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Research Article

Introduction to Soft Metric Preserving Functions

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Abstract: In this study, we aim to present the notion of soft metric preserving functions (SMPFs) which allows us to transform a soft metric into another one. We study some properties of SMPFs and investigate some characterizations to decide whether a soft function is soft metric preserving or not. Then, we show that the soft topology induced by soft metric was not preserved under SMPFs, present the stronger concept for these functions and also research the relationships of this concept with continuity.

Keywords: Soft function, soft metric, completeness, metric preserving function.

1. Introduction

In mathematical analysis and topology, metric spaces are fundamental structures that provide a rigorous way to measure distances between elements. A metric space is defined by a set paired with a distance function (metric) that satisfies the following conditions: non-negativity, identity of indiscernibles, symmetry, and triangle inequality. In the study of metric spaces, understanding how functions interact with the underlying metric structure is essential for many areas of mathematics, including topology, analysis, and geometry. When we use of a metric space endowed with a given metric d, it is often useful to exchange d for a different metric which is more suitable for our purposes. This possibility is crucial for applications where the integrity of the distance relationships must be maintained, such as in data analysis, machine learning, and various forms of geometric transformations. The importance of this concept can be interpreted that a particular property is fulfilled or not by a subset or a mapping when verifying such a property is a difficult task in the metric space. Actually, the first studies of this type functions was shown by Wilson [34]. The notion of metric preserving function (MPF) was given in [7] as follows: $f:\mathbb{R}^+\to\mathbb{R}^+$ is called a MPF if $d_f:U\times U\to\mathbb{R}^+$ by $d_f(u_1,u_2)=f(d(u_1,u_2))$ for all $u_1,u_2\in U$ is a metric on U whenever (U,d) is a metric space. A function $f:\mathbb{R}^+\to\mathbb{R}^+$ is said to be amenable if $f(u)=0\Leftrightarrow u=0$. Also, f is called subadditive if $f(u_1)+f(u_2)\geq f(u_1+u_2)$ for all $u_1,u_2\in\mathbb{R}^+$. If the topology

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generated by transformed metric coincides with the topology generated by original metric to be transformed, then f is called a strong metric preserving function (S-MPF). The metric preserving functions have applications in various mathematical and practical contexts such as geometry and topology, computer graphics and image processing, data analysis and machine learning, physics and engineering. Some quality research related to this concept in the settings of different views can be found in [4–6, 12, 13, 15, 18, 21, 23–27, 30].

In the literature, different set theories have been presented after Zadeh [35] introduced the fuzzy set theory since there are situations that the traditional classical methods do not capable of solving complex problems, especially including uncertain data, in many fields such as engineering, economics, environmental sciences, computer sciences, medical sciences, and etc. One of these theories is the soft set (SS) theory given by Molodtsov [22] and defined as a parameterized family of sets where the parameter takes value over an arbitrary set. This theory has been applied in different areas successfully since this notion was initiated [3, 8, 9, 19, 28, 31–33]. Further, Majumdar and Samanta [20] gave the the idea of soft mappings and studied images and inverse images of crisp sets and SSs under soft mappings. In 2012, Das and Samanta [10] defined the notion of the soft element (SE), and in particular, the soft real number which is interpreted as the extension of fuzzy number in the sense of Dubois and Prade [14].

Classical metric spaces may not be suitable for dealing with problems involving uncertainties, vagueness, or imprecision, which are common in real-world applications. To overcome these limitations, the concept of a soft metric space (SMS) has been introduced by the authors of [11] as an extension of classical metric spaces. SMSs integrate the ideas of SSs and fuzzy sets to handle uncertainty and imprecision more effectively. SSs provide a flexible mathematical framework for modeling situations where traditional methods struggle, especially in cases involving incomplete or partially known data. The primary distinction between classical metric spaces and SMSs lies in the nature of the metric itself. In classical metric spaces, the distance between any two points is a single, exact number. In contrast, in SMS we represent this distance as a ”soft value,” which is effectively a set of possible values or a range that encapsulates the uncertainty or fuzziness inherent in the measurement. The authors of [11] also study the topological properties of these spaces and gave Banach's fixed point theorem and Cantor's intersection theorem. Some different types of fixed point theorems in the soft setting for SMSs can be found in [1, 2, 16, 17]. Recently, Taşköprü and Altıntaş [29] described the soft functions by means of SEs as particular soft mappings in the sense of [20].

In this paper, we study on the soft mappings given by Taşköprü and Altıntaş [29]. We introduce the notion of SMPFs that can be taken as soft generalization of the metric preserving functions. We study some features of SMPFs and investigate some characterizations that allow us practical usefulness in the applications. Then, we show that the soft topology generated by soft metric was not preserved under SMPFs, present the stronger concept for these functions and also investigate the relationships of this concept with continuity.

2. Preliminaries

In this part, we recall some necessary notions such as SS, SE, soft function, soft metric and soft topology that will be used in the other sections. Suppose that U is an universal set, A is a non-empty set of parameters and $\mathfrak{B}(\mathbb{R})$ is the collection of all non-empty bounded subsets of the set $\mathbb{R}$.

Definition 2.1 *[10, 19, 22] A pair (F,A) is said to be SS over U if F is a mapping of A into the set of all subsets of U (i.e., $F:A\to P(U)$). We denote SS (F,A) by F shortly.*

The complement of SS F is denoted by F^c where $F^c:A\to P(U)$ is a mapping given by $F^c(\gamma)=U\setminus F(\gamma)$ for all $\gamma\in A$. SS F over U is said to be

(1) A null SS and denoted by Φ if $F(\gamma)=\varnothing$ for all $\gamma\in A$,

(2) An absolute SS and denoted by $\widetilde{U}$ if $F(\gamma)=U$ for all $\gamma\in A$.

We will denote by $S(\widetilde{U})$ the collection of all SSs F for which $F(\gamma)\neq\varnothing$ for any $\gamma\in A$.

Also, all mappings $\epsilon:A\to U$ are said to be SEs of U. SE ϵ is said to belong to SS F and denoted by $\epsilon\tilde{\in}F$ if $\epsilon(\gamma)\in F(\gamma)$ for all $\gamma\in A$. Here, we note that any family of SEs of SS can generate a soft subset of this SS. We will denote SS constructed from a collection $\mathscr{B}$ of SEs by $SS(\mathscr{B})$. Also, we will denote the collection of SEs of SS F by $SE(F)$.

Specially, a soft real set is a mapping $F:A\to\mathfrak{B}(\mathbb{R})$. If F is a single-valued function on A taking values in $\mathbb{R}$, then F is called SE of $\mathbb{R}$ or a soft real number. If F is a single-valued function on A taking values in $\mathbb{R}^+$, then F is said to be a non-negative soft real number. The set of all non-negative soft real numbers is denoted by $\mathbb{R}(A)^*$. Also, the notations $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are used to denote soft real numbers whereas $\bar{u}$, $\bar{v}$, $\bar{w}$ are used to denote a special type of soft real numbers such as $\bar{u}(\gamma)=u$ for all $\gamma\in A$ which is called a constant soft real number. For instance, $\bar{0}$ is the soft real number where $\bar{0}(\gamma)=0$ for all $\gamma\in A$. The collection of all non-negative constant soft real numbers is denoted by $\overline{\mathbb{R}(A)^*}$.

Definition 2.2 *[10] The soft orderings are defined for soft real numbers $\widetilde{u_1}$, $\widetilde{u_2}$ as follows:*

(1) $\widetilde{u_1}\tilde{\leq}\widetilde{u_2}$ if $\widetilde{u_1}(\gamma)\leq\widetilde{u_2}(\gamma)$, for all $\gamma\in A$,

(2) $\widetilde{u_1}\tilde{\geq}\widetilde{u_2}$ if $\widetilde{u_1}(\gamma)\geq\widetilde{u_2}(\gamma)$, for all $\gamma\in A$,

(3) $\widetilde{u_1}\tilde{<}\widetilde{u_2}$ if $\widetilde{u_1}(\gamma)<\widetilde{u_2}(\gamma)$, for all $\gamma\in A$,

(4) $\widetilde{u_1}\widetilde{>}\widetilde{u_2}$ *if* $\widetilde{u_1}(\gamma) > \widetilde{u_2}(\gamma)$, *for all* $\gamma \in A$.

Definition 2.3 *[11] Let* $F, G \in S(\widetilde{U})$.

(1) The complement of F *is denoted by* $F^{\complement}$ *and defined by* $F^{\complement} = SS(\mathscr{B})$ *where* $\mathscr{B} = \{\widetilde{u_1}\widetilde{\in}\widetilde{U} : \widetilde{u_1}\widetilde{\notin}F^c\}$.

(2) F *is said to be a soft subset of* G *and denoted by* $F \Subset G$ *if every SE of* F *is also SE of* G.

(3) The union of F *and* G *is denoted by* $F \Cup G$ *and defined by* $F \Cup G = SS(\mathscr{B})$ *where* $\mathscr{B} = \{\widetilde{u}\widetilde{\in}\widetilde{U} : \widetilde{u}\widetilde{\in}F$ *or* $\widetilde{u}\widetilde{\in}G\}$, *i.e.,* $F \Cup G = SS(SE(F) \cup SE(G))$.

(4) The intersection of F *and* G *is denoted by* $F \Cap G$ *and defined by* $F \Cap G = SS(\mathscr{B})$ *where* $\mathscr{B} = \{\widetilde{u}\widetilde{\in}\widetilde{U} : \widetilde{u}\widetilde{\in}F$ *and* $\widetilde{u}\widetilde{\in}G\}$, *i.e.,* $F \Cap G = SS(SE(F) \cap SE(G))$.

Definition 2.4 *[29] A soft mapping from* U *to* V *with parameter set* A *is denoted by the mapping* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$.

If $\{f_\gamma : \gamma \in A\}$ *is a collection of crisp mapping from* U *to* V, *then* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$ *is a soft mapping such that* $f(\widetilde{u})(\gamma) = f_\gamma(\widetilde{u}(\gamma))$ *for all* $\gamma \in A$. *Hence, every parameterized family of crisp mappings can be taken as a soft mapping. However, the converse of this statement is not satisfied in general as seen in [29].*

Theorem 2.5 *[29] If the soft mapping* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$ *satisfies the following condition (F), then* $f_\gamma : U \to V$ *defined by* $f_\gamma(\widetilde{u}(\gamma)) = f(\widetilde{u})(\gamma)$ *is a function.*

(F) $f(\widetilde{u})(\gamma) : \widetilde{u}(\gamma) = u$ *is a single-point set for all* $u \in U$ *and* $\gamma \in A$.

Definition 2.6 *[29] A soft mapping* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$ *is called a soft function if* f *satisfies the condition (F).*

A soft function f *is called injective if* $\widetilde{u_1} = \widetilde{u_2}$ *whenever* $f(\widetilde{u_1}) = f(\widetilde{u_2})$ *and surjective if* $f(SE(\widetilde{U})) = SE(\widetilde{V})$. *It is obvious that a soft function* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$ *is injective (surjective) if and only if* $f_\gamma : U \to V$ *is injective (surjective) for all* $\gamma \in A$.

We also note that a soft function is a special soft mapping in the sense of [20].

Definition 2.7 *[11] A soft metric on SS* $\widetilde{U}$ *is a mapping* $d : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ *fulfilling the following axioms:*

(SM1) $\widetilde{u_1} = \widetilde{u_2}$ *iff* $d(\widetilde{u_1}, \widetilde{u_2}) = \bar{0}$,

(SM2) $d(\widetilde{u_1}, \widetilde{u_2}) = d(\widetilde{u_2}, \widetilde{u_1})$,

(SM3) $d(\widetilde{u_1}, \widetilde{u_2})\widetilde{\leq}d(\widetilde{u_1}, \widetilde{u_3}) + d(\widetilde{u_3}, \widetilde{u_2})$ *for all* $\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3} \in SE(\widetilde{U})$.

SS $\widetilde{U}$ *with a soft metric* d *on* $\widetilde{U}$ *is called SMS and denoted by the triplet* $(\widetilde{U},d,A)$ *or* $(\widetilde{U},d)$ *for short. If there exists a* $\bar{k}\in\overline{\mathbb{R}(A)^*}$ *such that* $d(\widetilde{u_1},\widetilde{u_2})\widetilde{\leq}\bar{k}$ *for all* $\widetilde{u_1},\widetilde{u_2}\in\mathbb{R}(A)^*$, *then* $(\widetilde{U},d)$ *is called a bounded SMS.*

Example 2.8 *[11] If* $\{d_\gamma,\gamma\in A\}$ *is a parameterized family of crisp metrics on a set* U, *then the mapping* $d:SE(\widetilde{U})\times SE(\widetilde{U})\to\mathbb{R}(A)^*$ *defined by* $d(\widetilde{u_1},\widetilde{u_2})(\gamma)=d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))$, *for all* $\gamma\in A$ *and* $\widetilde{u_1},\widetilde{u_2}\in SE(\widetilde{U})$, *is a soft metric on* $\widetilde{U}$.

Result 2.9 *[11] Let* $\widetilde{U}=\overline{\mathbb{R}(A)^*}$ *and define the mapping* $d:SE(\widetilde{U})\times SE(\widetilde{U})\to\mathbb{R}(A)^*$ *by* $d(\bar{u_1},\bar{u_2})=|\bar{u_1}-\bar{u_2}|$ *for all* $\bar{u_1},\bar{u_2}\in\overline{\mathbb{R}(A)^*}$. *Then,* $(\widetilde{U},d)$ *is SMS.*

Proposition 2.10 *[11] If* $(\widetilde{U},d)$ *is SMS, then the mapping* $d_\gamma:U\times U\to\mathbb{R}^+$ *defined by* $d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))=d(\widetilde{u_1},\widetilde{u_2})(\gamma)$, *for all* $\gamma\in A$, *is a metric on* U *provided that* d *satisfies the following condition:*

(SM4) $\{d(\widetilde{u_1},\widetilde{u_2})(\gamma):\widetilde{u_1}(\gamma)=a,\widetilde{u_2}(\gamma)=b\}$ *is a singelton set for all* $(a,b)\in U\times U$ *and* $\gamma\in A$.

Definition 2.11 *[11] Let* $(\widetilde{U},d)$ *be SMS,* $\widetilde{u}\in SE(\widetilde{U})$ *and* $\widetilde{r}\in\mathbb{R}(A)^*$. *Then, the subset* $B(\widetilde{u},\widetilde{r})=\{\widetilde{v}\in SE(\widetilde{U}):d(\widetilde{u},\widetilde{v})\widetilde{<}\widetilde{r}\}$ *of* $SE(\widetilde{U})$ *is called an open disc centered at* $\widetilde{u}$ *with radius* $\widetilde{r}$ *and SS* $B(\widetilde{u},\widetilde{r})$ *is called a soft open disc with the center* $\widetilde{u}$ *and radius* $\widetilde{r}$.

Let $\mathscr{B}\subset SE(\widetilde{U})$. *Then,* $\mathscr{B}$ *is called open with respect to* d *if for all* $\widetilde{u}\in\mathscr{B}$ *there exists* $\widetilde{r}\in SE(\widetilde{U})$ *such that* $B(\widetilde{u},\widetilde{r})\subset\mathscr{B}$. *SS* $F\in S(\widetilde{U})$ *is said to be soft open with respect to* d *if there exist a collection* $\mathscr{B}$ *of SEs of* F *such that* $\mathscr{B}$ *is open with respect to* d *and* $F=SS(\mathscr{B})$.

Proposition 2.12 *[11] If* $(\widetilde{U},d)$ *is SMS satisfying the condition (SM4), then for every open disc* $B(\widetilde{u},\widetilde{r})$ *in SMS* $(\widetilde{U},d)$, $SS(B(\widetilde{u},\widetilde{r}))(\gamma)=B(\widetilde{u}(\gamma),\widetilde{r}(\gamma))$ *is an open disc in* (U,d_γ) *for all* $\gamma\in A$.

