = \beta(r) = \log r$ and $w(z) = \exp z$, then $\rho_{(\alpha,\beta)}[g]_w = \rho(g)$ and $\lambda_{(\alpha,\beta)}[g]_w = \lambda(g)$.

Now one may give the definitions of generalized relative Nevanlinna hyper order (α,β) and generalized relative Nevanlinna logarithmic order (α,β) of an analytic function g with respect to another entire function w in the unit disc U in the following way:

Definition 3. *The generalized relative Nevanlinna hyper order* (α, β) *denoted by* $\overline{\rho}_{(\alpha,\beta)}[g]_w$ *and generalized relative Nevanlinna hyper lower order* (α, β) *denoted by* $\overline{\lambda}_{(\alpha,\beta)}[g]_w$ *of an analytic function* g *with respect to entire function* w *in the unit disc* U *are defined as:*

$$\overline{\rho}_{(\alpha,\beta)}[g]_w = \limsup_{r\to 1} \frac{\alpha(\log\left(T_w^{-1}\left(T_g(r)\right)\right))}{\beta\left(\frac{1}{1-r}\right)} \text{ and } \overline{\lambda}_{(\alpha,\beta)}[g]_w = \liminf_{r\to 1} \frac{\alpha(\log\left(T_w^{-1}\left(T_g(r)\right)\right))}{\beta\left(\frac{1}{1-r}\right)}.$$

Definition 4. *The generalized relative Nevanlinna logarithmic order* (α, β) *denoted by* $\underline{\rho}_{(\alpha,\beta)}[g]_w$ *and generalized relative Nevanlinna logarithmic lower order* (α, β) *denoted by* $\underline{\lambda}_{(\alpha,\beta)}[g]_w$ *of an analytic function* g *with respect to entire function* w *in the unit disc* U *are defined as:*

$$\underline{\rho}_{(\alpha,\beta)}[g]_w = \limsup_{r\to 1} \frac{\alpha\left(T_w^{-1}\left(T_g(r)\right)\right)}{\beta\left(\log\left(\frac{1}{1-r}\right)\right)} \text{ and } \underline{\lambda}_{(\alpha,\beta)}[g]_w = \liminf_{r\to 1} \frac{\alpha\left(T_w^{-1}\left(T_g(r)\right)\right)}{\beta\left(\log\left(\frac{1}{1-r}\right)\right)}.$$

Now in order to refine the growth scale namely the generalized relative Nevanlinna order (α, β), we introduce the definitions of another growth indicators, called generalized relative Nevanlinna type (α, β) and generalized relative Nevanlinna lower type (α, β) respectively of an analytic function g with respect to entire function w in the unit disc U which are as follows:

Definition 5. *The generalized relative Nevanlinna type* (α, β) *and generalized relative Nevanlinna lower type* (α, β) *of an analytic function* g *with respect to entire function* w *in the unit disc* U *having finite positive generalized relative Nevanlinna order* (α, β) $\left(0 < \rho_{(\alpha,\beta)}[g]_w < \infty\right)$ *are defined as :*

$$\begin{aligned}
\sigma_{(\alpha,\beta)}[g]_w &= \limsup_{r\to 1} \frac{\exp(\alpha(T_w^{-1}\left(T_g\left(r\right)\right)))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho_{(\alpha,\beta)}[g]_w}} \\
\text{and } \overline{\sigma}_{(\alpha,\beta)}[g]_w &= \liminf_{r\to 1} \frac{\exp(\alpha(T_w^{-1}\left(T_g\left(r\right)\right)))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho_{(\alpha,\beta)}[g]_w}}.
\end{aligned}$$

It is obvious that $0 \le \overline{\sigma}_{(\alpha,\beta)}[g]_w \le \sigma_{(\alpha,\beta)}[g]_w \le \infty$.

Analogously, to determine the relative growth of two analytic functions in the unit disc U having same non zero finite generalized relative Nevanlinna lower order (α, β), one can introduced the definition of generalized relative Nevanlinna weak type (α, β) and generalized relative Nevanlinna upper weak type (α, β) of an analytic function g with respect to entire function w in the unit disc U of finite positive generalized relative Nevanlinna lower order (α, β) , $\lambda_{(\alpha,\beta)}[g]_w$ in the following way:

Definition 6. *The generalized Nevanlinna upper weak type* (α, β) *and generalized Nevanlinna weak type* (α, β) *of an analytic function* g *with respect to entire function* *w in the unit disc U having finite positive generalized relative Nevanlinna lower order* (α, β) $\left(0 < \lambda_{(\alpha,\beta)}[g]_w < \infty\right)$ *are defined as :*

$$\overline{\tau}_{(\alpha,\beta)}[g]_w = \limsup_{r\to 1} \frac{\exp(\alpha(T_w^{-1}(T_g(r))))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\lambda_{(\alpha,\beta)}[g]_w}}$$

$$\text{and } \tau_{(\alpha,\beta)}[g]_w = \liminf_{r\to 1} \frac{\exp(\alpha(T_w^{-1}(T_g(r))))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\lambda_{(\alpha,\beta)}[g]_w}}.$$

It is obvious that $0 \le \tau_{(\alpha,\beta)}[g]_w \le \overline{\tau}_{(\alpha,\beta)}[g]_w \le \infty$.

In this paper we study some growth properties relating to the composition of two analytic functions in the unit disc on the basis of generalized relative Nevanlinna order (α, β), generalized relative Nevanlinna hyper order (α, β), generalized relative Nevanlinna logarithmic order (α, β), generalized relative Nevanlinna type (α, β) and generalized relative Nevanlinna weak type (α, β) as compared to the growth of their corresponding left and right factors. Also the standard definitions and notations relating to the theory of entire functions are not explained here, as those are available in [1], [3] and [4].

2. Main Results

In this section, the main results of the paper are presented.

Theorem 1. *Let g be an analytic function and h, w and k be non-constant entire functions in the unit disc U such that* $0 < \lambda_{(\alpha,\beta)}[g(h)]_w \le \rho_{(\alpha,\beta)}[g(h)]_w < \infty$ *and* $0 < \lambda_{(\alpha,\beta)}[g]_k \le \rho_{(\alpha,\beta)}[g]_k < \infty$. *Then*

$$\frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k} \le \liminf_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \le \min\left\{\frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}, \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}\right\}$$

$$\le \max\left\{\frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}, \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}\right\} \le \limsup_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \le \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}.$$

Proof. From the definitions of $\lambda_{(\alpha,\beta)}[g(h)]_w$, $\rho_{(\alpha,\beta)}[g(h)]_w$, $\lambda_{(\alpha,\beta)}[g]_k, \rho_{(\alpha,\beta)}[g]_k$ and we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\alpha\left(T_w^{-1}(T_{g(h)}(r))\right) \geqslant \left(\lambda_{(\alpha,\beta)}[g(h)]_w - \varepsilon\right)\beta((1-r)^{-1}), \tag{1}$$

$$\alpha\left(T_w^{-1}(T_{g(h)}(r))\right) \le \left(\rho_{(\alpha,\beta)}[g(h)]_w + \varepsilon\right)\beta((1-r)^{-1}), \tag{2}$$

$$\alpha\left(T_k^{-1}(T_g(r))\right) \geqslant \left(\lambda_{(\alpha,\beta)}[g]_k - \varepsilon\right)\beta((1-r)^{-1}) \tag{3}$$

$$\text{and } \alpha\left(T_k^{-1}(T_g(r))\right) \le \left(\rho_{(\alpha,\beta)}[g]_k + \varepsilon\right)\beta((1-r)^{-1}). \tag{4}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity,

$$\alpha\left(T_w^{-1}(T_{g(h)}(r))\right) \le \left(\lambda_{(\alpha,\beta)}[g(h)]_w + \varepsilon\right)\beta((1-r)^{-1}), \tag{5}$$

$$\alpha\left(T_w^{-1}(T_{g(h)}(r))\right) \geqslant \Big(\rho_{(\alpha,\beta)}[g(h)]_w - \varepsilon\Big)\,\beta((1-r)^{-1}), \tag{6}$$

$$\alpha\left(T_k^{-1}(T_g(r))\right) \leq \left(\lambda_{(\alpha,\beta)}[g]_k + \varepsilon\right)\beta((1-r)^{-1}) \tag{7}$$

$$\text{and } \alpha\left(T_k^{-1}(T_g(r))\right) \geqslant \Big(\rho_{(\alpha,\beta)}[g]_k - \varepsilon\Big)\,\beta((1-r)^{-1}). \tag{8}$$

Now from (1) and (4) it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geqslant \frac{\lambda_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\rho_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geqslant \frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}, \tag{9}$$

which is the first part of the theorem.

Combining (5) and (3) , we have for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\lambda_{(\alpha,\beta)}[g(h)]_w + \varepsilon}{\lambda_{(\alpha,\beta)}[g]_k - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}. \tag{10}$$

Again from (1) and (7), for a sequence of values of $\frac{1}{1-r}$ tending to infinity, we get

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geq \frac{\lambda_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\lambda_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geq \frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}. \tag{11}$$

Now, it follows from (3) and (2) , for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[g(h)]_w + \varepsilon}{\lambda_{(\alpha,\beta)}[g]_k - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}. \tag{12}$$

Which is the last part of the theorem.

Now from (2) and (8), it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[g(h)]_w + \varepsilon}{\rho_{(\alpha,\beta)}[g]_k - \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}. \tag{13}$$

So combining (4) and (6), we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geqslant \frac{\rho_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\rho_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geqslant \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}. \tag{14}$$

So, the second part of the theorem follows from (10) and (13) ,the third part is trivial and fourth part follows from (11) and (14).

Thus the theorem follows from (9), (10), (11), (12), (13) and (14). ☐

Remark 1. *If we take "$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$" instead of "$0 < \lambda_{(\alpha,\beta)}[g]_k \leq \rho_{(\alpha,\beta)}[g]_k < \infty$" and other conditions remain same, the conclusion of Theorem 1 remains true with "$\lambda_{(\alpha,\beta)}[g]_k$", "$\rho_{(\alpha,\beta)}[g]_k$" and "$\alpha\left(T_k^{-1}(T_g(r))\right)$" replaced by "$\lambda_{(\alpha,\beta)}[h]_k$", "$\rho_{(\alpha,\beta)}[h]_k$" and "$\alpha\left(T_k^{-1}(T_h(r))\right)$" respectively in the denominator.*

Theorem 2. *Let g be an analytic function and h, w and k be non-constant entire functions in the unit disc U such that $0 < \lambda_{(\alpha,\beta)}[g]_k \leq \rho_{(\alpha,\beta)}[g]_k < \infty$ and $\lambda_{(\alpha,\beta)}[g(h)]_w = \infty$. Then*

$$\lim_{r\to 1} \frac{\alpha(T_w^{-1}(T_{g(h)}(r)))}{\alpha(T_k^{-1}(T_g(r)))} = \infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of $\frac{1}{1-r}$ tending to infinity

$$\alpha(T_w^{-1}(T_{g(h)}(r))) \leq \Delta \cdot \alpha(T_k^{-1}(T_g(r))). \tag{15}$$

Again from the definition of $\rho_{(\alpha,\beta)}[g]_k$, it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\alpha(T_k^{-1}(T_g(r))) \leq (\rho_{(\alpha,\beta)}[g]_k + \epsilon)\beta(\frac{1}{1-r}). \tag{16}$$

From (15) and (16), for a sequence of values of r tending to 1,we have

$$\alpha(T_w^{-1}(T_{g(h)}(r))) \leq \Delta(\rho_{(\alpha,\beta)}[g]_k + \epsilon)\beta(\frac{1}{1-r})$$

$$i.e., \ \frac{\alpha(T_w^{-1}(T_{g(h)}(r)))}{\beta(\frac{1}{1-r})} \leq \Delta(\rho_{(\alpha,\beta)}[g]_k + \epsilon)$$

$$i.e., \ \liminf_{r\to 1} \frac{\alpha(T_w^{-1}(T_{g(h)}(r)))}{\beta(\frac{1}{1-r})} = \lambda_{(\alpha,\beta)}[g(h)]_w < \infty.$$

This is a contradiction. □

Thus the theorem follows.

Remark 2. *If we take "$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$" instead of "$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$" and other conditions remain same, the conclusion of Theorem 2 remains true with "$\alpha(T_k^{-1}(T_g(r)))$" replaced by "$\alpha(T_k^{-1}(T_h(r)))$" in the denominator.*

Remark 3. *Theorem 2 and Remark 2 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta)}[g(h)] = \infty$" is replaced by "$\rho_{(\alpha,\beta)}[g(h)] = \infty$" and the other conditions remain the same.*

We may now state the following theorem without proof based on Definition 3.

Theorem 3. *Let g be an analytic function and h, w and k be non-constant entire functions in U such that $0 < \overline{\lambda}_{(\alpha,\beta)}[g(h)]_w \leq \overline{\rho}_{(\alpha,\beta)}[g(h)]_w < \infty$ and $0 < \overline{\lambda}_{(\alpha,\beta)}[g]_k \leq \overline{\rho}_{(\alpha,\beta)}[g]_k < \infty$. Then*

$$\frac{\overline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\overline{\rho}_{(\alpha,\beta)}[g]_k} \leq \liminf_{r\to 1} \frac{\alpha(\log\left(T_w^{-1}\left(T_{g(h)}(r)\right)\right))}{\alpha(\log\left(T_k^{-1}(T_g(r))\right))} \leq \min\left\{\frac{\overline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\overline{\lambda}_{(\alpha,\beta)}[g]_k}, \frac{\overline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\overline{\rho}_{(\alpha,\beta)}[g]_k}\right\}$$

$$\leq \max\left\{\frac{\overline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\overline{\lambda}_{(\alpha,\beta)}[g]_k}, \frac{\overline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\overline{\rho}_{(\alpha,\beta)}[g]_k}\right\} \leq \limsup_{r\to 1} \frac{\alpha(\log\left(T_w^{-1}\left(T_{g(h)}(r)\right)\right))}{\alpha(\log\left(T_k^{-1}(T_g(r))\right))} \leq \frac{\overline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\overline{\lambda}_{(\alpha,\beta)}[g]_k}.$$

Remark 4. *If we take "$0 < \overline{\lambda}_{(\alpha,\beta)}[h]_k \leq \overline{\rho}_{(\alpha,\beta)}[h]_k < \infty$" instead of "$0 < \overline{\lambda}_{(\alpha,\beta)}[g]_k \leq \overline{\rho}_{(\alpha,\beta)}[g]_k < \infty$" and other conditions remain same, the conclusion of Theorem 3 remains true with "$\overline{\lambda}_{(\alpha,\beta)}[g]_k$", "$\overline{\rho}_{(\alpha,\beta)}[g]_k$" and "$\alpha(\log\left(T_k^{-1}(T_g(r))\right))$" replaced by "$\overline{\lambda}_{(\alpha,\beta)}[h]_k$", "$\overline{\rho}_{(\alpha,\beta)}[h]_k$" and "$\alpha(\log\left(T_k^{-1}(T_h(r))\right))$" respectively in the denominator.*

We may now state the following theorem without proof based on Definition 4.

Theorem 4. *Let g be an analytic function and h, w and k be non-constant entire functions in the unit disc U such that $0 < \underline{\lambda}_{(\alpha,\beta)}[g(h)]_w \leq \underline{\rho}_{(\alpha,\beta)}[g(h)]_w < \infty$ and $0 < \underline{\lambda}_{(\alpha,\beta)}[g]_k \leq \underline{\rho}_{(\alpha,\beta)}[g]_k < \infty$. Then*

$$\frac{\underline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\underline{\rho}_{(\alpha,\beta)}[g]_k} \le \liminf_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \le \min\left\{\frac{\underline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\underline{\lambda}_{(\alpha,\beta)}[g]_k}, \frac{\underline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\underline{\rho}_{(\alpha,\beta)}[g]_k}\right\}$$
$$\le \max\left\{\frac{\underline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\underline{\lambda}_{(\alpha,\beta)}[g]_k}, \frac{\underline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\underline{\rho}_{(\alpha,\beta)}[g]_k}\right\} \le \limsup_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \le \frac{\underline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\underline{\lambda}_{(\alpha,\beta)}[g]_k}.$$

Remark 5. *If we take "$0 < \underline{\lambda}_{(\alpha,\beta)}[h]_k \le \underline{\rho}_{(\alpha,\beta)}[h]_k < \infty$" instead of "$0 < \underline{\lambda}_{(\alpha,\beta)}[g]_k \le \underline{\rho}_{(\alpha,\beta)}[g]_k < \infty$" and other conditions remain same, the results of Theorem 4 remain true with "$\underline{\lambda}_{(\alpha,\beta)}[g]_k$", "$\underline{\rho}_{(\alpha,\beta)}[g]_k$" and "$\alpha\left(T_k^{-1}(T_g(r))\right)$" replaced by "$\underline{\lambda}_{(\alpha,\beta)}[h]_k$", "$\underline{\rho}_{(\alpha,\beta)}[h]_k$" and "$\alpha\left(T_k^{-1}(T_h(r))\right)$" respectively in the denominator.*

Theorem 5. *Let g be an analytic function and h, w and k be non-constant entire functions in the unit disc U such that $0 < \overline{\sigma}_{(\alpha,\beta)}[g(h)]_w \le \sigma_{(\alpha,\beta)}[g(h)]_w < \infty$, $0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \le \sigma_{(\alpha,\beta)}[g]_k < \infty$ and $\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$. Then*

$$\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k} \le \liminf_{r\to 1} \frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \le \min\left\{\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}, \frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k}\right\}$$
$$\le \max\left\{\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}, \frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k}\right\} \le \limsup_{r\to 1} \frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \le \frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}.$$

Proof. From the definitions of $\sigma_{(\alpha,\beta)}[g]_k, \overline{\sigma}_{(\alpha,\beta)}[g]_k, \sigma_{(\alpha,\beta)}[g(h)]_w$ and $\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w$, we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right))) \ge \left(\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w - \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g(h)]_w}, \quad (17)$$
$$\exp(\alpha(T_k^{-1}\left(T_g(r)\right))) \le \left(\sigma_{(\alpha,\beta)}[g]_k + \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g]_k}, \quad (18)$$
$$\exp(\alpha(T_k^{-1}\left(T_g(r)\right))) \ge \left(\overline{\sigma}_{(\alpha,\beta)}[g]_k - \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g]_k}, \quad (19)$$
$$\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right))) \le \left(\sigma_{(\alpha,\beta)}[g(h)]_w + \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g(h)]_w}. \quad (20)$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity, we get that

$$\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right))) \le \left(\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w + \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g(h)]_w}, \quad (21)$$
$$\exp(\alpha(T_k^{-1}\left(T_g(r)\right))) \le \left(\overline{\sigma}_{(\alpha,\beta)}[g]_k + \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g]_k}, \quad (22)$$
$$\exp(\alpha(T_k^{-1}\left(T_g(r)\right))) \ge \left(\sigma_{(\alpha,\beta)}[g]_k - \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g]_k}, \quad (23)$$
$$\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right))) \geqslant (\sigma_{(\alpha,\beta)}[g(h)]_w - \varepsilon)(\exp(\beta((1-r)^{-1})))^{\rho_{(\alpha,\beta)}[g(h)]_w}. \quad (24)$$

Now from (17), (18) and the condition $\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$, it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \geqslant \frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\sigma_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_w^{-1}\left(T_g(r)\right)))}\geqslant\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_w}. \tag{25}$$

Combining (21) and (19) and the condition $\rho_{(\alpha,\beta)}[g(h)]_w=\rho_{(\alpha,\beta)}[g]_k$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w+\varepsilon}{\overline{\sigma}_{(\alpha,\beta)}[g]_k-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}. \tag{26}$$

Now from (17), (22) and the condition $\rho_{(\alpha,\beta)}[g(h)]_w=\rho_{(\alpha,\beta)}[g]_k$, we obtain for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\geq\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w-\varepsilon}{\overline{\sigma}_{(\alpha,\beta)}[g]_k+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\geq\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}. \tag{27}$$

In view of the condition $\rho_{(\alpha,\beta)}[g(h)]_w=\rho_{(\alpha,\beta)}[g]_k$, it follows from (19) and (20) for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\sigma_{(\alpha,\beta)}[g(h)]_w+\varepsilon}{\overline{\sigma}_{(\alpha,\beta)}[g]_k-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}. \tag{28}$$

Now from (20), (23) and the condition $\rho_{(\alpha,\beta)}[g(h)]_w=\rho_{(\alpha,\beta)}[g]_k$, it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\sigma_{(\alpha,\beta)}[g(h)]_w+\varepsilon}{\sigma_{(\alpha,\beta)}[g]_k-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k}. \tag{29}$$

So combining (18) and (24) and in view of the condition $\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \geqslant \frac{\sigma_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\sigma_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \geqslant \frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k}. \tag{30}$$

Thus the theorem follows from (25) , (26) , (27), (28) , (29) and (30) . □

Remark 6. *If we take "*$0 < \overline{\sigma}_{(\alpha,\beta)}[h]_k \leq \sigma_{(\alpha,\beta)}[h]_k < \infty$*" and "*$\rho_{(\alpha,\beta)}[g(h)]_w = \rho^{(\alpha,\beta)}[h]_k$*" instead of "*$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$*" and other conditions remain same, the results of Theorem 5 remain true with "*$\sigma_{(\alpha,\beta)}[g]_k$*", "*$\overline{\sigma}_{(\alpha,\beta)}[g]_k$*" and "*$\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))$*" replaced by "*$\sigma_{(\alpha,\beta)}[h]_k$*", "*$\overline{\sigma}_{(\alpha,\beta)}[h]_k$*" and "*$\exp(\alpha(T_k^{-1}\left(T_h(r)\right)))$*" respectively in the denominator.*

Remark 7. *If we take "*$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\rho_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$*" instead of "*$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$*" and other conditions remain same, the results of Theorem 5 remain true with "*$\sigma_{(\alpha,\beta)}[g]_k$*" and "*$\overline{\sigma}_{(\alpha,\beta)}[g]_k$*" replaced by "*$\overline{\tau}_{(\alpha,\beta)}[g]_k$*" and "*$\tau_{(\alpha,\beta)}[g]_k$*" respectively in the denominator.*

Remark 8. *If we take "*$0 < \tau_{(\alpha,\beta)}[h]_k \leq \overline{\tau}_{(\alpha,\beta)}[h]_k < \infty$*" and "*$\rho_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[h]_k$*" instead of "*$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$*" and other conditions remain same, the results of Theorem 5 remain true with "*$\overline{\sigma}_{(\alpha,\beta)}[g]_k$*", "*$\sigma_{(\alpha,\beta)}[g]_k$*" and "*$\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))$*" replaced by "*$\tau_{(\alpha,\beta)}[h]_k$*", "*$\overline{\tau}_{(\alpha,\beta)}[h]_k$*" and "*$\exp(\alpha(T_k^{-1}\left(T_h(r)\right)))$*" respectively in the denominator.*

Now in the line of Theorem 5 , one can easily prove the following theorem using the notion of generalized Nevanlinna weak type and therefore the proof is omitted.

Theorem 6. *Let g be a analytic function and h, w and k be non-constant entire functions in the unit disc U such that $0 < \tau_{(\alpha,\beta)}[g(h)]_w \leq \overline{\tau}_{(\alpha,\beta)}[g(h)]_w < \infty$, $0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$ and $\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$. Then*

$$\frac{\tau_{(\alpha,\beta)}[g(h)]_w}{\overline{\tau}_{(\alpha,\beta)}[g]_k} \leq \liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \leq \min\left\{\frac{\tau_{(\alpha,\beta)}[g(h)]_w}{\tau_{(\alpha,\beta)}[g]_k}, \frac{\overline{\tau}_{(\alpha,\beta)}[g(h)]_w}{\overline{\tau}_{(\alpha,\beta)}[g]_k}\right\}$$

$$\leq \max\left\{\frac{\tau_{(\alpha,\beta)}[g(h)]_w}{\tau_{(\alpha,\beta)}[g]_k}, \frac{\overline{\tau}_{(\alpha,\beta)}[g(h)]_w}{\overline{\tau}_{(\alpha,\beta)}[g]_k}\right\} \leq \limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \leq \frac{\overline{\tau}_{(\alpha,\beta)}[g(h)]_w}{\tau_{(\alpha,\beta)}[g]_k}.$$

Remark 9. *If we take "*$0 < \tau_{(\alpha,\beta)}[h]_k \leq \overline{\tau}_{(\alpha,\beta)}[h]_k < \infty$*" and "*$\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[h]_k$*" instead of "*$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$*" and other conditions remain same, the results of Theorem 6 remain true* *with "$\tau_{(\alpha,\beta)}[g]_k$", "$\overline{\tau}_{(\alpha,\beta)}[g]_k$" and "$\exp(\alpha(T_k^{-1}(T_g(r))))$" replaced by "$\tau_{(\alpha,\beta)}[h]_k$", "$\overline{\tau}_{(\alpha,\beta)}[h]_k$" and "$\exp(\alpha(T_k^{-1}(T_h(r))))$" respectively in the denominator.*

Remark 10. *If we take "$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$" instead of "$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$" and other conditions remain same, the results of Theorem 6 remain true with "$\tau_{(\alpha,\beta)}[g]_k$" and "$\overline{\tau}_{(\alpha,\beta)}[g]_k$" replaced by "$\overline{\sigma}_{(\alpha,\beta)}[g]_k$" and "$\sigma_{(\alpha,\beta)}[g]_k$" respectively in the denominator.*

Remark 11. *If we take "$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[h]_k$" instead of "$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$" and other conditions remain same, the results of Theorem 6 remain true with "$\tau_{(\alpha,\beta)}[g]_k$", "$\overline{\tau}_{(\alpha,\beta)}[g]_k$" and "$\exp(\alpha(T_k^{-1}(T_g(r))))$" replaced by "$\overline{\sigma}_{(\alpha,\beta)}[h]_k$", "$\sigma_{(\alpha,\beta)}[h]_k$" and "$\exp(\alpha(T_k^{-1}(T_h(r))))$" respectively in the denominator.*

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k-FREE NUMBERS AND INTEGER PARTS OF αp

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Abstract. In this note, we obtain asymptotic results on integer parts of αp that are free of kth powers of primes, where p is a prime number and α is a positive real number.

1. Introduction and Statement of Results

Let α and β be real numbers such that $\alpha > 0$. Let $\lfloor x \rfloor$ denote the largest integer not greater than x. Sequences of the form $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ are called Beatty sequences. A Beatty sequence is said to be homogeneous if $\beta = 0$. Beatty sequences have been attracting a lot of attention since they can be viewed as analogues of arithmetic progressions, therefore they show up in a broad context. The interested reader is referred to [1, 2, 4–6, 8–11, 14–16, 19, 24].

Let $k \geqslant 2$ be an integer. An integer is said to be *k-free* if it is not divisible by a kth power of a prime. Very recently in [3], an asymptotic formula with an explicit error term is obtained for k-free values of homogeneous Beatty sequences at prime arguments (i.e. sequences of the form $\{\lfloor \alpha p \rfloor\}_{p=2}^{\infty}$) provided that α is of finite type (see Definition 1). This result can be viewed as a natural analogue of the result of Mirsky [20]. In this article, we pursue this result and obtain two asymptotic formulas that are of the same flavour. The results we present here are well applicable to non-homogeneous Beatty sequences.