Proposition 2.13 *[11] If* $(\widetilde{U},d)$ *is SMS satisfying the condition (SM4), then* $F\in S(\widetilde{U})$ *is a soft open set with respect to* d *if and only if* $F(\gamma)$ *is open in* (U,d_γ) *for all* $\gamma\in A$.

Definition 2.14 *[29] Let* $\tau\subset S(\widetilde{U})$ *be a family of SSs over* U. *Then,* τ *is said to be a soft topology on* $\widetilde{U}$ *if the following axioms are satisfied:*

(ST1) $\Phi,\ \widetilde{U}\in\tau$.

(ST2) If $F,G\in\tau$, *then* $F\Cap G\in\tau$.

(ST3) If $F_i\in\tau$ *for all* $i\in I$, *then* $\Cup_{i\in I}F_i\in\tau$.

The triplet $(\widetilde{U},\tau,A)$ *is called a soft topological space.*

Remark 2.15 *[11] Let $(\widetilde{U},d)$ be SMS satisfying the condition (SM4). Then, the collection τ of all soft open sets with respect to d form a soft topology on $\widetilde{U}$, this topology is called soft metric topology and denoted by τ_d.*

In the following proposition, we note that the condition "$F(\gamma)\cap G(\gamma)\neq\varnothing$" is not necessary unlike [29] when we demand a crisp topology τ_γ on U whenever $(\widetilde{U},\tau,A)$ is a soft topology.

Proposition 2.16 *If $(\widetilde{U},\tau,A)$ is a soft topology, then $\tau_\gamma=\{F(\gamma):F\in\tau\}$, for all $\gamma\in A$, is a crisp topology on U.*

Proof We have $\varnothing,U\in\tau_\gamma$ for all $\gamma\in A$ since Φ, $\widetilde{U}\in\tau$. Let $F_1(\gamma),F_2(\gamma)\in\tau_\gamma$. Then, we have $F_1,F_2\in\tau$. Also, if $F_1(\gamma)\cap F_2(\gamma)=\varnothing$, then it is clear that $F_1(\gamma)\cap F_2(\gamma)\in\tau_\gamma$. Otherwise, if $F_1(\gamma)\cap F_2(\gamma)\neq\varnothing$, then we have $F_1\Cap F_2\neq\Phi$ which follows that $(F_1\Cap F_2)(\gamma)=\{\widetilde{u_1}(\gamma):\widetilde{u_1}(\gamma)\in F_1(\gamma)\ \textit{and}\ \widetilde{u_1}(\gamma)\in F_2(\gamma)\}=F_1(\gamma)\cap F_2(\gamma)$. Since $F_1\Cap F_2$, then we obtain $F_1(\gamma)\cap F_2(\gamma)\in\tau_\gamma$. Finally, let $F_i(\gamma)\in\tau_\gamma$ for all $i\in I$ which means that $F_i\in\tau$ for all $i\in I$ and so, $\Cup_{i\in I}F_i\in\tau$. Also, we know that $(\Cup_{i\in I}F_i)(\gamma)=\bigcup_{i\in I}F_i(\gamma)$ which concludes the proof since this implies that $\bigcup_{i\in I}F_i(\gamma)\in\tau_\gamma$. □

3. Soft Metric Preserving Functions

In this part, we introduce the notion of SMPF which let us obtain a new SMS from the existing SMS. Then, we obtain some features of this type of soft function and so, we give some characterizations of these functions. Also, we present that the topology generated by soft metric was not preserved under SMPFs.

Definition 3.1 *Let $(\widetilde{U},d)$ be SMS and $f:\mathbb{R}(A)^*\to\mathbb{R}(A)^*$ be a soft function. Define a mapping $d_f:SE(\widetilde{U})\times SE(\widetilde{U})\to\mathbb{R}(A)^*$ by $d_f(\widetilde{u_1},\widetilde{u_2})=f(d(\widetilde{u_1},\widetilde{u_2}))$ for all $\widetilde{u_1},\widetilde{u_2}\in SE(\widetilde{U})$. Then, the function f is said to be SMPF if the mapping d_f is a soft metric on $SE(\widetilde{U})$.*
For example, we can obtain a bounded SMS from a given SMS $(\widetilde{U},d)$ such as $d(\widetilde{u_1},\widetilde{u_2})\to\frac{d(\widetilde{u_1},\widetilde{u_2})}{\overline{1}+d(\widetilde{u_1},\widetilde{u_2})}$ for all $\widetilde{u_1},\widetilde{u_2}\in SE(\widetilde{U})$. So, the function $f:\mathbb{R}(A)^*\to\mathbb{R}(A)^*$ defined by $f(\widetilde{u_1})=\frac{\widetilde{u_1}}{\overline{1}+\widetilde{u_1}}$ is SMPF.

Let us denote by $\widetilde{\mathscr{O}}$ the set of all soft functions $f:\mathbb{R}(A)^*\to\mathbb{R}(A)^*$ satisfy the condition

$$f(\widetilde{u_1})=\overline{0}\Leftrightarrow\widetilde{u_1}=\overline{0}$$

whenever $\widetilde{u_1}\in\mathbb{R}(A)^*$, i.e., $\widetilde{\mathscr{O}}=\{f:\mathbb{R}(A)^*\to\mathbb{R}(A)^*|f^{-1}(\overline{0})=\overline{0}\}$. The elements of $\widetilde{\mathscr{O}}$ are called soft amenable functions.

Proposition 3.2 *If* $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *is SMPF, then* $f \in \widetilde{\mathcal{O}}$*.*

Proof Let $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be SMPF and consider SMS $(\widetilde{U},d)$ where

$$d(\widetilde{u_1},\widetilde{u_2})(\gamma) = e(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))$$

for all $\widetilde{u_1},\widetilde{u_2} \in SE(\widetilde{U})$. To show that $f(\widetilde{u_1}) = \bar{0} \Leftrightarrow \widetilde{u_1} = \bar{0}$, first assume $f(\widetilde{u_1}) = \bar{0}$. Then, we can write

$$f(\widetilde{u_1}) = f(d(\widetilde{u_1},\bar{0})) = d_f(\widetilde{u_1},\bar{0}) = \bar{0} \Rightarrow \widetilde{u_1} = \bar{0}$$

since d_f is a soft metric on $\widetilde{U}$. The other side is obvious. □

Definition 3.3 *A function* $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *is called soft subadditive if* f *satisfies the following inequality:*

$$f(\widetilde{u_1}+\widetilde{u_2}) \leq f(\widetilde{u_1}) + f(\widetilde{u_2})$$

for all $\widetilde{u_1},\widetilde{u_2} \in SE(\widetilde{U})$*.*

Proposition 3.4 *If* $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *is SMPF, then* f *is subadditive.*

Proof Let $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be SMPF and consider SMS $(\widetilde{U},d)$ where

$$d(\widetilde{u_1},\widetilde{u_2})(\gamma) = e(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))$$

for all $\widetilde{u_1},\widetilde{u_2} \in SE(\widetilde{U})$. Then, we have

$$\begin{aligned} f(\widetilde{u_1}+\widetilde{u_2}) &= f(d(\widetilde{u_1}+\widetilde{u_2},\bar{0})) = d_f(\widetilde{u_1}+\widetilde{u_2},\bar{0}) \leq d_f(\widetilde{u_1}+\widetilde{u_2},\widetilde{u_2}) + d_f(\widetilde{u_2},\bar{0}) \\ &= f(d(\widetilde{u_1}+\widetilde{u_2},\widetilde{u_2})) + f(d(\widetilde{u_2},\bar{0})) = f(\widetilde{u_1}) + f(\widetilde{u_2}). \end{aligned}$$

□

Remark 3.5 *The converse implication of the above proposition may not be satisfied. Consider the soft mapping* $f(\widetilde{u_1}) = \bar{1}$ *for all* $\widetilde{u_1} \in \mathbb{R}(A)^*$*. Then, it is obvious that* f *is soft subadditive but* f *is not SMPF since* $f \notin \widetilde{\mathcal{O}}$*.*

Proposition 3.6 *If* $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *is soft subadditive, non-decreasing and* $f \in \widetilde{\mathcal{O}}$*, then* f *is SMPF.*

Proof Let $(\widetilde{U},d)$ be SMS. Now, we need to show that $d_f: SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ defined by $d_f(\widetilde{u_1},\widetilde{u_2}) = f(d(\widetilde{u_1},\widetilde{u_2}))$ is a soft metric on $\widetilde{U}$. Since f is non-decreasing and $d(\widetilde{u_1},\widetilde{u_2}) \widetilde{\geq} \bar{0}$ for all $\widetilde{u_1}, \widetilde{u_2} \in \mathbb{R}(A)^*$, we obtain $d_f(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\geq} f(\bar{0}) = \bar{0}$. Assume that $d_f(\widetilde{u_1}, \widetilde{u_2}) = \bar{0}$. Then, we have $d(\widetilde{u_1}, \widetilde{u_2}) = \bar{0}$ which means that $\widetilde{u_1} = \widetilde{u_2}$ since d is a soft metric on $\widetilde{U}$. Let $\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3} \in \mathbb{R}(A)^*$.

$$\begin{aligned} d_f(\widetilde{u_1}, \widetilde{u_2}) &= f(d(\widetilde{u_1}, \widetilde{u_2})) \widetilde{\leq} f(d(\widetilde{u_1}, \widetilde{u_3}) + d(\widetilde{u_3}, \widetilde{u_2})) \ (since\ f\ is\ non-decreasing) \\ &\widetilde{\leq} f(d(\widetilde{u_1}, \widetilde{u_3})) + f(d(\widetilde{u_3}, \widetilde{u_2})) = d_f(\widetilde{u_1}, \widetilde{u_3}) + d_f(\widetilde{u_3}, \widetilde{u_2}) \ (since\ f\ is\ subadditive). \end{aligned}$$

As a result, d_f is a soft metric on $\widetilde{U}$ and so, f is SMPF. □

Example 3.7 *Let $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ defined by*

$$f(\widetilde{u_1}) = \begin{cases} \bar{0}, & \widetilde{u_1} = \bar{0} \\ \bar{1}, & \widetilde{u_1} \neq \bar{0} \end{cases}.$$

Then, f is SMPF since f is soft subadditive, non-decreasing and $f \in \widetilde{\mathcal{O}}$.

Remark 3.8 *The transferred SMS $(\widetilde{U}, d_f)$ may not satisfy the condition (SM4) even if SMS $(\widetilde{U}, d)$ satisfies the condition (SM4) when f is SPMF.*

Example 3.9 *Let $U = \{a, b\}$, $A = \{\gamma, \mu\}$ and $SE(\widetilde{U}) = \{\widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3}, \widetilde{v_4}\}$ where $\widetilde{v_1}(\gamma) = a$, $\widetilde{v_1}(\mu) = a$, $\widetilde{v_2}(\gamma) = a$, $\widetilde{v_2}(\mu) = b$, $\widetilde{v_3}(\gamma) = b$, $\widetilde{v_3}(\mu) = a$, $\widetilde{v_4}(\gamma) = b$ and $\widetilde{v_4}(\mu) = b$. Consider the soft metric $d : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ given by $d(\widetilde{u_1}, \widetilde{u_2})(\gamma) = |\widetilde{u_1}(\gamma) - \widetilde{u_2}(\gamma)|$. Then, it is easily seen that $(\widetilde{U}, d)$ satisfies the condition (SM4). Take SMPF $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ defined by*

$$f(\widetilde{u_1}) = \begin{cases} \bar{0}, & \widetilde{u_1} = \bar{0} \\ \bar{1}, & \widetilde{u_1} \neq \bar{0} \end{cases}.$$

Now, we obtain the mapping $d_f : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ as

$$d_f(\widetilde{u_1}, \widetilde{u_2}) = f(d(\widetilde{u_1}, \widetilde{u_2})) = \begin{cases} \bar{0}, & \widetilde{u_1} = \widetilde{u_2} \\ \bar{1}, & \widetilde{u_1} \neq \widetilde{u_2} \end{cases}$$

which is a discrete soft metric on $\widetilde{U}$. However, for $(a, a) \in U \times U$ and $\gamma \in A$, we have $\{d(\widetilde{u_1}, \widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = a\} = \{0, 1\}$ which is not a singelton set. Hence, $(\widetilde{U}, d_f)$ does not satisfy the condition (SM4).

Proposition 3.10 *If SPMF $f : SE(\widetilde{U}) \to SE(\widetilde{U})$ is surjective, then the transferred SMS $(\widetilde{U}, d_f)$ satisfies the condition (SM4) when the soft metric $(\widetilde{U}, d)$ satisfies the condition (SM4).*

Proof Assume that there exists a point $(a,b) \in U \times U$ and a parameter $\gamma \in A$ such that $\{d_f(\widetilde{u_1},\widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = b\}$ is not a singelton set. Hence, there are k_1, k_2 $(k_1 \neq k_2)$ such that $k_1, k_2 \in \{d_f(\widetilde{u_1},\widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = b\}$. This means that $f_\gamma(d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))) = k_1$ and $f_\gamma(d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))) = k_2$ whenever $\widetilde{u_1}(\gamma) = a$ and $\widetilde{u_2}(\gamma) = b$. We know that f_γ is surjective for all $\gamma \in A$ since f is a soft surjective function. Hence, there exists a point m_1, m_2 such that $d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma)) = m_1$ and $d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma)) = m_2$. Therefore, we have that there exists SE $\widetilde{v_1} \neq \widetilde{u_1}$ such that $\widetilde{v_1}(\gamma) = \widetilde{u_1}(\gamma) = a$ or $\widetilde{y_1} \neq \widetilde{u_2}$ such that $\widetilde{y_1}(\gamma) = \widetilde{u_2}(\gamma) = b$ since d_γ is a crisp metric on U. This follows that $\{d(\widetilde{u_1},\widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = b\}$ is not a singelton set and so, we obtain a contradiction. As a result, $(\widetilde{U}, d_f)$ satisfies the condition (SM4). □

Definition 3.11 *Let $\widetilde{k},\widetilde{l},\widetilde{m} \in \mathbb{R}(A)^*$. A triplet $(\widetilde{k},\widetilde{l},\widetilde{m})$ is said to be soft triangular if*

$$\widetilde{k} \widetilde{\leq} \widetilde{l} + \widetilde{m}, \quad \widetilde{l} \widetilde{\leq} \widetilde{k} + \widetilde{m} \quad \text{and} \quad \widetilde{m} \widetilde{\leq} \widetilde{k} + \widetilde{l}.$$

Proposition 3.12 *If $(\widetilde{U},d)$ is SMS and $\widetilde{u_1},\widetilde{u_2},\widetilde{u_3} \in \widetilde{U}$, then $(d(\widetilde{u_1},\widetilde{u_2}), d(\widetilde{u_2},\widetilde{u_3}), d(\widetilde{u_1},\widetilde{u_3}))$ is a soft triangular triplet.*

Theorem 3.13 *Let $f \in \widetilde{\mathcal{O}}$ and $\widetilde{k},\widetilde{l},\widetilde{m} \in SE(\widetilde{U})$. Then, the followings are equivalent:*

(i) f is SMPF.

(ii) If $(\widetilde{k},\widetilde{l},\widetilde{m})$ is a soft triangular triplet, then $(f(\widetilde{k}), f(\widetilde{l}), f(\widetilde{m}))$ is soft triangular triplet.

Proof $(i) \Rightarrow (ii)$: Let f be SMPF and $(\widetilde{k},\widetilde{l},\widetilde{m})$ be a soft triangular triplet. Consider SMS $(\widetilde{U},d)$ where

$$d(\widetilde{u_1},\widetilde{u_2})(\gamma) = e(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))$$

for all $\widetilde{u_1},\widetilde{u_2} \in SE(\widetilde{U})$. Since $(\widetilde{k},\widetilde{l},\widetilde{m})$ is a soft triangular triplet, then we can find $\widetilde{u},\widetilde{v},\widetilde{w} \in \mathbb{R}(A)^*$ such that $d(\widetilde{u},\widetilde{v}) = \widetilde{k}$, $d(\widetilde{v},\widetilde{w}) = \widetilde{l}$ and $d(\widetilde{u},\widetilde{w}) = \widetilde{m}$. Then, we obtain

$$f(\widetilde{k}) = f(d(\widetilde{u},\widetilde{v})) = d_f(\widetilde{u},\widetilde{v}) \widetilde{\leq} d_f(\widetilde{u},\widetilde{w}) + d_f(\widetilde{w},\widetilde{v}) = f(\widetilde{l}) + f(\widetilde{m})$$

and, with similar way, $f(\widetilde{l}) \widetilde{\leq} f(\widetilde{k}) + f(\widetilde{m})$ and $f(\widetilde{m}) \widetilde{\leq} f(\widetilde{k}) + f(\widetilde{l})$. Hence, we conclude that $(f(\widetilde{k}), f(\widetilde{l}), f(\widetilde{m}))$ is soft triangular triplet.

$(ii) \Rightarrow (i)$: Let $f \in \widetilde{\mathcal{O}}$ and $(\widetilde{U},d)$ be SMS. Since $(d(\widetilde{u_1},\widetilde{u_2}), d(\widetilde{u_1},\widetilde{u_2}), \bar{0})$ is a soft triangular triplet, we obtain $(f(d(\widetilde{u_1},\widetilde{u_2})), f(d(\widetilde{u_1},\widetilde{u_2})), f(\bar{0}))$ is soft triangular triplet which means that $d_f(\widetilde{u_1},\widetilde{u_2}) \widetilde{\geq} \bar{0}$ for all $\widetilde{u_1},\widetilde{u_2} \in \mathbb{R}(A)^*$. Also, it is clear that $d_f(\widetilde{u_1},\widetilde{u_2}) = \bar{0} \Leftrightarrow \widetilde{u_1} = \widetilde{u_2}$ since $f \in \widetilde{\mathcal{O}}$. Finally, since $(d(\widetilde{u_1},\widetilde{u_2}), d(\widetilde{u_2},\widetilde{u_3}), d(\widetilde{u_1},\widetilde{u_3}))$ is a soft triangular triplet, from hypothesis, we also obtain that $d_f(\widetilde{u_1},\widetilde{u_2}) \widetilde{\leq} d_f(\widetilde{u_1},\widetilde{u_3}) + d_f(\widetilde{u_3},\widetilde{u_2})$ which concludes that f is SMPF. □

Proposition 3.14 *Let* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *be a soft function associated with the family of functions* $\{f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+, \gamma \in A\}$*. If* f *is SMPF, then* f_γ *is a MPF for all* $\gamma \in A$*.*

Proof Let us consider SMS $(\overline{\mathbb{R}(A)^*}, d)$ where $d(\bar{x},\bar{y}) = |\bar{x}-\bar{y}|$ for all $\bar{x},\bar{y} \in \overline{\mathbb{R}(A)^*}$. We will show that f_γ is amenable, non-decreasing and subadditive function for all $\gamma \in A$ (We refer to [7] for the notion of metric preserving function).

Let us take $\gamma \in A$.

(Amenable:) Let $x \in \mathbb{R}^+$. Then, we have the followings:

$$\begin{aligned} f_\gamma(x) = 0 \quad &\Leftrightarrow \quad 0 = f_\gamma(\bar{x}(\gamma)) = f_\gamma(d(\bar{x},\bar{0})(\gamma)) = f(d(\bar{x},\bar{0}))(\gamma) \\ &\Leftrightarrow \quad f(d(\bar{x},\bar{0})) = \bar{0} \Leftrightarrow d(\bar{x},\bar{0}) = \bar{0} \Leftrightarrow \bar{x} = \bar{0} \Leftrightarrow x = 0. \end{aligned}$$

(Non-decreasing:) Let $x \le y$ for any $x,y \in \mathbb{R}^+$.

$$\begin{aligned} f_\gamma(x) &= f_\gamma(\bar{x}(\gamma)) = f_\gamma(d(\bar{x},\bar{0})(\gamma)) = f(d(\bar{x},\bar{0}))(\gamma) \\ &\le f(d(\bar{y},\bar{0}))(\gamma) = f_\gamma(d(\bar{y},\bar{0})(\gamma)) = f_\gamma(\bar{y}(\gamma)) = f_\gamma(y). \end{aligned}$$

(Subadditive:) Let $x,y \in \mathbb{R}^+$.

$$\begin{aligned} f_\gamma(x+y) &= f_\gamma(\overline{x+y}(\gamma)) = f_\gamma(d(\overline{x+y},\bar{0})(\gamma)) = f(d(\overline{x+y},\bar{0}))(\gamma) \\ &\le f(d(\overline{x+y},\bar{y}))(\gamma) + f(d(\bar{y},\bar{0}))(\gamma) \\ &= f_\gamma(d(\overline{x+y},\bar{y})(\gamma)) + f_\gamma(d(\bar{y},\bar{0})(\gamma)) \\ &= f_\gamma(\bar{x}(\gamma)) + f_\gamma(\bar{y}(\gamma)) = f_\gamma(x) + f_\gamma(y). \end{aligned}$$

□

Remark 3.15 *The converse of the above proposition may not be true. Let us consider the example given in [11] as follows:*

Let $U = \{a,b\}$, $A = \{\gamma,\mu\}$ *and* $SE(\widetilde{U}) = \{\widetilde{v}_1,\widetilde{v}_2,\widetilde{v}_3,\widetilde{v}_4\}$ *where* $\widetilde{v}_1(\gamma) = a$, $\widetilde{v}_1(\mu) = a$, $\widetilde{v}_2(\gamma) = a$, $\widetilde{v}_2(\mu) = b$, $\widetilde{v}_3(\gamma) = b$, $\widetilde{v}_3(\mu) = a$, $\widetilde{v}_4(\gamma) = b$ *and* $\widetilde{v}_4(\mu) = b$*. Consider the discrete metric* $d : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ *given by*

$$d(\widetilde{u}_1,\widetilde{u}_2) = \begin{cases} \bar{0}, & \widetilde{u}_1 = \widetilde{u}_2 \\ \bar{1}, & \widetilde{u}_1 \neq \widetilde{u}_2 \end{cases}.$$

Take the function $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *defined by* $f_\gamma(x) = \frac{x}{1+x}$ *for all* $x \in \mathbb{R}^+$ *and* $\gamma \in A$ *and the soft function* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *given by* $f(\widetilde{u}_1) = \frac{\widetilde{u}_1}{\bar{1}+\widetilde{u}_1}$*. Here, it is obvious that* $f(\widetilde{u}_1)(\gamma) = f_\gamma(\widetilde{u}_1(\gamma))$ *for all* $\widetilde{u}_1 \in SE(\widetilde{U})$ *and* $\gamma \in A$*. Also,* f_γ *is a MPF and* f *is SMPF. However, the mapping*

$d_f : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ *defined by* $d_f(\widetilde{u_1}, \widetilde{u_2}) = f(d(\widetilde{u_1}, \widetilde{u_2}))$ *for all* $\widetilde{u_1}, \widetilde{u_2} \in SE(\widetilde{U})$, *is not a soft metric on* $\widetilde{U}$ *since* $d_f(\widetilde{v_1}, \widetilde{v_1})(\gamma) = f_\gamma(0)$, $d_f(\widetilde{v_1}, \widetilde{v_1})(\gamma) = f_\gamma(1)$ *and* $f_\gamma(0) \neq f_\gamma(1)$.

Proposition 3.16 *If* $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *is a MPF for all* $\gamma \in A$, *then the soft function* f *is SMPF when SMS* $(\widetilde{U}, d)$ *satisfies the condition (SM4).*

Proof Let $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be a MPF for all $\gamma \in A$. Suppose that $(\widetilde{U}, d)$ is SMS satisfying the condition (SM4). Then, we know that the mapping $d_\gamma : U \times U \to \mathbb{R}^+$ defined by $d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma)) = d(\widetilde{u_1}, \widetilde{u_2})(\gamma)$, for all $\gamma \in A$, is a metric on U. Now, we show that $d_f : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ defined by $d_f(\widetilde{u_1}, \widetilde{u_2}) = f(d(\widetilde{u_1}, \widetilde{u_2}))$ is a soft metric on $\widetilde{U}$.
Let $\gamma \in A$.

$$d_f(\widetilde{u_1}, \widetilde{u_2})(\gamma) = f(d(\widetilde{u_1}, \widetilde{u_2}))(\gamma) = f_\gamma(d(\widetilde{u_1}, \widetilde{u_2})(\gamma)) = f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))) \geq f_\gamma(0) = 0$$

. So, we have $d_f(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\geq} \overline{0}$ since γ is an arbitrary chosen parameter.
(SM1) Let $\gamma \in A$ and $d_f(\widetilde{u_1}, \widetilde{u_2}) = \overline{0}$. Then, we have

$$0 = f(d(\widetilde{u_1}, \widetilde{u_2}))(\gamma) = f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))) \Rightarrow d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma)) = 0 \Rightarrow \widetilde{u_1}(\gamma) = \widetilde{u_2}(\gamma)$$

which means that $\widetilde{u_1} = \widetilde{u_2}$ since γ is an arbitrary parameter. It is clear that $d_f(\widetilde{u_1}, \widetilde{u_2}) = \overline{0}$ when $\widetilde{u_1} = \widetilde{u_2}$.
(SM2) It is obvious from the definitions.
(SM3) Let $\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3} \in \mathbb{R}(A)^*$. Then, we obtain

$$\begin{aligned} d_f(\widetilde{u_1}, \widetilde{u_2})(\gamma) &= f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))) \leq f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_3}(\gamma)) + d_\gamma(\widetilde{u_3}(\gamma), \widetilde{u_2}(\gamma))) \\ &\leq f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_3}(\gamma))) + f_\gamma(d_\gamma(\widetilde{u_3}(\gamma), \widetilde{u_2}(\gamma))) \\ &= f(d(\widetilde{u_1}, \widetilde{u_3}))(\gamma) + f(d(\widetilde{u_3}, \widetilde{u_2}))(\gamma) = d_f(\widetilde{u_1}, \widetilde{u_3})(\gamma) + d_f(\widetilde{u_3}, \widetilde{u_2})(\gamma) \end{aligned}$$

which follows that $d_f(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\leq} d_f(\widetilde{u_1}, \widetilde{u_3}) + d_f(\widetilde{u_3}, \widetilde{u_2})$ as required. □

In the following example, we notice that the topology generated by the transformed SMS may not be equivalent to the topology generated by SMS to be transformed.

Example 3.17 *Consider the soft metric* $d : \overline{\mathbb{R}(A)^*} \times \overline{\mathbb{R}(A)^*} \to \mathbb{R}(A)^*$ *given by* $d(\overline{x}, \overline{y})(\gamma) = |\overline{x}(\gamma) - \overline{y}(\gamma)|$ *where* A *is a non-empty parameter set. Then, it is easily seen that* $(\widetilde{U}, d)$ *satisfies the condition (SM4). Take SMPF* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *defined by*

$$f(\widetilde{u_1}) = \begin{cases} \overline{0}, & \widetilde{u_1} = \overline{0} \\ \overline{1}, & \widetilde{u_1} \neq \overline{0} \end{cases}.$$

Now, we obtain the mapping $d_f : \overline{\mathbb{R}(A)^*} \times \overline{\mathbb{R}(A)^*} \to \mathbb{R}(A)^*$ *as*

$$d_f(\overline{x},\overline{y}) = f(d(\overline{x},\overline{y})) = \begin{cases} \overline{0}, & \overline{x} = \overline{y} \\ \overline{1}, & \overline{x} \neq \overline{y} \end{cases}$$

which is a discrete soft metric on $\overline{\mathbb{R}(A)^*}$ *satisfying the condition (SM4). Then, we have*

$$B_d(\widetilde{k},\widetilde{r}) = \{\overline{x} : d(\overline{x},\widetilde{k}) \widetilde{<} \widetilde{r}\} \Rightarrow SS(B_d(\widetilde{k},\widetilde{r}))(\gamma) = B_{d_\gamma}(a,r) = (a-r,a+r)$$

and

$$B_{d_f}(\widetilde{k},\widetilde{r}) = \{\overline{x} : d_f(\overline{x},\widetilde{k}) \widetilde{<} \widetilde{r}\} \Rightarrow SS(B_{d_f}(\widetilde{k},\widetilde{r}))(\gamma) = B_{d_{f_\gamma}}(\overline{x}(\gamma),\overline{y}(\gamma)) = \begin{cases} \{a\}, & \widetilde{r}(\gamma) \leq 1 \\ \mathbb{R}^+, & \widetilde{r}(\gamma) > 1 \end{cases}$$

which means that $(\tau_d)_\gamma \neq (\tau_{d_f})_\gamma$ *and so, we have that* $\tau_d \neq \tau_{d_f}$.

Definition 3.18 *Let* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *be a surjective SMPF.* f *is called strong SMPF (S-SMPF) if for each soft metric space* $(\widetilde{U},d)$ *satisfying the condition (SM4), the soft metrics* d *and* d_f *are topologically equivalent, i.e.,* $\tau_d = \tau_{d_f}$.

Proposition 3.19 *If* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *is S-SMPF, then* $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *is S-MPF for all* $\gamma \in A$.

Proof It is obvious from Proposition 3.14 and Definition 3.18. □

Proposition 3.20 *If* $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *is a surjective S-MPF for every* $\gamma \in A$*, then* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *is S-SMPF.*

Theorem 3.21 *Let* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *be SMPF. Then, the following assertions are equivalent:*

(i) f *is S-SMPF.*

(ii) f *is surjective and continuous.*

(iii) f *is surjective and continuous at* $\overline{0}$.

Proof $(i) \Rightarrow (ii)$ Let $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be S-SMPF, then from Proposition 3.19 $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is S-MPF for all $\gamma \in A$. Since f_γ, for all $\gamma \in A$, is continuous, then we have that f is continuous over $\mathbb{R}(A)^*$.

$(ii) \Rightarrow (i)$ Let $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be a surjective and continuous SMPF, then we have that $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is surjective and continuous for all $\gamma \in A$ which means that f_γ is S-MPF for all $\gamma \in A$. Hence, we conclude that f is S-SMPF.

$(ii) \Rightarrow (iii)$ This observation is clear.

$(iii) \Rightarrow (ii)$ Let f be surjective and continuous at $\bar{0}$. Then, $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is surjective and continuous at 0 for all $\gamma \in A$. This implies that f_γ is continuous over $\mathbb{R}^+$ and so, we have that f is continuous. □

4. Conclusion

Soft metric spaces provide a significant generalization of classical metric spaces by incorporating SS theory to handle uncertainty and vagueness. They retain the essential properties of metric spaces while allowing for a more flexible and nuanced representation of distances in situations where precise measurements are not possible. This makes them a valuable tool in a wide range of theoretical and practical applications, from decision-making to data analysis and beyond. As research in this area continues, further refinements and applications of soft metric spaces are likely to emerge, broadening their impact across multiple domains. On the other hand, metric preserving functions play a vital role in many areas of mathematics and its applications by ensuring that the metric structure of spaces is maintained under transformations. Whether in geometry, data analysis, or physics, these functions help maintain consistency in distance relationships, enabling meaningful interpretations and reliable results in various domains. Understanding these functions' properties and applications provide deeper insight into the structure and behavior of metric spaces and their transformations. This study includes an introduction to SMPFs and some characterizations of these types of functions by means of some properties of the soft functions. For future work, we plan to investigate soft contraction preserving functions which allow us to find the fixed point of functions on the transferred SMSs and also we research the soft partial metric preserving functions and their relationships with SMPFs.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Elif Güner]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%50).