Theorem 1. *Let $k \geq 2$ be an integer. Let $\{\alpha_i\}_{i=1}^{\ell}$ be a finite type subset of irrational numbers each greater than one. Assume that $\{\alpha_i\}_{i=1}^{\ell}$ satisfies (1) for some $\tau > 0$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and*

$$\pi(x, k, \boldsymbol{\alpha}) = \#\{p \leqslant x : \lfloor \alpha_i p \rfloor \text{ is } k\text{-free for each } i = 1, \ldots, \ell\}.$$

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Then the following asymptotic is satisfied:

$$\pi(x,k,\boldsymbol{\alpha}) = \frac{\pi(x)}{\zeta^{\ell}(k)} + O\left(x^{1-\frac{k-1}{(k-1+\ell)(3\tau+2)+k(\ell-1)\tau+k\ell}}e^{\frac{C\log x}{\log\log x}}\right)$$

for some constant $C = C(\alpha_1,\dots,\alpha_\ell)$ *and every large* x.

A nested version of Theorem 1 is given below.

Theorem 2. *Let* $k \geqslant 2$ *be an integer. Let* $\{\alpha_1\alpha_2, \alpha_2\}$ *be a finite type subset of irrational numbers each greater than zero. Then the following asymptotic is satisfied:*

$$\#\{p \leqslant x : \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \text{ is } k\text{-free}\} = \frac{\pi(x)}{\zeta(k)} + O(x^{1-\varepsilon})$$

for some $\varepsilon > 0$.

Here, the interested reader is invited to investigate the following problem: Let $\{\alpha_i\}_{i=1}^n$ be positive real numbers. Define

$$a_j = \prod_{i=1}^{j} \alpha_{n+1-i}.$$

Assuming that $\{a_1, a_2, \dots, a_n\}$ is of finite type (see Definition 1), show that

$$\#\{p \leqslant x : \lfloor a_n \lfloor a_{n-1} \cdots \lfloor a_1 p \rfloor \rfloor \rfloor \text{ is } k\text{-free}\} = \frac{\pi(x)}{\zeta(k)} + O(x^{1-\varepsilon})$$

for some $\varepsilon > 0$. It might also be fruitful to investigate the possible power saving in the error term above.

1.1. **Preliminaries and Notation.**

1.1.1. *Notation.* We recall that for functions F and G where G is real non-negative, the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leqslant \alpha G$ holds for some constant $\alpha > 0$. Further we use $F \sim G$ to indicate $(F/G)(x)$ tends to 1 as $x \to \infty$.

Given a real number x, we use the notation $\{x\}$ for the fractional part of x, the notation $\lfloor x \rfloor$ for the greatest integer not exceeding x and $e(x) = e^{2\pi i x}$.

We use $\|x\|$ to denote the distance from the real number x to the nearest integer. $\Lambda(n) = \log p$ if $n = p^r$ where p is a prime number (here and hereafter). Otherwise, $\Lambda(n) = 0$. $\mu(n)$ denotes the Mobius function. $\phi(n)$ denotes the Euler's totient function. $\tau(n)$ denotes the number of positive divisors of n. We also use $\pi(x)$ to denote the number of primes not more than x.

1.1.2. *Preliminaries.*

Definition 1. *An irrational number α is called of finite type τ, if*

$$\tau = \sup \left\{ \beta : \liminf_{\substack{q\to\infty \\ q\in\mathbb{N}}} q^{\beta} ||\alpha q|| = 0 \right\} < \infty.$$

If α is an irrational number of finite type τ, then by Dirichlet's approximation theorem (Lemma 2.1 of [25]) one has $\tau \geqslant 1$. The celebrated theorems of Khinchin [17] and of Roth [21,22] state that $\tau = 1$ for almost all (in the sense of the Lebesque measure) real numbers and for all irrational algebraic numbers respectively.

Definition 2. *A finite subset of real numbers $\{\beta_1, \beta_2, \dots, \beta_\ell\}$ is said to be of finite type if there is $\tau > 0$ such that the inequality*

$$||h_1\beta_1 + h_2\beta_2 + \cdots + h_\ell\beta_\ell|| < (\max\{1, |h_1|, \dots, |h_\ell|\})^{-\tau} \tag{1}$$

has only finitely many solutions for $h_i \in \mathbb{Z}$.

If $\{\beta_i\}_{i=1}^{\ell}$ satisfies (1) for some $\tau > 0$, then it follows from Dirichlet's theorem on rational approximations that $\tau \geqslant 1$. Furthermore, such a set is linearly independent over $\mathbb{Q}$.

Throughout this paper, we shall mostly use the weak form of the prime number theorem, that is

$$\pi(x) \sim \frac{x}{\log x}.$$

Lemma 1. *For every positive integer $n \geq 1$,*

$$\tau(n) < e^{\frac{C \log 5n}{\log\log 5n}}$$

for some constant $C > 0$.

Proof. Follows from [23, Theorem 2.11]. $\square$

Lemma 2. *If*

$$\left| \alpha - \frac{a}{q} \right| \leqslant \frac{1}{q^2}$$

for some integers a and q such that $(a, q) = 1$, then

$$\sum_{p\leqslant x} e(\alpha p) \ll x \log^3 x \left(q^{-\frac{1}{2}} + x^{-\frac{1}{5}} + q^{\frac{1}{2}} x^{-\frac{1}{2}} \right).$$

Proof. This follows in a standard way using the main result of [12, §25]. $\square$

Lemma 3 (Erdős-Turán-Koksma Inequality)**.** *If $\{\boldsymbol{x}_i\}_{i=1}^{N}$ is a finite sequence in $\mathbb{R}^\ell$, then for any $J \subseteq [0,1)^\ell$ that is a Cartesian product of subintervals of $[0,1)$ and any $H \geqslant 1$, we have*

$$\#\{1 \leqslant i \leqslant N : \ \boldsymbol{x}_i \in J \mod 1\} - |J|N \ll \frac{N}{H} + \sum_{0<||\boldsymbol{h}||\leqslant H} \frac{1}{r(\boldsymbol{h})} \left| \sum_{1\leqslant i\leqslant N} e(\langle \boldsymbol{h}, \boldsymbol{x}_i \rangle) \right|.$$

Here $|J|$ denotes the ℓ-dimensional Lebesgue measure of J, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^\ell$ and we set $||\boldsymbol{h}|| = \max_{1 \leqslant i \leqslant \ell}\{|h_i|\}$ and

$$r(\boldsymbol{h}) = \prod_{i=1}^{\ell} \max\{|h_i|, 1\} \tag{2}$$

for all $\boldsymbol{h} = (h_1, h_2, \ldots, h_\ell) \in \mathbb{Z}^\ell$. Moreover, the implied constant depends only on ℓ.

Proof. For the proof see [18]. □

The following lemma is a classical result due to Vinogradov [26, Lemma 12].

Lemma 4. *Let α, β and Δ be real numbers such that*

$$0 < \Delta < \frac{1}{2} \quad \text{and} \quad \Delta \leqslant \beta - \alpha \leqslant 1 - \Delta.$$

Then there exists a periodic function $\Psi(x)$, with period 1, satisfying

(i) $\Psi(x) = 1$ *in the interval* $\alpha + \frac{1}{2}\Delta \leqslant x \leqslant \beta - \frac{1}{2}\Delta$,

(ii) $\Psi(x) = 0$ *in the interval* $\beta + \frac{1}{2}\Delta \leqslant x \leqslant 1 + \alpha - \frac{1}{2}\Delta$,

(iii) $0 \leqslant \Psi(x) \leqslant 1$ *in the remainder of the interval* $\alpha - \frac{1}{2}\Delta \leqslant x \leqslant 1 + \alpha - \frac{1}{2}\Delta$,

(iv) $\Psi(x)$ *has a Fourier expansion of the form*

$$\Psi(x) = \sum_{h=-\infty}^{\infty} a_h e(hx),$$

where

$$|a_h| \leqslant c \cdot \min\left\{|h|^{-1}, |h|^{-2}\Delta^{-1}\right\}$$

for every $|h| \geqslant 1$ and some c fixed. Furthermore, $a_0 = \beta - \alpha$.

2. Proof of The Main Results

2.1. **Proof of Theorem 1.** Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and

$$\pi(x, k, \boldsymbol{\alpha}) = \#\{p \leqslant x : \lfloor \alpha_i p \rfloor \text{ is } k\text{-free for each } i = 1, \ldots, \ell\}.$$

Let $\mathcal{I}_k$ denote the characteristic function of k-free integers. Since

$$\mathcal{I}_k(n) = \sum_{d^k | n} \mu(d), \tag{3}$$

we have

$$\begin{aligned}
&\pi(x,k,\boldsymbol{\alpha})\\
&\quad= \sum_{p\leqslant x}\mathcal{I}_k(\lfloor\alpha_1 p\rfloor)\cdots\mathcal{I}_k(\lfloor\alpha_\ell p\rfloor)\\
&\quad= \sum_{p\leqslant x}\left(\sum_{d_1^k\mid\lfloor\alpha_1 p\rfloor}\mu(d_1)\right)\cdots\left(\sum_{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor}\mu(d_\ell)\right)\\
&\quad= \sum_{p\leqslant x}\sum_{\substack{(d_1,\ldots,d_\ell)\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\\
&\quad= \sum_{(d_1,\ldots,d_\ell)}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}}1\\
&\quad= \sum_{\substack{(d_1,\ldots,d_\ell)\\ d_i\leqslant z\\ i=1,\ldots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}}1 \;+\; \sum_{\substack{(d_1,\ldots,d_\ell)\\ d_i>z\\ \text{for some } i=1,\ldots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}}1,
\end{aligned}$$

where $z \leqslant x^{1/k}$ will be chosen later. It follows from Lemma 1 that for all $i = 1, 2, \ldots, \ell$ there exists a positive constant $c_i = c_i(\alpha_i)$ depending on α_i such that

$$\tau(\lfloor\alpha_i p\rfloor) \ll e^{\frac{c_i \log x}{\log\log x}}$$

for every $p \leqslant x$. Then, for all $i = 1, 2, \ldots, \ell$ and $p \leq x$

$$\tau(\lfloor\alpha_i p\rfloor) \ll e^{\frac{c \log x}{\log\log x}}, \tag{4}$$

where $c = \max\{c_1, \ldots, c_\ell\}$. Set $C = c(\ell-1)$. Then, by (4) and using partial summation in the last step, we get

$$\begin{aligned}
&\sum_{\substack{(d_1,\ldots,d_\ell)\\ d_i>z\\ \text{for some } i=1,\ldots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}}1\\
&\quad< \sum_{\substack{(d_1,\ldots,d_\ell)\\ d_1>z}}\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}}1+\cdots+\sum_{\substack{(d_1,\ldots,d_\ell)\\ d_\ell>z}}\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}}1\\
&\quad= \sum_{p\leqslant x}\left(\sum_{\substack{d_1^k\mid\lfloor\alpha_1 p\rfloor\\ d_1>z}}1\right)\cdots\left(\sum_{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor}1\right)+\cdots+\sum_{p\leqslant x}\left(\sum_{d_1^k\mid\lfloor\alpha_1 p\rfloor}1\right)\cdots\left(\sum_{\substack{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor\\ d_\ell>z}}1\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{p\leqslant x}\left(\sum_{\substack{d_1^k\mid\lfloor\alpha_1 p\rfloor\\ d_1>z}}1\right)\left(\prod_{i=2}^{\ell}\tau(\lfloor\alpha_i p\rfloor)\right)+\cdots+\sum_{p\leqslant x}\left(\sum_{\substack{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor\\ d_\ell>z}}1\right)\left(\prod_{i=1}^{\ell-1}\tau(\lfloor\alpha_i p\rfloor)\right)\\
&\ll e^{\frac{C\log x}{\log\log x}}\left(\sum_{p\leqslant x}\sum_{\substack{d_1^k\mid\lfloor\alpha_1 p\rfloor\\ d_1>z}}1+\cdots+\sum_{p\leqslant x}\sum_{\substack{d_\ell^k\mid\lfloor\alpha_\ell p\rfloor\\ d_\ell>z}}1\right)\\
&\ll e^{\frac{C\log x}{\log\log x}}\left(\sum_{d_1>z}\sum_{\substack{p\leqslant x\\ d_1^k\mid\lfloor\alpha_1 p\rfloor}}1+\cdots+\sum_{d_\ell>z}\sum_{\substack{p\leqslant x\\ d_\ell^k\mid\lfloor\alpha_\ell p\rfloor}}1\right)\\
&\leqslant e^{\frac{C\log x}{\log\log x}}\left(\sum_{d_1>z}\sum_{\substack{m\leqslant\alpha_1 x\\ d_1^k\mid m}}1+\cdots+\sum_{d_\ell>z}\sum_{\substack{m\leqslant\alpha_\ell x\\ d_\ell^k\mid m}}1\right)\\
&\leqslant e^{\frac{C\log x}{\log\log x}}\left(\sum_{d_1>z}\frac{\alpha_1 x}{d_1^k}+\cdots+\sum_{d_\ell>z}\frac{\alpha_\ell x}{d_\ell^k}\right)\ll\frac{e^{\frac{C\log x}{\log\log x}}x}{z^{k-1}}.
\end{aligned}
$$

Therefore,

$$
\pi(x,k,\boldsymbol{\alpha})=\sum_{\substack{(d_1,\ldots,d_\ell)\\ d_i\leqslant z\\ i=1,\ldots,\ell}}\mu(d_1)\cdots\mu(d_\ell)\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}}1+O\left(\frac{e^{\frac{C\log x}{\log\log x}}x}{z^{k-1}}\right). \tag{5}
$$

Next, we will study the sum above appearing in (5) which runs over all tuples $(d_1,\ldots,d_\ell)$ of positive integers where $d_i\leqslant z$ for all $i=1,\ldots,\ell$.
So, let $\mathbf{d}=(d_1,\ldots,d_\ell)$ be such a tuple and set

$$
D=\prod_{j=1}^{\ell}d_j^k,\qquad D_i=\prod_{\substack{j=1\\ j\neq i}}^{\ell}d_j^k\quad\text{and}\quad\mathcal{I}_{\mathbf{d}}=\left[0,\frac{1}{d_1^k}\right)\times\cdots\times\left[0,\frac{1}{d_\ell^k}\right) \tag{6}
$$

for all $i=1,\ldots,\ell$. For a positive integer i, let p_i denote the ith prime. Observing that

$$
\lfloor\alpha p\rfloor\equiv 0\pmod d\text{ if and only if }\left\{\frac{\alpha p}{d}\right\}<\frac{1}{d}, \tag{7}
$$

we have

$$\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}} 1 = \sum_{\substack{p\leqslant x\\ \lfloor\alpha_i p\rfloor\equiv 0 \pmod{d_i^k}\\ i=1,\ldots,\ell}} 1 = \sum_{\substack{p\leqslant x\\ \left\{\frac{\alpha_i p}{d_i^k}\right\}<\frac{1}{d_i^k}\\ i=1,\ldots,\ell}} 1 = \sum_{\substack{p\leqslant x\\ \left(\left\{\frac{\alpha_1 p}{d_1^k}\right\},\ldots,\left\{\frac{\alpha_\ell p}{d_\ell^k}\right\}\right)\in\mathcal{I}_{\mathbf{d}}}} 1 \tag{8}$$
$$= \#\{i\leqslant\pi(x):\mathbf{t}_i\in\mathcal{I}_{\mathbf{d}}\},$$

where

$$\mathbf{t}_i = \left(\left\{\frac{\alpha_1 p_i}{d_1^k}\right\},\ldots,\left\{\frac{\alpha_\ell p_i}{d_\ell^k}\right\}\right).$$

It follows from Erdős-Turán-Koksma Inequality that for all $H\geqslant 1$,

$$\begin{aligned}
&\#\{i\leqslant\pi(x):\mathbf{t}_i\in\mathcal{I}_{\mathbf{d}}\} - \frac{\pi(x)}{d_1^k\cdots d_\ell^k}\\
&\ll \frac{\pi(x)}{H} + \sum_{0<\|\mathbf{h}\|\leqslant H}\frac{1}{r(\mathbf{h})}\left|\sum_{i\leqslant\pi(x)} e\left(\langle\mathbf{h},\mathbf{t}_i\rangle\right)\right| \\
&\ll \frac{\pi(x)}{H} + \sum_{0<\|\mathbf{h}\|\leqslant H}\frac{1}{r(\mathbf{h})}\left|\sum_{p\leqslant x} e\left(\frac{h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell}{D}\cdot p\right)\right|.
\end{aligned}\tag{9}$$

Next, we shall prove the following lemma.

Lemma 5.

$$\begin{aligned}
&\sum_{p\leqslant x} e\left(\frac{h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell}{D}\cdot p\right)\\
&\qquad\ll x\log^3 x\left(x^{-\frac{1}{2(\tau+1)}}\left(\max\{|h_1|D_1,\ldots,|h_\ell|D_\ell\}\right)^{\frac{\tau}{2(\tau+1)}}D^{\frac{1}{2(\tau+1)}} + x^{-\frac{1}{5}}\right)
\end{aligned}$$

uniformly for all $\mathbf{h}=(h_1,\ldots,h_\ell)\in\mathbb{Z}^\ell$ *such that* $||\mathbf{h}||>0$*, where* D_i *and* D *are defined in* (6).

Proof. Since $\{\alpha_i\}_{i=1}^{\ell}$ satisfies (1) for some $\tau>0$, there exists a positive constant $A\geq 1$ such that

$$(\max\{|h_1|,\ldots,|h_\ell|\})^{-\tau}\leqslant A\ ||h_1\alpha_1+h_2\alpha_2+\cdots+h_\ell\alpha_\ell|| \tag{10}$$

for all $(h_1,\ldots,h_\ell)\in\mathbb{Z}^\ell$ such that $\max_{1\leqslant i\leqslant\ell}\{|h_i|\}>0$. Let $\mathbf{h}=(h_1,\ldots,h_\ell)\in\mathbb{Z}^\ell$ be such a tuple and set

$$m_{\mathbf{h}} = \frac{h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell}{D}.$$

Let $1\leqslant Q<x/2$ to be determined later. By Dirichlet's rational approximation theorem, there exists $\dfrac{r}{q}\in\mathbb{Q}$ such that $1\leqslant q\leqslant\dfrac{x}{Q}$ and

$$\left|m_{\mathbf{h}}-\frac{r}{q}\right|<\frac{Q}{qx}.$$

So,

$$\|q(h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell)\| < \frac{QD}{x}. \tag{11}$$

On the other hand, it follows from (10) that

$$\|q(h_1D_1\alpha_1+\cdots+h_\ell D_\ell\alpha_\ell)\| \geqslant A^{-1}q^{-\tau}(\max\{|h_1D_1|,\ldots,|h_\ell D_\ell|\})^{-\tau}. \tag{12}$$

Combining (11) and (12), we get

$$q \geqslant \frac{x^{\frac{1}{\tau}}}{\max\{|h_1D_1|,\ldots,|h_\ell D_\ell|\}A^{\frac{1}{\tau}}D^{\frac{1}{\tau}}Q^{\frac{1}{\tau}}}. \tag{13}$$

Then it follows from Lemma 2 that

$$\sum_{p\leqslant x} e\left(m_{\mathbf{h}}\cdot p\right) \ll x\log^3 x\left(x^{-\frac{1}{2\tau}}M^{\frac{1}{2}}D^{\frac{1}{2\tau}}Q^{\frac{1}{2\tau}}+x^{-\frac{1}{5}}+Q^{-\frac{1}{2}}\right), \tag{14}$$

where for the sake of brevity we set $M=\max\{|h_1D_1|,\ldots,|h_\ell D_\ell|\}$. By [13, Lemma 2.4], there exists $1\leqslant Q<x/2$ such that the left hand side of (14) is

$$\ll x\log^3 x\left(x^{-\frac{1}{2(\tau+1)}}M^{\frac{\tau}{2(\tau+1)}}D^{\frac{1}{2(\tau+1)}}+x^{-\frac{1}{2\tau}}M^{\frac{1}{2}}D^{\frac{1}{2\tau}}+x^{-\frac{1}{5}}\right).$$

At this point, we can assume that $x^{-\frac{1}{2\tau}}M^{\frac{1}{2}}D^{\frac{1}{2\tau}}<1$, because otherwise the required upper bound holds trivially. Therefore, the second term is beaten by the first term giving the proof of Lemma 5. □

We next proceed by plugging this upper bound into (9). We also use the upper bound $|h_i|\leqslant H$ together with the upper bounds $D\leqslant z^{k\ell}$ and $D_i\leqslant z^{k(\ell-1)}$. Then the difference in the first line of (9) is

$$\ll \frac{\pi(x)}{H}+\left(x^{1-\frac{1}{2(\tau+1)}}H^{\frac{\tau}{2(\tau+1)}}z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}}\log^3 x+x^{\frac{4}{5}}\log^3 x\right)\left(\sum_{0<\|\mathbf{h}\|\leqslant H}\frac{1}{r(\mathbf{h})}\right). \tag{15}$$

Now, by (2)

$$\sum_{0<\|\mathbf{h}\|\leqslant H}\frac{1}{r(\mathbf{h})} \leqslant \sum_{0\leqslant\|\mathbf{h}\|\leqslant H}\frac{1}{\prod_{i=1}^{\ell}\left(\max\{|h_i|,1\}\right)} \leqslant \left(1+2\sum_{1\leqslant h\leqslant H}\frac{1}{h}\right)^{\ell} \ll \log^{\ell} H, \tag{16}$$

where in the last step we use integral test. Here we note that the implied constant depends on ℓ. Coupling (8), (9), (15) and (16), we arrive at

$$\left(\sum_{\substack{p\leqslant x\\ d_i^k\mid\lfloor\alpha_i p\rfloor\\ i=1,\ldots,\ell}} 1\right)-\frac{\pi(x)}{d_1^k\cdots d_\ell^k}$$

$$\ll \frac{\pi(x)}{H} + x^{1-\frac{1}{2(\tau+1)}} H^{\frac{\tau}{2(\tau+1)}} z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}} \log^{\ell} H \log^3 x + x^{\frac{4}{5}} \log^{\ell} H \log^3 x \quad (17)$$

for every $H \geqslant 1$ and every $(d_1, \dots, d_\ell)$ such that $d_i \leqslant z \leqslant x^{1/k}$ for each i. Noting $\pi(x) \ll x$ and choosing $1 \leqslant H \leqslant x$ by [13, Lemma 2.4], the left hand side of (17) is

$$\ll \log^{\ell+3} x \left(x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}} + x^{1-\frac{1}{2(\tau+1)}} z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}} + x^{\frac{4}{5}} \right).$$

On summing this over all tuples $(d_1, \dots, d_\ell)$ of positive integers where $d_i \leqslant z$ for all $i = 1, \dots, \ell$, we observe from (5) that for all $1 \leqslant z \leqslant x^{1/k}$,

$$\pi(x,k,\boldsymbol{\alpha}) - \pi(x) \sum_{\substack{(d_1,\dots,d_\ell) \\ d_i \leqslant z \\ i=1,\dots,\ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{d_1^k \cdots d_\ell^k}$$

$$\ll \log^{\ell+3} x \left(x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}+\ell} + x^{1-\frac{1}{2(\tau+1)}} z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}+\ell} + x^{\frac{4}{5}} z^{\ell} \right) + \frac{e^{\frac{C \log x}{\log\log x}} x}{z^{k-1}}.$$

Here,

$$\sum_{\substack{(d_1,\dots,d_\ell) \\ d_i \leqslant z \\ i=1,\dots,\ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{d_1^k \cdots d_\ell^k} = \left(\sum_{d \leqslant z} \frac{\mu(d)}{d^k} \right)^{\ell}$$

and using the following inequality

$$\left| \sum_{d \leqslant z} \frac{\mu(d)}{d^k} - \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \right| \leqslant \sum_{d>z} \frac{1}{d^k} \ll \frac{1}{z^{k-1}},$$

it follows by the mean value theorem that

$$\left(\sum_{d \leqslant z} \frac{\mu(d)}{d^k} \right)^{\ell} - \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \right)^{\ell} \ll \frac{1}{z^{k-1}}.$$

Therefore, the contribution of the sums running over $d_i \leqslant z$ for all $i = 1, \dots, \ell$ is

$$\frac{\pi(x)}{\zeta^{\ell}(k)} + O\left(\frac{\pi(x)}{z^{k-1}} \right)$$

yielding for all $1 \leqslant z \leqslant x^{1/k}$

$$\pi(x,k,\boldsymbol{\alpha}) - \frac{\pi(x)}{\zeta^{\ell}(k)}$$

$$\ll \log^{\ell+3} x \left(x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}+\ell} + x^{1-\frac{1}{2(\tau+1)}} z^{\frac{k(\ell-1)\tau+k\ell}{2(\tau+1)}+\ell} + x^{\frac{4}{5}} z^{\ell} \right) + \frac{e^{\frac{C \log x}{\log\log x}} x}{z^{k-1}}, \quad (18)$$

where $C = C(\ell, \boldsymbol{\alpha})$ is positive. On the right hand side of (18), the first term beats the third term as $\tau \geq 1$ and the second term whenever

$$z \leqslant x^{\frac{1}{k(\ell-1)\tau+k\ell}}$$

which one can assume since otherwise (18) holds trivially. Using now [13, Lemma 2.4] to choose optimal $z \leqslant x^{1/k}$, the left hand side of (18) is

$$\begin{aligned}
&\ll e^{\frac{C' \log x}{\log\log x}} \left(x^{1-\frac{1}{3\tau+2}} + x^{\frac{1}{k}} + x^{\frac{(k-1)(3\tau+1)+k(\ell-1)\tau+k\ell+\ell(3\tau+2)}{(k-1)(3\tau+2)+k(\ell-1)\tau+k\ell+\ell(3\tau+2)}} \right) \\
&\ll x^{1-\frac{k-1}{(k-1+\ell)(3\tau+2)+k(\ell-1)\tau+k\ell}} e^{\frac{C' \log x}{\log\log x}}
\end{aligned}$$

for some constant C' depending on ℓ and $\boldsymbol{\alpha}$, therefore the claim follows.