Author [Halis Aygün]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

A New Fractional Order School Academic Performance Model and Numerical Solutions

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Abstract: Academic achievement is defined as the degree to which a student has achieved a learning goal. It is typically measured through the utilisation of examinations, continuous assessments and grade point averages. The student's apprehension of failure can result in the accumulation of stress over time, which can consequently lead to a decline in academic achievement. Conversely, factors such as inadequate cognitive abilities, negative parental influence, familial circumstances and the physical and mental health of the child have been identified as the primary contributors to academic achievement. The present study proposes a novel fractional order mathematical model of academic achievement, comprising three compartments: students with above average achievement (S), students with average achievement (M) and students with below average achievement (B). The Caputo derivative definition was employed as the fractional derivative and a stability analysis of the fractional model was conducted. Numerical solutions were obtained via the Generalized Euler Method and their graphs were drawn.

Keywords: Fractional order school academic performance model, mathematical modeling, generalized Euler method, Caputo derivative, stability analysis.

1. Introduction

A substantial corpus of research has been dedicated to the study of students' behavior and learning, with a particular focus on personal characteristics such as intelligence, cognitive style, motivation, personality, self-concept, and locus of control. It has been noted by several institutions that certain factors related to students' behavior are perceived to contribute to academic failure. Consequently, the management of these institutions has adopted a serious approach to address these issues [20, 26]. Self-regulated learning strategies are important for individuals to be successful lifelong learners. It also provides them with the opportunity to manage their own learning processes [1, 17]. In particular, the impact of self-regulation skills on learners' acquisition of learning strategies merits consideration as an indisputable attribute. Self-regulated learning strategies

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refer to the capacity of individuals to exercise definitive control over their own learning processes, independently managing their learning in the absence of external influences. Consequently, self-regulation processes represent a subject of paramount significance within the domain of educational research [2, 4].

The primary issue that was addressed in the studies conducted during the 1980-1990 period pertains to the relationship between students' knowledge in a specific academic domain and their cognitive, meta-cognitive abilities, and motivation. The research undertaken during this era has yielded significant insights into the components of self-regulated learning. The cognitive and motivational strategies employed by students, the types of goals adapted for learning tasks and beliefs concerning the fulfillment of learning tasks, success and failure attributions have been identified as notable issues. The definition of self-regulatory learning strategies is generally understood to emphasize the learner's state of being active in terms of motivation, metacognition and behavior in the learning process, with the teacher playing a pivotal role in the acquisition of self-regulated learning strategies, particularly in the context of designing and implementing teaching activities in the classroom. Teachers can influence students' self-regulation skills through the strategies, methods and techniques they employ in the classroom. For instance, the development of meta-cognitive regulation skills in students who receive continuous teacher support is at a low level, underscoring the importance of teacher attention to their teaching activities. Additionally, the classroom atmosphere created by the teacher plays a pivotal role. Teachers who employ democratic, student-centered and active teaching practices in the classroom can positively influence students' motivation levels and their skills related to self-regulatory learning strategies [2, 4, 11].

Mathematical models can be defined as simplified representations of a real system or known process. They are utilised for the purpose of expressing observations or measurements of events, interactions and behaviours in a compact form, explaining them, predicting events or outcomes that have not yet been observed and designing systems that are intended to exhibit certain behaviours. The developed fractional order mathematical model will facilitate the determination of the main factors that play a role in determining the academic achievement levels of students. This will include the determination of the extent to which they are effective and their relations with each other, the level of academic failure in schools, and the obtaining of important findings on how to prevent failure [6, 7, 18, 21].

The employment of fractional order derivatives in the control theory of diverse physical and biological processes and dynamical systems has been shown to yield superior outcomes in comparison to the utilisation of integer order derivatives. One of the most significant reasons for this is that fractional order derivatives and integral definitions possess a memory property. Furthermore, the model remains identical, yet the fractional orders of the equations vary in each real-world application, thereby yielding specific and precise results for the pertinent problem. In population models, for instance, the future state of a population is contingent upon its past state, a phenomenon referred to as the 'memory effect'. The incorporation of a delay term or the utilisation of a fractional derivative within the model facilitates the analysis of the memory effect of the population. Fractional calculus, incorporating fractional derivatives and fractional integrals, has recently garnered heightened interest among researchers in the field. It has been determined that fractional operators offer a more precise and efficient characterisation of system behaviour in comparison to integer order derivatives. In view of the substantial advantages of fractional derivatives with regard to memory properties, the present system is modified by substituting the integer order time derivative with the Caputo fractional derivative [5–8, 11, 12, 18, 21].

The utilisation of fractional derivative operators, particularly non-local fractional derivatives, facilitates a more comprehensive investigation of these intricate systems. The majority of research domains pertain to supercomplex mechanisms comprising highly intricate and non-linear differential equations. To enhance comprehension, a range of fractional derivatives, encompassing singular and non-singular kernels, are employed. Through comparative analysis, the fractional derivative that yields the optimal result is identified and employed. To ensure the most accurate determination, real-life data are necessary, as the derivative exhibiting the closest behaviour to real-life data is determined as the derivative that gives the best result [3, 5, 8–10, 12–16, 19, 22–25, 27].

This paper is divided into four sections. The initial section outlines the significance of fractional mathematical modeling and the prevailing academic context within educational institutions. The second part of the paper presents the formation of a fractional order academic achievement model in schools, together with a mathematical analysis of the existence, uniqueness and non-negativity of the system, the Generalised Euler Method and a stability analysis of the model. The third section introduces a new application of the academic achievement model in fractional order schools, presents the numerical results and draws graphs. The fourth section concludes the paper.

2. Fractional Derivative and Fractional Order School Academic Performance Model

The most commonly used definitions of the fractional derivative are Riemann-Liouville, Caputo, Atangana-Baleanu and the conformable derivative. In this study, because the classical initial conditions are easily applicable and provide ease of calculation, the Caputo derivative operator was preferred and modeling was created. The definition of the Caputo fractional derivative is given below.

Definition 2.1 *[17] Let $f(t)$ be a function that is continuously differentiable n times. The value of the function $f(t)$ for α satisfying $n-1<\alpha<n$. The Caputo fractional derivative of order α*

of $f(t)$ is defined by

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f^{(n)}(x)\, dx.$$

Definition 2.2 *[17] The Riemann–Liouville (RL) fractional-order integral of a function $A(t) \in C_n$ $(n \geq -1)$ is given by*

$$J^{\gamma} A(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} A(s)\, ds, \quad J^0 A(t) = A(t).$$

Definition 2.3 *[17] The series expansion of the two-parameter Mittag–Leffler function for $a, b > 0$ is given by*

$$E_{a,b}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(ai+b)}.$$

2.1. Fractional Order School Academic Performance Model

The fractional order model of academic achievement in schools basically categorises a class into three main groups. The first one is students with above average achievement, the second one is students with average achievement and the third one is students with below average achievement. The expression of the academic achievement model in schools as a system of fractional differential equations is as follows:

$$\begin{aligned} \frac{d^{\alpha} S}{dt^{\alpha}} &= \mu N - \mu S - \beta S + \sigma S, \\ \frac{d^{\alpha} M}{dt^{\alpha}} &= \beta S - \mu M - \gamma M, \\ \frac{d^{\alpha} B}{dt^{\alpha}} &= \gamma M - \mu B - \delta B. \end{aligned} \tag{1}$$

Here $\frac{d^{\alpha}}{dt^{\alpha}}$ is the Caputo fractional derivative with respect to time t and $0 < \alpha \leq 1$. Initial values are given as,

$$S(0) = S_0, \; M(0) = M_0, \; B(0) = B_0$$

it is defined as. Since the society is divided into three compartments, $S + M + B = N$ with the derivation of all terms according to time

$$\frac{d^{\alpha} N}{dt^{\alpha}} = \frac{d^{\alpha} S}{dt^{\alpha}} + \frac{d^{\alpha} M}{dt^{\alpha}} + \frac{d^{\alpha} B}{dt^{\alpha}}.$$

In time-dependent phenomena, fractional order models are more realistic and accurate than integer order models because they possess a memory feature [3, 5, 8–10, 12–16, 19, 22–25, 27]. For $\alpha = 1$ in system (1), the fractional order differential equation is reduced to a full order differential equation. The compartment and parameters of the spell are shown in Table 1 and Table 2.

Table 1: Variables used in the systems and their meanings

Variables used in the systems	Meaning
$S(t)$	Students with above average achievement at time t
$M(t)$	Students with average achievement at time t
$B(t)$	Students with below average achievement at time t
$N(t)$	Total classroom

Table 2: Parameters and their meanings

Parameters	Meaning
β	Rate of negative teacher attitude
μ	Academic motivation rate
σ	Positive family relationship rate
δ	Negative family relationship rate
γ	Low rate of self-efficacy

The parameters defined in the model do not change with time. The N term was dimensionlessised and the new variables were created as follows:

$$s = \frac{S}{N}, \; m = \frac{M}{N}, \; b = \frac{B}{N}.$$

It is clear from here that $s + m + b = 1$. Thus, the new form of the academic achievement model in fractional order schools is written as follows:

$$\begin{aligned} D^{\alpha}s(t) &= \mu - \mu s(t) - \beta s(t) + \sigma s(t), \\ D^{\alpha}m(t) &= \beta s(t) - \mu m(t) - \gamma m(t), \\ D^{\alpha}b(t) &= \gamma m(t) - \mu b(t) - \delta b(t). \end{aligned} \tag{2}$$

2.2. Existence, Uniqueness and Non-Negativity of the System

We investigate the exintence and uniqueness of the solution of the fractional-order system (1) in the region $C \times [t_0, T]$ where

$$C = \{(S, M, B) \in R^3_+ : max\{| S |, | M |, | B |\} \le \Psi, min\{| S |, | M || B | \ge \Psi_0\} \tag{3}$$

and $T < +\infty$.

Theorem 2.4 *For each $H_0 = (S_0, M_0, B_0) \in C$, there exists a unique solution $H(t) \in C$ of the fractional-order system (1) with intial condition H_0, which is defined for all $t \ge 0$.*

Proof We denote $H=(S,M,B)$ and $\bar{H}=(\bar{S},\bar{M},\bar{B})$. Consider a mapping $X(H)=(X_1(H),X_2(H),X_3(H))$ and

$$\begin{aligned} X_1(H) &= \mu N-\mu S-\beta S+\sigma S, \\ X_2(H) &= \beta S-\mu M-\gamma M, \\ X_3(H) &= \gamma M-\mu B-\delta B. \end{aligned} \tag{4}$$

For any $H,\bar{H}\in C$, it follows from (4) that

$$\| X(H)-X(\bar{H}) \| = | X_1(H)-X_1(\bar{H}) | + | X_2(H)-X_2(\bar{H}) | + | X_3(H)-X_3(\bar{H}) | \tag{5}$$

and

$$\begin{aligned} | X_1(H)-X_1(\bar{H}) | &= | \mu N-\mu S-\beta S+\sigma S-\mu N+\mu\bar{S}+\beta\bar{S}-\sigma\bar{S} | \\ &= | -\mu(S-\bar{S})-\beta(S-\bar{S})+\sigma(S-\bar{S}) | \\ &\le \mu | S-\bar{S} | +\beta | S-\bar{S} | +\sigma | S-\bar{S} |, \end{aligned}$$

$$\begin{aligned} | X_2(H)-X_2(\bar{H}) | &= | \beta S-\mu M-\gamma M-\beta\bar{S}+\mu\bar{M}+\gamma\bar{M} | \\ &= | \beta(S-\bar{S})-\mu(M-\bar{M})-\gamma(M-\bar{M}) | \\ &\le \beta | S-\bar{S} | +\mu | M-\bar{M} | +\gamma | M-\bar{M} |, \end{aligned}$$

$$\begin{aligned} | X_3(H)-X_3(\bar{H}) | &= | \gamma M-\mu B-\delta B-\gamma\bar{M}+\mu\bar{B}+\delta\bar{B} | \\ &= | \gamma(M-\bar{M})-\mu(B-\bar{B})-\delta(B-\bar{B}) | \\ &\le \gamma | M-\bar{M} | +\mu | B-\bar{B} | +\delta | B-\bar{B} | . \end{aligned}$$

Then, (4) becomes,

$$\begin{aligned}\|X(H)-X(\bar{H})\| &\leq \mu|S-\bar{S}|+\beta|S-\bar{S}|+\sigma|S-\bar{S}|+\beta|S-\bar{S}| \\ &+\mu|M-\bar{M}|+\gamma|M-\bar{M}|+\gamma|M-\bar{M}| \\ &+\mu|B-\bar{B}|+\delta|B-\bar{B}| \\ &\leq (\mu+\sigma+2\beta)|S-\bar{S}|+(\mu+2\gamma)|M-\bar{M}|+(\mu+\gamma)|B-\bar{B}|,\end{aligned}$$

$$\| X(H)-X(\bar{H}) \| \leq L \| H-\bar{H} \|,$$

where $L = max(\mu+\sigma+2\beta, \mu+2\gamma, \mu+\gamma)$.

Therefore, $X(H)$ obeys Lipschitz condition which implies the existence and uniqueness of solution of the fractional-order system (1). □

Theorem 2.5 *For all $t \geq 0$, $S(0)=S_0 \geq 0$, $M(0)=M_0 \geq 0$, $B(0)=B_0 \geq 0$, the solution of the system (1) with initial conditions $(S(t), M(t), B(t)) \in R_+^3$ are not negative.*

Proof (Generalized Mean Value Theorem) Let $f(x) \in C[a,b]$ and $D^\alpha f(x) \in C[a,b]$ for $0<\alpha\leq 1$. Then, we have

$$f(x) = f(\alpha) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\epsilon)(x-a)^\alpha \tag{6}$$

with $0 \leq \epsilon \leq x$ for all $x \in (a,b]$.

The existence and and uniqueness of the solution of the system (1) in $(0,\infty)$ can be obtained via Generalized Mean Value Theorem. We need to show that the domain R_+^3 is positively invariant. Since

$$D^\alpha S = \mu N - \mu S - \beta S + \sigma S \geq 0,$$

$$D^\alpha M = \beta S - \mu M - \gamma M \geq 0,$$

$$D^\alpha B = \gamma M - \mu B - \delta B \geq 0$$

on each hyperplane bounding the nonnegative orthant, the vector field points into R_+^3. □

2.3. Stability Analysis of the Fractional Order School Academic Performance Model

Definition 2.6 *That the equilibrium point of the first-order difference equation system given as*

$$X_{t+1} = F(X_t) \tag{7}$$

is the point $\bar{X}$ that satisfies the equations $\bar{X} = F(\bar{X})$. Also, let us consider $J(\bar{X})$ to be the Jacobian matrix calculated at this equilibrium point. If the eigenvalues obtained from the equation

$det(J(\bar{X})-\lambda I)=0$ satisfy the conditions $\lambda_i \neq 1$ for $i=1,2,\ldots,n$ then this point is called hyperbolic equilibrium, otherwise it is called non-hyperbolic equilibrium [18].

In order to find the equilibrium point in the system (2), $D^{\alpha}s=0$, $D^{\alpha}m=0$, $D^{\alpha}b=0$ it is considered to be.

$E_0=(s_0,m_0,b_0)$ including,

$$E_0=\left(\frac{\mu}{\beta+\mu+\sigma},\frac{\mu\beta}{(\beta+\mu+\sigma)(\mu+\gamma)},\frac{\mu\beta\gamma}{(\beta+\mu+\sigma)(\mu+\gamma)(\mu+\delta)}\right) \tag{8}$$

the equilibrium point of the system is obtained. Jacobian matrix of the system at the equilibrium point

$$J(E_0)=\begin{bmatrix} -\beta-\mu+\sigma & 0 & 0 \\ \beta & -\mu-\gamma & 0 \\ 0 & \gamma & -\mu-\delta \end{bmatrix} \tag{9}$$

it is obtained. The eigenvalues obtained from the Jacobian matrix (9) are given below:

$$\lambda_1=-\beta-\mu+\sigma,$$

$$\lambda_2=-\mu-\gamma,$$

$$\lambda_3=-\mu-\delta$$

where $\beta,\mu,\delta,\sigma,\gamma$ are the parameters of positively defined real numbers. It is clear that $\lambda_2<0$ and $\lambda_3<0$. If $\lambda_1<0$, the equilibrium point of the system is locally asymptotically stable. If $\lambda_1>0$, the equilibrium point of the system is unstable. If $-\beta-\mu+\sigma<0$, $\sigma<\beta+\mu$ is.

$R_0=\frac{\sigma}{\beta+\mu}$ is the basic threshold rate, was determined. If $R_0<1$, academic achievement in schools will increase over time. If $R_0>1$, academic achievement in schools will decrease over time. The success level of students can be taken into consideration when planning studies. In the mathematical model developed for this study, the R_0 value is affected by parameters such as individual reasons (self-efficacy, self-esteem, motivation, etc.), and family-related reasons (parents' attitudes and behaviours, their participation in education, parents' education level, family socioeconomic level, etc.).

2.4. Generalized Euler Method

Generalised Euler Method was used to solve the initial value problem with Caputo fractional derivative. A significant proportion of mathematical models comprise non-linear systems, which can present a considerable challenge in terms of identifying solutions. In the majority of cases, analytical solutions cannot be obtained, necessitating the use of a numerical approach. One such approach is the Generalised Euler Method [24]. Let $D^{\alpha}y(t)=f(t,y(t)),y(0)=y_0,0<\alpha\leq 1,0<t<\alpha$

be the initial value problem. Let $[0,a]$ the interval over which we want to find the solution of the problem. For convenience, subdivide the $[0,a]$ into n subintervals $[t_j, t_{j+1}]$. Suppose that $y(t), D^{\alpha}y(t)$ and $D^{2\alpha}y(t)$ are continuous in range $[0,a]$ and using the generalized Taylor's formula, the following equality is obtained [24]:

$$y(t_1) = y(t_0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_0, y(t_0))$$

where $h = \frac{a}{n}$ for $j = 0, 1, \ldots, n-1$.

This process will be repeated to create an array. Let $t_j = t_{j+1} + h$ such that

$$y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_j, y(t_j))$$

for $j = 0, 1, .., n-1$ the generalized formula in the form is obtained. For each $k = 0, 1, \ldots, n-1$ with step size h,

$$\begin{aligned} D^{\alpha}S(t) &= \mu N - \mu S(k) - \beta S(k) + \sigma S(k), \\ D^{\alpha}M(t) &= \beta S(k) - \mu M(k) - \gamma M(k), \\ D^{\alpha}B(t) &= \gamma M(k) - \mu B(k) - \delta B(k). \end{aligned} \tag{10}$$

For $t \in [0,h)$, $\frac{t}{h} \in [0,1)$, we have

$$\begin{aligned} D^{\alpha}S(t) &= \mu N - \mu S(0) - \beta S(0) + \sigma S(0), \\ D^{\alpha}M(t) &= \beta S(0) - \mu M(0) - \gamma M(0), \\ D^{\alpha}B(t) &= \gamma M(0) - \mu B(0) - \delta B(0). \end{aligned} \tag{11}$$

The solution of (11) reduces to

$$\begin{aligned} S(1) &= S(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\mu N - \mu S(0) - \beta S(0) + \sigma S(0)), \\ M(1) &= M(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\beta S(0) - \mu M(0) - \gamma M(0)), \\ B(1) &= B(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\gamma M(0) - \mu B(0) - \delta B(0)). \end{aligned} \tag{12}$$

For $t \in [h, 2h)$, $\frac{t}{h} \in [1,2)$, we get

$$S(2)=S(1)+\frac{h^\alpha}{\Gamma(\alpha+1)}(\mu N-\mu S(1)-\beta S(1)+\sigma S(1)),$$
$$M(2)=M(1)+\frac{h^\alpha}{\Gamma(\alpha+1)}(\beta S(1)-\mu M(1)-\gamma M(1)), \tag{13}$$
$$B(2)=B(1)+\frac{h^\alpha}{\Gamma(\alpha+1)}(\gamma M(1)-\mu B(1)-\delta B(1)).$$

Repeating the process n times, we obtain

$$S(n+1)=S(n)+\frac{h^\alpha}{\Gamma(\alpha+1)}(\mu N-\mu S(n)-\beta S(n)+\sigma S(n)),$$
$$M(n+1)=M(n)+\frac{h^\alpha}{\Gamma(\alpha+1)}(\beta S(n)-\mu M(n)-\gamma M(n)), \tag{14}$$
$$B(n+1)=B(n)+\frac{h^\alpha}{\Gamma(\alpha+1)}(\gamma M(n)-\mu B(n)-\delta B(n)).$$

3. Numerical Simulation of Fractional Order School Academic Performance Model

In this section, numerical simulation and graphs of the academic achievement model in fractional order schools will be presented. Let us obtain the numerical simulation of the fractional order academic achievement model in schools using the generalized Euler Method. According to the data in [22], let us consider the following parameters.

Let $S=10$, $M=10$, $B=0$, $\beta=0.001$, $\mu=0.002$, $\gamma=0.021$, $\sigma=0.047$, $\delta=0.05$ and let the step size be $h=0.1$. Using the Euler method, the following Table 3 is obtained [22].

Table 3: The values of S, M and B at the moment t for $\alpha=1$

t	$S(t)$	$M(t)$	$B(t)$
0	10,00	10,00	0,00
1	10,04	9,97	0,02
2	10,09	9,95	0,04
3	10,14	9,93	0,06
4	10,19	9,91	0,08
5	10,24	9,89	0,10
6	10,29	9,86	0,12
7	10,34	9,84	0,14
8	10,38	9,82	0,16
9	10,43	9,80	0,18
10	10,48	9,78	0,20
11	10,53	9,76	0,22
12	10,59	9,73	0,24
13	10,64	9,71	0,26
14	10,69	9,69	0,28

Table 4: The values of S, M and B at the moment t for $\alpha=0.9$

t	$S(t)$	$M(t)$	$B(t)$
0	10,00	10,00	0,00
1	10,06	9,97	0,02
2	10,12	9,94	0,05
3	10,18	9,91	0,08
4	10,25	9,88	0,10
5	10,31	9,85	0,13
6	10,38	9,82	0,16
7	10,44	9,80	0,18
8	10,51	9,77	0,21
9	10,57	9,74	0,23
10	10,64	9,71	0,26
11	10,71	9,68	0,28
12	10,77	9,66	0,31
13	10,84	9,63	0,33
14	10,91	9,60	0,36

Table 5: The values of S, M and B at the moment t for $\alpha = 0.8$

t	$S(t)$	$M(t)$	$B(t)$
0	10,00	10,00	0,00
1	10,08	9,96	0,03
2	10,16	9,92	0,07
3	10,24	9,88	0,10
4	10,33	9,85	0,14
5	10,41	9,81	0,17
6	10,49	9,77	0,20
7	10,58	9,74	0,24
8	10,67	9,70	0,27
9	10,75	9,66	0,30
10	10,84	9,63	0,33
11	10,93	9,59	0,36
12	11,02	9,56	0,40
13	11,11	9,52	0,43
14	11,20	9,49	0,46

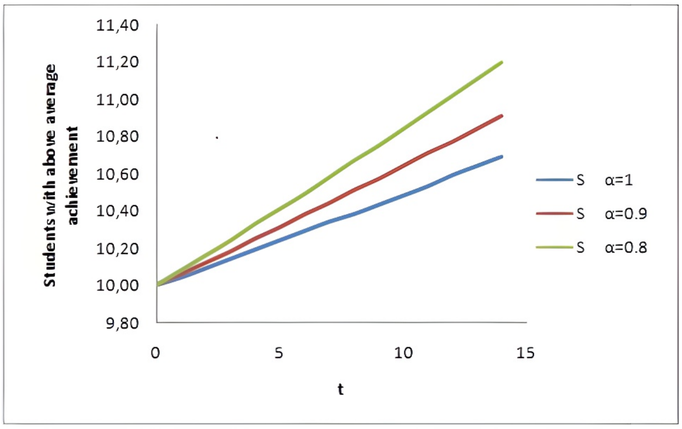

Figure 1: The graph of change of the S compartment model

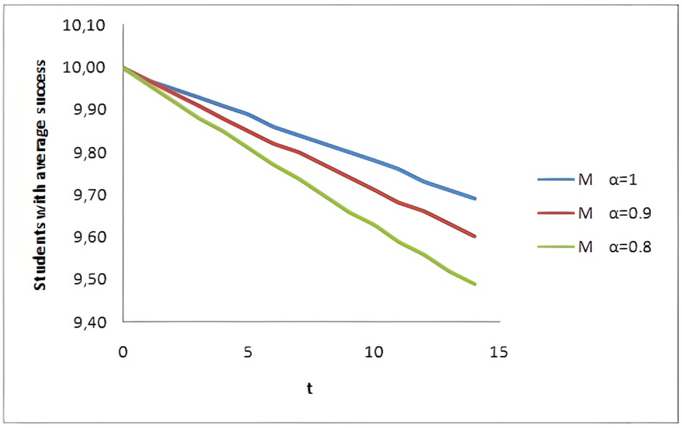

Figure 2: The graph of change of the M compartment model

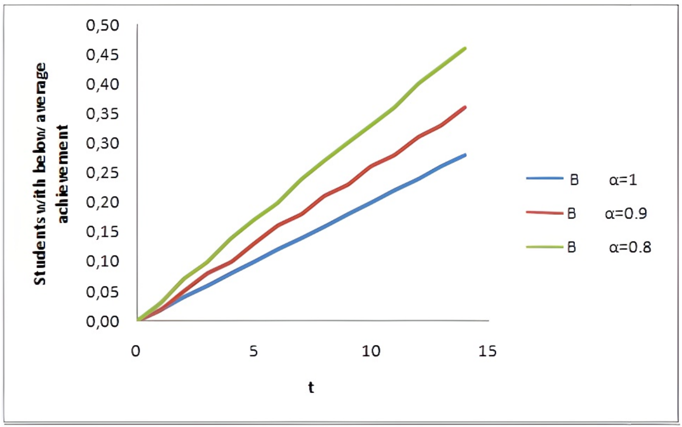

Figure 3: The graph of change of the B compartment model

Table 3, Table 4 and Table 5 show the changes of S, M and B are observed for different states of α.

By the above figures, we observe the following highlights:

* It is observed that the number of students with above average achievement progresses slowly over time (see Figure 1).

* It is observed that the average number of successful students is slowly decreasing over time (see Figure 2).

* It is observed that the number of students with below average achievement increases slowly over time (see Figure 3).

4. Conclusions and Comments

It is of great importance that students succeed academically, as this ensures that they are adequately prepared for the professional world and also has a significant impact on their social lives and future prospects. In the event of academic failure, students frequently encounter a range of emotional, cognitive and behavioural challenges. This study yielded a novel fractional order model that elucidates the factors influencing students' academic achievement levels. The model was then implemented numerically, and graphs were constructed using the numerical results obtained. The existence, uniqueness and non-negativity of the system were analysed mathematically. A stability analysis was performed by obtaining the equilibrium point of the fractional order model of Academic Achievement in Schools, and the number R_0, which is the basic threshold ratio, was found. The graphs obtained revealed that the number of students above the average achievement decreased slowly over time, the number of students with average achievement decreased slowly and the number of students below the average achievement increased slowly over time. In the subsequent models, novel characters may be incorporated into the existing framework of academic achievement, with the adjustment of grade point average score intervals. Furthermore, the variables influencing actors encompass learning speed, intelligence, gender, interests, personality traits, and readiness, among others. The mathematical model can be augmented with additional components. It is imperative to acknowledge the potential contributions of each study focusing on academic achievement, as they contribute unique values to the existing literature and lay the foundation for the development of new concepts.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Research Article

According to the Frenet Frame Spherical Indicators and Results on $\mathbb{E}^3$

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Abstract: In this study, we showed that the spherical indicator curve frames can correspond to a Bishop frame according to the Serret-Frenet frame of a regular curve.

Keywords: Serret-Frenet frame, Bishop frame, spherical indicator.

1. Introduction and Preliminaries

Curves are one of the critical areas of differential geometry. Space curves were defined as the intersection of two surfaces by Clairaut in the first quarter of the 18th century [9]. Frenet (1847) and Serret, without knowing each other, defined a frame using the derivatives of a regular curve. This frame was called the Serret-Frenet frame, referring to the two. Sometimes it is simply called the Frenet frame. The Frenet frame [7] in Euclidean space $\mathbb{E}^3$ is a frame obtained using the velocity and acceleration vectors of a regular curve. Let the velocity and acceleration vectors of the curve $\pi : I \longrightarrow \mathbb{E}^3$ be π' and π'', respectively. Accordingly, the orthonormal frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ obtained as

$$\mathbf{t} = \frac{\pi'}{\|\pi'\|}, \ \mathbf{b} = \frac{\pi' \wedge \pi''}{\|\pi' \wedge \pi''\|}, \ \mathbf{n} = \mathbf{b} \wedge \mathbf{t}$$

is the Frenet frame. Here, the vector fields $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are called the tangent vector field, the principal normal vector field and the binormal vector field of the curve π, respectively. If the curve π is unit speed ($\|\pi'\| = 1$), then

$$\mathbf{t} = \pi', \ \mathbf{n} = \frac{\pi''}{\|\pi''\|}, \ \mathbf{b} = \mathbf{t} \wedge \mathbf{n}.$$

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Derivative changes of the frame $\{\mathbf{t},\, \mathbf{n},\, \mathbf{b}\}$ are

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n}, \\ \mathbf{n}' &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \mathbf{b}' &= -\tau \mathbf{n}. \end{aligned}$$

Here κ and τ are called the first and second curvatures of the curve π, respectively, such that

$$\kappa = \frac{\|\pi' \wedge \pi''\|}{\|\pi''\|^3} \text{ and } \tau = \frac{\det(\pi', \pi'', \pi''')}{\|\pi' \wedge \pi''\|^2}. \tag{1}$$

The quintet $\{\mathbf{t},\, \mathbf{n},\, \mathbf{b},\, \kappa,\, \tau\}$ are called Frenet apparatus. Many studies have been done on this frame in geometry, physics and engineering. These studies have also been advanced in non-Euclidean spaces. Some of these studies are spherical indicators of curves. If $X_{\pi(s)} = X(\pi(s)) \in T_\pi \pi(s)$ the unit vector field X is said to be constrained to the curve π. If we take $X = \overrightarrow{PQ}$, while the point P flows on the curve π, the curve drawn by the unit sphere of the point Q is called the spherical indicator on the unite vector field X. Bilici [3] obtained spherical indicators of involute evolute curves with the help of the Frenet frame. Şenyurt and Çalışkan [10] studied the spherical indicators of timelike Bertrand curve pairs. Şenyurt and Demet [11] calculated the geodesic curvatures and natural lifts of the spherical indicators of timelike-spacelike Mannheim curve pairs. Ateş et al. [1] gave tubular surfaces obtained with spherical indicators. Çapın [5] calculated the arc lengths and geodesic curvatures of the spherical indices of curves in the Minkowski space $\mathbb{E}_1^3$. Kula and Yaylı [8] examined slant helices and their spherical indicators. Erkan and Yüce [6] studied the roles of Bézier curves in $\mathbb{E}^2$ and $\mathbb{E}^3$ with the help of Serret-Frenet and curvatures, both using and not using algorithms used in applied mathematics and computer engineering. Frenet frames on Riemannian manifolds have been also investigated, [1, 12].