2.2. **Proof of Theorem 2.** The proof will be similar to that of Theorem 1. We shall therefore be brief. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and define

$$\pi_{\boldsymbol{\alpha}}(x,k) = \#\{p \leqslant x : \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \text{ is } k\text{-free}\}.$$

Let $1 \leqslant z \leqslant x^{1/k}$ be a number to be determined. Using (3), it follows that

$$\pi_{\boldsymbol{\alpha}}(x,k) = \sum_{p \leqslant x} \sum_{d^k \mid \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor} \mu(d) = \sum_{p \leqslant x} \sum_{\substack{d^k \mid \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d \leqslant z}} \mu(d) + \sum_{p \leqslant x} \sum_{\substack{d^k \mid \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d > z}} \mu(d).$$

As we did before, we have

$$\sum_{p \leqslant x} \sum_{\substack{d^k \mid \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d > z}} \mu(d) \ll \frac{x}{z^{k-1}},$$

where the implied constant depends only on α_1 and α_2. This yields

$$\pi_{\boldsymbol{\alpha}}(x,k) = \sum_{p \leqslant x} \sum_{\substack{d^k \mid \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d \leqslant z}} \mu(d) + O\left(\frac{x}{z^{k-1}}\right).$$

We now proceed to derive the main term. Writing

$$\sum_{p \leqslant x} \sum_{\substack{d^k \mid \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \\ d \leqslant z}} \mu(d) = \sum_{d \leqslant z} \mu(d) \left(\left(\sum_{\substack{p \leqslant x \\ \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \equiv 0 \pmod{d^k}}} 1 \right) - \frac{\pi(x)}{d^k} \right) + \pi(x) \sum_{d \leqslant z} \frac{\mu(d)}{d^k},$$

and using partial summation to get

$$\sum_{d \leqslant z} \frac{\mu(d)}{d^k} = \frac{1}{\zeta(k)} + O\left(\frac{1}{z^{k-1}}\right),$$

one arrives at

$$\pi_{\boldsymbol{\alpha}}(x,k) = \frac{\pi(x)}{\zeta(k)} + O\left(\frac{x}{z^{k-1}} + \sum_{d\leqslant z}\left|\left(\sum_{\substack{p\leqslant x\\ \lfloor\alpha_1\lfloor\alpha_2 p\rfloor\rfloor\equiv 0 \pmod{d^k}}} 1\right) - \frac{\pi(x)}{d^k}\right|\right) \tag{19}$$

for any $1\leqslant z\leqslant x^{1/k}$. Let us now concentrate on the error term and proceed to show that it is $\ll x^{1-\varepsilon}$ for some $\varepsilon>0$. Using observation (7), together with Lemma 3 one ends up with

$$\left(\sum_{\substack{p\leqslant x\\ \lfloor\alpha_1\lfloor\alpha_2 p\rfloor\rfloor\equiv 0 \pmod{d^k}}} 1\right) - \frac{\pi(x)}{d^k} \ll \frac{\pi(x)}{H_1} + \sum_{1\leqslant|h_1|\leqslant H_1}\frac{1}{|h_1|}\left|\sum_{p\leqslant x} e\left(\frac{\alpha_1 h_1\lfloor\alpha_2 p\rfloor}{d^k}\right)\right|, \tag{20}$$

where H_1 is a positive number to be determined. So, it boils down to estimate the exponential sum above. To do this, we let K be a sufficiently large number and we write

$$\lfloor\alpha_2 p\rfloor = \alpha_2 p - \{\alpha_2 p\},$$

yielding

$$\sum_{p\leqslant x} e\left(\frac{\alpha_1 h_1\lfloor\alpha_2 p\rfloor}{d^k}\right) = \sum_{0\leqslant i\leqslant K-1}\sum_{p\in I_i(x)} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k} - \frac{\alpha_1 h_1\{\alpha_2 p\}}{d^k}\right), \tag{21}$$

where $I_i(x) = \left\{p\leqslant x : \frac{i}{K}\leqslant\{\alpha_2 p\}<\frac{i+1}{K}\right\}$. Since

$$e(t) = 1 + O(|t|)$$

uniformly for all $t\in\mathbb{R}$, we have

$$e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k} - \frac{\alpha_1 h_1\{\alpha_2 p\}}{d^k}\right) = e\left(-\frac{\alpha_1 h_1 i}{Kd^k}\right)\left(e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) + O\left(\frac{|h_1|}{Kd^k}\right)\right)$$

if $p\in I_i(x)$. Therefore, the left hand side of (21) is

$$\ll \frac{|h_1|\pi(x)}{Kd^k} + \sum_{0\leqslant i\leqslant K-1}\left|\sum_{p\in I_i(x)} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right)\right|. \tag{22}$$

Given $0\leqslant i\leqslant K-1$, let $\beta_i = i/K$, $\gamma_i=(i+1)/K$ and $0<\Delta<1/K$ be a number to be chosen. By Lemma 4, there exists a periodic function $\Psi_i(x)$, with period 1, satisfying

(i) $\Psi_i(x)=1$ in the interval $\beta_i+\frac{1}{2}\Delta\leqslant x\leqslant\gamma_i-\frac{1}{2}\Delta$,

(ii) $\Psi_i(x)=0$ in the interval $\gamma_i+\frac{1}{2}\Delta\leqslant x\leqslant 1+\beta_i-\frac{1}{2}\Delta$,

(iii) $0 \leqslant \Psi_i(x) \leqslant 1$ in the remainder of the interval $\beta_i - \frac{1}{2}\Delta \leqslant x \leqslant 1 + \beta_i - \frac{1}{2}\Delta$,

(iv) $\Psi_i(x)$ has a Fourier expansion of the form

$$\Psi_i(x) = \sum_{h=-\infty}^{\infty} a_h e(hx),$$

where $a_0 = 1/K$ and

$$|a_h| \leqslant c \cdot \min\left\{|h|^{-1}, |h|^{-2}\Delta^{-1}\right\}$$

for every $|h| \geqslant 1$ and some c fixed.

Let $\psi_i(x)$ be 1 if $\beta_i \leqslant \{x\} \leqslant \gamma_i$ and $\psi_i(x) = 0$ otherwise. It follows that $\Psi_i(x)$ and $\psi_i(x)$ agree on $[0,1]$ except possibly for two subintervals of $[0,1]$ of length $\leqslant \Delta$. Therefore,

$$\sum_{p \in I_i(x)} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) = \sum_{p \leqslant x} \Psi_i(\alpha_2 p) e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) + O\left(\sum_{\substack{p \leqslant x \\ \{\alpha_2 p\} \in I}} 1\right) \tag{23}$$

where I is a union of two intervals and is of length Δ. Since α_2 is of finite type, following the proof of Theorem 5.1 in [8] together with a partial summation argument, it follows that for some $0 < \varepsilon'' < 1/5$, one has

$$\sum_{\substack{p \leqslant x \\ \{\alpha_2 p\} \in I}} 1 = \Delta\pi(x) + O\left(x^{1-\varepsilon''}\right), \tag{24}$$

uniformly for all $0 < \Delta < 1/K$. Therefore, we see that the left hand side of (23) is

$$= \frac{1}{K} \sum_{p \leqslant x} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) + O\left(\sum_{|h_2| > 0} |a_{h_2}| \left|\sum_{p \leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1 + \alpha_2 h_2 d^k)p}{d^k}\right)\right| + \Delta\pi(x) + x^{1-\varepsilon''}\right).$$

Letting H_2 be a positive integer to be determined, we split the sum running over h_2 at H_2. For $|h_2| > H_2$, estimating the innermost exponential sum by $\pi(x)$, and using the upper bounds $a_h \ll 1/(\Delta h^2)$ and $a_h \ll 1/|h|$, we obtain that the left hand side of (23) is

$$= \frac{1}{K} \sum_{p \leqslant x} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right) + O\left(\sum_{0 < |h_2| \leqslant H_2} \frac{1}{|h_2|} \left|\sum_{p \leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1 + \alpha_2 h_2 d^k)p}{d^k}\right)\right|\right) + O\left(\frac{\pi(x)}{\Delta H_2} + \Delta\pi(x) + x^{1-\varepsilon''}\right).$$

Plugging this upper bound into (22) yields that

$$\begin{aligned}&\sum_{p\leqslant x} e\left(\frac{\alpha_1 h_1\lfloor\alpha_2 p\rfloor}{d^k}\right)\\ &\ll \left|\sum_{p\leqslant x} e\left(\frac{\alpha_1\alpha_2 h_1 p}{d^k}\right)\right| + \sum_{i\leqslant K}\sum_{0<|h_2|\leqslant H_2}\frac{1}{|h_2|}\left|\sum_{p\leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1+\alpha_2 h_2 d^k)p}{d^k}\right)\right| \qquad (25)\\ &+\frac{\pi(x)K}{\Delta H_2}+\Delta K\pi(x)+Kx^{1-\varepsilon''}+\frac{|h_1|\pi(x)}{Kd^k}.\end{aligned}$$

We are therefore left with the estimation of

$$\sum_{p\leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1+\alpha_2 h_2 d^k)p}{d^k}\right), \qquad (26)$$

whenever $\max\{|h_1|,|h_2|\}>0$. To estimate the exponential sum, by Dirichlet's theorem we pick up a rational number a/q satisfying

$$\left|\frac{(\alpha_1\alpha_2 h_1+\alpha_2 h_2 d^k)}{d^k}-\frac{a}{q}\right|<\frac{1}{qx^{1-\kappa}}$$

with $1\leqslant q\leqslant x^{1-\kappa}$, where $0<\kappa<1$ is to be determined. Since $\{\alpha_1\alpha_2,\alpha_2\}$ is of finite type, similar to how we obtain (13)

$$\frac{x^{\frac{1-\kappa}{\tau}}}{d^{\frac{k}{\tau}}\max\{|h_1|,|h_2 d^k|\}}\ll q\leqslant x^{1-\kappa}$$

for some $\tau\geqslant 1$. Then by Lemma 2, the exponential sum (26) is

$$\ll x\log^3 x\left((\max\{|h_1|,|h_2 d^k|\})^{\frac12} d^{\frac{k}{2\tau}} x^{-\frac{1-\kappa}{2\tau}}+x^{-\frac15}+x^{-\frac{\kappa}{2}}\right).$$

At this point, we assume that $0<\max\{|h_1|,|h_2|\}\leqslant x^{\varepsilon'}$ where ε' is a sufficiently small number to be determined in terms of κ. Then,

$$\sum_{p\leqslant x} e\left(\frac{(\alpha_1\alpha_2 h_1+\alpha_2 h_2 d^k)p}{d^k}\right)\ll\left(d^{\frac{k\tau+k}{2\tau}}x^{1-\frac{1-\kappa}{2\tau}+\frac{\varepsilon'}{2}}+x^{\frac45}+x^{1-\frac{\kappa}{2}}\right)\log^3 x, \qquad (27)$$

uniformly for

$$0<\max\{|h_1|,|h_2|\}\leqslant x^{\varepsilon'}.$$

Plugging the upper bound (27) into (25), we arrive at

$$\begin{aligned}\sum_{p\leqslant x} e\left(\frac{\alpha_1 h_1\lfloor\alpha_2 p\rfloor}{d^k}\right)&\ll K\left(d^{\frac{k\tau+k}{2\tau}}x^{1-\frac{1-\kappa}{2\tau}+\frac{\varepsilon'}{2}}+x^{\frac45}+x^{1-\frac{\kappa}{2}}\right)\log^4 x\\ &\qquad+\frac{\pi(x)K}{\Delta H_2}+\Delta K\pi(x)+Kx^{1-\varepsilon''}+\frac{H_1\pi(x)}{Kd^k},\end{aligned}$$

uniformly for $|h_1| \leqslant x^{\varepsilon'}$, provided that $H_2 \leqslant x^{\varepsilon'}$, $0 < \kappa < 1$, $0 < \Delta < 1/K$ and K is sufficiently large. Plugging this upper bound into (20) and summing over $d \leqslant z$, we see that the error term in (19) is

$$\ll \frac{xz}{H_1} + K\left(z^{1+\frac{k\tau+k}{2\tau}} x^{1-\frac{1-\kappa}{2\tau}+\frac{\varepsilon'}{2}} + zx^{\frac{4}{5}} + zx^{1-\frac{\kappa}{2}}\right)\log^5 x$$
$$+\left(\frac{zxK}{\Delta H_2} + z\Delta Kx + zKx^{1-\varepsilon''} + \frac{H_1 x}{K}\right)\log x + \frac{x}{z^{k-1}} \quad (28)$$

provided that $0 < H_1, H_2 \leqslant x^{\varepsilon'}$, $0 < \kappa < 1$, $0 < \Delta < 1/K$ and K is sufficiently large. We now make all unspecified constants explicit. For $0 < \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 < 1$ to be determined, we set

$$K = x^{\varepsilon_1}, H_1 = x^{\varepsilon_2}, H_2 = x^{\varepsilon_3}, \Delta = x^{-\varepsilon_4} \text{ and } z = x^{\varepsilon_5},$$

where $0 < \varepsilon_5 \leqslant 1/k$ (this assumption is from the beginning of the proof). Examining each term in (28), the right hand side of (28) is $\ll x^{1-\varepsilon}$ for some $\varepsilon > 0$, if the following inequalities are satisfied:

(1) $\varepsilon_5 < 1/k$,
(2) $\varepsilon_2, \varepsilon_3 < \varepsilon'$,
(3) $\varepsilon_5 < \varepsilon_2 < \varepsilon_1$,
(4) $\varepsilon_1 + \varepsilon_5 < \min\{\varepsilon_4, \varepsilon'', \kappa/2\}$,
(5) $\varepsilon_1 + \varepsilon_4 + \varepsilon_5 < \varepsilon_3$,
(6) $\varepsilon_1 + \varepsilon_5(1 + \frac{k\tau+k}{2\tau}) + \frac{\varepsilon'}{2} < \frac{1-\kappa}{2\tau}$,

where $\varepsilon'' < 1/5$ is a fixed positive number defined in (24), $\tau \geqslant 1$ is a fixed number and $0 < \kappa < 1$ and $0 < \varepsilon' < 1$ are to be chosen. We choose $\kappa = 2/5$ and $\varepsilon' = (1-\kappa)/(2\tau)$. Then since $\varepsilon'' < 1/5$, we assume that $\varepsilon_4 < \varepsilon''$ so that the fourth inequality becomes equivalent to $\varepsilon_1 + \varepsilon_5 < \varepsilon_4$. We next choose $\varepsilon_3 < \varepsilon'$ and $\varepsilon_4 < \min\{\varepsilon_3, \varepsilon''\}$ and $\varepsilon_1 < \min\{\varepsilon_4, \varepsilon_3 - \varepsilon_4, (1-\kappa)/(4\tau)\}$. Finally, we choose $\varepsilon_2 < \min\{\varepsilon_1, \varepsilon'\}$ and

$$\varepsilon_5 < \min\left\{\varepsilon_2, \varepsilon_4 - \varepsilon_1, \varepsilon_3 - \varepsilon_1 - \varepsilon_4, \frac{1}{k}, \frac{2\tau}{(k+2)\tau + k}\left(\frac{1-\kappa}{4\tau} - \varepsilon_1\right)\right\},$$

completing the proof.

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MODIFIED-LINDLEY DISTRIBUTION AND ITS APPLICATIONS TO THE REAL DATA

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Abstract. In this paper, a new three-parameter lifetime distribution is proposed by mixing modified Weibull and generalized gamma distributions. The point estimation on the distribution parameters are discussed through several estimators. The interval estimation is also studied with two methods based on asymptotic normality and likelihood ratio. A Monte Carlo simulation study is performed to evaluate the biases and mean square errors behaviors of point estimates for a different sample of size. A simulation study is also conducted to investigate the coverage probabilities of confidence intervals. The distribution modeling analyses are provided based on several real data sets to demonstrate the fitting ability of the introduced distribution.

1. Introduction

The Lindley (L) distribution is introduced in [18] with cumulative distribution function (cdf) and probability density function (pdf),

$$F_L(x;\theta) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x},$$

and

$$f_L(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \ x > 0$$

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respectively, and $\theta > 0$ is a parameter.

L distribution can be represented as a mixture of two distributions with pdf

$$f_L(x;\theta) = p\, f_E(x;\theta) + (1-p)\, f_G(x;\theta), \tag{1}$$

where $f_E(x;\theta) = \theta e^{-\theta x}$ and $f_G(x;\theta) = \theta^2 x e^{-\theta x}$ are the pdfs of exponential and gamma distributions respectively and $p = \frac{\theta}{1+\theta}$ is the mixing proportion of distributions. Since the L distribution is IFR, it is unsuitable for modelling the data that obeys the non-linear hazard rate structure. [12] introduced the power Lindley (PL) distribution, which generalizes the Lindley distribution with the following pdf

$$\begin{aligned} f_{PL}(x;\alpha,\theta) &= \tfrac{\alpha\theta^2}{\theta+1}(1+x^\alpha)\, x^{\alpha-1} e^{-\theta x^\alpha},\ x>0,\ \alpha,\theta>0, \\ &= p\, f_W(x;\alpha,\theta) + (1-p)\ f_{GG}(x;\alpha,\theta), \end{aligned}$$

where $f_W(x;\alpha,\theta) = \alpha\theta x^{\alpha-1}e^{-\theta x^\alpha}$ and $f_{GG}(x;\alpha,\theta) = \alpha\theta^2 x^{2\alpha-1}e^{-\theta x^\alpha}$ are the pdfs of Weibull and generalized gamma (GG) distributions respectively and $p = \frac{\theta}{1+\theta}$ is the mixing proportion of distributions. [12] investigated properties of the PL distribution with an application and outlined that the PL distribution is a better model than the other L and exponential based distribution.

Moreover, several generalizations have been proposed in the literature in order to increase the flexibility and usefulness of the L model. Some of them are: generalized Lindley (GL) [32], exponentiated Lindley (EL) [22], discrete Lindley [11], extended Lindley [6], beta Lindley [20,21], exponentiated power Lindley (EPL) [31], odd log logistic power Lindley [1], odd log-logistic Lindley Poisson [24], odd Burr Lindley [3], binomial discrete Lindley [16], Weibull-Lindley [4] and generalized power Lindley [15] among others.

This paper aims to introduce a new flexible distribution that generalizes the L and PL distributions with the same structure of (1). Furthermore, we are also motivated to propose a new L distribution because introduced model has various pdf shapes as well as non-monotone hazard rate function (hrf) shapes unlike L and PL models.

The paper is organized as follows: In Section 2, a new lifetime distribution is proposed and several distributional properties are discussed. Several point estimation methods are discussed for the distribution parameters in Section 3. In Section 4, the interval estimation is considered with two well-known methods. The Section 5 close the paper with three distribution modeling analyses based on real data.

2. Modified Lindley Distribution and Some Properties

A random variable X has a Modified Lindley (MoL) distribution if its pdf is given by

$$f_{MoL}(x;\boldsymbol{\Xi}) = \tfrac{\theta^2}{\theta+1}\left[(\alpha+\beta x)\, e^{-\theta x^\alpha\left(e^{\beta x}-1\right)+\beta x} + \alpha x^\alpha\right] x^{\alpha-1} e^{-\theta x^\alpha},\ x>0,$$

where $\alpha, \beta, \theta > 0$ are parameters and $\mathbf{\Xi} = (\alpha, \beta, \theta)$. Indeed MoL distribution is a mixture of two distribution with the following representation:

$$f\left(x; \mathbf{\Xi}\right) = p\, g_1\left(x; \alpha, \beta, \theta\right) + (1-p)\; g_2\left(x; \alpha, \theta\right),\; x > 0,\; \alpha, \beta, \theta > 0, \tag{2}$$

where $p = \frac{\theta}{\theta+1}$ is the weighting parameter of the distributions, $g_1\left(x; \alpha, \beta, \theta\right)$ is the pdf of Modified Weibull (MW) distribution introduced in [17], with the following pdf

$$g_1\left(x; \alpha, \beta, \theta\right) = \theta\left(\alpha + \beta x\right) x^{\alpha-1} e^{\beta x - \theta x^{\alpha} \exp(\beta x)},\; x > 0,$$

and $g_2\left(x; \alpha, \theta\right)$ is the pdf of a GG distribution introduced in [28], with the following pdf

$$g_2\left(x; \alpha, \theta\right) = \alpha \theta^2 x^{2\alpha-1} e^{-\theta x^{\alpha}},\; x > 0.$$

From (2), we see that the MoL distribution is a two-component mixture of MW and GG distributions with weighting parameter p. We denote the MoL distribution with parameter $\mathbf{\Xi}$ by $MoL(\mathbf{\Xi})$.
While $\beta \to 0$, MoL distribution reduces to the PL distribution. While $\beta \to 0$ and $\alpha \to 1$, it is reduced to L distribution.

The cdf and hrf of the MoL distribution are

$$F\left(x; \mathbf{\Xi}\right) = 1 - \frac{1 + \theta x^{\alpha} + \theta e^{-\theta x^{\alpha} e^{\beta x} + \theta x^{\alpha}}}{\theta + 1} e^{-\theta x^{\alpha}}, x > 0 \tag{3}$$

and

$$h\left(x; \mathbf{\Xi}\right) = \frac{\theta^2 x^{\alpha-1}\left\{\left(\alpha + \beta x\right) e^{-\theta x^{\alpha}\left(e^{\beta x} - 1\right) + \beta x} + \alpha x^{\alpha}\right\}}{1 + \theta x^{\alpha} + \theta e^{-\theta x^{\alpha} e^{\beta x} + \theta x^{\alpha}}}, x > 0$$

respectively. The plots of the pdf and hrf are given in Figure 1 to identify their possible shapes. These figures show that the MoL distribution can be unimodal, bimodal, decreasing and firstly decreasing then unimodal shaped. On the other hand, the hrf of MoL can be both monotone and non-monotone structures.

In distribution theory, stochastic ordering is an essential measure for evaluating the comparative behavior of random variables. It is known that $X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{st} Y$, see [25]. For more information about stochastic ordering with different applications, one can see [27]. Likelihood ratio ordering is shortly defined as follow: X is less than Y in the likelihood ratio order (denoted by $X <_{lr} Y$) if $f_X\left(x\right) / \ f_Y\left(x\right)$ increases in x over the union of the supports of X and Y .

Theorem 1. *If X $\sim MoL(\alpha, \beta, \theta_1)$ and Y $\sim MoL(\alpha, \beta, \theta_2)$ and $\theta_1 < \theta_2$, then $X <_{lr} Y$.*

Proof. See Appendix.

Corollary 1. *If X $\sim MoL(\alpha, \beta, \theta_1)$ and Y $\sim MoL(\alpha, \beta, \theta_2)$ and $\theta_1 < \theta_2$ then $X <_{hr} Y$ and $X <_{st} Y$.*

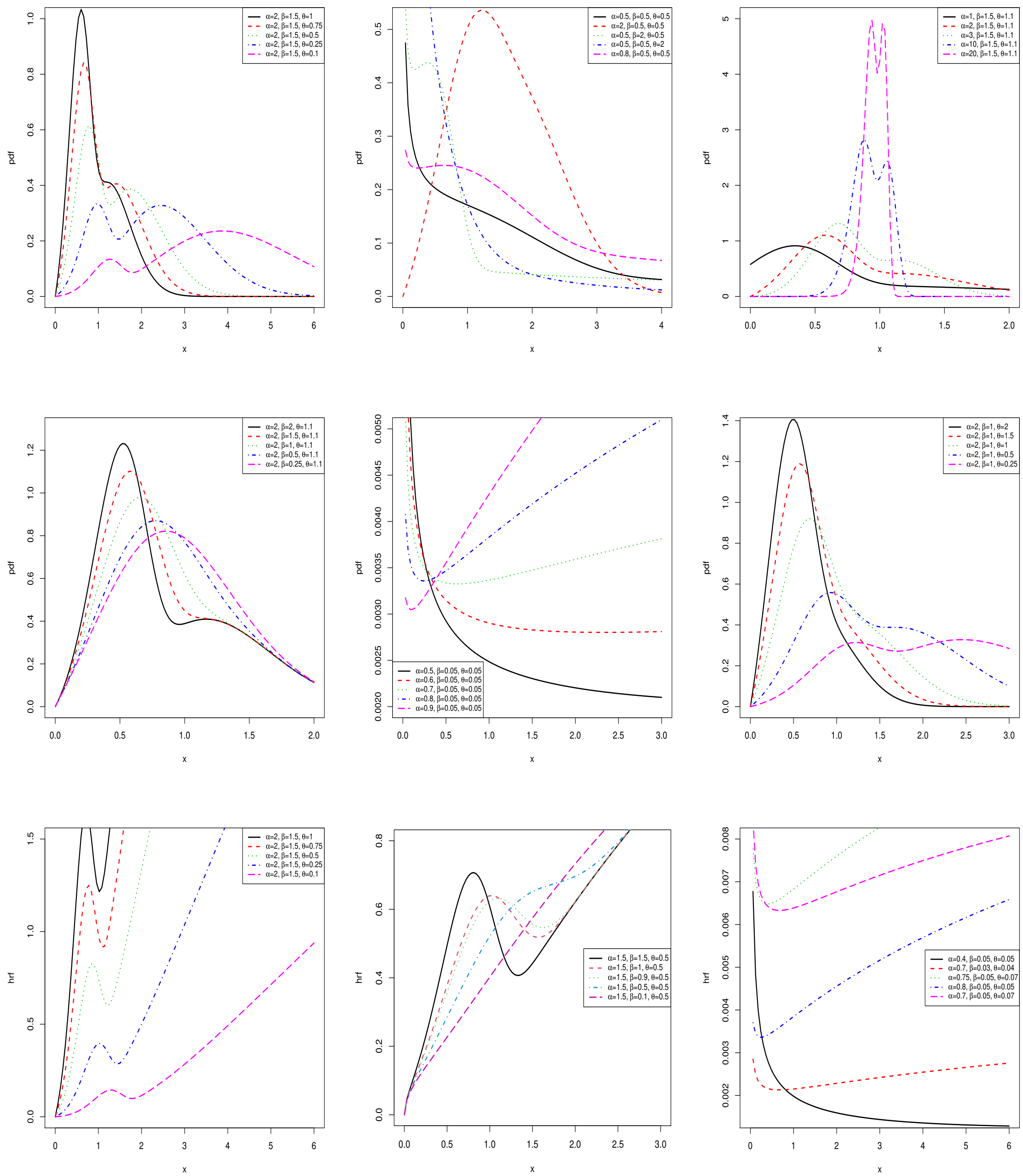

FIGURE 1. Possible pdf and hrf plots of MoL distribution

Theorem 2. *For $r \in \mathbb{N}^{+}$, the raw moments of $MoL(\boldsymbol{\Xi})$ are given by*

$$\mu_r' = E\left(X^r\right) = \frac{1}{\theta+1}\left[\Gamma\left(r/\alpha+2\right)\theta^{-r/\alpha} + \sum_{i_1,\ldots,i_r=1}^{\infty} A_{i_1,\ldots,i_r}\Gamma\left(s_r/\alpha+1\right)\theta^{1-s_r/\alpha}\right]. \tag{4}$$

Proof. See Appendix.

Corollary 2. *The mean and rth central moment of the $MoL(\boldsymbol{\Xi})$ are given, respectively, by*

$$\mu = \frac{1}{\theta+1}\left[\Gamma\left(1/\alpha+2\right)\theta^{-1/\alpha} + \sum_{i=1}^{\infty} a_i\Gamma\left(i/\alpha+1\right)\theta^{1-i/\alpha}\right], \tag{5}$$

and

$$\mu_r = E(X-\mu)^r = \sum_{h=0}^{r}(-1)^h \binom{r}{h} \mu^r \mu_{r-h}'. \tag{6}$$

Using (6), the skewness and kurtosis coefficients can be obtained by

$$\sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2},$$

respectively. The mean, variance, skewness and kurtosis are computed for some choices of parameters and given in Table 1. From Table 1, it is seen that the coefficient of kurtosis can take negative and positive values. This shows that the distribution has a flexible structure in data modeling. In addition, it is seen that the new distribution is flatter than the normal distribution. When θ increases, the kurtosis coefficient increases and the variance decreases. $E(X)$ decreases when the parameter β increases.

Table 1. The mean, variance, coefficients of skewness and kurtosis for some choices of parameters

		$\theta = 0.9$				$\theta = 1.5$			
α	β	$E(X)$	$Var(X)$	Skewness	Kurtosis	$E(X)$	$Var(X)$	Skewness	Kurtosis
0.9	0.9	1.5597	3.0668	2.2407	9.9440	0.7844	0.8065	2.5707	12.4062
	1.5	1.5104	3.1444	2.2221	9.7165	0.7466	0.8203	2.6166	12.4272
	5	1.4181	3.3299	2.1388	9.1238	0.6687	0.7341	2.5844	11.8172
1.5	0.9	1.1363	0.6088	1.1457	4.2521	0.7436	0.2664	1.3223	5.0558
	1.5	1.0910	0.6416	1.1738	4.1289	0.7042	0.2750	1.4381	5.1801
	5	0.9906	0.7491	1.1068	3.6418	0.6099	0.3188	1.5008	4.8333
5	0.9	0.9740	0.0536	-0.1046	2.6206	0.8486	0.0419	0.0095	2.7336
	1.5	0.9444	0.0614	0.0376	2.3197	0.8166	0.0466	0.2250	2.5554
	5	0.8473	0.1046	0.1864	1.5997	0.7086	0.0764	0.5842	1.9761

3. Point Estimation

In this section, the maximum likelihood, least square, weighted least square, Anderson-Darling, Cramer-von Mises, and maximum product spacing methods are discussed to estimate the MoL distribution parameters. It is noticed that these estimates are also used in [2], [13], [14], [29], [30] among others. Let $X_1, X_2, \ldots, X_n$ be a random sample from the MoL($\boldsymbol{\Xi}$) distribution with realizations $x_1, x_2, \ldots, x_n$. Furthermore, $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding ordered statistics with realizations $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. Then the log likelihood function can be written by

$$\begin{aligned}\ell(\boldsymbol{\Xi}) &= 2n\log\theta - n\log(\theta+1) + (\alpha-1)\sum_{i=1}^{n}\log(x_i) \\ &-\sum_{i=1}^{n}\log\left[(\alpha+\beta x_i)\exp\left\{-\theta x_i^{\alpha}\left(e^{\beta x_i}-1\right)-\beta x_i\right\}+\alpha x_i^{\alpha}\right]. \end{aligned} \tag{7}$$

Hence, the maximum likelihood estimate (MLE) $\widehat{\boldsymbol{\Xi}}$ of $\boldsymbol{\Xi}$ is written by

$$\widehat{\boldsymbol{\Xi}} = \arg\max_{\boldsymbol{\Xi}} \ell(\boldsymbol{\Xi}). \tag{8}$$

The maximum product spacing estimate (MPSE) was proposed by [9]. The MPSE $\widehat{\boldsymbol{\Xi}}_{MPS}$ of parameter $\boldsymbol{\Xi}$ are achieved by maximizing

$$MPS(\boldsymbol{\Xi}) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F(x_{(i)};\boldsymbol{\Xi}) - F(x_{(i-1)};\boldsymbol{\Xi})\right], \tag{9}$$

where, F is MoL cdf given in (3) and $F(x_{(0)};\boldsymbol{\Xi}) = 0$ and $F(x_{(n+1)};\boldsymbol{\Xi}) = 1$. Note that the MPSE can be written by

$$\widehat{\boldsymbol{\Xi}}_{MPS} = \arg\max_{\boldsymbol{\Xi}}\ MPS(\boldsymbol{\Xi}). \tag{10}$$

The least square estimate (LSE) $\widehat{\boldsymbol{\Xi}}_{LSE}$ of parameter $\boldsymbol{\Xi}$ are obtained by minimizing the function

$$LS(\boldsymbol{\Xi}) = \sum_{i=1}^{n}\left(F\left(x_{(i)};\boldsymbol{\Xi}\right) - \frac{i}{n+1}\right)^2, \tag{11}$$

where F is MoL cdf given in (3). Hence, LSE of $\boldsymbol{\Xi}$ is given by

$$\widehat{\boldsymbol{\Xi}}_{LSE} = \arg\min_{\boldsymbol{\Xi}} LS(\boldsymbol{\Xi}). \tag{12}$$

The weighted least square estimate (WLSE) $\widehat{\boldsymbol{\Xi}}_{WLSE}$ of $\boldsymbol{\Xi}$ are obtained by minimizing

$$WLS(\boldsymbol{\Xi}) = \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(F\left(x_{(i)};\boldsymbol{\Xi}\right) - \frac{i}{n+1}\right)^2. \tag{13}$$

Then the WLSE of Ξ is presented by

$$\widehat{\Xi}_{WLSE} = \arg\min_{\Xi} WLS(\Xi). \tag{14}$$

The Anderson-Darling (ADE) type estimate $\widehat{\Xi}_{AD}$ of parameters Ξ are obtained by minimizing

$$ADD\left(\Xi\right) = -n - \sum_{i=1}^{n} \tfrac{2i-1}{n} \left[\log F\left(x_{(i)};\Xi\right) + \log\left\{1 - F\left(x_{(n+1-i)};\Xi\right)\right\}\right]. \tag{15}$$

The ADE of Ξ is written by

$$\widehat{\Xi}_{AD} = \arg\min_{\Xi} ADD(\Xi). \tag{16}$$

The Cramer-von Mises (CVME) type estimate, $\widehat{\Xi}_{CVM}$ of parameter Ξ are obtained by minimizing

$$CVM\left(\Xi\right) = \tfrac{1}{12n} + \sum_{i=1}^{n} \left[F\left(x_{(i)};\Xi\right) - \frac{2i-1}{2n}\right]^2. \tag{17}$$

The CVME of Ξ is given by

$$\widehat{\Xi}_{CVM} = \arg\min_{\Xi} CVM(\Xi). \tag{18}$$

In order to achieve the values of estimates, the R functions such as `constrOptim`, `optim` or `maxLik` can be used.

The simulation study is performed for the bias and mean square errors (MSEs) of estimates and the results are presented by graphically. We consider $N = 1000$ trials of size $n = 20, 25, \ldots, 1000$ from the MoL distribution with true parameter $\Xi = (5, 5, 2)$. All estimates are achieved by using `constrOptim` routine in the R. The simulation results are presented in Figs. 2-4. Figs. 2-4 show that all estimates are consistent since the MSEs decrease to zero for large sample size. The CVME and MPSE have the maximum amount of the biases for all parameters while CVME and WLSE have the maximum MSEs for all parameters. On the other hand, MPSE is the best estimator according to MSEs for small sample size. It is noticed that the MPSE and MLE has almost same MSEs for moderate and large sample size cases. The ADE and LSE have the lowest bias for all parameters. As a final comment on the simulation study, we recommend that the MLE or MPSE should be used to estimate the parameters.

4. Interval Estimation of MoL Distribution Parameters

In this section, the confidence intervals (CIs) are discussed for the parameters a, β and θ. In general, CIs are constructed by using MLE based on pivotal quantities through the asymptotic normality(AN) property of MLE. These CIs are most

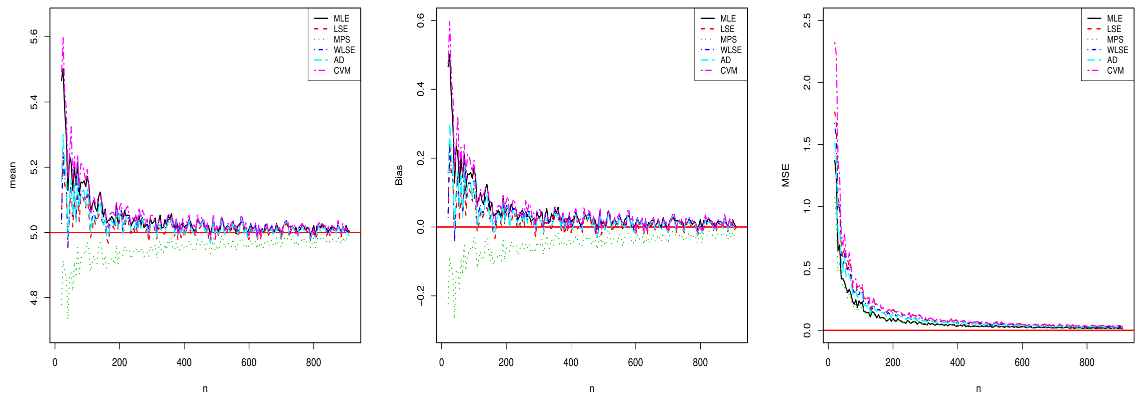

FIGURE 2. The empirical means, biases and MSEs of the parameter α

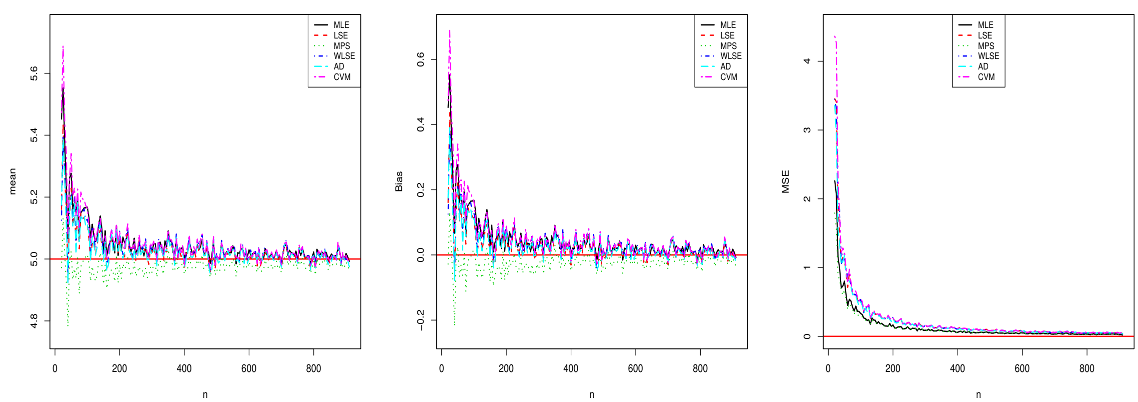

FIGURE 3. The empirical means, biases and MSEs of the parameter β

popular in many fields and they are commonly used in statistical software. The AN of MLE can be written by

$$\widehat{\boldsymbol{\Xi}} \xrightarrow{d} N_3\left(\boldsymbol{\Xi}, \mathbb{I}^{-1}\left(\boldsymbol{\Xi}\right)\right),$$

where $\widehat{\boldsymbol{\Xi}}$ is MLE of $\boldsymbol{\Xi}$ given in (8) and $\mathbb{I}\left(\boldsymbol{\Xi}\right)$ is Fisher Information matrix. Using this fact, the $100 \times (1-\gamma)\,\%$ AN CIs of parameters α, β and θ are constructed,

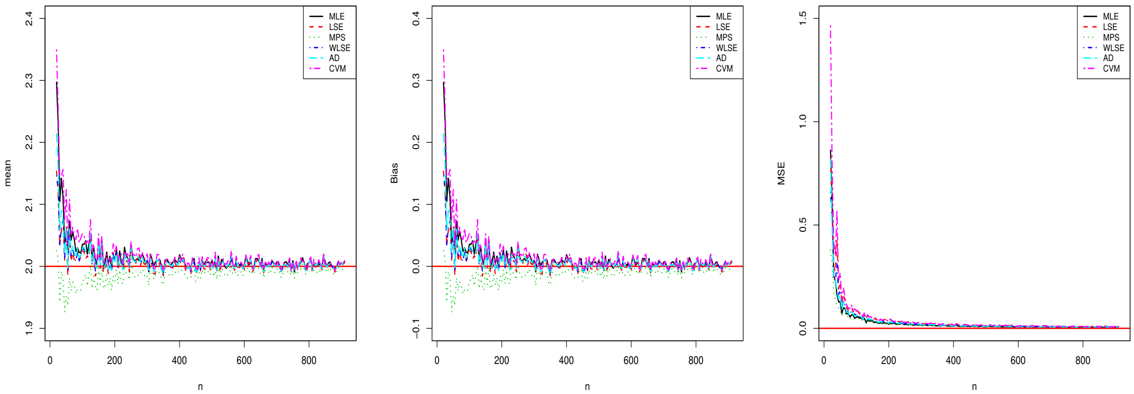

FIGURE 4. The empirical means, biases and MSEs of the parameter θ

respectively, by

$$\begin{aligned}
&\widehat{\alpha} \pm z_{1-\frac{\gamma}{2}} \times se\left(\widehat{\alpha}\right),\\
&\widehat{\beta} \pm z_{1-\frac{\gamma}{2}} \times se\left(\widehat{\beta}\right),\\
&\widehat{\theta} \pm z_{1-\frac{\gamma}{2}} \times se\left(\widehat{\theta}\right),
\end{aligned}$$

where z_a, is the a^{th} quantile of the standard normal distribution, $se\left(\widehat{\alpha}\right)$, $se\left(\widehat{\beta}\right)$ and $se\left(\widehat{\theta}\right)$ are the roots of the diagonal member of $\mathbb{I}^{-1}\left(\widehat{\boldsymbol{\Xi}}\right)$ which is a consistent estimate of $\mathbb{I}^{-1}\left(\boldsymbol{\Xi}\right)$ and the $se\left(\cdot\right)$ stands for standard error.

There is another method called uncorrected likelihood ratio (ULR). It is noticed that AN and ULR CIs are asymptotically equivalent [10].

Under usual regularity assumptions on the likelihood function, if the α is true parameter, then $-2\log\left(\ell\left(\alpha,\widetilde{\boldsymbol{\lambda}}\right)-\ell\left(\widehat{\boldsymbol{\Xi}}\right)\right)$ distributed as $\chi^2_{(1)}$, where $\boldsymbol{\lambda}=(\beta,\theta)$ are the nuisance parameters, ℓ is the log-likelihood function as in (7), $\widehat{\boldsymbol{\Xi}}$ is the joint MLE of (α,β,θ) given in (8), $\widetilde{\boldsymbol{\lambda}}=\left(\widetilde{\beta},\widetilde{\theta}\right)$ is the restricted MLE of $\boldsymbol{\lambda}$ given a fixed value of α. Using this fact, $100\times(1-\gamma)\,\%$ ULR CI limits α_L and α_U that satisfy

$$\ell\left(\alpha,\widetilde{\boldsymbol{\lambda}}\right)=\underbrace{\ell\left(\widehat{\boldsymbol{\Xi}}\right)-\frac{1}{2}\chi^2_{(1)}\left(1-\alpha\right)}_{\text{LR Bound}} \tag{19}$$

with $\alpha_L < \alpha$ and $\alpha_U > \alpha$, where $\chi^2_{(1)}(a)$ is the a^{th} quantile of the χ^2 distribution with 1 degrees of freedom. The $100 \times (1-\gamma)\,\%$ ULR CIs can be produced in the same manner for the other parameters β and θ.

In the simulation study, 5000 trials are used to predict the coverage probabilities (CPs) of the AN and ULR CIs. The nominal level is fixed at 0.95. In order to get CPs of ULR CIs, there is no need to obtain the CIs limits. It is possible that the CPs of ULR CIs can be simulated by a likelihood ratio test on the true parameter. The simulated CPs of these intervals are given in Table 2. Let us discuss the true parameter cases $\Xi = (1,1,0.5), (1,1,2.5), (5,5,2),\ (1,2,3), (3,0.5,1.5)$ and $(2,1,0.25)$. From Table 2, it is observed that the CPs of ULR reach to the desired level when the all sample of size discussed here (say $n \geq 50$) for all parameters. However, the CPs of AN can not reach the desired level for small sample of size case especially for parameter β. The CPs reach the nominal level when the sample of size increases (say $n \geq 250$ or $n \geq 500$ according to selected true parameters). Under discussion given here, it is indicated that ULR CIs powerful tool to construct the CIs for the MoL parameters.

5. Real-life Data Analysis

In this section, we provide three applications to the real data sets to demonstrate empirically the potentiality of the proposed model. All data sets, we compare the MoL model with MW, PL, GL, EPL, EL and L models. In order to reveal the best model, the estimated log-likelihood values $\ell(\widehat{\boldsymbol{\Xi}})$, Akaike information criteria (AIC), consistent Akaike information criteria (CAIC), Kolmogorov-Smirnov (KS), Cramer von Mises (W^*) and Anderson-Darling (A^*) goodness of-fit statistics are computed for all models.

The first data set represents the times between successive failures (in thousands of hours) in events of secondary reactor pumps studied by [5], [19] and [26]. The data are: 2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

The second data for breaking stress of carbon fibers of 50 mm length (GPa) was studied in [23].The data are: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

The third data reported in [7] which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consisting of survival times (in years) for 45 patients are: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

TABLE 2. The CPs of AN and ULR CIs

True parameter				AN			ULR		
α	β	θ	n	α	β	θ	α	β	θ
1	1	0.5	50	0.9480	0.8698	0.9390	0.9472	0.9412	0.9480
			100	0.9462	0.8848	0.9354	0.9474	0.9366	0.9444
			250	0.9488	0.9294	0.9472	0.9494	0.9430	0.9508
			500	0.9476	0.9394	0.9500	0.9468	0.9438	0.9494
			1000	0.9502	0.9484	0.9520	0.9500	0.9510	0.9526
1	1	2.5	50	0.9514	0.8840	0.9688	0.9374	0.9544	0.9556
			100	0.9524	0.9174	0.9618	0.9464	0.9534	0.9468
			250	0.9484	0.9386	0.9514	0.9466	0.9492	0.9422
			500	0.9470	0.9470	0.9494	0.9480	0.9520	0.9450
			1000	0.9488	0.9470	0.9486	0.9488	0.9498	0.9464
5	5	2	50	0.9480	0.9428	0.9472	0.9420	0.9392	0.9456
			100	0.9464	0.9444	0.9500	0.9444	0.9430	0.9458
			250	0.9464	0.9498	0.9482	0.9452	0.9498	0.9472
			500	0.9548	0.9506	0.9522	0.9534	0.9502	0.9510
			1000	0.9554	0.9544	0.9518	0.9554	0.9544	0.9502
1	2	3	50	0.9474	0.8976	0.9712	0.9396	0.9368	0.9422
			100	0.9468	0.9310	0.9602	0.9414	0.9496	0.9454
			250	0.9496	0.9380	0.9566	0.9464	0.9428	0.9488
			500	0.9428	0.9462	0.9472	0.9430	0.9474	0.9442
			1000	0.9544	0.9482	0.9506	0.9554	0.9478	0.9508
3	0.5	1.5	50	0.9276	0.8896	0.9426	0.9326	0.9426	0.9698
			100	0.9324	0.9120	0.9360	0.9422	0.9516	0.9674
			250	0.9444	0.9490	0.9502	0.9548	0.9680	0.9688
			500	0.9422	0.9514	0.9592	0.9526	0.9608	0.9582
			1000	0.9492	0.9638	0.9580	0.9542	0.9558	0.9486
2	1	0.25	50	0.9580	0.8750	0.9472	0.9580	0.9500	0.9584
			100	0.9572	0.8886	0.9510	0.9614	0.9480	0.9586
			250	0.9444	0.9164	0.9428	0.9458	0.9376	0.9452
			500	0.9444	0.9390	0.9464	0.9446	0.9504	0.9466
			1000	0.9538	0.9478	0.9510	0.9540	0.9494	0.9510

We give the summary statistics of the data sets in Table 3. The first and third data sets have right skewness as well as the second data set has the left skewness.

TABLE 3. Some summary statistics of the data sets

Data set	Mean	Median	Standard Deviation	Skewness	Kurtosis
I	1.5780	0.6140	1.9306	1.3643	3.5445
II	2.7600	2.8350	0.8914	-0.1314	3.2230
III	1.3410	0.8410	1.2466	0.9721	2.6638

Tables 4-6 list the MLEs, standard errors, $\ell(\widehat{\boldsymbol{\Xi}})$ and goodness-of-fits statistics from the fitted models. Tables 4-6 show that the MoL model can be chosen as the best model based on all criteria. In addition, we give the parameter estimation results and goodness-of-fit statistics of the MoL distribution based on other estimation methods in Table 7. Figures 5-7 show the fitted densities, cdfs and probability-probability (P-P) plots of the MoL model. We also sketch the P-P plots of others models in Figures 8-10. From Figures 8-10, we clearly show that the MoL model fits this data set better than the other models.

TABLE 4. MLEs, standard erros of the estimates (in parentheses), $\hat{\ell}$, goodness-of-fits statistics and related p-values [in parentheses] for the first data set

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$-\hat{\ell}$	AIC	$CAIC$	KS	A^*	W^*
MoL	0.8148	1.7119	0.9419	31.3782	68.7565	70.0196	0.0785	0.1881	0.0204
	(0.1582)	(2.0132)	(0.2169)				[0.9967]	[0.9934]	[0.9972]
MW	0.7922	0.0093	0.7517	32.5082	71.0165	72.2796	0.1198	0.4141	0.0639
	(0.1925)	(0.0850)	(0.2199)				[0.8575]	[0.8330]	[0.7939]
PL	0.7253		1.1948	32.7476	69.4952	70.0952	0.1189	0.4279	0.0643
	(0.1129)		(0.2119)				[0.8628]	[0.8190]	[0.7918]
GL	0.7457	0.00016	0.4728	32.7592	71.5184	72.7815	0.1379	0.5236	0.0889
	(0.1885)	(0.0116)	(0.1659)				[0.7293]	[0.7209]	[0.6462]
EL	0.6130		0.7251	33.4889	70.9779	71.5779	0.1558	0.7059	0.1246
	(0.1647)		(0.1782)				[0.5784]	[0.5521]	[0.4799]
EPL	0.2770	11.5880	3.7238	31.8359	69.6718	70.9349	0.0963	0.2264	0.0253
	(0.2404)	(32.7190)	(2.9916)				[0.9691]	[0.9814]	[0.9903]
L			0.9575	35.3054	72.6107	72.8013	0.2439	2.2967	0.3821
			(0.1504)				[0.1085]	[0.0640]	[0.0798]

In Table 8, 95% AN and ULR confidence limits of the parameters are presented for the all data sets. In general the limits of AN and ULR intervals are close to each other. Figure 11 demonstrate the ULR intervals for the third real data.

TABLE 5. MLEs, standard erros of the estimates (in parentheses), $\hat{\ell}$, goodness-of-fits statistics and related p-values [in parentheses] for the second data set

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$-\hat{\ell}$	AIC	$CAIC$	KS	A^*	W^*
MoL	2.7039	0.7905	0.0967	85.6419	177.2838	177.6709	0.0741	0.3956	0.0620
	(0.2170)	(0.4304)	(0.0257)				[0.8607]	[0.8526]	[0.8025]
MW	2.1535	0.4302	0.0228	85.9866	177.9732	178.3603	0.0905	0.5266	0.0838
	(1.0622)	(0.3581)	(0.0092)				[0.6519]	[0.7192]	[0.6719]
PL	2.5099		0.1241	85.8055	175.6111	175.8015	0.0790	0.4651	0.0819
	(0.2088)		(0.0311)				[0.8051]	[0.7820]	[0.6824]
GL	6.9574	8.0712	2.7905	90.9276	187.8552	188.2423	0.1318	1.2861	0.2420
	(1.4779	(21.2598)	(0.4860)				[0.2014]	[0.2368]	[0.1991]
EL	7.0411		1.2461	93.7970	191.5939	191.7844	0.1470	1.8375	0.3284
	(1.6730)		(0.1090)				[0.1154]	[0.1132]	[0.1124]
EPL	3.1439	0.6238	0.0458	85.4258	176.8516	177.2387	0.0772	0.4094	0.0683
	(0.8257)	(0.3149)	(0.0585)				[0.8258]	[0.8388]	[0.7638]
L			0.5903	122.3841	246.7681	246.8306	0.2977	10.6922	2.0914
			(0.0532)				[0.0000]	[0.0000]	[0.0000]

TABLE 6. MLEs, standard erros of the estimates (in parentheses), $\hat{\ell}$, goodness-of-fits statistics and related p-values [in parentheses] for the third data set

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	$-\hat{\ell}$	AIC	$CAIC$	KS	A^*	W^*
MoL	1.1610	2.5389	0.8263	55.8323	117.6647	118.250	0.0661	0.3437	0.0393
	(0.1378)	(0.7903)	(0.1309)				[0.9819]	[0.9015]	[0.9388]
MW	0.9677	0.0620	0.6529	57.9942	121.9885	122.5738	0.1116	0.5700	0.0864
	(0.2047)	(0.1235)	(0.1702)				[0.5958]	[0.6758]	[0.6577]
PL	0.9465		1.1351	58.4028	120.8056	121.0913	0.1104	0.5656	0.0845
	(0.1076)		(0.1465)				[0.6033]	[0.6801]	[0.6683]
GL	1.0931	0.8896	0.0991	58.0862	122.1725	122.7578	0.1110	0.5482	0.0842
	(0.2256)	(0.4456)	(0.7380)				[0.5967]	[0.6973]	[0.6702]
EL	0.9412		1.0656	58.4784	120.9568	121.2425	0.1196	0.6498	0.1015
	(0.1919)		(0.1693)				[0.5026]	[0.6013]	[0.5794]
EPL	0.6579	2.0911	1.8562	58.1167	122.2333	122.8187	0.0972	0.4736	0.0702
	(0.3390)	(2.2867)	(1.1306)				[0.7521]	[0.7729]	[0.7530]
L			1.1004	58.5231	119.0461	119.1391	0.1304	0.7721	0.1253
			(0.1249)				[0.3964]	[0.5007]	[0.4758]

TABLE 7. The different estimations results of the MoL model parameters for the data sets

	Data set-I						Data set-II						Data set-III					
Method	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	A^*	W^*	KS	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	A^*	W^*	KS	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	A^*	W^*	KS
LSE	0.6773	1.5845	0.9559	0.1593	0.0132	0.0677	2.9451	1.9433	0.0748	0.4434	0.0668	0.0701	1.0068	2.0508	0.8411	0.2629	0.0301	0.0626
WLSE	0.7205	1.8431	0.9376	0.1554	0.0131	0.0689	2.5905	0.0144	0.1124	0.5642	0.1054	0.0898	1.0586	2.1764	0.8467	0.2873	0.0369	0.0658
AD	0.7659	1.9442	0.9314	0.1417	0.0131	0.0595	2.8612	1.4052	0.0815	0.3394	0.0408	0.0661	1.0537	2.1279	0.8455	0.2619	0.0292	0.0642
CVM	0.7315	1.6954	0.9514	0.1481	0.0124	0.0666	2.9952	1.6715	0.0712	0.3852	0.0354	0.0585	1.0479	2.1197	0.8354	0.2673	0.0283	0.0683

TABLE 8. Confidence limits for parameters α, β and θ based on AN and ULR for the data sets

Data	AN			ULR		
	α	β	θ	α	β	θ
Data set-I	$(0.5051, 1.1247)$	$(-2.2231, 5.6466)$	$(0.5167, 1.3669)$	$(0.5628, 1.1177)$	$(0, 4.6380)$	$(0.5967, 1.3985)$
Data set-II	$(2.2759, 3.1253)$	$(-0.0451, 1.6400)$	$(0.0465, 0.1470)$	$(2.1268, 3.1366)$	$(0, 3.2554)$	$(0.0565, 0.1991)$
Data set-III	$(0.8909, 1.4311)$	$(0.9898, 4.0880)$	$(0.5695, 1.0830)$	$(0.9069, 1.4431)$	$(1.1613, 3.9695)$	$(0.5983, 1.1126)$