Many frames can be obtained from one curve. One of them is the Bishop frame. A Bishop frame [4] $\{\mathbf{t},\, \mathbf{n}_1,\, \mathbf{n}_2\}$ on the curve π that rotates about the tangent vector $\mathbf{t}$ by an angle x is

$$\begin{aligned} \mathbf{t} &= \mathbf{t}, \\ \mathbf{n} &= \mathbf{n}_1 \cos x + \mathbf{n}_2 \sin x, \\ \mathbf{b} &= -\mathbf{n}_1 \sin x + \mathbf{n}_2 \cos x. \end{aligned}$$

The derivative change of this frame is

$$\begin{aligned} \mathbf{t}' &= k_1\mathbf{n}_1 + k_2\mathbf{n}_2, \\ \mathbf{n}_1' &= -k_1\mathbf{t}, \\ \mathbf{n}_2' &= -k_2\mathbf{t}, \\ k_1 &= \kappa\cos x, \\ k_2 &= \kappa\sin x, \\ \tau &= x'. \end{aligned}$$

Here, the quintet $\{\mathbf{t},\, \mathbf{n}_1,\, \mathbf{n}_2,\, k_1,\, k_2\}$ are called Bishop apparatus.

In this study, we examined the spherical indicator curve frames using angles according to the Serret-Frenet frame of a regular curve. We showed that these frames can correspond to a Bishop frame. We expressed and proved the results. We reinforced the study with an example.

2. According to the Frenet Frame Spherical Indicators and Results

Let $\{\mathbf{t},\, \mathbf{n},\, \mathbf{b},\, \kappa \neq 0,\, \tau \neq 0\}$ be the Frenet apparatus of a unit speed curve

$$\begin{aligned} \pi : \quad J &\longmapsto \mathbb{E}^3 \\ s &\longmapsto \pi(s). \end{aligned}$$

The Darboux vector and the pol vector of this curve are

$$\begin{aligned} \mathbf{w} &= \tau\mathbf{t} + \kappa\mathbf{b}, \\ \mathbf{c} &= \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi, \end{aligned}$$

respectively. Here

$$\cos\phi = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \tag{2}$$

$$\sin\phi = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} \tag{3}$$

and ϕ are the angles between the pole vector c and the binormal vector $\mathbf{b}$.

From now on, unless we state otherwise, we will consider a curve π as a curve with a unit speed and curvatures $\kappa \neq 0, \tau \neq 0$.

Theorem 2.1 *Let the Frenet apparatuses of a curve* $\pi : J \longrightarrow \mathbb{E}^3$ *be* $\{\mathbf{t},\, \mathbf{n},\, \mathbf{b},\, \kappa,\, \tau\}$ *and the* *tangents indicator curve* $\pi_{\mathbf{t}} = \mathbf{t}$ *be the Frenet apparatuses* $\{\mathbf{t}_{\mathbf{t}}, \mathbf{n}_{\mathbf{t}}, \mathbf{b}_{\mathbf{t}}, \kappa_{\mathbf{t}}, \tau_{\mathbf{t}}\}$. *Therefore*

$$\begin{aligned} \mathbf{t}_{\mathbf{t}} &= \mathbf{n}, \\ \mathbf{n}_{\mathbf{t}} &= -\mathbf{t}\cos\phi + \mathbf{b}\sin\phi, \\ \mathbf{b}_{\mathbf{t}} &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi, \\ \kappa_{\mathbf{t}} &= \sec\phi, \\ \tau_{\mathbf{t}} &= \frac{\phi'}{\kappa}. \end{aligned}$$

Here, $\phi' = \dfrac{\mathrm{d}\phi}{\mathrm{d}s}$.

Proof On condition that $\dfrac{\mathrm{d}\pi_{\mathbf{t}}}{\mathrm{d}s} = \dfrac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \pi_{\mathbf{t}}'$,

$$\begin{aligned} \pi_{\mathbf{t}}' &= \kappa\mathbf{n}, \\ \pi_{\mathbf{t}}'' &= -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}, \\ \pi_{\mathbf{t}}''' &= -3\kappa\kappa'\mathbf{t} + \left(\kappa'' - \kappa^3 - \kappa\tau^2\right)\mathbf{n} + 2\left(\kappa'\tau + \kappa\tau'\right)\kappa\tau\mathbf{b}. \end{aligned}$$

Using Equation (1), we obtain the first and second curvatures of the curve $\pi_{\mathbf{t}} = \mathbf{t}$ is

$$\kappa_{\mathbf{t}} = \frac{\|\pi_{\mathbf{t}}' \wedge \pi_{\mathbf{t}}''\|}{\|\pi_{\mathbf{t}}''\|^3} = \sec\phi$$

and

$$\tau_{\mathbf{t}} = \frac{\det\left(\pi_{\mathbf{t}}', \pi_{\mathbf{t}}'', \pi_{\mathbf{t}}'''\right)}{\|\pi_{\mathbf{t}}' \wedge \pi_{\mathbf{t}}''\|^2} = \frac{\phi'}{\kappa},$$

respectively. If we take the derivative of the curve $\pi_{\mathbf{t}} = \mathbf{t}$ with respect to its arc parameter $s_{\mathbf{t}}$,

$$\frac{\mathrm{d}\pi_{\mathbf{t}}}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}}\kappa\mathbf{n}.$$

If so,

$$\frac{\mathrm{d}\pi_{\mathbf{t}}}{\mathrm{d}s_{\mathbf{t}}} = \mathbf{t}_{\mathbf{t}} = \mathbf{n}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}} = \frac{1}{\kappa}. \tag{4}$$

On the other hand, if we use (2), (3) and (4), we have

$$\mathbf{n}_{\mathbf{t}} = \frac{\dfrac{\mathrm{d}\mathbf{t}_{\mathbf{t}}}{\mathrm{d}s_{\mathbf{t}}}}{\left\|\dfrac{\mathrm{d}\mathbf{t}_{\mathbf{t}}}{\mathrm{d}s_{\mathbf{t}}}\right\|} = -\mathbf{t}\cos\phi + \mathbf{b}\sin\phi,$$

and

$$\mathbf{b_t} = \mathbf{t_t} \wedge \mathbf{n_t} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi.$$

According to these, the proof ends. □

Corollary 2.2 *On the tangent indicator curve* $\pi_{\mathbf{t}} = \mathbf{t}$, *there is a Bishop frame* $\{\mathbf{n}, -\mathbf{t}, \mathbf{b}\}$ *that rotates about the tangent vector* $\mathbf{t_t} = \mathbf{n}$ *by an angle* ϕ *and the following equations exist*

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s_{\mathbf{t}}} &= a_1(-\mathbf{t}) + a_2\mathbf{b}, \\
\frac{\mathrm{d}(-\mathbf{t})}{\mathrm{d}s_{\mathbf{t}}} &= -a_1\mathbf{n}, \\
\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s_{\mathbf{t}}} &= -a_2\mathbf{n}, \\
a_1 &= 1, \\
a_2 &= \tan\phi,
\end{aligned}$$

where a_1 *and* a_2 *are the first and second curvatures of the Bishop frame* $\{\mathbf{n}, -\mathbf{t}, \mathbf{b}\}$, *respectively.*

Proof It is seen from Theorem 2.1 that the frame $\{\mathbf{n}, -\mathbf{t}, \mathbf{b}\}$ is a Bishop frame. We have

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s_{\mathbf{t}}} &= \frac{d\mathbf{n}}{ds}\frac{ds}{ds_{\mathbf{t}}} = (-\kappa\mathbf{t} + \tau\mathbf{b})\frac{1}{\kappa} \\
&= -\mathbf{t} + \left(\frac{\tau}{\kappa}\right)\mathbf{b},
\end{aligned}$$

$$\frac{\mathrm{d}(-\mathbf{t})}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}(-\mathbf{t})}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}} = -\kappa\mathbf{n}\frac{1}{\kappa} = -\mathbf{n},$$

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}} = -\frac{\tau}{\kappa}\mathbf{n}.$$

Therefore

$$\begin{aligned}
a_1 &= -1, \\
a_2 &= \frac{\tau}{\kappa} = \tan\phi.
\end{aligned}$$

If so, the proof ends. □

Theorem 2.3 *For a curve* $\pi : J \longrightarrow \mathbb{E}^3$, *let apparatuses of the tangents indicator curve* $\pi_{\mathbf{t}} = \mathbf{t}$ *be* $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}, \kappa_{\mathbf{t}}, \tau_{\mathbf{t}}\}$ *and let apparatuses of the principal normal indicator curve* $\pi_{\mathbf{t}} = \mathbf{t}$ *be*

$\{\mathbf{t_n}, \mathbf{n_n}, \mathbf{b_n}, \kappa_\mathbf{n}, \tau_\mathbf{n}\}$. *There are the following equations*

$$\begin{aligned}
\mathbf{t_n} &= \mathbf{n_t}, \\
\mathbf{n_n} &= \mathbf{b_t}\cos\omega - \mathbf{t_t}\sin\omega, \\
\mathbf{b_n} &= \mathbf{b_t}\sin\omega + \mathbf{t_t}\cos\omega, \\
\kappa_\mathbf{n} &= \sqrt{1+\left(\frac{\phi'}{\|\mathbf{w}\|}\right)^2}, \\
\tau_\mathbf{n} &= -\frac{\omega'}{\|\mathbf{w}\|},
\end{aligned}$$

where $\cos\omega = \dfrac{\sqrt{\kappa_n^2-1}}{\kappa_\mathbf{n}}$, $\sin\omega = \dfrac{1}{\kappa_\mathbf{n}}$ *and the angle* ω *is the angle between vectors* $\mathbf{b_t}$ *and* $\mathbf{n_n}$.

Proof On condition that $\dfrac{d\pi_\mathbf{n}}{ds} = \dfrac{d\mathbf{n}}{ds} = \pi'_\mathbf{n}$,

$$\begin{aligned}
\pi'_\mathbf{n} &= -\kappa\mathbf{t}+\tau\mathbf{b}, \\
\pi''_\mathbf{n} &= -\kappa'\mathbf{t} - \left(\kappa^2+\tau^2\right)\mathbf{n}+\tau'\mathbf{b}, \\
\pi'''_\mathbf{n} &= \left[-\kappa'' + \left(\kappa^2+\tau^2\right)\kappa\right]\mathbf{t} - 3\left(\kappa'\tau+\kappa\tau'\right)\mathbf{n} + \left[\tau''+\left(\kappa^2+\tau^2\right)\kappa\right]\mathbf{b}.
\end{aligned}$$

Using Equation (1), the first and second curvatures of the curve $\pi_\mathbf{n} = \mathbf{n}$ are obtained as

$$\kappa_\mathbf{n} = \frac{\|\pi'_\mathbf{n} \wedge \pi''_\mathbf{n}\|}{\|\pi''_\mathbf{n}\|^3} = \sqrt{1+\left(\frac{x'}{\|\mathbf{w}\|}\right)^2} \tag{5}$$

and

$$\tau_\mathbf{n} = \frac{\det\left(\pi'_\mathbf{n}, \pi''_\mathbf{n}, \pi'''_\mathbf{n}\right)}{\|\pi'_\mathbf{n} \wedge \pi''_\mathbf{n}\|^2} = -\frac{\omega'}{\|\mathbf{w}\|}, \tag{6}$$

respectively. If we take the derivative of the curve $\pi_\mathbf{n} = \mathbf{n}$ with respect to its arc parameter $s_\mathbf{n}$,

$$\frac{d\pi_\mathbf{n}}{ds_\mathbf{n}} = \frac{d\mathbf{n}}{ds_\mathbf{n}} = \frac{d\mathbf{n}}{ds}\frac{ds}{ds_\mathbf{n}} = \frac{ds}{ds_\mathbf{n}}\left(-\kappa\mathbf{t}+\tau\mathbf{b}\right)$$

and

$$\frac{ds}{ds_\mathbf{n}} = \frac{1}{\sqrt{\kappa^2+\tau^2}} = \frac{1}{\|\mathbf{w}\|}. \tag{7}$$

If so,

$$\frac{d\pi_\mathbf{n}}{ds_\mathbf{n}} = \mathbf{t_n} = -\mathbf{t}\cos\phi + \mathbf{b}\sin\phi = \mathbf{n_t}.$$

On the other hand, if we use (5), (6) and (7), we have

$$\mathbf{n_n} = \frac{\dfrac{d\mathbf{t_n}}{ds_\mathbf{n}}}{\left\|\dfrac{d\mathbf{t_n}}{ds_\mathbf{n}}\right\|} = \frac{\sqrt{\kappa_n^2 - 1}}{\kappa_\mathbf{n}}(\mathbf{t}\sin\phi + \mathbf{b}\cos\phi) - \frac{1}{\kappa_\mathbf{n}}\mathbf{n}.$$

If we say $\cos\omega = \dfrac{\sqrt{\kappa_n^2 - 1}}{\kappa_\mathbf{n}}$, $\sin\omega = \dfrac{1}{\kappa_\mathbf{n}}$, we get

$$\mathbf{n_n} = \mathbf{b_t}\cos\omega - \mathbf{t_t}\sin\omega,$$

and

$$\mathbf{b_n} = \mathbf{t_n} \wedge \mathbf{n_n} = \mathbf{b_t}\sin\omega + \mathbf{t_t}\cos\omega.$$

□

Corollary 2.4 *The frame* $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}\}$ *is a Bishop frame rotating about the tangent vector* $\mathbf{t_n} = \mathbf{n_t}$ *by an angle* $-\omega$ *on the principal normals indicator curve* $\pi_\mathbf{n} = \mathbf{n}$. *We have the following equations*

$$\begin{aligned} \frac{d\mathbf{n_t}}{ds_\mathbf{n}} &= b_1\mathbf{b_t} - b_2\mathbf{t_t}, \\ \frac{d\mathbf{b_t}}{ds_\mathbf{n}} &= -b_1\mathbf{n_t}, \\ \frac{d\mathbf{t_t}}{ds_\mathbf{n}} &= b_2\mathbf{n_t}, \\ b_1 &= \frac{\phi'}{\|\mathbf{w}\|}, \\ b_2 &= -1, \end{aligned}$$

where b_1 *and* b_2, $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}\}$ *are the first and second Bishop curvatures of the Bishop frame, respectively.*

Proof It is seen from Theorem 2.3 that the frame $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}\}$ is a Bishop frame. We have

$$\begin{aligned} \frac{d\mathbf{n_t}}{ds_\mathbf{n}} &= \frac{d\mathbf{t_n}}{ds_\mathbf{n}} \\ &= \kappa_\mathbf{n}\mathbf{n_n} \\ &= \kappa_\mathbf{n}[\mathbf{b_t}\cos\omega - \mathbf{t_t}\sin\omega] \\ &= \kappa_\mathbf{n}\mathbf{b_t}\cos\omega - \kappa_\mathbf{n}\mathbf{t_t}\sin\omega, \end{aligned}$$

$$\frac{d\mathbf{b_t}}{ds_\mathbf{n}} = \frac{d(\mathbf{t}\sin\phi + \mathbf{b}\cos\phi)}{ds}\frac{ds}{ds_\mathbf{n}} = -\frac{\phi'}{\|\mathbf{w}\|}\mathbf{n_t},$$

$$\frac{d\mathbf{t_t}}{ds_\mathbf{n}} = \frac{d\mathbf{n}}{ds}\frac{ds}{ds_\mathbf{n}} = \mathbf{n_t}.$$

Therefore

$$\begin{aligned} b_1 &= \kappa_{\mathbf{n}} \cos(\omega) = \kappa_{\mathbf{n}} \frac{\sqrt{\kappa_n^2 - 1}}{\kappa_{\mathbf{n}}} = \frac{\phi'}{\|w\|}, \\ b_2 &= -\kappa_{\mathbf{n}} \sin(\omega) = -\kappa_{\mathbf{n}} \frac{1}{\kappa_{\mathbf{n}}} = -1. \end{aligned}$$

Thus, the proof is completed. □

Theorem 2.5 *Let the Frenet apparatuses of a curve* $\pi : J \longrightarrow \mathbb{E}^3$ *be* $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau\}$ *and the Frenet apparatuses of the binormal indicator curve* $\pi_{\mathbf{b}} = \mathbf{b}$ *be* $\{\mathbf{t_b}, \mathbf{n_b}, \mathbf{b_b}, \kappa_{\mathbf{b}}, \tau_{\mathbf{b}}\}$. *We have*

$$\begin{aligned} \mathbf{t_b} &= -\mathbf{n}, \\ \mathbf{n_b} &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi, \\ \mathbf{b_b} &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi, \\ \kappa_{\mathbf{b}} &= \csc\phi, \\ \tau_{\mathbf{b}} &= -\frac{\phi'}{\tau}. \end{aligned}$$

Here, the angle ϕ *is the angle between vectors* $\mathbf{t}$ *and* $\mathbf{n_b}$.