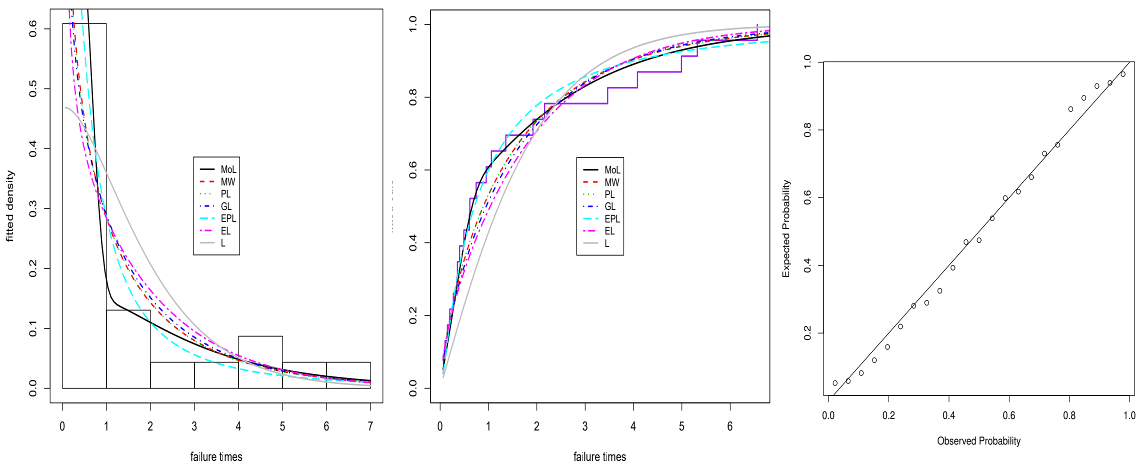

FIGURE 5. The fitted plots for the first data set

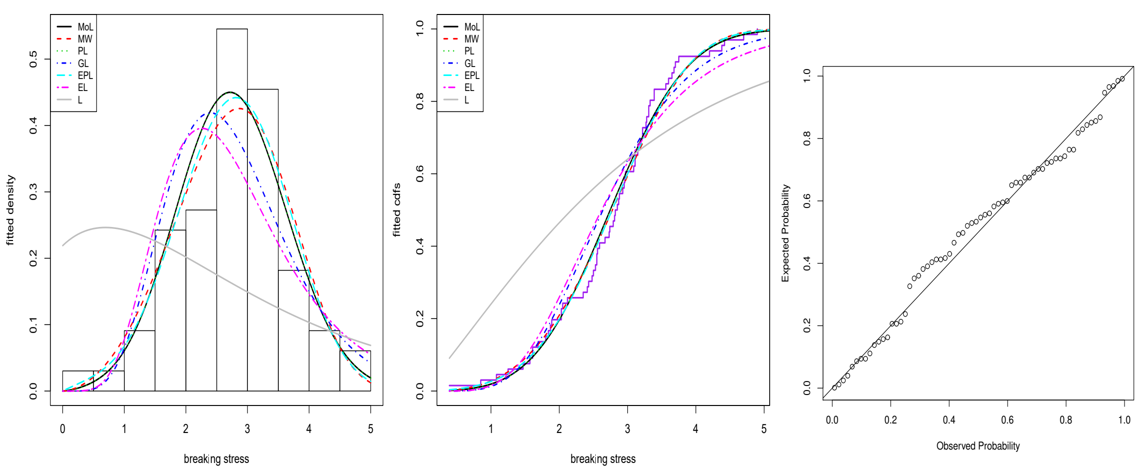

FIGURE 6. The fitted plots for the second data set

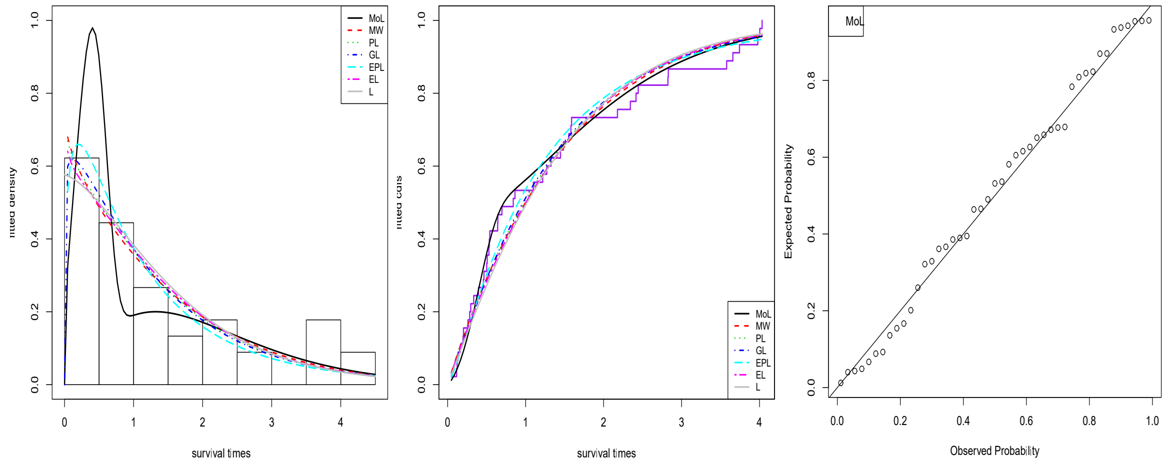

FIGURE 7. The fitted plots for the third data set

Author Contribution Statements The authors jointly worked on the results and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

Appendix

Proof of Theorem 1.
The pdf of the X is

$$f(x) = \frac{\theta}{\theta+1} g_1(x;\alpha,\beta,\theta) + \frac{1}{\theta+1} g_2(x;\alpha,\beta,\theta)$$

Get the $W(x)$ density ratio of MoL distribution in two parts as $W_1(x)$ and $W_2(x)$. If $W_1(x)$ and $W_2(x)$ density ratios are increasing functions in x, the $W(x)$ density ratio is also an increasing function of x. The $W_1(x)$ and $W_2(x)$ ratios are given by

$$W_1(x) = \frac{g_1(x;\alpha,\beta,\theta_1)}{g_1(x;\alpha,\beta,\theta_2)}$$

and

$$W_2(x) = \frac{g_2(x;\alpha,\beta,\theta_1)}{g_2(x;\alpha,\beta,\theta_2)}$$

where $g_1(x;\alpha,\beta,\theta)$ and $g_2(x;\alpha,\beta,\theta)$ are the pdfs of MW and GG distributions respectively. Firstly, the MW density ratio is given by

$$W_1(x) = \frac{g_1(x;\alpha,\beta,\theta_1)}{g_1(x;\alpha,\beta,\theta_2)} = \frac{\theta_1 \exp(\beta x - \theta_1 x^{\alpha} \exp(\beta x))}{\theta_2 \exp(\beta x - \theta_2 x^{\alpha} \exp(\beta x))}$$

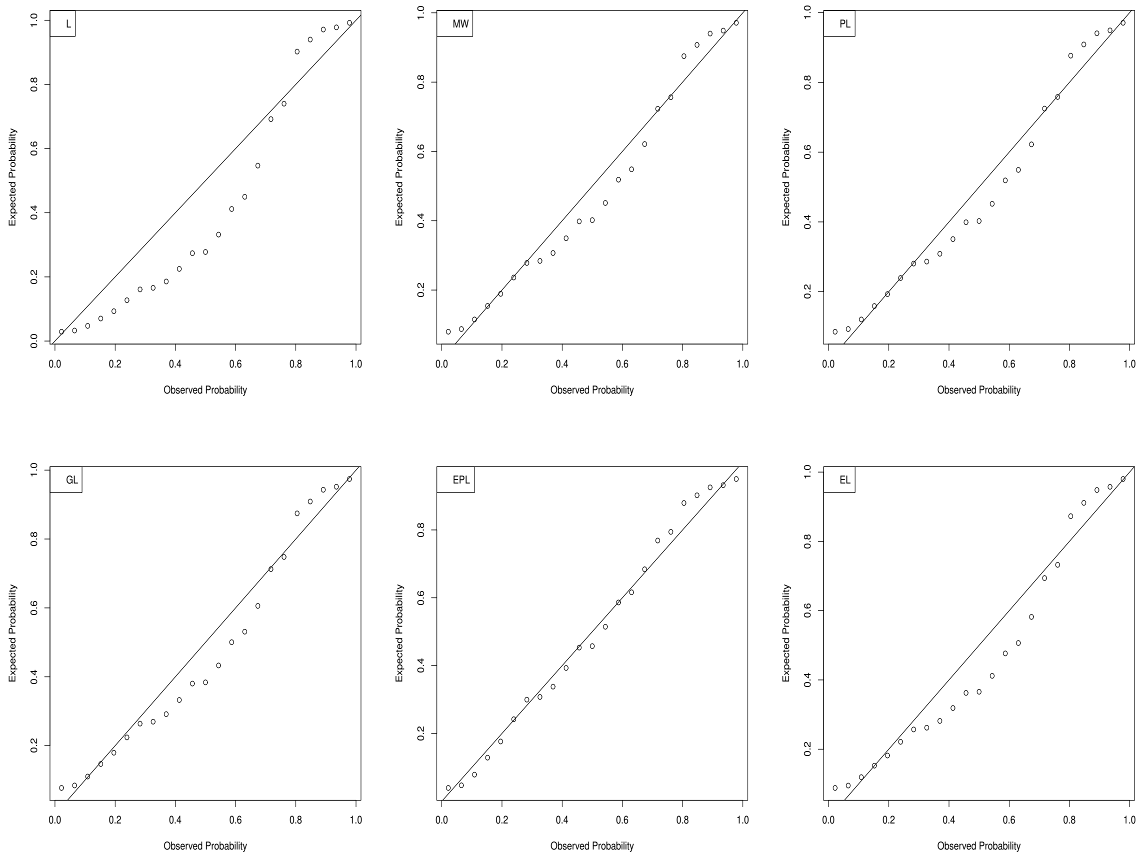

FIGURE 8. The PP plots for the first data set

Taking the derivative with respect to x,

$$W_1'(x) = -\frac{\overbrace{\theta_1 x^{\alpha} \exp(\beta x)(\alpha + \beta x)\exp(\beta x - \theta_1 x^{\alpha}\exp(\beta x))}^{>0}(\theta_1 - \theta_2)}{\underbrace{x\theta_2 \exp(\beta x - \theta_2 x^{\alpha}\exp(\beta x))}_{>0}}$$

for $\theta_1 < \theta_2$, $-((\theta_1 - \theta_2))$ is greater than zero. So $W_1'(x) > 0$ when $\theta_1 < \theta_2$ is taken. $W_1(x)$ is an increasing function in x. Secondly, the same steps are applied

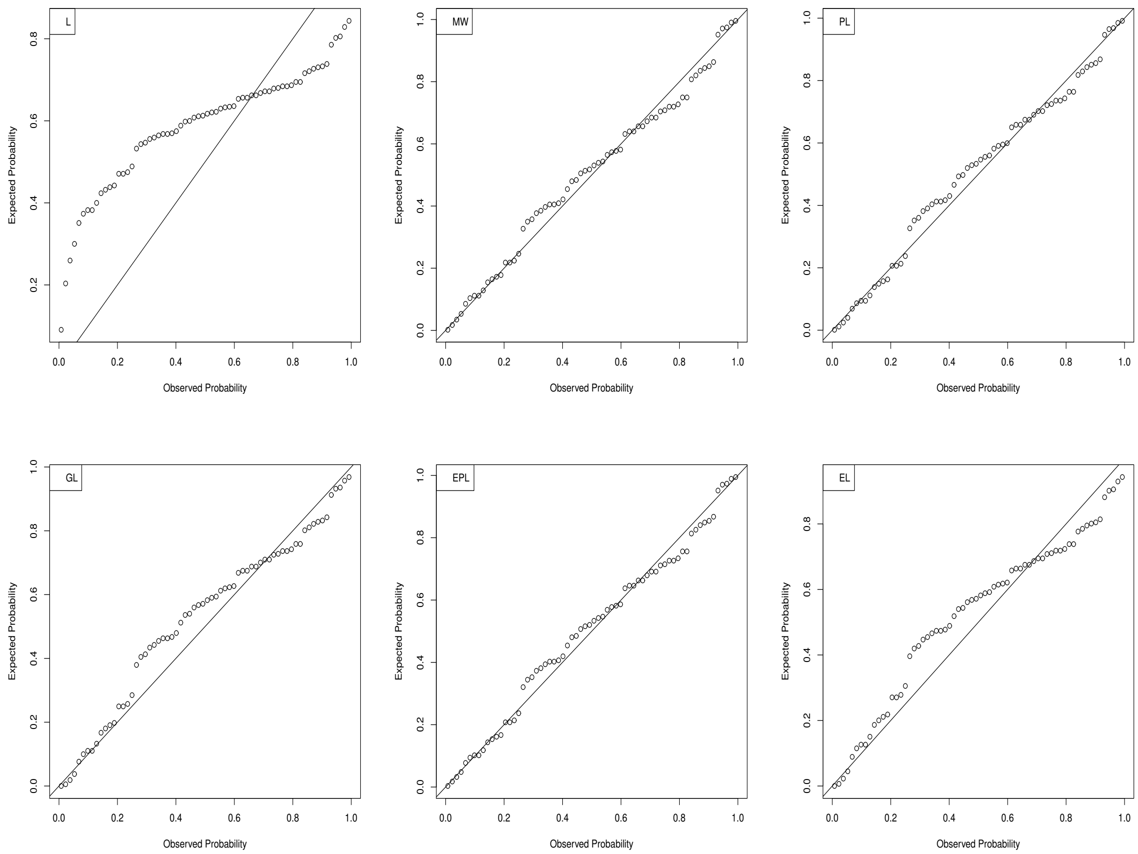

FIGURE 9. The PP plots for the second data set

for GG density ratio. The GG density ratio is given by

$$W_2(x) = \frac{g_2(x;\alpha,\beta,\theta_1)}{g_2(x;\alpha,\beta,\theta_2)} = \frac{\theta_1^2 \exp(-\theta_1 x^\alpha)}{\theta_2^2 \exp(-\theta_2 x^\alpha)}$$

Taking the derivative with respect to x,

$$W_2'(x) = -\frac{\overbrace{\theta_1^2 \exp(-\theta_1 x^\alpha)\, x^\alpha \alpha}^{>0}(\theta_1 - \theta_2)}{\underbrace{x\theta_2^2 \exp(-\theta_2 x^\alpha)}_{>0}}$$

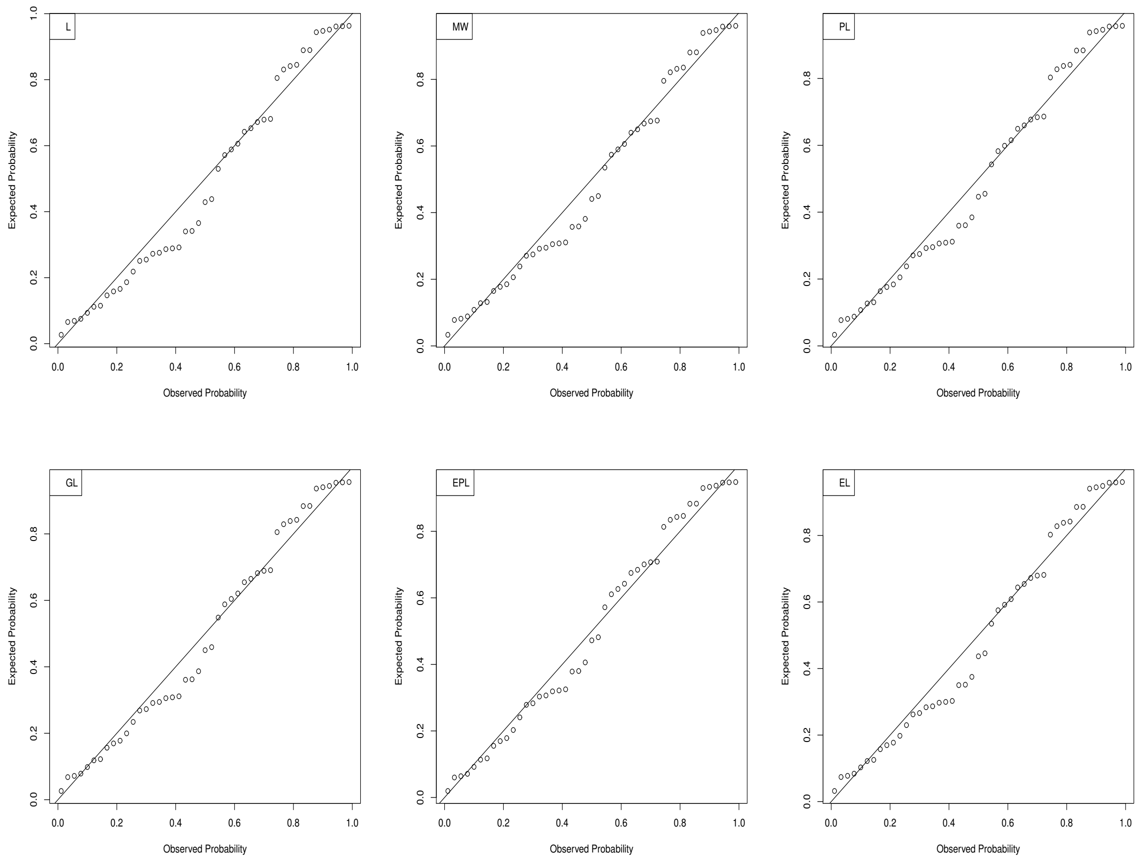

FIGURE 10. The PP plots for the third data set

for $\theta_1 < \theta_2$, $-((\theta_1 - \theta_2))$ is greater than zero. So $W_2'(x) > 0$ when $\theta_1 < \theta_2$. $W_2(x)$ is an increasing function in x. Since both $W_1(x)$ and $W_2(x)$ are increasing functions in x, $W(x) = W_1(x) + W_2(x)$ is also an increasing function in x. The proof is completed.

Proof of Theorem 2
Using the fact that mixed representation MoL pdf given in (2), the rth moment, $\mu_r' = E(X^r)$, of the MoL distribution can be written by

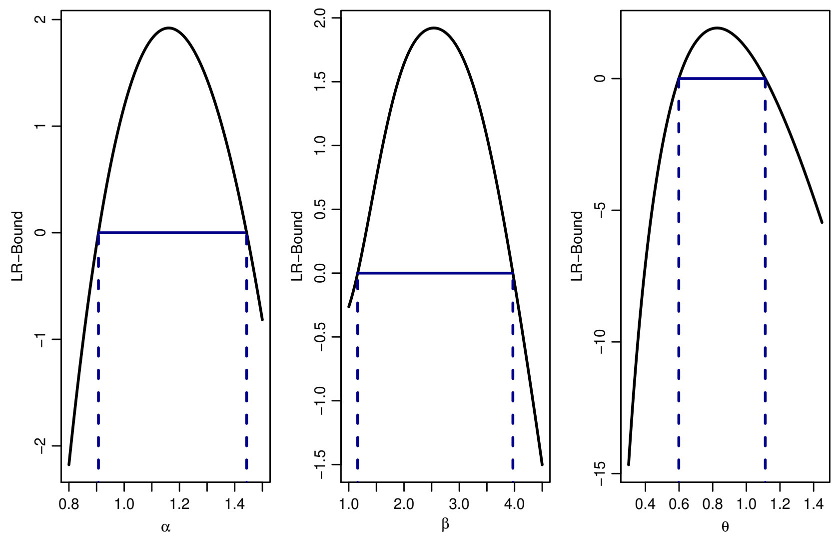

FIGURE 11. ULR confidence limits for parameters α, β and θ for the real data III

$$\mu_r' = \frac{\theta}{\theta+1} E\left(X_{MW}^r\right) + \frac{1}{\theta+1} E\left(X_{GG}^r\right), \tag{20}$$

where

$$E\left(X_{GG}^r\right) = \Gamma\left(r/\alpha+2\right)\theta^{-r/\alpha} \tag{21}$$

is the rth moment of GG distribution and

$$E\left(X_{MW}^r\right) = \sum_{i_1,\ldots,i_r=1}^{\infty} A_{i_1,\ldots,i_r}\Gamma\left(s_r/\alpha+1\right)\theta^{-s_r/\alpha}, \tag{22}$$

is the rth moment of the MW distribution [8] with

$$A_{i_1,\ldots,i_r} = a_{i_1},\ldots,a_{i_r}, s_r = i_1+\cdots+i_r$$

and

$$a_i = (-1)^{i+1} i^{i-2}\beta^{i-1}\left[\alpha^{i-1}(i-1)!\right]^{-1}.$$

The proof is completed by using (21) and (22) in (20),

$$\mu_r' = \frac{1}{\theta+1}\left[\Gamma\left(r/\alpha+2\right)\theta^{-r/\alpha} + \sum_{i_1,\ldots,i_r=1}^{\infty} A_{i_1,\ldots,i_r}\Gamma\left(s_r/\alpha+1\right)\theta^{1-s_r/\alpha}\right].$$

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SOME GROUP ACTIONS AND FIBONACCI NUMBERS

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Abstract. The Fibonacci sequence has many interesting properties and studied by many mathematicians. The terms of this sequence appear in nature and is connected with combinatorics and other branches of mathematics. In this paper, we investigate the orbit of a special subgroup of the modular group. Taking

$$T_c := \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix} \in \Gamma_0(c^2), c \in \mathbb{Z}, c \neq 0,$$

we determined the orbit

$$\{T_c^r(\infty) : r \in \mathbb{N}\}.$$

Each rational number of this set is the form $P_r(c)/Q_r(c)$, where $P_r(c)$ and $Q_r(c)$ are the polynomials in $\mathbb{Z}[c]$. It is shown that $P_r(1)$, and $Q_r(1)$ the sum of the coefficients of the polynomials $P_r(c)$ and $Q_r(c)$ respectively, are the Fibonacci numbers, where

$$P_r(c) = \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+1}$$

and

$$Q_r(c) = \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+2}.$$

1. Introduction

The modular group theory plays an important role in many areas of mathematics, such as number theory, graph theory, automorphic function theory and combinatorics. A natural action of the modular group on extended rationals, yields interesting results. In [4], by using this action, Jones et. al. studied the suborbital graphs known as the Farey graph for the modular group. Kader et al. studied the

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suborbital graphs for the extended modular group in [11]. Değer et. al. investigate some results on continued fractions in suborbital graphs [1]. In [8, 9], Keskin searched the suborbital graphs for the normalizer of $\Gamma_0(n)$. Güler et. al. examined relations between elliptic elements and circuits in graphs for the normalizer of $\Gamma_0(n)$ in $PSL(2,\mathbb{R})$ which turns to be a very important group in the studies of moonshine [2].

Some results in these studies are directly related to the number theory. Köroğlu et. al. obtained interesting results about the Fibonacci numbers and the suborbital graphs by means of the action of a special subgroup of the modular group on extended rationals [7]. Güler et. al. studied on solutions of congruence equations that come from the action of the normalizer of $\Gamma_0(n)$ via suborbital graphs [3].

On the other hand, it is known that Pascal and Fibonacci numbers are crucial subjects in combinatorics [5]. In [10], Falcon and Plaza obtained some results about Fibonacci sequence and Pascal's triangle.

The aim of the paper is to examine the action of a special subgroup of the modular group on the extended rationals. With the idea of this group action, some interesting results are obtained about the number theory. Many properties of Fibonacci numbers are deduced and associated with the so-called Pascal's triangle mentioned.

2. Modular Group

Let $PSL(2,\mathbb{R})$ denote the group of all linear fractional transformations

$T : z \to \frac{az+b}{cz+d}$, where a, b, c and d are real and $ad - bc = 1$.

In terms of the matrix representation, the elements of $PSL(2,\mathbb{R})$ correspond to the matrices

$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

These matrix representations are composed of the special linear group denoted by $SL(2,\mathbb{R})$. The modular group denoted by Γ is the subgroup of $SL(2,\mathbb{R})$ consisting of the 2×2 matrices having integer entries. Furthermore, the modular group is generated by the matrices

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

with defining relationships $x^2 = y^3 = -I$, where I is the identity matrix. Here, x and y are cyclic matrices of order two and three, respectively. And we can write

$$\Gamma =< x, y > .$$

We remark that something very related to the trace $Tr(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) := |a + d|$ will be of great use in the classification. Note that, an element of modular group is called elliptic, parabolic or hyperbolic if its trace $Tr(\cdot) < 2$, $Tr(\cdot) = 2$ or $Tr(\cdot) > 2$ respectively. Important subgroups of the modular group Γ, called congruence subgroups, are given by imposing congruence relations on the associated matrices. One of them is

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \; mod \, n \right\}.$$

3. The Action of Γ on $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$

Any element of $\hat{\mathbb{Q}}$ (the extended rational numbers set) can be written as a reduced fraction $\frac{x}{y}$, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$; since $\frac{x}{y} = \frac{-x}{-y}$, this representation is unique. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action $z \to \frac{az+b}{cz+d}$ of Γ on $\hat{\mathbb{Q}}$ now becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax + by}{cx + dy}.$$

Note that as

$$c(ax + by) - a(cx + dy) = -y$$

and

$$d(ax + by) - b(cx + dy) = x,$$

it follows that $(ax+by, cx+dy) = 1$ and so $(ax+by)/(cx+dy)$ is a reduced fraction.

4. Main Calculations

In this section, we investigate the action of a special subgroup of the congruence subgroup $\Gamma_0(c^2)$ on extended rationals for some integer $c \neq 0$. Here, we use the action of the group generated by the commutator of the elements $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ on $\hat{\mathbb{Q}}$. Let

$$xyx^{-1}y^{-1} = \begin{pmatrix} c^2 + c + 1 & -c \\ c^2 & 1 - c \end{pmatrix}.$$

Since $Tr(T_c) = c^2 + 2 > 2$, we can say the element T_c is hyperbolic element of modular group for $c \neq 0$.

Proposition 1. *The fixed points of the element T_c are,*

$$\frac{c+2}{2c} \pm \frac{\sqrt{c^2 + 4}}{2c}. \tag{1}$$

Furthermore, T_c generates an infinitely ordered subgroup $< T_c >$ whose elements are in congruence subgroup $\Gamma_0(c^2)$. At the same time, the group $< T_c >$ generated by T_c is a subgroup of commutator subgroup of modular group. Also, $T_c(\infty) = \frac{c^2+c+1}{c^2}$ is an element of $\hat{\mathbb{Q}}$.

Proposition 2. *The group* $\Gamma_0(c^2)$ *acts on the set* $M := \{\frac{x}{c^2y} : x, y \in \mathbb{Z}, gcd(x, yc) = 1, y \neq 0\} \cup \{\infty\}$ *transitively.*

Note that, if $y = 0$ *and* $x \neq 0$ *then we assumed that,* $\frac{x}{c^2y} = \frac{x}{0} = \infty$ *such as the definition of extended rationals in [4].*

Proof. For arbitrary $x, y \in \mathbb{Z},\ gcd(x, yc) = 1$, there exists $T = \begin{pmatrix} x & * \\ yc^2 & * \end{pmatrix} \in \Gamma_0(c^2)$. such that $T(\infty) = \frac{x}{yc^2}$. This completes the proof. □

We interested in sequence of natural powers of the number $T_c(\infty)$ denoted by $\{T_c^r(\infty)\}$, where $r \in \mathbb{N}$. Clearly $\{T_c^r(\infty)\} \subset M \cup \{\infty\}$. Hence, there is some element of $\Gamma_0(c^2)$ such that its orbit coincidence the terms of the sequence $\{T_c^r(\infty)\}$. The following theorem show us how $\{T_c^r(\infty)\}$ sequence proceeds.

Theorem 1. *Let* $T_c = \begin{pmatrix} c^2 + c + 1 & -c \\ c^2 & 1 - c \end{pmatrix}$, *with* $c \in \mathbb{Z}$. *Suppose*

$$T_c^r(\infty) := \frac{P_r(c)}{Q_r(c)}.$$

Then

$$P_r := P_r(c) = \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+1}, \tag{2}$$

$$Q_r := Q_r(c) = \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+2}. \tag{3}$$

Since Theorem 1 includes the combinatorial identities we frequently use some combinatorial basics such as,

$$\binom{r}{s} = \binom{r-1}{s-1} + \binom{r-1}{s} \tag{4}$$

so-called the Pascal Identity for integers $1 \leq s \leq r$.

Before the proof of the theorem, we give the following lemma.

Lemma 1. *Assume that the identities* (2) *and* (3) *are true for any* $r > 1$. *Then, we have*

$$c^2 P_r - cQ_r = \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s+2}. \tag{5}$$

Proof. By using (4) and other properties of the combinatorial theory we get proof easily, as follow:

$$c^2 P_r - cQ_r = c^2 \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s} + c^2 \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+1}$$

$$
\begin{aligned}
&- c\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+2}\\
&= \sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2} + \sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+3}\\
&- \sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+3}\\
&= \sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}.
\end{aligned}
$$

□

Now we give the proof of Theorem 1 by using the mathematical induction method.

Proof. For $r = 1$, it is clear that

$$T_c(\infty) = \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c^2+c+1 \\ c^2 \end{pmatrix}.$$

So, $P_1 = c^2 + c + 1$ and $Q_1 = c^2$. This shows that (2) and (3) are true for $r = 1$. As

$$T_c^{r+1}(\infty) = \frac{P_{r+1}}{Q_{r+1}} \tag{6}$$

and

$$
\begin{aligned}
T_c^{r+1}(\infty) = T_c(T_c^r(\infty)) &= \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix}\begin{pmatrix} P_r \\ Q_r \end{pmatrix}\\
&= \begin{pmatrix} c^2P_r + cP_r + P_r - cQ_r \\ c^2P_r - cQ_r + Q_r \end{pmatrix},
\end{aligned} \tag{7}
$$

we get

$$P_{r+1} = c^2aP_r + cP_r + P_r - cQ_r \tag{8}$$

and

$$Q_{r+1} = c^2P_r - cQ_r + Q_r. \tag{9}$$

Now assume that (2) and (3) are true for any $r > 1$. We will show that (2) and (3) are true for $r + 1$. To complete the proof, by using Lemma 1, it can be shown that the following two equations can be obtained from the identities (8) and (9).

$$P_{r+1} = \sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2} + (c+1)P_r, \tag{10}$$

$$Q_{r+1} = \sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2} + Q_r. \tag{11}$$

Indeed, if we begin with the right side of the equation (10), then we obtain desired results as follow.

$$\begin{aligned}
&\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}+(c+1)P_r \\
&=\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2} \\
&\quad+(c+1)\left[\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s}+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+1}\right] \\
&=\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}+c^{2r+1}+\sum_{s=1}^{r}\binom{2r-s}{s}c^{2r-2s+1} \\
&\quad+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+2}+\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s} \\
&\quad+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+1}.
\end{aligned} \tag{12}$$

From the equation (12), we obtain

$$\begin{aligned}
&\sum_{s=0}^{r+1}\binom{2r-s+2}{s}c^{2r-2s+2} \\
&=\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s+2}+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+2} \\
&\quad+\sum_{s=0}^{r}\binom{2r-s}{s}c^{2r-2s},
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\sum_{s=1}^{r+1}\binom{2r-s+2}{s-1}c^{2r-2s+3} \\
&=\sum_{s=1}^{r}\binom{2r-s}{s}c^{2r-2s+1}+\sum_{s=1}^{r}\binom{2r-s}{s-1}c^{2r-2s+1}+c^{2r+1}.
\end{aligned} \tag{14}$$

So, by using (13) and (14) we have

$$P_{r+1}=\sum_{s=0}^{r+1}\binom{2r-s+2}{s}c^{2r-2s+2}+\sum_{s=1}^{r+1}\binom{2r-s+2}{s-1}c^{2r-2s+3}.$$

Hence, the equation (2) is true for $r+1$.

By using Lemma 1, we get

$$
\begin{aligned}
Q_{r+1} = c^2 P_r - cQ_r + Q_r &= \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s+2} + Q_r \\
&= \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s+2} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+2} \\
&= c^{2r+2} + \sum_{s=1}^{r} \binom{2r-s}{s} c^{2r-2s+2} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+2} \\
&= c^{2r+2} + \sum_{s=1}^{r} \left[\binom{2r-s}{s} + \binom{2r-s}{s-1} \right] c^{2r-2s+2} \\
&\overset{(4)}{=} c^{2r+2} + \sum_{s=1}^{r} \binom{2r-s+1}{s} c^{2r-2s+2} \\
&= \sum_{s=0}^{r} \binom{2r-s+1}{s} c^{2r-2s+2} \\
&= \sum_{s=1}^{r+1} \binom{2r-s+2}{s-1} c^{2r-2s+4} = Q_{r+1}.
\end{aligned}
$$

This implies that (3) is true for $r+1$.

$\square$

5. Pascal Numbers and Fibonacci Sequence

In this section, we give some useful informations for Fibonacci numbers related to our results in this study. The Fibonacci numbers F_r are given by the recurrence in [6];

$$F_1 = F_2 = 1,\ F_{r+2} = F_{r+1} + F_r,\ r \geq 1.$$

Thus, the first few Fibonacci numbers are

$$1,\ 1,\ 3,\ 5,\ 8,\ ,13,\ 21,\ \ldots$$

Also, the elegant formula is

$$F_{r+1} = \sum_{s=0}^{\lfloor r/2 \rfloor} \binom{r-s}{s} \tag{15}$$

where $\lfloor r/2 \rfloor$ denotes the largest integer less than or equal to $r/2$ [6].

We consider coefficients of the polynomials P_r and Q_r as shown below in first five terms of P_r and Q_r. Furthermore we investigate that these coefficents are related to the Pascal triangle.

$$P_1 = c^2 + c + 1$$

$$P_2 = c^4 + c^3 + 3c^2 + 2c + 1$$

$$
\begin{aligned}
P_3 &= c^6 + c^5 + 5c^4 + 4c^3 + 6c^2 + 3c + 1 \\
P_4 &= c^8 + c^7 + 7c^6 + 6c^5 + 15c^4 + 10c^3 + 10c^2 + 4c + 1 \\
P_5 &= c^{10} + c^9 + 9c^8 + 8c^7 + 28c^6 + 21c^5 + 35c^4 + 20c^3 + 15c^2 + 5c + 1 \\
&\vdots
\end{aligned}
$$

Each sequences of numerators obtained from this action consists of numbers in Pascal 2-Triangle as shown in Table 1. For example, second sequences of numerators $(1, 1, 3, 2, 1)$ are located by bold numbers in Table 1.

Table 1. The Pascal 2-Triangle

					1					
					1					
				1		1				
				1		**2**				
			$\nearrow$	$\searrow$	$\nearrow$	$\searrow$				
			1		**3**		**1**			
			1		4		3			
		1		5		6		1		
		1		6		10		4		
	1		7		15		10		1	
	1		8		21		20		5	
1		9		28		35		15		1
1		10		36		56		35		6

Proposition 3. *Sum of all coefficients of P_r gives the $(2r+2)-th$ Fibonacci number denoted by F_{2r+2}, i.e. $P_r(1) = F_{2r+2}$.*

Proof. We remark that sum of all coefficients of P_r is $P_r(1)$. So, by using the identities (4) and (15) we have desired result as follows:

$$
\begin{aligned}
P_r(1) &= 1 + \sum_{s=1}^{r} \left[\binom{2r-s}{s-1} + \binom{2r-s}{s} \right] \\
&\overset{(4)}{=} 1 + \sum_{s=1}^{r} \binom{2r-s+1}{s} \\
&= \sum_{s=0}^{r} \binom{2r-s+1}{s} \\
&= \sum_{s=0}^{\lfloor (2r+1)/2 \rfloor} \binom{2r-s+1}{s} \overset{(15)}{=} F_{2r+2}.
\end{aligned}
$$

□

Now, we consider the coefficients of the polynomial Q_r. These coefficients are written at the sub rows in Pascal 2-triangle in Table 1. For example, 1, 3, 1 and 1, 6, 10, 4. So the first five terms of Q_r are listed as follow:

$$\begin{aligned}
Q_1 &= c^2\\
Q_2 &= c^4 + 2c^2\\
Q_3 &= c^6 + 4c^4 + 3c^2\\
Q_4 &= c^8 + 6c^6 + 10c^4 + 4c^2\\
Q_5 &= c^{10} + 8c^8 + 21c^6 + 20c^4 + 5c^2\\
&\vdots
\end{aligned}$$

Proposition 4. *Sum of all coefficients of Q_r gives the $2r-th$ Fibonacci number, i.e. $Q_r(1) = F_{2r}$.*

Proof.

$$\begin{aligned}
Q_r(1) &= \sum_{s=1}^{r} \binom{2r-s}{s-1} = \sum_{j=0}^{r-1} \binom{2r-(j+1)}{j}\\
&= \sum_{j=1}^{r-1} \binom{2r-1-j}{j} = \sum_{j=0}^{\lfloor (2r-1)/2 \rfloor} \binom{2r-1-j}{j}\\
&\overset{(15)}{=} F_{2r}.
\end{aligned}$$

□

Proposition 5. $P_r(-1) = F_{2r-1}$.

Proof.

$$
\begin{aligned}
P_r(-1) &= \sum_{s=0}^{r}\binom{2r-s}{s} - \sum_{s=1}^{r}\binom{2r-s}{s-1} = \sum_{s=0}^{r}\binom{2r-s}{s} - \sum_{s=0}^{r-1}\binom{2r-s-1}{s} \\
&= 1 + \sum_{s=0}^{r-1}\binom{2r-s}{s} - \sum_{s=0}^{r-1}\binom{2r-s-1}{s} \\
&= 1 + \sum_{s=0}^{r-1}\left[\binom{2r-s}{s} - \binom{2r-s-1}{s}\right] \\
&\overset{(4)}{=} 1 + \sum_{s=1}^{r-1}\binom{2r-s-1}{s-1} = 1 + \sum_{s=0}^{r-2}\binom{2(r-1)-s}{s} \\
&= 1 + \sum_{s=0}^{u-1}\binom{2u-s}{s} = \sum_{s=0}^{u}\binom{2u-s}{s} \overset{(15)}{=} F_{2u+1} = F_{2r-1}.
\end{aligned}
$$

□

Proposition 6. *Sum of the coefficients of odd order terms of P_r is F_{2r}.*

Proof. We remark that the sum of the coefficients of odd order terms of P_r is $\frac{P_r(1)-P_r(-1)}{2}$. Therefore, by using Proposition 3, Proposition 5 and recurrence relations of Fibonacci numbers, we obtain the desired result as follows:

$$
\begin{aligned}
\frac{P_r(1)-P_r(-1)}{2} &= \frac{F_{2r+2}-F_{2r-1}}{2} = \frac{F_{2r}+F_{2r+1}-F_{2r-1}}{2} \\
&= \frac{F_{2r-1}+2F_{2r}-F_{2r-1}}{2} = F_{2r}.
\end{aligned}
$$

□

Also considering the scope of this study, we can also talk about $k-$Fibonacci numbers. Let $k \neq 0$ be an integer and $F_{k,0}=0$, $F_{k,1}=1$, and $F_{k,n}=kF_{k,n-1}+F_{k,n-2}$ for $n \geq 2$. The sequence $(F_{k,n})$ is called $k-$Fibonacci sequence. A few terms of this sequence are

$$0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, k^5+4k^3+3k, k^6+5k^4+6k^2+1, \ldots$$

In [10], it is proved that

$$F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1-i}{i}k^{n-1-2i} \text{ for } n \geq 2.$$

That is,

$$F_{k,n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-i}{i}k^{n-2i} \text{ for } n \geq 1.$$

Considering this result for $k = c$ and $n = r$, we can give the following two conjectures:

$$Q_r = Q_r(c) = cF_{c,2r}$$

and

$$P_r = P_r(c) = F_{c,2r} + F_{c,2r+1}.$$

6. Conclusion

In this paper, we examined the action of a special subgroup of the congruence subgroup on $\hat{\mathbb{Q}}$. Using this action we obtained some results on Pascal and Fibonacci numbers via the modular group. The results obtained are important for the fields of number theory and combinatorics. Further, it has also been observed that

$$\infty \to T_c(\infty) \to T_c^2(\infty) \to \cdots \to T_c^r(\infty) \to T_c^{r+1}(\infty) \to \cdots$$

is an infinitely long path in the suborbital graph $G(\infty, \frac{c^2+c+1}{c^2})$. Hence, this action is related to suborbital graphs theory which firstly studied by Jones et. al. in the reference [4]. This relationship can be examined in the future studies.

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ASSOCIATED CURVES OF A FRENET CURVE IN THE DUAL LORENTZIAN SPACE

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Abstract. In this work, we firstly introduce notions of principal directed curves and principal donor curves which are associated curves of a Frenet curve in the dual Lorentzian space $\mathbb{D}_1^3$. We give some relations between the curvature and the torsion of a dual principal directed curve and the curvature and the torsion of a dual principal donor curve. We show that the dual principal directed curve of a dual general helix is a plane curve and obtain the equation of dual general helix by using position vector of plane curve. Then we show that the principal donor curve of a circle in $\mathbb{D}^2$ or a hyperbola in $\mathbb{D}_1^2$ and the principal directed curve of a slant helix in $\mathbb{D}_1^3$ are a helix and general helix, respectively. We explain with an example for the second case. Finally, according to causal character of the principal donor curve of principal directed rectifying curve in $\mathbb{D}_1^3$, we show this curve to correspond to any timelike or spacelike ruled surface in Minkowski 3−space $\mathbb{R}_1^3$.

1. Introduction

It is very interesting to study curves in both dual space $\mathbb{D}^3$ and dual Lorentzian space $\mathbb{D}_1^3$. Because a differentiable curve on dual unit sphere in $\mathbb{D}^3$ represents a ruled surface in Euclidean 3−space $\mathbb{R}^3$ with the aid of the E. Study mapping. Similarly, a differentiable curve on dual pseudo hyperbolic space $\mathbb{H}_0^2$ in $\mathbb{D}_1^3$ corresponds to a timelike ruled surface in Minkowski 3−space $\mathbb{R}_1^3$ and the timelike (resp. spacelike) curve on dual pseudo sphere $\mathbb{S}_1^2$ in $\mathbb{D}_1^3$ corresponds to any spacelike (resp. timelike) ruled surface in $\mathbb{R}_1^3$. Therefore, we can say something about ruled surfaces in $\mathbb{R}^3$ or $\mathbb{R}_1^3$ when examining curves in $\mathbb{D}^3$ or $\mathbb{D}_1^3$, respectively [9, 16–18].

Keywords. Dual Lorentzian space, associated curves, dual general helix, dual slant helix, principal directed rectifying curve, ruled surface.

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In this paper, we examine associated curves of a Frenet curve in $\mathbb{D}_1^3$ and show these curves to correspond to any timelike or spacelike ruled surfaces in Minkowski 3−space $\mathbb{R}_1^3$. For this purpose, we recall the fundamental properties of $\mathbb{R}_1^3$ and $\mathbb{D}_1^3$.

$\mathbb{R}_1^3$ is the 3−dimensional Lorentzian space (or Minkowski 3−space) with symmetric, bilinear and non-degenerate metric given by

$$\langle u, v\rangle = -u_1v_1 + u_2v_2 + u_3v_3$$

for vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in Euclidean 3−space $\mathbb{R}^3$. In $\mathbb{R}_1^3$, the Lorentzian vector product of u and v is defined by

$$u \times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

We know that a vector and a curve have three different categories, namely, spacelike, timelike and null, depending on their causal characters. Then a vector u is said to be spacelike, timelike or null (lightlike) if $\langle u, u\rangle > 0$ (or $u = 0$), $\langle u, u\rangle < 0$, $\langle u, u\rangle = 0$ (and $u \neq 0$), respectively. Similarly, a curve γ is called spacelike, timelike or null (lightlike) if its velocity vector is spacelike, timelike or null vector, respectively. We also state that Frenet curves are timelike curves and spacelike curves with spacelike or timelike principal normal vector. Lastly, a surface is named non-degenerate (or degenerate) if induced metric on its tangent plane is non-degenerate (or degenerate). The pseudo sphere of radius $r > 0$ in $\mathbb{R}_1^3$ denoted by

$$S_1^2 = \{p \in \mathbb{R}_1^3 : \langle p, p\rangle = r^2,\ r > 0\}$$

and the pseudo hyperbolic space of radius $r > 0$ in $\mathbb{R}_1^3$ denoted by

$$H_0^2 = \{p \in \mathbb{R}_1^3 : \langle p, p\rangle = -r^2,\ r > 0\}$$

are non-degenerate surfaces [2, 12, 13].

A number expressed as

$$\widehat{a} = a + \xi a^* \text{ or } \widehat{a} = (a, a^*)$$

is called a dual number for $\forall a, a^* \in \mathbb{R}$ and the set of all dual numbers is indicated by $\mathbb{D}$, where ξ is called as dual unit with properties

$$\xi \neq 0,\ 0\xi = \xi 0 = 0,\ 1\xi = \xi 1 = \xi,\ \xi^2 = 0.$$

Equality and some operations on $\mathbb{D}$ are defined as follows:

i) Equality: $\widehat{a} = \widehat{b}$ for $\widehat{a} = a + \xi a^*$, $\widehat{b} = b + \xi b^*$ iff $a = b$ and $a^* = b^*$.

ii) Addition: $\widehat{a} + \widehat{b} = (a + \xi a^*) + (b + \xi b^*) = (a + b) + \xi(a^* + b^*)$.

iii) Multiplication: $\widehat{ab} = (a + \xi a^*)(b + \xi b^*) = ab + \xi(ab^* + a^*b)$.

iv) Division: $\frac{\widehat{a}}{\widehat{b}} = \frac{a}{b} + \xi(\frac{a^*b - ab^*}{b^2})$, $b \neq 0$.

We note that $\mathbb{D}$ is a commutative ring according to the above addition and multiplication operations. Also f on $\mathbb{D}$ is defined by

$$f(\widehat{a}) = f(a + \xi a^*) = f(a) + \xi a^* f'(a),$$

where $f^{'}$ represents the derivative of f . For example,

$$\sin(\widehat{a}) = \sin(a + \xi a^*) = \sin a + \xi a^* \cos a$$

(see [9, 17, 19] for more details).

A dual vector $\widehat{x}$ is an ordered triple of dual numbers $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ and also a dual vector $\widehat{x}$ has the form $\widehat{x} = x + \xi x^*$ for $\forall x = (x_1, x_2, x_3)$, $x^* = (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3$, where x and x^* are the real and dual parts of $\widehat{x}$, respectively. The set of all dual vectors which is denoted as $\mathbb{D}^3$ is a module on the ring $\mathbb{D}$. The Lorentzian inner product of dual vectors $\widehat{x}$ and $\widehat{y}$ is defined by

$$\langle \widehat{x}, \widehat{y} \rangle = \langle x, y \rangle + \xi(\langle x, y^* \rangle + \langle x^*, y \rangle).$$

The dual space $\mathbb{D}^3$ together with this Lorentzian inner product is called dual Lorentzian space and it is represented by $\mathbb{D}^3_1$. The causal characterization of a dual vector $\widehat{x} = x + \xi x^*$ depends on the causal characterization of x, that is a dual vector $\widehat{x}$ is called to be spacelike, timelike, null (lightlike) if the vector x is spacelike, timelike, null (lightlike), respectively. The Lorentzian vector product of dual vectors $\widehat{x} = (\widehat{x_1}, \widehat{x}_2, \widehat{x}_3)$ and $\widehat{y} = (\widehat{y_1}, \widehat{y}_2, \widehat{y}_3)$ in $\mathbb{D}^3_1$ is defined by

$$\widehat{x} \times \widehat{y} = (\widehat{x}_3\widehat{y}_2 - \widehat{x}_2\widehat{y}_3, \widehat{x}_3\widehat{y}_1 - \widehat{x}_1\widehat{y}_3, \widehat{x}_1\widehat{y}_2 - \widehat{x}_2\widehat{y}_1).$$

If $x \neq 0$, then the norm of $\widehat{x}$ is given by

$$\|\widehat{x}\| = \sqrt{|< \widehat{x}, \widehat{x} >|} = \|x\| + \xi \frac{< x, x^* >}{\|x\|^2}.$$

A dual vector $\widehat{x}$ with norm $1+\xi 0 = (1, 0) \in \mathbb{D}$ is called a dual unit vector. Therefore, dual pseudo sphere and dual pseudo hyperbolic space are defined by

$$\mathbb{S}^2_1 = \left\{ \widehat{x} = x + \xi x^* | \ \ \|\widehat{x}\| = (1, 0) \ ; \ x, \ x^* \in \mathbb{R}^3_1 \text{ and the vector } \widehat{x} \text{ is spacelike} \right\}$$

and

$$\mathbb{H}^2_0 = \left\{ \widehat{x} = x + \xi x^* | \ \ \|\widehat{x}\| = (1, 0) \ ; \ x, \ x^* \in \mathbb{R}^3_1 \text{ and the vector } \widehat{x} \text{ is timelike} \right\},$$

respectively.

Let $\widehat{\gamma}(\sigma) = \gamma(\sigma) + \xi \gamma^*(\sigma)$ be a dual curve with parameter $\sigma \in \mathbb{R}$ in $\mathbb{D}^3_1$. The real curve $\gamma(\sigma)$ is called the (real) indicatrix of $\widehat{\gamma}(\sigma)$. If every $\gamma(\sigma)$ and $\gamma^*(\sigma)$ are differentiable, then $\widehat{\gamma}(\sigma)$ is differentiable in $\mathbb{D}^3_1$. The dual arc length of the dual curve $\widehat{\gamma}$ is given by

$$\widehat{s} = \int_0^s \|\widehat{\gamma}(\sigma)'\| \, d\sigma = \int_0^s \|\gamma(\sigma)'\| \, d\sigma + \xi \int_0^s < t, \gamma^*(\sigma) > d\sigma = s + \xi s^*,$$

where s and t is arclength and the unit tangent vector of γ, respectively. As in $\mathbb{R}^3_1$ we call timelike dual curves and spacelike dual curves with spacelike or timelike dual principal normal vector as dual Frenet curves (or Frenet curves in $\mathbb{D}^3_1$). Assume that $\widehat{\gamma}$ is a reparametrization curve with the parametrization s of the indicatrix. Hence the dual Frenet formulae for the dual unit speed Frenet curve $\widehat{\gamma}$ are

$$\frac{d}{d\widehat{s}}\begin{bmatrix}\widehat{t}\\ \widehat{n}\\ \widehat{b}\end{bmatrix}=\begin{bmatrix}0 & \widehat{\kappa} & 0\\ -\varepsilon_0\varepsilon_1\widehat{\kappa} & 0 & \widehat{\tau}\\ 0 & -\varepsilon_1\varepsilon_2\widehat{\tau} & 0\end{bmatrix}\begin{bmatrix}\widehat{t}\\ \widehat{n}\\ \widehat{b}\end{bmatrix}, \tag{1}$$

such that $< t,t >= \varepsilon_0 = \pm 1$, $< n,n >= \varepsilon_1 = \pm 1$ and $< b,b >= \varepsilon_2 = \pm 1$, where

$$\begin{array}{rccl}\widehat{\kappa}: & \mathbb{R} & \to & \mathbb{D}\\ & s & \to & \widehat{\kappa}(s)=\kappa(s)+\xi\kappa^*(s)\end{array}$$

is nowhere pure dual curvature and

$$\begin{array}{rccl}\widehat{\tau}: & \mathbb{R} & \to & \mathbb{D}\\ & s & \to & \widehat{\tau}(s)=\tau(s)+\xi\tau^*(s)\end{array}$$

is nowhere pure dual torsion [4, 14, 16–20].

Let $\widehat{\gamma}$ be a dual unit speed Frenet curve in $\mathbb{D}^3_1$ and $\widehat{W}$ be a dual unit vector field along $\widehat{\gamma}$. The curve $\widehat{\gamma}_0$ in $\mathbb{D}^3_1$ is called the $\widehat{W}-$directional dual curve of $\widehat{\gamma}$ if the dual unit tangent vector $\widehat{t}_0$ of $\widehat{\gamma}_0$ is equal to $\widehat{W}$. Moreover $\widehat{\gamma}$ is called the $\widehat{W}-$donor dual curve of $\widehat{\gamma}_0$. Thus, we can define three different dual curves by special selection of $\widehat{W}$:

i) If $\widehat{W}=\widehat{t}$, then $\widehat{t}_0=\widehat{t}$. In this case $\widehat{\gamma}$ and $\widehat{\gamma}_0$ are the same dual curves.

ii) If $\widehat{W}=\widehat{n}$, then $\widehat{t}_0=\widehat{n}$. In this case $\widehat{\gamma}_0$ is called the dual principal directional curve of $\widehat{\gamma}$ and $\widehat{\gamma}$ is called the dual principal donor curve of $\widehat{\gamma}_0$.

iii) If $\widehat{W}=\widehat{b}$ then $\widehat{t}_0=\widehat{b}$. In this case $\widehat{\gamma}_0$ is called the dual binormal directional curve of $\widehat{\gamma}$ and $\widehat{\gamma}$ is called the dual binormal donor curve of $\widehat{\gamma}_0$ [1, 7, 8, 11].

In this paper, we obtain firstly some relations between the curvature and the torsion of a principal directed curve and the curvature and the torsion of a principal donor curve in $\mathbb{D}^3_1$. We see that the principal directed curve of a dual general helix is a plane curve and give the equation of a dual general helix by using position vector of a plane curve. Then we show that the principal donor curve of a circle in $\mathbb{D}^2$ or a hyperbola in $\mathbb{D}^2_1$ is a dual helix and we also obtain that the principal directed curve of a dual slant helix is a dual general helix. We give an example for simple closed dual slant helix. Finally, according to causal character of the principal donor curve of a principal directed rectifying curve in $\mathbb{D}^3_1$, we show that this curve to correspond to any timelike or spacelike ruled surface in Minkowski $3-$space $\mathbb{R}^3_1$.

2. Principal Directional and Principal Donor Curves of a Frenet Curve in $\mathbb{D}^3_1$

In this section, we examine principal directional and principal donor curves of a Frenet curve in the dual Lorentzian space $\mathbb{D}^3_1$. Firstly, we state that the causal characterization of a curve $\widehat{\gamma}$ in $\mathbb{D}^3_1$ depends on the causal characterization of a curve

γ which is the real part of $\widehat{\gamma}$. Then we give the following Lemma from Lemma 3.1 in [8].

Lemma 1. *There is no timelike dual general helix or spacelike dual general helix with spacelike principal normal that provides the condition* $\left|\frac{\widehat{\tau}}{\widehat{\kappa}}\right| = (1,0)$ *in the dual Lorentzian space* $\mathbb{D}_1^3$.

Now we give the following theorem which expresses the relationship between the dual curvature and torsion of $\widehat{\gamma}(s)$ and the dual curvature and torsion of $\widehat{\gamma}_0$ which is the principal direction of $\widehat{\gamma}$.