Proof If we take the derivative of the curve $\pi_{\mathbf{b}} = \mathbf{b}$ with respect to its arc parameter $s_{\mathbf{b}}$,

$$\frac{d\pi_{\mathbf{b}}}{ds_{\mathbf{b}}} = \frac{d\mathbf{b}}{ds}\frac{ds}{ds_{\mathbf{b}}} = -\tau\mathbf{n}\frac{ds}{ds_{\mathbf{b}}}.$$

For this reason

$$\mathbf{t_b} = -\mathbf{n} \text{ ve } \frac{ds}{ds_{\mathbf{b}}} = \frac{1}{\tau}. \tag{8}$$

Accordingly

$$\kappa_{\mathbf{b}} = \csc\phi,$$

and

$$\mathbf{n_b} = \mathbf{t}\cos\phi - \mathbf{b}\sin\phi.$$

On the other hand, we obtain

$$\mathbf{b_b} = \mathbf{t_b} \wedge \mathbf{n_b} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi.$$

Also, if we consider Equation (8),

$$\tau_{\mathbf{b}} = \frac{d(-\phi)}{ds_{\mathbf{c}}} = \frac{d(-\phi)}{ds}\frac{ds}{ds_{\mathbf{c}}} = -\frac{\phi'}{\tau}.$$

□

Corollary 2.6 *The frame* $\{\mathbf{t}, -\mathbf{n}, \mathbf{b}\}$ *is a Bishop frame rotating about the tangent vector* $\pi_{\mathbf{b}} = \mathbf{b}$ *by an angle* $-\phi$ *on the binormals indicator curve* $\pi_{\mathbf{n}} = \mathbf{n}$. *We have the following equations*

$$\begin{aligned}
\frac{\mathrm{d}(-\mathbf{n})}{\mathrm{d}s_{\mathbf{b}}} &= c_1\mathbf{t} - c_2\mathbf{b}, \\
\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s_{\mathbf{b}}} &= -c_1(-\mathbf{n}), \\
\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s_{\mathbf{b}}} &= c_2(-\mathbf{n}), \\
c_1 &= \cot\phi, \\
c_2 &= -1,
\end{aligned}$$

where c_1 *and* c_2, $\{\mathbf{t}, -\mathbf{n}, \mathbf{b}\}$ *are the first and second Bishop curvatures of the Bishop frame, respectively.*

Proof It is seen from Theorem 2.5 that the frame $\{\mathbf{t}, -\mathbf{n}, \mathbf{b}\}$ is a Bishop frame. We have

$$\begin{aligned}
\frac{\mathrm{d}(-\mathbf{n})}{\mathrm{d}s_{\mathbf{b}}} &= \frac{\mathrm{d}(-\mathbf{n})}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = (\kappa\mathbf{t} - \tau\mathbf{b})\frac{1}{\tau}, \\
&= \left(\frac{\kappa}{\tau}\right)\mathbf{t} - \mathbf{b}, \\
\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s_{\mathbf{b}}} &= \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = -(-\mathbf{n})\cot\phi, \\
\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s_{\mathbf{b}}} &= \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = -\mathbf{n}.
\end{aligned}$$

If so,

$$\begin{aligned}
c_1 &= \frac{\kappa}{\tau} = \cot\phi, \\
c_2 &= -1.
\end{aligned}$$

On the other hand, if we consider (8),

$$\tau_{\mathbf{b}} = \frac{\mathrm{d}\phi}{\mathrm{d}s_{\mathbf{b}}} = \frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = \phi'\frac{1}{\tau}.$$

□

Theorem 2.7 *Let the Frenet apparatuses of a curve* $\pi : J \longrightarrow \mathbb{E}^3$ *be* $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau\}$ *and let the Frenet apparatuses of the spherical indicator curve of the pol vector* $\pi_{\mathbf{c}} = \mathbf{c}$ *be* $\{\mathbf{t}_{\mathbf{c}}, \mathbf{n}_{\mathbf{c}}, \mathbf{b}_{\mathbf{c}}, \kappa_{\mathbf{c}}, \tau_{\mathbf{c}}\}$.

We have

$$
\begin{aligned}
\mathbf{t_c} &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi, \\
\mathbf{n_c} &= \mathbf{n}\cos\theta - [\mathbf{t}\sin\phi + \mathbf{b}\cos\phi]\sin\theta, \\
\mathbf{b_c} &= \mathbf{n}\sin\theta + [\mathbf{t}\sin\phi + \mathbf{b}\cos\phi]\cos\theta, \\
\kappa_{\mathbf{c}} &= \sqrt{1 + \left(\frac{\|\mathbf{w}\|}{\phi'}\right)^2}, \\
\tau_{\mathbf{c}} &= -\frac{\theta'}{\phi'},\ \phi \neq 0.
\end{aligned}
$$

Here, the angle θ *is the angle between vectors* $\mathbf{n}$ *and* $\mathbf{n_c}$, *and* $\cos\theta = \dfrac{\|w\|}{\sqrt{(\phi')^2 + \|\mathbf{w}\|^2}}$, $\sin\theta = \dfrac{\phi'}{\sqrt{(\phi')^2 + \|\mathbf{w}\|^2}}$.

Proof If we take the derivative of the curve $\pi_{\mathbf{c}} = \mathbf{c} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi$ with respect to its arc parameter $s_{\mathbf{c}}$,

$$\frac{d\pi_{\mathbf{c}}}{ds_{\mathbf{c}}} = \frac{d\mathbf{c}}{ds}\frac{ds}{ds_{\mathbf{c}}} = \phi'(\mathbf{t}\cos\phi - \mathbf{b}\sin\phi)\frac{ds}{ds_{\mathbf{c}}}$$

and provided that $\phi' \neq 0$,

$$\mathbf{t_c} = \mathbf{t}\cos\phi - \mathbf{b}\sin\phi \text{ ve } \frac{ds}{ds_{\mathbf{c}}} = \frac{1}{\phi'}. \tag{9}$$

Since

$$\frac{d\mathbf{t_c}}{ds_{\mathbf{c}}} = \kappa_{\mathbf{c}}\mathbf{n_c} = -\frac{d(\mathbf{t}\cos\phi - \mathbf{b}\sin\phi)}{ds}\frac{ds}{ds_{\mathbf{b}}} = -(\mathbf{t}\sin\phi + \mathbf{b}\cos\phi) + \frac{\|\mathbf{w}\|}{\phi'}\mathbf{n},$$

$$\kappa_{\mathbf{c}} = \sqrt{1 + \left(\frac{\|\mathbf{w}\|}{\phi'}\right)^2},$$

$$\mathbf{n_c} = \mathbf{n}\cos\theta - [\mathbf{t}\sin\phi + \mathbf{b}\cos\phi]\sin\theta, \tag{10}$$

and

$$\mathbf{b_c} = \mathbf{t_c} \wedge \mathbf{n_c} = \mathbf{n}\sin\theta + [\mathbf{t}\sin\phi + \mathbf{b}\cos\phi]\cos\theta. \tag{11}$$

On the other hand, if we consider (9),

$$\tau_{\mathbf{c}} = \frac{d(-\theta)}{ds_{\mathbf{c}}} = \frac{d(-\theta)}{ds}\frac{ds}{ds_{\mathbf{c}}} = -\frac{\theta'}{\phi'}.$$

□

Corollary 2.8 *The frame*

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi, \\ \mathbf{v}_2 &= \mathbf{n}, \\ \mathbf{v}_3 &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi \end{aligned}$$

is the pol vector field indicator curve $\pi_{\mathbf{c}} = \mathbf{c}$, *a Bishop frame rotating about the tangent vector* $\mathbf{v}_1 = \mathbf{t}\cos\phi - \mathbf{b}\sin\phi$ *by an angle* $-\theta$. *We have*

$$\begin{aligned} \frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}s_{\mathbf{c}}} &= d_1\mathbf{v}_2 - d_2\mathbf{v}_3, \\ \frac{\mathrm{d}\mathbf{v}_2}{\mathrm{d}s_{\mathbf{c}}} &= -d_1\mathbf{v}_1, \\ \frac{\mathrm{d}\mathbf{v}_3}{\mathrm{d}s_{\mathbf{c}}} &= d_2\mathbf{v}_1, \\ d_1 &= \frac{\|\mathbf{w}\|}{\phi'}, \\ d_2 &= -1, \end{aligned}$$

where d_1 *and* d_2 *are the first and second Bishop curvatures of the Bishop frame* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, *respectively.*

Proof If we use the following equations

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi, \\ \mathbf{v}_2 &= \mathbf{n}, \\ \mathbf{v}_3 &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi, \end{aligned}$$

with (9), (10) and (11), we obtain

$$\begin{aligned} \mathbf{t}_{\mathbf{c}} &= \mathbf{v}_1, \\ \mathbf{n}_{\mathbf{c}} &= \mathbf{v}_2\cos\theta - \mathbf{v}_3\sin\theta, \\ \mathbf{b}_{\mathbf{c}} &= \mathbf{v}_2\sin\theta + \mathbf{v}_3\cos\theta. \end{aligned}$$

This shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a Bishop frame. Accordingly,

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}s_{\mathbf{c}}} &= \frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{c}}} \\
&= \frac{\mathrm{d}\left(\mathbf{t}\cos\phi - \mathbf{b}\sin\phi\right)}{\mathrm{d}s}\frac{1}{\phi'} \\
&= \frac{\|\mathbf{w}\|}{\phi'}\mathbf{n} - \left(\mathbf{t}\sin\phi + \mathbf{b}\cos\phi\right) \\
&= \frac{\|\mathbf{w}\|}{\phi'}\mathbf{v}_2 - \mathbf{v}_3,
\end{aligned}$$

$$\frac{\mathrm{d}\mathbf{v}_2}{\mathrm{d}s_{\mathbf{c}}} = \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{c}}} = -\frac{\|\mathbf{w}\|}{\phi'}\mathbf{v}_1,$$

$$\frac{\mathrm{d}\mathbf{v}_3}{\mathrm{d}s_{\mathbf{c}}} = \frac{\mathrm{d}\mathbf{v}_3}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{c}}} = \mathbf{v}_1.$$

Therefore

$$\begin{aligned}
d_1 &= \frac{\|\mathbf{w}\|}{\phi'}, \\
d_2 &= -1.
\end{aligned}$$

□

Example 2.9 *Let a curve π be defined as*

$$\begin{aligned}
\pi : J &\longmapsto \mathbb{E}^3 \\
t &\longmapsto \pi(t) = \left(\frac{2t^3}{3}, t^2, t\right)
\end{aligned}$$

in $\mathbb{E}^3$. The Frenet apparatuses of the curve π are

$$\begin{aligned}
\mathbf{t} &= \frac{1}{2t^2+1}\left(2t^2, 2t, 1\right), \\
\mathbf{n} &= \frac{1}{2\left(2t^2+1\right)^2}\left(8t^3+4t, -8t^4+2, -8t^3-4t\right), \\
\mathbf{b} &= \frac{1}{2\left(2t^2+1\right)}\left(-2, 4t, -4t^2\right), \\
\kappa &= \frac{2}{\left(2t^2+1\right)^2}, \\
\tau &= \frac{-2}{\left(2t^2+1\right)^2}.
\end{aligned}$$

From (2) and (3), it is obtained that

$$\begin{aligned}\cos\phi &= \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} = \frac{1}{\sqrt{2}},\\ \sin\phi &= \frac{\tau}{\sqrt{\kappa^2+\tau^2}} = -\frac{1}{\sqrt{2}}.\end{aligned}$$

Accordingly, we can easily calculate the following apparatuses:

If the Frenet apparatuses of the tangent indicator curve $\pi_{\mathbf{t}} = \mathbf{t}$ are $\{\mathbf{t}_{\mathbf{t}}, \mathbf{n}_{\mathbf{t}}, \mathbf{b}_{\mathbf{t}}, \kappa_{\mathbf{t}}, \tau_{\mathbf{t}}\}$, then from Theorem 2.1

$$\begin{aligned}\mathbf{t}_{\mathbf{t}} &= \mathbf{n} = \frac{1}{2\left(2t^2+1\right)^2}\left(8t^3+4t, -8t^4+2, -8t^3-4t\right),\\ \mathbf{n}_{\mathbf{t}} &= -\mathbf{t}\cos\phi + \mathbf{b}\sin\phi = -\frac{1}{2\sqrt{2}\left(2t^2+1\right)}\left(4t^2-2, 8t, -4t^2+2\right),\\ \mathbf{b}_{\mathbf{t}} &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right),\\ \kappa_{\mathbf{t}} &= \sec\phi = \sqrt{2},\\ \tau_{\mathbf{t}} &= \frac{\phi'}{\kappa} = 0.\end{aligned}$$

If the Frenet apparatuses of the tangent indicator curve $\pi_{\mathbf{n}} = \mathbf{n}$ are $\{\mathbf{t}_{\mathbf{n}}, \mathbf{n}_{\mathbf{n}}, \mathbf{b}_{\mathbf{n}}, \kappa_{\mathbf{n}}, \tau_{\mathbf{n}}\}$, then from Theorem 2.3

$$\begin{aligned}\mathbf{t}_{\mathbf{n}} &= \mathbf{n}_{\mathbf{t}} = -\frac{1}{2\sqrt{2}\left(2t^2+1\right)}\left(4t^2-2, 8t, -4t^2+2\right),\\ \mathbf{n}_{\mathbf{n}} &= \mathbf{b}_{\mathbf{t}}\cos\omega - \mathbf{t}_{\mathbf{t}}\sin\omega = -\frac{1}{2\left(2t^2+1\right)^2}\left(8t^3+4t, -8t^4+2, -8t^3-4t\right),\\ \mathbf{b}_{\mathbf{n}} &= \mathbf{b}_{\mathbf{t}}\sin\omega + \mathbf{t}_{\mathbf{t}}\cos\omega = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right),\\ \kappa_{\mathbf{n}} &= \sqrt{1+\left(\frac{\phi'}{\|\mathbf{w}\|}\right)^2} = 1,\\ \tau_{\mathbf{n}} &= -\frac{\omega'}{\|\mathbf{w}\|} = 0.\end{aligned}$$