Theorem 1. *Let* $\widehat{\gamma}$ *be a dual unit speed Frenet curve with the dual curvature* $\widehat{\kappa}$ *and the dual torsion* $\widehat{\tau}$ *and* $\widehat{\gamma}_0$ *be the principal directional curve of* $\widehat{\gamma}$ *in* $\mathbb{D}_1^3$. *Then the dual curvature* $\widehat{\kappa}_0$ *and the dual torsion* $\widehat{\tau}_0$ *of* $\widehat{\gamma}_0$ *is*

$$\widehat{\kappa}_0 = \sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}, \quad \widehat{\tau}_0 = \frac{\widetilde{\varepsilon}_2\varepsilon_1\varepsilon_2\widehat{\kappa}^2}{\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right), \tag{2}$$

where $\varepsilon_0 =< t,t >$, $\varepsilon_1 =< n,n >$, $\varepsilon_2 =< b,b >$, $\widetilde{\varepsilon}_1 =< n_0,n_0 >$ *and* $\widetilde{\varepsilon}_2 =< b_0,b_0 >$ *such that* $\{t,n,b\}$ *and* $\{t_0,n_0,b_0\}$ *Frenet frames of the curves* γ *and* γ_0, *respectively.*

Proof. Since $\widehat{\gamma}_0$ is the principal direction curve of a dual unit speed Frenet curve $\widehat{\gamma}$, the equations $\widehat{t}_0 = \widehat{n}$ and $\frac{d\widehat{t}_0}{d\widehat{s}} = \frac{d\widehat{n}}{d\widehat{s}}$ are provided. Considering the dual Frenet formulae (1) we have

$$\frac{d\widehat{t}_0}{d\widehat{s}} = -\varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{t} + \widehat{\tau}\widehat{b}$$

and

$$\widehat{\kappa}_0^2 < \widehat{n}_0, \widehat{n}_0 >= \varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2.$$

Therefore, we obtain

$$< n_0, n_0 >= \frac{\varepsilon_0\kappa^2 + \varepsilon_2\tau^2}{\kappa_0^2}. \tag{3}$$

The dual curvature of $\widehat{\gamma}$ is also

$$\widehat{\kappa}_0 = \sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}.$$

Thus, the dual Frenet vectors along $\widehat{\gamma}_0$ are

$$\widehat{t}_0 = \widehat{n},\ \ \widehat{n}_0 = \frac{-\varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{t} + \widehat{\tau}\widehat{b}}{\sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}},\ \widehat{b}_0 = \widetilde{\varepsilon}_0\widetilde{\varepsilon}_1\frac{\widehat{\kappa}\widehat{b} + \varepsilon_1\varepsilon_2\widehat{\tau}\widehat{t}}{\sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}}. \tag{4}$$

By taking differentiation of equation (4) with respect to $\widehat{s}$ and this is written in the equation

$$\widehat{\tau}_0 = -\widetilde{\varepsilon}_2 < \frac{d\widehat{b}_0}{d\widehat{s}}, \widehat{n}_0 >,$$

we have

$$\widehat{\tau}_0 = -\widetilde{\varepsilon}_2 < \left(\varepsilon_0\varepsilon_2 \frac{\widehat{\kappa}\left(\widehat{\kappa}\frac{d\widehat{\tau}}{d\widehat{s}} - \widehat{\tau}\frac{d\widehat{\kappa}}{d\widehat{s}}\right)}{\left(\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2+\varepsilon_2\widehat{\tau}^2\right)\right)^{\frac{3}{2}}}\right)\widehat{t} - \left(\varepsilon_1\varepsilon_2 \frac{\widehat{\tau}\left(\widehat{\kappa}\frac{d\widehat{\tau}}{d\widehat{s}} - \widehat{\tau}\frac{d\widehat{\kappa}}{d\widehat{s}}\right)}{\left(\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2+\varepsilon_2\widehat{\tau}^2\right)\right)^{\frac{3}{2}}}\right)\widehat{b}, \frac{-\varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{t}+\widehat{\tau}\widehat{b}}{\sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2+\varepsilon_2\widehat{\tau}^2\right)}} > .$$

Then we get

$$\widehat{\tau}_0 = \frac{\widetilde{\varepsilon}_2\varepsilon_1\varepsilon_2\widehat{\kappa}^2}{\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right).$$

□

We can write the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ of $\widehat{\gamma}$ in terms of the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ of $\widehat{\gamma}_0$ in the following theorem:

Theorem 2. *Let $\widehat{\gamma}$ be a dual unit speed spacelike Frenet curve having a spacelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and $\widehat{\gamma}_0$ with the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ be a spacelike principal direction of $\widehat{\gamma}$ in $\mathbb{D}^3_1$.*
(a) If $\kappa > |\tau|$, then $\widehat{\gamma}_0$ is a spacelike dual curve with spacelike dual principal normal. Then the curvature and the torsion of principal donor curve of $\widehat{\gamma}_0$ are

$$\widehat{\kappa}(s) = \widehat{\kappa}_0(s)\cosh(\int \widehat{\tau}_0(s)d\widehat{s}), \qquad \widehat{\tau}(s) = \widehat{\kappa}_0(s)\sinh(\int \widehat{\tau}_0(s)d\widehat{s}) \tag{5}$$

(b) If $\kappa < |\tau|$, then $\widehat{\gamma}_0$ is a spacelike dual curve with timelike dual principal normal. Then the curvature and the torsion of principal donor curve of $\widehat{\gamma}_0$ are

$$\widehat{\kappa}(s) = \widehat{\kappa}_0(s)\sinh(\int \widehat{\tau}_0(s)d\widehat{s}), \qquad \widehat{\tau}(s) = -\widehat{\kappa}_0(s)\cosh(\int \widehat{\tau}_0(s)d\widehat{s}) \tag{6}$$

Proof. (*a*) If $\kappa > |\tau|$, as a result of (3), $\widehat{\gamma}_0$ is a spacelike dual curve with spacelike dual principal normal. Then by using (2) the curvature and the torsion functions of $\widehat{\gamma}_0$ are,

$$\widehat{\kappa}_0^2(s) = \widehat{\kappa}^2(s) - \widehat{\tau}^2(s), \quad \widehat{\tau}_0(s) = \frac{\widehat{\kappa}^2(s)}{\widehat{\kappa}^2(s) - \widehat{\tau}^2(s)}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right) \tag{7}$$

respectively. Firstly we replace $\frac{\widehat{\tau}}{\widehat{\kappa}}$ in the second equation of (7) with $\widehat{f}$. Then the second equation of (7) is rewritten as

$$\widehat{\tau}_0(s) = \frac{1}{1 - \left(\frac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)^2}\frac{d\widehat{f}(s)}{d\widehat{s}} = \frac{1}{1 - \widehat{f}^2(s)}\frac{d\widehat{f}(s)}{d\widehat{s}},$$

where

$$\widehat{f}(s) = f(s) + \xi f^*(s) = \frac{\tau(s)}{\kappa(s)} + \xi(\frac{\tau^*(s)}{\kappa(s)} - \frac{\tau(s)\kappa^*(s)}{\kappa^2(s)}).$$

On the other hand, since $\kappa > |\tau|$, $|f(s)|$ is less than 1. Thus, we get that

$$\int \widehat{\tau}_0(s)d\widehat{s} = \int \frac{\frac{d\widehat{f}(s)}{d\widehat{s}}}{1 - \widehat{f}^2(s)}d\widehat{s} = \tanh^{-1}\widehat{f}(s) + \widehat{c},$$

where $\widehat{c}$ is dual constant. If we take $\widehat{c} = 0$ without breaking the generality, then we obtain

$$\widehat{f}(s) = \tanh\left(\int \widehat{\tau}_0(s)d\widehat{s}\right).$$

By using $\widehat{f} = \frac{\widehat{\tau}}{\widehat{\kappa}}$ we obtain

$$\widehat{\tau}(s) = \tanh\left(\int \widehat{\tau}_0(s)d\widehat{s}\right)\widehat{\kappa}(s).$$

If this equation is written in place of the first equation of (7) and the necessary arrangements are made, then both equations of (5) are obtained.

(b) The proof is similar to the proof of the statement (a). □

Similarly, we can write Theorem 3 and Theorem 4.

Theorem 3. *Let $\widehat{\gamma}$ be a dual unit speed spacelike Frenet curve having a timelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and $\widehat{\gamma}_0$ with the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ be a timelike principal direction of $\widehat{\gamma}$ in $\mathbb{D}^3_1$. Then the dual curvature and the dual torsion of principal donor curve of $\widehat{\gamma}_0$ are*

$$\widehat{\kappa}(s) = \widehat{\kappa}_0(s)\cos(\int \widehat{\tau}_0(s)d\widehat{s}), \qquad \widehat{\tau}(s) = -\widehat{\kappa}_0(s)\sin(\int \widehat{\tau}_0(s)d\widehat{s}). \tag{8}$$

Theorem 4. *Let $\widehat{\gamma}$ be a dual unit speed timelike Frenet curve with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and $\widehat{\gamma}_0$ with the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ be principal direction of $\widehat{\gamma}$ in $\mathbb{D}^3_1$.*
(a) If $\kappa < |\tau|$, then $\widehat{\gamma}_0$ is a spacelike dual curve with spacelike dual principal normal. Then the dual curvature and the dual torsion of principal donor curve of $\widehat{\gamma}_0$ are

$$\widehat{\kappa}(s) = \widehat{\kappa}_0(s)\sinh(\int \widehat{\tau}_0(s)d\widehat{s}), \qquad \widehat{\tau}(s) = \widehat{\kappa}_0(s)\cosh(\int \widehat{\tau}_0(s)d\widehat{s}). \tag{9}$$

(b) If $\kappa > |\tau|$, then $\widehat{\gamma}_0$ is a spacelike dual curve with timelike dual principal normal. Then the dual curvature and the dual torsion of principal donor curve of $\widehat{\gamma}_0$ are

$$\widehat{\kappa}(s) = \widehat{\kappa}_0(s)\cosh(\int \widehat{\tau}_0(s)d\widehat{s}), \qquad \widehat{\tau}(s) = -\widehat{\kappa}_0(s)\sinh(\int \widehat{\tau}_0(s)d\widehat{s}). \tag{10}$$

3. Principal Directional Curves of General Helices in $\mathbb{D}^3_1$

In this section, we show that principal directional curves of general helices in $\mathbb{D}^3_1$ is plane curves. Then we obtain the position vectors of dual general helices with the aid of this plane curves (see [5, 12] for general helix in $\mathbb{R}^3_1$).

Theorem 5. *A dual unit speed Frenet curve $\widehat{\gamma}$ in $\mathbb{D}^3_1$ is a general helix iff the principal directional curve of $\widehat{\gamma}$ is a plane curve.*

Proof. Since $\widehat{\gamma}$ is a dual unit speed Frenet curve, we will only give the proof for a spacelike dual Frenet curve with timelike principal normal.

($\Rightarrow$) Let $\widehat{\gamma}(s)$ be a dual unit speed Frenet curve with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and $\widehat{\gamma}_0$ be the principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}_1^3$. Then it is clear that

$$\frac{\widehat{\tau}}{\widehat{\kappa}} = -\tan\left(\int \widehat{\tau}_0(s)d\widehat{s}\right) \tag{11}$$

from the equation (8). By taking derivative of (11) with respect to $\widehat{s}$ we have

$$\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right) = -\widehat{\tau}_0(s)\sec^2\left(\int \widehat{\tau}_0(s)d\widehat{s}\right) = 0.$$

Since $\sec^2\left(\int \widehat{\tau}_0(s)d\widehat{s}\right) \neq 0$, we say that $\widehat{\tau}_0(s) = 0$. Then $\widehat{\gamma}_0$ is a plane curve in $\mathbb{D}_1^3$.

($\Leftarrow$) Let $\widehat{\gamma}_0$ which is principal directional curve of $\widehat{\gamma}$ be a plane curve in $\mathbb{D}_1^3$. Then $\widehat{\tau}_0 = 0$. As a result of $\widehat{\kappa} \neq 0$, $\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right) = 0$ and $\frac{\widehat{\tau}}{\widehat{\kappa}}$ is a dual constant from (2). Consequently the Frenet curve $\widehat{\gamma}$ is a general helix in $\mathbb{D}_1^3$. $\square$

Similarly, we can also prove in case $\widehat{\gamma}$ is a timelike curve or a spacelike curve with spacelike principal normal $\mathbb{D}_1^3$.

Theorem 6. *Let $\widehat{\gamma}$ be a spacelike plane curve with the dual curvature $\widehat{\kappa}$ in $\mathbb{D}_1^3$.*
(a) *If the principal normal vector of $\widehat{\gamma}$ in $\mathbb{D}_1^3$ is a spacelike, then the position vector of $\widehat{\gamma}$ is given by*

$$\widehat{\gamma}(s) = \int \left(0, \cos\left(\int \widehat{\kappa}(s)d\widehat{s}\right), \sin\left(\int \widehat{\kappa}(s)d\widehat{s}\right)\right) d\widehat{s}, \tag{12}$$

(b) *If the principal normal vector of $\widehat{\gamma}$ in $\mathbb{D}_1^3$ is a timelike, then the position vector of $\widehat{\gamma}$ is given by*

$$\widehat{\gamma}(s) = \int \left(\sinh\left(\int \widehat{\kappa}(s)d\widehat{s}\right), \cosh\left(\int \widehat{\kappa}(s)d\widehat{s}\right), 0\right) d\widehat{s}. \tag{13}$$

Proof. Let $\widehat{\gamma}$ be a spacelike plane curve with the dual curvature $\widehat{\kappa}$ in $\mathbb{D}_1^3$. Since $\widehat{\gamma}$ is a spacelike dual curve, $\left\langle \widehat{t}, \widehat{t} \right\rangle = (1, 0)$. On the other hand if we consider the dual Frenet formulae (1) and $\widehat{\theta} = \int \widehat{\kappa}(s)d\widehat{s}$, then the following statements hold:
(a) If the principal normal vector of $\widehat{\gamma}$ is spacelike, then $\widehat{t}(s) = \left(0, \cos\widehat{\theta}, \sin\widehat{\theta}\right)$. Therefore, we have the equation (12).
(b) If the principal normal vector of $\widehat{\gamma}$ is timelike, then $\widehat{t}(s) = \left(\sinh\widehat{\theta}, \cosh\widehat{\theta}, 0\right)$. Therefore, we have the equation (13). $\square$

Theorem 7. *The position vector $\widehat{\gamma}$ of a timelike plane curve with the dual curvature $\widehat{\kappa}$ in $\mathbb{D}_1^3$ is given by*

$$\widehat{\gamma}(s) = \int \left(\cosh\left(\int \widehat{\kappa}(s)d\widehat{s}\right), \sinh\left(\int \widehat{\kappa}(s)d\widehat{s}\right), 0\right) d\widehat{s}. \tag{14}$$

Theorem 8. *Let $\widehat{\gamma}$ be a dual unit speed spacelike general helix having a spacelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau} = \widehat{m}\widehat{\kappa}$ for dual constant $\widehat{m} = m + \xi m^*$ in $\mathbb{D}_1^3$.*
(a) If $\frac{|\tau|}{\kappa} = |m| < 1$, then the position vector $\widehat{\gamma}$ is given by

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{1-\widehat{m}^2}} \int \left(\widehat{m}, \sin\left(\sqrt{1-\widehat{m}^2} \int \widehat{\kappa}(s) d\widehat{s} \right), -\cos\left(\sqrt{1-\widehat{m}^2} \int \widehat{\kappa}(s) d\widehat{s} \right) \right) d\widehat{s}, \tag{15}$$

and the principal directional curve of $\widehat{\gamma}$ is a spacelike plane curve with a spacelike principal normal in $\mathbb{D}^2$,
(b) If $\frac{|\tau|}{\kappa} = |m| > 1$ then the position vector $\widehat{\gamma}$ is given by

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{\widehat{m}^2-1}} \int \left(\cosh\left(\sqrt{\widehat{m}^2-1} \int \widehat{\kappa}(s) d\widehat{s} \right), \sinh\left(\sqrt{\widehat{m}^2-1} \int \widehat{\kappa}(s) d\widehat{s} \right), \widehat{m} \right) d\widehat{s}. \tag{16}$$

and the principal directional curve of $\widehat{\gamma}$ is a spacelike plane curve with a timelike principal normal in $\mathbb{D}_1^2$.

Proof. Let $\widehat{\gamma}_0$ be principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}_1^3$. $\widehat{\gamma}_0$ is a spacelike dual curve because $\widehat{\gamma}$ has a spacelike principal normal.
In case (a) we can say that $\widehat{\gamma}_0$ has the dual Frenet vectors,

$$\begin{cases} \widehat{t_0}(s) = \left(0, \cos\left[\int \widehat{\kappa}_0(s) d\widehat{s}\right], \sin\left[\int \widehat{\kappa}_0(s) d\widehat{s}\right]\right) \\ \widehat{n}_0(s) = \left(0, -\sin\left[\int \widehat{\kappa}_0(s) d\widehat{s}\right], \cos\left[\int \widehat{\kappa}_0(s) d\widehat{s}\right]\right) \\ \widehat{b}_0(s) = (1, 0, 0) \end{cases}$$

by using (12). If we consider the equation (11) and $0 < |m| < 1$, then the equations

$$\widehat{\kappa}(s) = \frac{\widehat{\kappa}_0(s)}{\sqrt{1-\widehat{m}^2}} \quad \text{and} \quad \widehat{\tau}(s) = \widehat{m}\widehat{\kappa}(s)$$

are hold. From (4) and $\widehat{\kappa}_0(s) = \widehat{\kappa}(s)\sqrt{1-\widehat{m}^2}$, the dual unit tangent vector $\widehat{t}$ is obtained as

$$\widehat{t} = \frac{1}{\sqrt{1-\widehat{m}^2}} \left(\widehat{m}, \sin\left[\sqrt{1-\widehat{m}^2} \int \widehat{\kappa}(s) d\widehat{s} \right], -\cos\left[\sqrt{1-\widehat{m}^2} \int \widehat{\kappa}(s) d\widehat{s} \right] \right).$$

Hence, if $\frac{|\tau|}{\kappa} = |m| < 1$, then a spacelike general helix with a spacelike principal normal in $\mathbb{D}_1^3$ is given by the equation (15).
(b) The proof is similar to the proof of the statement (a). □

Similarly, we have Theorem 9 and Theorem 10.

Theorem 9. *Let $\widehat{\gamma}$ be a dual unit speed spacelike general helix having timelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau} = \widehat{m}\widehat{\kappa}$ for dual* *constant* $\widehat{m} = m + \xi m^*$ *in* $\mathbb{D}_1^3$. *The position vector of* $\widehat{\gamma}$ *is given by*

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{1+\widehat{m}^2}} \int \left(\sinh\left[\sqrt{1+\widehat{m}^2}\int \widehat{\kappa}(s)d\widehat{s}\right], \cosh\left[\sqrt{1+\widehat{m}^2}\int \widehat{\kappa}(s)d\widehat{s}\right], -\widehat{m} \right) d\widehat{s} \tag{17}$$

and the principal directional curve of $\widehat{\gamma}$ *is a timelike plane curve in* $\mathbb{D}_{1.}^2$.

Theorem 10. *Let* $\widehat{\gamma}$ *be a dual unit speed timelike general helix with the dual curvature* $\widehat{\kappa}$ *and the dual torsion* $\widehat{\tau} = \widehat{m}\widehat{\kappa}$ *for dual constant* $\widehat{m} = m + \xi m^*$ *in* $\mathbb{D}_1^3$.
(a) *If* $\frac{|\tau|}{\kappa} = |m| > 1$, *then the position vector of* $\widehat{\gamma}$ *is given by*

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{\widehat{m}^2-1}} \int \left(\widehat{m}, \sin\left[\sqrt{\widehat{m}^2-1}\int \widehat{\kappa}(s)d\widehat{s}\right], -\cos\left[\sqrt{\widehat{m}^2-1}\int \widehat{\kappa}(s)d\widehat{s}\right] \right) d\widehat{s} \tag{18}$$

and the principal directional curve of $\widehat{\gamma}$ *is a spacelike plane curve with spacelike principal normal in* $\mathbb{D}^2$,
(b) *If* $\frac{|\tau|}{\kappa} = |m| < 1$, *then the position vector of* $\widehat{\gamma}$ *is given by*

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{1-\widehat{m}^2}} \int \left(\cosh\left[\sqrt{1-\widehat{m}^2}\int \widehat{\kappa}(s)d\widehat{s}\right], \sinh\left[\sqrt{1-\widehat{m}^2}\int \widehat{\kappa}(s)d\widehat{s}\right], \widehat{m} \right) d\widehat{s} \tag{19}$$

and the principal directional curve of $\widehat{\gamma}$ *is a spacelike plane curve with timelike principal normal in* $\mathbb{D}_1^2$.

Taking into consideration the above three theorems, the following three results are obtained:

Corollary 1. *Let* $\widehat{\gamma}$ *be a dual unit speed spacelike Frenet curve with a spacelike principal normal and* $\widehat{\gamma}_0$ *be a spacelike principal directional curve of* $\widehat{\gamma}$ *in* $\mathbb{D}_1^3$. *Then* $\widehat{\gamma}_0$ *is a plane curve in* $\mathbb{D}^2$ *or* $\mathbb{D}_1^2$ *iff* $\widehat{\gamma}$ *is a general helix in* $\mathbb{D}_1^3$ *with inequalities* $\kappa > |\tau|$ *or* $\kappa < |\tau|$, *respectively. Furthermore* $\widehat{\gamma}_0$ *is a circle in* $\mathbb{D}^2$ *or spacelike hyperbola in* $\mathbb{D}_1^2$ *if and only if* $\widehat{\gamma}$ *is a helix in* $\mathbb{D}_1^3$ *with* $\kappa > |\tau|$ *or a helix* $\kappa < |\tau|$ *,respectively.*

Corollary 2. *Let* $\widehat{\gamma}$ *be a dual unit speed spacelike Frenet curve with a timelike principal normal and* $\widehat{\gamma}_0$ *be a timelike principal directional curve of* $\widehat{\gamma}$ *in* $\mathbb{D}_1^3$. *Then* $\widehat{\gamma}_0$ *is a plane curve iff* $\widehat{\gamma}$ *is a general helix in* $\mathbb{D}_1^3$. *Furthermore* $\widehat{\gamma}_0$ *is a timelike hyperbola if and only if* $\widehat{\gamma}$ *is a helix in* $\mathbb{D}_1^3$.

Corollary 3. *Let* $\widehat{\gamma}$ *be a dual unit speed timelike Frenet curve and* $\widehat{\gamma}_0$ *be a spacelike principal directional curve of* $\widehat{\gamma}$ *in* $\mathbb{D}_1^3$. *Then* $\widehat{\gamma}_0$ *is a plane curve in* $\mathbb{D}^2$ *or* $\mathbb{D}_1^2$ *iff* $\widehat{\gamma}$ *is a general helix in* $\mathbb{D}_1^3$ *with inequalities* $\kappa < |\tau|$ *or* $\kappa > |\tau|$, *respectively. Furthermore* $\widehat{\gamma}_0$ *is a circle in* $\mathbb{D}^2$ *or a spacelike hyperbola in* $\mathbb{D}_1^2$ *iff* $\widehat{\gamma}$ *is a helix in* $\mathbb{D}_1^3$ *with* $\kappa < |\tau|$ *or* $\kappa > |\tau|$, *respectively.*

Consequently, the general helices are characterized in $\mathbb{D}_1^3$ according to the associated curve as follows:

Theorem 11. *A general helix in* $\mathbb{D}_1^3$ *is the principal donor curve of some planar curves.*

4. Principal Directional Curves of Slant Helices in $\mathbb{D}_1^3$

In this section, we examine the causal characters of general helices which are principal directional curves of slant helices according to causal characters of slant helices in $\mathbb{D}_1^3$. We are state that the connections between general helices and slant helices in $\mathbb{D}_1^3$ as follows:

Let $\widehat{\gamma}$ be a Frenet curve in $\mathbb{D}_1^3$ and $\widehat{W}$ be a dual unit vector along the dual Frenet curve $\widehat{\gamma}$. If $\widehat{W}$ has a constant dual angle with a constant dual vector $\widehat{V}$ along $\widehat{\gamma}$, then the tangent vector of $\widehat{\gamma}_0$, which is the $\widehat{W}-$directional curve of $\widehat{\gamma}$, also has a constant dual angle with $\widehat{V}$ along $\widehat{\gamma}$. Conversely, if the dual unit tangent vector of the Frenet curve $\widehat{\gamma}_0$ in $\mathbb{D}_1^3$ makes a constant dual angle with the constant vector $\widehat{V}$ in $\mathbb{D}_1^3$ then $\widehat{\gamma}$ is the $\widehat{W}-$donor curve of $\widehat{\gamma}_0$.

In the expression given above, we take principal normal vector instead of $\widehat{W}$ along $\widehat{\gamma}$ in $\mathbb{D}_1^3$. Then $\widehat{\gamma}$ is a dual slant helix (slant helix in $\mathbb{D}_1^3$) that is the principal normal vector of $\widehat{\gamma}$ makes a constant dual angle with a constant vector $\widehat{V}$ in $\mathbb{D}_1^3$ iff the principal directional curve of $\widehat{\gamma}$ is a general helix in $\mathbb{D}_1^3$ that is the dual unit tangent vector of $\widehat{\gamma}_0$ makes a constant dual angle with a constant vector $\widehat{V}$ in $\mathbb{D}_1^3$. On the other hand, a slant helix is the principal donor curve of a general helix and a general helix is the principal directional curve of a slant helix in $\mathbb{D}_1^3$ (see [3, 15] for slant helices)

Now let $\widehat{\gamma}_0$ be a spacelike general helix having a spacelike dual principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}_0 = \widehat{c}\widehat{\kappa}_0$ for dual constant $\widehat{c}$ in $\mathbb{D}_1^3$. Then the spacelike principal donor curve $\widehat{\gamma}_1$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa}_1 = \widehat{\kappa}_0(s)\cosh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_1 = \widehat{\kappa}_0(s)\sinh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. A timelike principal donor curve $\widehat{\gamma}_2$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa_2} = \widehat{\kappa}_0(s)\sinh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_2 = \widehat{\kappa}_0(s)\cosh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. The dual Frenet curves $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ hold the equations of slant helices:

$$\frac{\widehat{\kappa}_1^2}{\left(\widehat{\kappa}_1^2-\widehat{\tau}_1^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_1}{\widehat{\kappa}_1}\right) = \frac{\cosh^2\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]}{\widehat{\kappa}_0}\frac{d}{d\widehat{s}}\left(\tanh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right) = \widehat{c} \tag{20}$$

and

$$\frac{-\widehat{\kappa}_2^2}{\left(\widehat{\tau}_2^2-\widehat{\kappa}_2^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_2}{\widehat{\kappa}_2}\right) = -\frac{\sinh^2\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]}{\widehat{\kappa}_0}\frac{d}{d\widehat{s}}\left(\coth\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right) = \widehat{c}, \tag{21}$$

respectively.

Let $\widehat{\gamma}_0$ be a spacelike general helix having a timelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}_0 = \widehat{c}\widehat{\kappa}_0$ for dual constant $\widehat{c}$ in $\mathbb{D}_1^3$. Then the spacelike principal donor curve $\widehat{\gamma}_3$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa}_3 = \widehat{\kappa}_0(s)\sinh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_3 = -\widehat{\kappa}_0(s)\cosh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. The timelike principal donor curve $\widehat{\gamma}_4$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa}_4 = \widehat{\kappa}_0(s)\cosh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_4 = -\widehat{\kappa}_0(s)\sinh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. The dual Frenet curves $\widehat{\gamma}_3$ and $\widehat{\gamma}_4$ hold the equations of slant helices:

$$\frac{\widehat{\kappa}_3^2}{\left(\widehat{\tau}_3^2-\widehat{\kappa}_3^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_3}{\widehat{\kappa}_3}\right)=\frac{\sinh^2\left[\widehat{c}\int\widehat{\kappa}_0 d\widehat{s}\right]}{\widehat{\kappa}_0(s)}\frac{d}{d\widehat{s}}\left(-\coth\left[\widehat{c}\int\widehat{\kappa}_0 d\widehat{s}\right]\right)=\widehat{c} \tag{22}$$

and

$$\frac{-\widehat{\kappa}_4^2}{\left(\widehat{\kappa}_4^2-\widehat{\tau}_4^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_4}{\widehat{\kappa}_4}\right)=-\frac{\cosh^2\left[\widehat{c}\int\widehat{\kappa}_0(s) d\widehat{s}\right]}{\widehat{\kappa}_0(s)}\frac{d}{d\widehat{s}}\left(-\tanh\left[\widehat{c}\int\widehat{\kappa}_0(s) d\widehat{s}\right]\right)=\widehat{c}, \tag{23}$$

respectively.

Finally, let $\widehat{\gamma}_0$ be a timelike general helix with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}_0 = \widehat{c}\widehat{\kappa}_0$ for dual constant $\widehat{c}$ in $\mathbb{D}^3_1$. Then the principal donor curve $\widehat{\gamma}_5$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa}_5 = \widehat{\kappa}_0(s)\cos\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_5 = -\widehat{\kappa}_0(s)\sin\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. The dual Frenet curve $\widehat{\gamma}_5$ hold the equation of slant helix:

$$\frac{-\widehat{\kappa}_5^2}{\left(\widehat{\tau}_5^2+\widehat{\kappa}_5^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_5}{\widehat{\kappa}_5}\right)=\frac{-\cos^2\left[\widehat{c}\int\widehat{\kappa}_0(s) d\widehat{s}\right]}{\widehat{\kappa}_0(s)}\frac{d}{d\widehat{s}}\left(-\tan\left[\widehat{c}\int\widehat{\kappa}_0(s) d\widehat{s}\right]\right)=\widehat{c}. \tag{24}$$

The value of a dual slant helix equation is called the dual slant helix constant. Then we can write following proposition:

Proposition 1. *Let $\widehat{\gamma}_0(s)$ be a general helix with the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ and $\widehat{\gamma}$ be the principal donor curve of $\widehat{\gamma}_0$ in $\mathbb{D}^3_1$. Then $\widehat{\gamma}$ is a dual slant helix with the dual slant helix constant $\frac{\widehat{\tau}_0}{\widehat{\kappa}_0}$.*

In the previous section, general helices were constructed in $\mathbb{D}^3_1$ with the help of plane curves. The above methods gave idea to construct slant helix with the help of general helices in $\mathbb{D}^3_1$. Now, by using the method in the third chapter the slant helices will be constructed from the general helices in $\mathbb{D}^3_1$.

Theorem 12. *Let $\widehat{\gamma}$ be a dual unit speed spacelike slant helix having a spacelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ in $\mathbb{D}^3_1$ and $\widehat{c} = c + \xi c^*$ be a dual slant helix constant.*

(a) *If $\kappa > |\tau|$ and $|c| < 1$, then the position vector $\widehat{\gamma}$ is given by*

$$\begin{aligned}
\widehat{\gamma}(s) \;\;=\;\; & -\int \Big(-\tfrac{\sinh\left[\widehat{c}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}}, \\
& \cosh\left[\widehat{c}\widehat{K}_1(s)\right]\cos\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right] - \tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\sin\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}}, \\
& \cosh\left[\widehat{c}\widehat{K}_1(s)\right]\sin\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right] \\
& + \tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\cos\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}} \Big) d\widehat{s},
\end{aligned} \tag{25}$$

where $\widehat{K}_1(s) = \int \sqrt{\widehat{\kappa}^2(s) - \widehat{\tau}^2(s)}d\widehat{s}$.

(b) *If $\kappa > |\tau|$ and $|c| > 1$ then the position vector $\widehat{\gamma}$ is given by*

$$\begin{aligned}
\widehat{\gamma}(s) \;\;=\;\; & -\int \Big(\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\sinh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_1(s)\right] \\
& - \tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\cosh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_1(s)\right]}{\sqrt{\widehat{c}^2-1}}, \sinh\left[\widehat{c}\widehat{K}_1(s)\right]\cosh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_1(s)\right] \\
& - \tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\sinh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_1(s)\right]}{\sqrt{\widehat{c}^2-1}}, - \tfrac{\cosh\left[\widehat{c}\widehat{K}_1(s)\right]}{\sqrt{\widehat{c}^2-1}} \Big) d\widehat{s},
\end{aligned} \tag{26}$$

where $\widehat{K}_1(s) = \int \sqrt{\widehat{\kappa}^2(s) - \widehat{\tau}^2(s)}d\widehat{s}$.

(c) *If $\kappa < |\tau|$ then the position vector $\widehat{\gamma}$ is given by*

$$\begin{aligned}
\widehat{\gamma}(s) \;\;=\;\; & -\int \Big(\sinh\left[\widehat{c}\widehat{K}_2(s)\right]\cosh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_2(s)\right] \\
& - \tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\sinh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_2(s)\right]}{\sqrt{1+\widehat{c}^2}}, \; \sinh\left[\widehat{c}\widehat{K}_2(s)\right]\cosh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_2(s)\right] \\
& - \tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\sinh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_2(s)\right]}{\sqrt{1+\widehat{c}^2}}, \; \tfrac{\cosh\left[\widehat{c}\widehat{K}_2(s)\right]}{\sqrt{1+\widehat{c}^2}})d\widehat{s} \Big),
\end{aligned} \tag{27}$$

where $\widehat{K}_2(s) = \int \sqrt{\widehat{\tau}^2(s) - \widehat{\kappa}^2(s)}d\widehat{s}$.

Proof. Let $\widehat{\gamma}_0$ the principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}^3_1$. Since $\widehat{\gamma}_0$ is a general helix with the dual torsion $\widehat{\tau}_0 = \widehat{c}\widehat{\kappa}_0$ for dual constant $\widehat{c}$.
(a) From the equation (15) we obtain

$$\begin{cases}
\widehat{t}_0(s) \;=\; \tfrac{1}{\sqrt{1-\widehat{c}^2}}\left(\widehat{c}, \; \sin\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right],\right. \\
\qquad\qquad \left. -\cos\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right), \\
\widehat{n}_0(s) \;=\; \left(0, \; \cos\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right], \; \sin\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right), \\
\widehat{b}_0(s) \;=\; \tfrac{1}{\sqrt{1-\widehat{c}^2}}\left(1, \; -\widehat{c}\sin\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right],\right. \\
\qquad\qquad \left. \widehat{c}\cos\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right).
\end{cases} \tag{28}$$

On the other hand from (4) it is clear that

$$\widehat{t} = -\cosh\left[\int\widehat{\tau}_0(s)d\widehat{s}\right]\widehat{n}_0 + \sinh\left[\int\widehat{\tau}_0(s)d\widehat{s}\right]\widehat{b}_0.$$

If we take into consideration the equation (28) then the dual unit tangent vector $\widehat{t}$ of $\widehat{\gamma}$ can be written as

$$\begin{aligned}\widehat{t} \;=\; & \Big(\tfrac{\sinh[\int \widehat{\tau}_0(s)d\widehat{s}]}{\sqrt{1-\widehat{c}^2}}, \\ & -\cosh\left[\int \widehat{\tau}_0(s)d\widehat{s}\right]\cos\left[\sqrt{1-\widehat{c}^2}\int \widehat{\kappa}_0(s)d\widehat{s}\right] - \tfrac{\widehat{c}\sinh[\int \widehat{\tau}_0(s)d\widehat{s}]\sin\left[\sqrt{1-\widehat{c}^2}\int \widehat{\kappa}_0(s)d\widehat{s}\right]}{\sqrt{1-\widehat{c}^2}}, \\ & -\cosh\left[\int \widehat{\tau}_0(s)d\widehat{s}\right]\sin\left[\sqrt{1-\widehat{c}^2}\int \widehat{\kappa}_0(s)d\widehat{s}\right] + \tfrac{\widehat{c}\sinh[\int \widehat{\tau}_0(s)d\widehat{s}]\cos\left[\sqrt{1-\widehat{c}^2}\int \widehat{\kappa}_0(s)d\widehat{s}\right]}{\sqrt{1-\widehat{c}^2}}\Big).\end{aligned}$$

By using the equations $\widehat{K}_1(s) = \int \widehat{\kappa}_0(s)d\widehat{s} = \int \sqrt{\widehat{\kappa}^2(s) - \widehat{\tau}^2(s)}d\widehat{s}$ and $\widehat{c}\widehat{K}_1(s) = \int \widehat{\tau}_0(s)d\widehat{s}$ we have

$$\begin{aligned}\widehat{t} \;=\; & \Big(\tfrac{\sinh\left[\widehat{c}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}}, \\ & -\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\cos\left(\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right) - \tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\sin\left(\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right)}{\sqrt{1-\widehat{c}^2}}, \\ & -\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\sin\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right] + \tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\cos\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}}\Big).\end{aligned} \tag{29}$$

If we take into consideration $\widehat{t} = \frac{d\widehat{\gamma}(s)}{d\widehat{s}}$ and integrate both sides of the equation (29) with respect to $\widehat{s}$, then we get (25).
The proofs of (b) and (c) are similar to the proof of the statement (a). □

Similarly, we have Theorem 13 and Theorem 14.

Theorem 13. *Let $\widehat{\gamma}$ be a dual unit speed spacelike slant helix having a timelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ in $\mathbb{D}^3_1$ and $\widehat{c} = c + \xi c^*$ be a dual slant helix constant.*
(a) *If $|c| > 1$ then the position vector $\widehat{\gamma}$ is given by*

$$\begin{aligned}\widehat{\gamma}(s) \;=\; & \int\Big(\tfrac{\sin\left[\widehat{c}\widehat{K}_3(s)\right]}{\sqrt{\widehat{c}^2-1}}, \\ & \tfrac{\widehat{c}\sin\left[\widehat{c}\widehat{K}_3(s)\right]\cos\left[\sqrt{\widehat{c}^2-1}\widehat{K}_3(s)\right]}{\sqrt{\widehat{c}^2-1}} - \cos\left[\widehat{c}\widehat{K}_3(s)\right]\sin\left[\sqrt{\widehat{c}^2-1}\widehat{K}_3(s)\right], \\ & \tfrac{\widehat{c}\sin\left[\widehat{c}\widehat{K}_3(s)\right]\sin\left[\sqrt{\widehat{c}^2-1}\widehat{K}_3(s)\right]}{\sqrt{\widehat{c}^2-1}} + \cos\left[\widehat{c}\widehat{K}_3(s)\right]\cos\left[\sqrt{\widehat{c}^2-1}\widehat{K}_3(s)\right]\Big)d\widehat{s},\end{aligned} \tag{30}$$

where $\widehat{K}_3(s) = \int \sqrt{\widehat{\kappa}^2(s) + \widehat{\tau}^2(s)}d\widehat{s}$.

(b) *If $|c| < 1$ then the position vector $\widehat{\gamma}$ is given by*

$$\begin{aligned}\widehat{\gamma}(s) \;=\; & \int\Big(\cos\left[\widehat{c}\widehat{K}_3(s)\right]\sinh\left[\sqrt{1-\widehat{c}^2}\widehat{K}_3(s)\right] \\ & +\tfrac{\widehat{c}\sin\left[\widehat{c}\widehat{K}_3(s)\right]\cosh\left[\sqrt{1-\widehat{c}^2}\widehat{K}_3(s)\right]}{\sqrt{1-\widehat{c}^2}}, \cos\left[\widehat{c}\widehat{K}_3(s)\right]\cosh\left[\sqrt{1-\widehat{c}^2}\widehat{K}_3(s)\right] \\ & +\tfrac{\widehat{c}\sin\left[\widehat{c}\widehat{K}_3(s)\right]\sinh\left[\sqrt{1-\widehat{c}^2}\widehat{K}_3(s)\right]}{\sqrt{1-\widehat{c}^2}}, \tfrac{\sin\left[\widehat{c}\widehat{K}_3(s)\right]}{\sqrt{1-\widehat{c}^2}}\Big)d\widehat{s}\end{aligned} \tag{31}$$

where $\widehat{K}_3(s) = \int \sqrt{\widehat{\kappa}^2(s) + \widehat{\tau}^2(s)}d\widehat{s}$.

Theorem 14. *Let $\widehat{\gamma}(s)$ be a dual unit speed timelike slant helix with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ in $\mathbb{D}_1^3$ and $\widehat{c} = c + \xi c^*$ be a dual slant helix constant.*
(a) If $\kappa < |\tau|$ and $|c| < 1$ then the position vector $\widehat{\gamma}$ is given by

$$\begin{aligned} \widehat{\gamma}(s) \quad = \quad & \int \Bigg(\tfrac{\sinh\left[\widehat{c}\widehat{K}_2(s)\right]}{\sqrt{1-\widehat{c}^2}}, \cos\left(\sqrt{1-\widehat{c}^2}\widehat{K}_2(s)\right)\cosh\left[\widehat{c}\widehat{K}_2(s)\right] \\ & -\tfrac{\widehat{c}}{\sqrt{1-\widehat{c}^2}}\sin\left(\sqrt{1-\widehat{c}^2}\widehat{K}_2(s)\right)\sinh\left[\widehat{c}\widehat{K}_2(s)\right], \\ & \sin\left[\sqrt{1-\widehat{c}^2}\widehat{K}_2(s)\right]\cosh\left[\widehat{c}\widehat{K}_2(s)\right] \\ & +\tfrac{\widehat{c}}{\sqrt{1-\widehat{c}^2}}\cos\left[\sqrt{1-\widehat{c}^2}\widehat{K}_2(s)\right]\sinh\left[\widehat{c}\widehat{K}_2(s)\right]\Bigg) d\widehat{s} \end{aligned} \tag{32}$$

where $\widehat{K}_2(s) = \int \sqrt{\widehat{\tau}^2(s) - \widehat{\kappa}^2(s)}d\widehat{s}$.

(b) If $\kappa < |\tau|$ and $|c| > 1$ then $\widehat{\gamma}$ can denoted by

$$\begin{aligned} \widehat{\gamma}(s) \quad = \quad & \int \Bigg(\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\sinh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_2(s)\right] - \tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\cosh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_2(s)\right]}{\sqrt{\widehat{c}^2-1}}, \\ & \sinh\left[\widehat{c}\widehat{K}_2(s)\right]\cosh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_2(s)\right] - \tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\sinh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_2(s)\right]}{\sqrt{\widehat{c}^2-1}}, \\ & -\tfrac{\cosh\left[\widehat{c}\widehat{K}_2(s)\right]}{\sqrt{\widehat{c}^2-1}})\Bigg) d\widehat{s}, \end{aligned} \tag{33}$$

where $\widehat{K}_2(s) = \int \sqrt{\widehat{\tau}^2(s) - \widehat{\kappa}^2(s)}d\widehat{s}$.

(c) If $\kappa > |\tau|$ then the position vector $\widehat{\gamma}$ is given by

$$\begin{aligned} \widehat{\gamma}(s) \quad = \quad & \int (\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\cosh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_1(s)\right] \\ & -\tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\sinh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1+m^2}}, \ \cosh\left[\widehat{c}\widehat{K}_1(s)\right]\sinh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_1(s)\right] \\ & -\tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\cosh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1+\widehat{c}^2}}, \ \tfrac{\sinh\left[\widehat{c}\widehat{K}_1(s)\right]}{\sqrt{1+\widehat{c}^2}})d\widehat{s}, \end{aligned} \tag{34}$$

where $\widehat{K}_1(s) = \int \sqrt{\widehat{\kappa}^2(s) - \widehat{\tau}^2(s)}d\widehat{s}$.

In Theorem 11, general helices in $\mathbb{D}_1^3$ were characterized according to the associated curve. Similarly, the characterization of slant helices in $\mathbb{D}_1^3$ is given as follows:

Theorem 15. *A slant helix in $\mathbb{D}_1^3$ is the second principal donor curve of some plane curves.*

A Frenet curve $\widehat{\gamma}$ in $\mathbb{D}_1^3$ is called a circular slant helix or hyperbolic slant helix if the second principal directional curve of $\widehat{\gamma}$ a circle in $\mathbb{D}^2$ or a hyperbola in $\mathbb{D}_1^2$, respectively. These curves are called simple dual curves.

Now we will deal with simple closed slant helices in $\mathbb{D}_1^3$. Taking into consideration the equations (25)-(27) and (30)-(34), we can state that there are no closed simple dual slant helices given by (25)-(27) and (31)$-$(34). Therefore we only interest a closed simple dual slant helix given by (30).

Remark 1. *Let $\widehat{\gamma}$ be a spacelike circular slant helix providing the equation (30) and its first principal directional curve of $\widehat{\gamma}_0$ and its second principal directional curve of $\widehat{\gamma}_1$ be a helix with $\frac{|\tau_0|}{\kappa_0}=|c|>1$ and a circle with radius $\widehat{r}$ in $\mathbb{D}_1^3$, respectively. Since the dual curvature of $\widehat{\gamma}_1$ is $\widehat{\kappa}_1=\frac{1}{\widehat{r}}$, the dual curvature $\widehat{\kappa}_0$ is expressed by $\widehat{\kappa}_0=\frac{1}{\widehat{r}\sqrt{\widehat{c}^2-1}}$. Thus the dual function $\widehat{K}_3$ in (30) is given by*

$$\widehat{K}_3(s)=\int\sqrt{\widehat{\kappa}^2+\widehat{\tau}^2}d\widehat{s}=\int\widehat{\kappa}_0 d\widehat{s}=\frac{\widehat{s}}{\widehat{r}\sqrt{\widehat{c}^2-1}}.$$

Therefore, by the a simple integration we can give that $\widehat{\gamma}$ is closed iff $\frac{c}{\sqrt{c^2-1}}$ is rational. Similarly, it appears that other simple dual slant helices are not closed.

Example 1. *A spacelike circular dual slant helix*

$$\widehat{\gamma}(s)=\gamma(s)+\xi\gamma^*(s) \tag{35}$$

of (30) can be denoted by

$$\begin{aligned}\gamma(s) \;=\;& -r\left(\frac{1}{c}\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right],\right.\\ &\left(2c^2-1\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]+2c\sqrt{c^2-1}\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right],\\ &\left.\left(2c^2-1\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right]-2c\sqrt{c^2-1}\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\right)\end{aligned} \tag{36}$$

and

$$\begin{aligned}\gamma^*(s) \;=\;& \left(\frac{rc^*-r^*c}{c^2}\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]+\frac{cr^*-c^*r-c^3r^*}{rc(c^2-1)^{\frac{3}{2}}}s\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right],\right.\\ &\left(csr^*\left(1-c^2\right)+c^*sr\left(1-2c^2\right)\right)\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\\ &+\left(\frac{r^*s}{r}+\frac{2cc^*s}{c^2-1}\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right]\\ &+\left(r^*-2r^*c^2-4cc^*r\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\\ &-2c\left(\frac{cc^*r+c^2r^*-r^*}{\sqrt{c^2-1}}\right)\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right],\\ &\left(csr^*\left(1-c^2\right)+c^*sr\left(1-2c^2\right)\right)\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right]\\ &-\left(\frac{r^*s}{r}+\frac{2cc^*s}{c^2-1}\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\\ &+\left(r^*-2r^*c^2-4cc^*r\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right]\\ &\left.+2c\left(\frac{cc^*r+c^2r^*-r^*}{\sqrt{c^2-1}}\right)\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\right).\end{aligned} \tag{37}$$

If we put $c = \frac{3}{2\sqrt{2}}$ for dual constant $\widehat{c} = c + \xi c^*$ and $\widehat{r} = (1, 0)$, then the closed condition $\frac{c}{\sqrt{c^2-1}} = 3$ is provided and an example of a spacelike closed circular dual slant helix with timelike principal normal is given by

$$\begin{aligned}\widehat{\gamma}_1(s) &= -\left(\tfrac{2\sqrt{2}\cos[3s]}{3}, \tfrac{5\cos[3s]\cos[s]+3\sin[3s]\sin[s]}{4}, \tfrac{5\cos[3s]\sin[s]-3\sin[3s]\cos[s]}{4}\right)\\ &+\xi c^*\left(\tfrac{8\cos[3s]}{9} - \tfrac{64s\sin[3s]}{3}, \tfrac{-5s\sin[3s]\cos[s]}{4} + 12\sqrt{2}s\cos[3s]\sin[s]\right.\\ &-3\sqrt{2}\cos[3s]\cos[s] - \tfrac{9\sqrt{2}\sin[3s]\sin[s]}{2}, \tfrac{-5s\sin[3s]\sin[s]}{4} - 12\sqrt{2}s\cos[3s]\cos[s]\\ &\left.-3\sqrt{2}\cos[3s]\sin[s] + \tfrac{9\sqrt{2}\sin[3s]\cos[s]}{2}\right).\end{aligned}$$

If we put $c = 2$ for dual constant $\widehat{c} = c + \xi c^*$ and $\widehat{r} = (1, 0)$, then the closed condition $\frac{c}{\sqrt{c^2-1}} = \frac{2}{\sqrt{3}}$ is not provided and the an example of a spacelike non-closed circular dual slant helix with timelike principal normal is given by

$$\begin{aligned}\widehat{\gamma}_2(s) &= -\left(\frac{\cos\left[\frac{2s}{\sqrt{3}}\right]}{2}, 7\cos[s]\cos\left[\frac{2s}{\sqrt{3}}\right] + 4\sqrt{3}\sin\left[\frac{2s}{\sqrt{3}}\right]\sin[s],\right.\\ &\left.7\cos\left[\frac{2s}{\sqrt{3}}\right]\sin[s] - 4\sqrt{3}\sin\left[\frac{2s}{\sqrt{3}}\right]\cos[s]\right) + \xi c^*\left(\frac{\cos\left[\frac{2s}{\sqrt{3}}\right]}{4} - s\frac{\sin\left[\frac{2s}{\sqrt{3}}\right]}{6\sqrt{3}},\right.\\ &-7s\sin\left[\frac{2s}{\sqrt{3}}\right]\cos[s] + \frac{4s}{3}\cos\left[\frac{2s}{\sqrt{3}}\right]\sin[s] - 8\cos\left[\frac{2s}{\sqrt{3}}\right]\cos[s]\\ &-\frac{8}{\sqrt{3}}\sin\left[\frac{2s}{\sqrt{3}}\right]\sin[s], -7s\sin\left[\frac{2s}{\sqrt{3}}\right]\sin[s] - \frac{4s}{3}\cos\left[\frac{2s}{\sqrt{3}}\right]\cos[s]\\ &\left.-8\cos\left[\frac{2s}{\sqrt{3}}\right]\sin[s] + \frac{8}{\sqrt{3}}\sin\left[\frac{2s}{\sqrt{3}}\right]\cos[s]\right).\end{aligned}$$

Corollary 4. *The closed simple slant helix $\widehat{\gamma}$ given by (35) whose real part (36) and dual part (37) in $\mathbb{D}_1^3$ is a spacelike circular slant helix with timelike principal normal having slant helix constant $\widehat{c} = c + \xi c^*$ providing the condition $\frac{c}{\sqrt{c^2-1}}$ is rational.*

5. Principal Directed Rectifying Curve in $\mathbb{D}_1^3$

In this section, we examine the principal directed rectifying curve whose the position vector always lie in rectifying plane of its principal donor curve in $\mathbb{D}_1^3$ (see [6, 10, 14, 19] for rectifying curve). We show that a principal directional rectifying curve in $\mathbb{D}_1^3$ corresponds to a spacelike or a timelike ruled surface in $\mathbb{R}_1^3$ depending on causal characters of its principal donor curves.

Theorem 16. *Let $\widehat{\gamma}_0$ be a pseudo spherical Frenet curve (a Frenet curve lies on $\mathbb{S}_1^2$ or $\mathbb{H}_0^2$) and $\widehat{\gamma}$ be a principal donor curve of $\widehat{\gamma}_0$ in $\mathbb{D}_1^3$. Then $\widehat{\gamma}_0$ is a principal directed rectifying curve.*

Proof. Let $\widehat{\gamma}_0$ be a pseudo spherical Frenet curve and $\widehat{\gamma}$ be a principal donor curve of $\widehat{\gamma}_0$ in $\mathbb{D}_1^3$. According to the dual Frenet frame of $\widehat{\gamma}$, the position vector of $\widehat{\gamma}_0$ is written as

$$\widehat{\gamma}_0(s) = \widehat{\lambda}(s)\widehat{t}(s) + \widehat{\mu}(s)\widehat{n}(s) + \widehat{\beta}(s)\widehat{b}(s), \tag{38}$$

for some dual functions $\widehat{\lambda}$, $\widehat{\mu}$ and $\widehat{\beta}$. Since $\frac{d\widehat{\gamma}_0}{d\widehat{s}} = \widehat{t}_0$, we have

$$\widehat{t}_0 = \widehat{n} = \left(\frac{d\widehat{\lambda}}{d\widehat{s}} - \varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{\mu}\right)\widehat{t} + \left(\widehat{\kappa}\widehat{\lambda} + \frac{d\widehat{\mu}}{d\widehat{s}} - \varepsilon_1\varepsilon_2\widehat{\tau}\widehat{\beta}\right)\widehat{n} + \left(\widehat{\mu}\widehat{\tau} + \frac{d\widehat{\beta}}{d\widehat{s}}\right)\widehat{b}.$$

Thus the system of equations

$$\begin{cases} \frac{d\widehat{\lambda}}{d\widehat{s}} - \varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{\mu} = 0 \\ \widehat{\kappa}\widehat{\lambda} + \frac{d\widehat{\mu}}{d\widehat{s}} - \varepsilon_1\varepsilon_2\widehat{\tau}\widehat{\beta} = 1 \\ \widehat{\mu}\widehat{\tau} + \frac{d\widehat{\beta}}{d\widehat{s}} = 0 \end{cases} \tag{39}$$

is formed. Since $\widehat{\gamma}_0$ is a pseudo spherical Frenet curve, taking into consideration the equation (38) we obtain

$$\varepsilon_0\widehat{\lambda}(s)^2 + \varepsilon_1\widehat{\mu}(s)^2 + \varepsilon_2\widehat{\beta}(s)^2 = \mp\widehat{r}^2.$$

If we take derivative of this last equation with respect to $\widehat{s}$, then we get

$$\varepsilon_0\widehat{\lambda}\frac{d\widehat{\lambda}}{d\widehat{s}} + \varepsilon_1\widehat{\mu}\frac{d\widehat{\mu}}{d\widehat{s}} + \varepsilon_2\widehat{\beta}\frac{d\widehat{\beta}}{d\widehat{s}} = 0 \tag{40}$$

is denoted. By using the equations (39) and (40) it is clear that $\widehat{\mu}(s) = 0$. Hence we can rewrite the equation (38) as

$$\widehat{\gamma}_0(s) = \widehat{\lambda}(s)\widehat{t}(s) + \widehat{\beta}(s)\widehat{b}(s).$$

So the position vector of $\widehat{\gamma}_0(s)$ lies in the rectifying plane of $\widehat{\gamma}$ which is the principal donor curve of $\widehat{\gamma}_0$. Therefore, $\widehat{\gamma}_0$ is principal directed rectifying curve. □

Theorem 17. *Let $\widehat{\gamma}$ be a Frenet curve with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and a pseudo spherical Frenet curve $\widehat{\gamma}_0$ be principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}^3_1$. Then the position vector of $\widehat{\gamma}_0$ lies in the normal plane $S_p\left\{\widehat{n}_0, \widehat{b}_0\right\}$ and the position vector of $\widehat{\gamma}_0$ is given by*

$$\widehat{\gamma}_0(s) = -\frac{\widetilde{\varepsilon}_1(\varepsilon_1\widehat{c}_1\widehat{\kappa} + \varepsilon_2\widehat{c}_2\widehat{\tau})}{(\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right))^{3/2}}\widehat{n}_0(s) + \frac{\widehat{c}_1\widehat{\tau} + \varepsilon_0\varepsilon_1\widehat{c}_2\widehat{\kappa}}{(\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right))^{3/2}}\widehat{b}_0(s) \tag{41}$$

for dual constants $\widehat{c}_1$ and $\widehat{c}_2$.

Proof. Let $\widehat{\gamma}$ is a Frenet curve with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and a pseudo spherical Frenet curve $\widehat{\gamma}_0$ be principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}^3_1$. We know that the dual curve $\widehat{\gamma}_0$ lies on the rectifying plane of $\widehat{\gamma}$. Then the position vector of $\widehat{\gamma}_0$ can be written by

$$\widehat{\gamma}_0(s) = \widehat{\lambda}\widehat{t}(s) + \widehat{\beta}\widehat{b}(s) \tag{42}$$

for dual functions $\widehat{\lambda}$ and $\widehat{\beta}$. If we take derivative of the equation (42) with respect to $\widehat{s}$, then we have

$$\widehat{n} = \frac{d\widehat{\lambda}}{d\widehat{s}}\widehat{t}(s) + (\widehat{\lambda}\widehat{\kappa} - \varepsilon_1\varepsilon_2\widehat{\beta}\widehat{\tau})\widehat{n}(s) + \frac{d\widehat{\beta}}{d\widehat{s}}\widehat{b}(s).$$

From the last equation it is clear that $\widehat{\lambda} = \widehat{c}_1$ and $\widehat{\beta} = \widehat{c}_2$ are dual constants. Therefore, we obtain (41) by using (4). $\square$

Corollary 5. *Let $\widehat{\gamma}$ be a spacelike Frenet curve with a spacelike principal normal in $\mathbb{D}^3_1$. Then the principal directed rectifying curve of $\widehat{\gamma}$ corresponds to a timelike ruled surface in $\mathbb{R}^3_1$.*

Corollary 6. *Let $\widehat{\gamma}$ be a spacelike Frenet curve with a timelike principal normal in $\mathbb{D}^3_1$. Then the principal directed rectifying curve of $\widehat{\gamma}$ corresponds to a spacelike ruled surface in $\mathbb{R}^3_1$.*

Corollary 7. *Let $\widehat{\gamma}$ be a timelike Frenet curve in $\mathbb{D}^3_1$. Then the principal directed rectifying curve of $\widehat{\gamma}$ corresponds to a timelike ruled surface in $\mathbb{R}^3_1$.*

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