If the Frenet apparatuses of the tangent indicator curve $\pi_{\mathbf{b}} = \mathbf{b}$ are $\{\mathbf{t}_{\mathbf{b}}, \mathbf{n}_{\mathbf{b}}, \mathbf{b}_{\mathbf{b}}, \kappa_{\mathbf{b}}, \tau_{\mathbf{b}}\}$, *then from Theorem 2.5*

$$\begin{aligned}
\mathbf{t_b} &= -\mathbf{n} = -\frac{1}{2\left(2t^2+1\right)^2}\left(8t^3+4t,-8t^4+2,-8t^3-4t\right),\\
\mathbf{n_b} &= \mathbf{t}\cos\phi-\mathbf{b}\sin\phi = \frac{1}{2\sqrt{2}\left(2t^2+1\right)}\left(4t^2-2,8t,-4t^2+2\right),\\
\mathbf{b_b} &= \mathbf{t}\sin\phi+\mathbf{b}\cos\phi = \left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right),\\
\kappa_{\mathbf{b}} &= \csc\phi = -\sqrt{2},\\
\tau_{\mathbf{b}} &= 0.
\end{aligned}$$

Since $\pi_{\mathbf{c}} = \mathbf{c} = \mathbf{t}\sin\phi+\mathbf{b}\cos\phi = \left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$, *the spherical indicator of the pole vector* $\pi_{\mathbf{c}} = \mathbf{c}$ *is a point.*

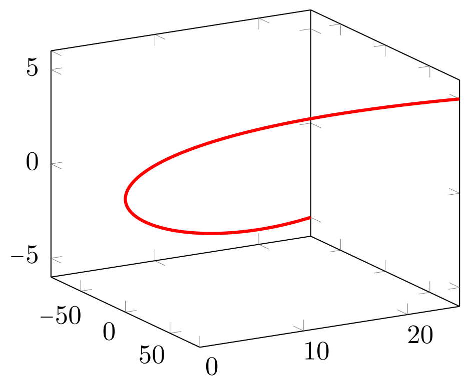

Figure 1: The curve π

Conclusion

Curves are a subject that is used in many fields such as science, engineering, computer design, astronomy studies, and geography. Examining curves means examining the changes in curves. These changes are called the differential geometry of curves. The characterization of curves can be examined with the differential of curves. A lot of work has been done on this subject so far. We have given the sources related to these in the previous sections. Sometimes it is easier to give an idea about a curve with the help of spherical indicators. In this way, spherical indicators of curves are also important. In the studies so far, spherical indicators have been examined with the help of the curvatures of their curves. In this study, we examined spherical indicators depending on the angle between the tangent vector field of a curve and the Darboux vector field. We saw that with this technique, operations and calculations become simpler. In addition, in this study, we showed that spherical indicators (tangent spherical indicators, primary normal spherical indicators, binormal spherical indicators) correspond to a Bishop frame according to the Frenet frame of a

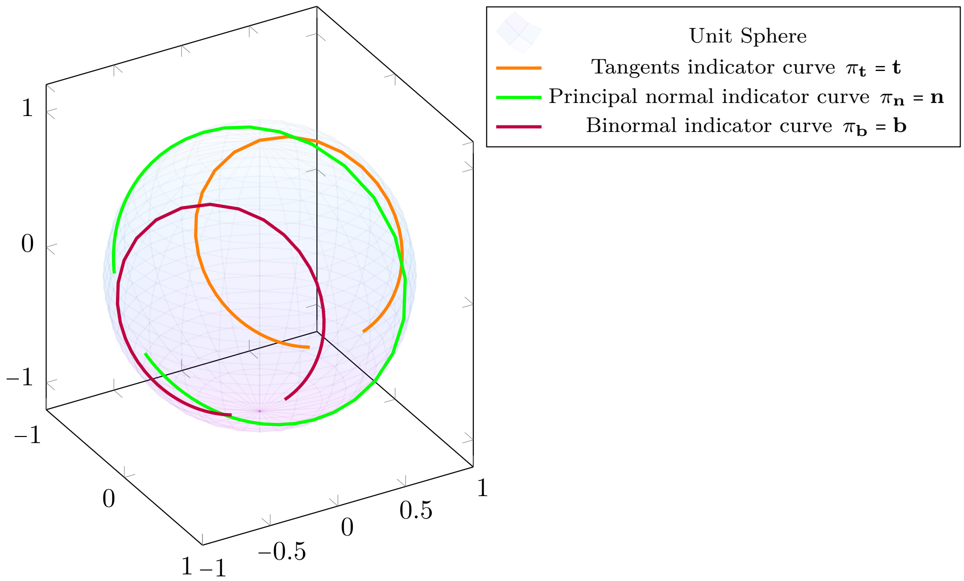

Figure 2: Spherical indicators

regular curve. We could not fully achieve our goals with this study due to lack of time. We could not examine the indicators of a regular curve according to the Darboux frame and the Sabban conflict. These will be addressed in other studies later.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Abdullah Yıldırım]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%75).

Author [Ali Toktimur]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%25).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

Fixed Point Theorems for Generalized Integral Type Weak-Contraction Mappings in Convex Modular Metric Spaces

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Abstract: Fixed point theory in convex modular metric spaces has seen significant advancements due to its broad applicability in various fields. In 2020, Chaira et al. [5] extended fixed point theorems for weak contraction mappings within partially ordered modular metric spaces. Subsequently, in 2023, Mithun et al. [14] established fixed point theorems for integral-type weak contraction mappings in modular metric spaces. Building on these foundational results, this paper investigates fixed point results for four mappings under integral-type contraction conditions in convex modular metric spaces.

Keywords: Fixed point, Δ_2-condition, convex modular metric spaces.

1. Introduction

Let (X, d) be a metric space, and let $T : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $Tx = x$. A mapping $T : X \to X$ is said to be a contraction if there exists a constant $\alpha \in [0, 1)$ such that $d(Tu, Tv) \le \alpha d(u, v)$ for all $u, v \in X$. The Banach Contraction Principle, introduced by Banach in 1922, asserts that such a mapping has a unique fixed point in X if X is a complete metric space. This principle is foundational in fixed point theory and has inspired extensive research, leading to numerous extensions and generalizations under various contractive conditions. One significant generalization is the concept of modular metric spaces, which extends the traditional notion of metric spaces. Modular spaces on linear spaces were first introduced by Nakano in 1950 [11]. Later, in 2010, Chistyakov [6] developed the framework of modular metric spaces, also known as parameterized metric spaces, by incorporating a time parameter. More recently, Khamsi and Kozlowski [9] introduced a fixed point theorem in modular function spaces in 2015, further advancing this field. To continue exploring fixed point theorem in metric modular space, follow those articles [2, 3, 8, 10, 13].

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In this paper, we present new fixed point results for four mappings satisfying integral-type contraction conditions in convex modular metric spaces.

2. Preliminaries

Let X be a non-empty set and $\omega : (0,\infty) \times X \times X \to [0,\infty]$ a function defined by:

$$\omega_\tau(x,y) = \omega(\tau,x,y)$$

for all $x, y \in X$ and $\tau > 0$.

Definition 2.1 [1, 7, 9] *A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be modular metric on X if it satisfies the following conditions:*

(1) $\omega_\tau(x,y) = 0$ if and only if $x = y$ for all x, y in X and for all $\tau > 0$;

(2) $\omega_\tau(x,y) = \omega_\tau(y,x)$ for all x, y in X and for all $\tau > 0$;

(3) $\omega_{\tau+\nu}(x,y) \leq \omega_\tau(x,z) + \omega_\nu(z,y)$ for all $x,y,z \in X$, for all $\tau, \nu > 0$.

The pair (X,ω) is said to be modular metric space.

Proposition 2.2 *If the condition* (1) *is satisfy for some $\tau > 0$, then ω is called regular modular space.*

Proposition 2.3 [14] *If $\omega_{\tau+\nu}(x,y) \leq \frac{\tau}{\tau+\nu}\omega_\tau(x,z) + \frac{\nu}{\tau+\nu}\omega_\nu(z,y)$ for all $x,y,z \in X$ and for all $\tau,\nu > 0$, then ω is said to be a convex modular metric.*

If $0 < \nu < \tau$, then for the modular metric ω on a set X, the function $\tau \to \omega_\tau(x,y)$ is non-increasing on $(0,\infty)$ since, for any $x, y \in X$,

$$\omega_\tau(x,y) \leq \omega_{\tau-\nu}(x,x) + \omega_\nu(x,y) = \omega_\nu(x,y).$$

Definition 2.4 [12] *Let (X,ω) be a modular metric space and fix $z_0 \in X$. Set*

$$X_\omega = X_\omega(z_0) = \{z \in X : \omega_\tau(z,z_0) \to 0 \ as \ \tau \to \infty\},$$

$$X^*_\omega = X^*_\omega(z_0) = \{z \in X : \omega_\tau(z,z_0) < \infty \ for \ \tau > 0\};$$

then the two linear spaces X_ω and X^*_ω are called modular spaces centered at z_0.

Proposition 2.5 *In case of some metric modular ω on X, if $\omega_\tau(x,y) = \omega_\nu(x,y) < \infty$ for all $x,y \in X$ and for all $\tau,\nu > 0$, then there exists a function $\rho(x,y)$ defined by $\rho(x,y) = \omega_\tau(x,y)$ is a metric on X.*

Definition 2.6 [1] *Let ω be a modular metric on a set X. Then*

(1) *A sequence $\{x_n\} \subset X_\omega$ is called ω-convergent to some $x \in X_\omega$ if and only if $\lim_{n\to\infty} \omega_1(x_n, x) = 0$ and x is called the ω-limit of $\{x_n\}$.*

(2) *A sequence $\{x_n\} \subset X_\omega$ is ω-Cauchy if for $m, n \in \mathbb{N}$ such that $\lim_{m,n\to\infty} \omega_1(x_m, x_n) = 0$.*

(3) *A set $W \subset X_\omega$ is ω-closed if ω-limit of any ω-convergent sequence of W is in W.*

(4) *A subset $W \subset X_\omega$ is ω-complete if any ω-Cauchy sequence in W is ω-convergent in W.*

Definition 2.7 [7] *Let ω is a modular metric on X, then ω satisfies Δ_2-condition or simply ω is Δ_2 if for a given sequence $\{x_n\} \subset X_\omega$ and for $x \in X_\omega$, for some $\tau > 0$, $\lim_{n\to\infty} \omega_\tau(x_n, x) = 0$ implies $\lim_{n\to\infty} \omega_\tau(x_n, x) = 0$ for all $\tau > 0$.*

Definition 2.8 [5] *Let ω is a modular metric on X, then ω satisfies the Δ_2-type condition if there exists a positive number k_1 such that $\omega_{\frac{\tau}{2}}(x, y) \le k_1 \omega_\tau(x, y)$ for all $x, y \in X_\omega$ and for all $\tau > 0$.*

Lemma 2.9 [5] *If ω satisfies the Δ_2-type condition, then ω satisfies Δ_2-condition.*

Lemma 2.10 [5] *Let $\{x_n\}$ be a sequence in X_ω and $\tau > 0$. If ω satisfies Δ_2-type condition, then $\{x_n\}$ is ω-Cauchy if and only if $\lim_{m,n\to\infty} \omega_\tau(x_m, x_n) = 0$.*

Lemma 2.11 [5] *If ω holds Δ_2-type condition, then ω is regular.*

Lemma 2.12 [5] *Let ω be a modular metric on X. If a sequence $\{x_n\} \subset X$ is not ω-Cauchy, then there exists $\epsilon > 0$ and two sub-sequence of integers $\{m_k\}$ and $\{n_k\}$ such that*

$$for\ m_k > n_k \ge k, \quad \omega_1(x_{n_k}, x_{m_k}) \ge \epsilon \ and\ \omega_1(x_{n_k}, x_{m_k - 1}) < \epsilon.$$

Lemma 2.13 [5] *Let (X, ω) be a modular metric space and $r, s \in \mathbb{N}^*$ such that ω holds Δ_2-type condition. If a sequence $\{x_n\}$ is not ω-Cauchy in X, then there exists $\epsilon > 0$ and two sub-sequence of integers $\{m_k\}$ and $\{n_k\}$ such that*

$$for\ m_k > n_k \ge k, \quad \omega_{2^r}(x_{n_k}, x_{m_k}) \ge \epsilon \ and\ \omega_{\frac{1}{2^s}}(x_{n_k}, x_{m_k - 1}) < \epsilon.$$

Lemma 2.14 [5] *Let ω be a modular metric on X such that ω satisfies Δ_2-condition. If a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} \omega_1(x_n, x_{n+1}) = 0$ then, $\{x_n\}$ is said to be a ω-Cauchy .*

Theorem 2.15 [14] *Let (X,ω) be a convex modular space and F be a non-empty complete subset of X such that ω satisfies the Δ_2-type condition. Let f, g : $(F,\omega) \to (F,\omega)$ be two functions satisfying the following:*

$$\int_0^{\gamma_1\{\omega_1(fx,gy)\}} \lambda(t)dt \leq \int_0^{\upsilon\{\Omega(x,y)\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x,y)\}} \lambda(t)dt; \tag{1}$$

with

$$\int_0^{\gamma_1(t)} \lambda(z)dz - \int_0^{\upsilon(t)} \lambda(z)dz + \int_0^{\phi(t)} \lambda(z)dz > 0;$$

for all

$$r > 0,\ \lim_{t\to r} \int_0^{\gamma_1(t)} \lambda(z)dz - \lim_{t\to r} \int_0^{\upsilon(t)} \lambda(z)dz + \liminf_{t\to r} \int_0^{\phi(t)} \lambda(z)dz > 0;$$

where $\gamma_1 \in \Gamma$, $\upsilon \in \Upsilon$, $\phi \in \Phi$ and $\lambda \in \Lambda$, and

$$\Omega(x,y) = \max\Big\{\omega_1(x,fx), \omega_1(y,gy), \omega_1(x,y), \omega_2(fx,y), \omega_2(x,gy)\Big\}. \tag{2}$$

Then f and g have a unique fixed point in F.

3. Main Results

From reference [4], we consider $\Lambda = \{\lambda | \lambda : \mathbb{R}^+ \to \mathbb{R}^+\}$ which is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ such that

(a) $\int_0^{\epsilon} \lambda(t)dt > 0$ for each $\epsilon > 0$,

(b) $\int_0^{a+b} \lambda(t)dt \leq \int_0^{a} \lambda(t)dt + \int_0^{b} \lambda(t)dt$.

Lemma 3.1 [4] *Let $\lambda \in \Lambda$ and $\{s_n\}$ be a non-negative sequence with $\lim_{n\to\infty} s_n = s$, then*

$$\lim_{n\to\infty} \int_0^{s_n} \lambda(t)dt = \int_0^{s} \lambda(t)dt.$$

Lemma 3.2 [4] *Let $\lambda \in \Lambda$ and $\{s_n\}$ be a non-negative sequence. Then*

$$\lim_{n\to\infty} \int_0^{s_n} \lambda(t)dt = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} s_n = 0.$$

Consider three classes of functions Γ, Υ and Φ are as follows:

(a) $\Gamma = \Big\{\gamma : [0,\infty) \to [0,\infty)$ such that (i) γ is strictly increasing; (ii) $\lim_{t\to r} \gamma(t) > 0$ for $r > 0$ and $\lim_{t\to 0^+} \gamma(t) = 0$; (iii) $\gamma(t) = 0$ if and only if $t = 0\Big\}$.

(b) $\Upsilon = \Big\{\upsilon : [0,\infty) \to [0,\infty)$ such that (i) υ is non-decreasing; (ii) $\lim_{t\to r}\upsilon(t) > 0$ for $r > 0$ and $\lim_{t\to 0^+}\upsilon(t) = 0$; (iii) $\upsilon(t) = 0$ if and only if $t = 0\Big\}$.

(c) $\Phi = \Big\{\phi : [0,\infty) \to [0,\infty)$ such that (i) $\lim_{t\to r} inf \phi(t) > 0$ for all $r > 0$; (ii) $\phi(t) \to 0 \Rightarrow t \to 0$; (iii) $\phi(t) = 0$ if and only if $t = 0\Big\}$.

Theorem 3.3 *Let (X,ω) be a convex modular space and F be a non-empty complete subset of X such that ω satisfies the Δ_2-type condition. Let P, Q, R, $S : (F,\omega) \to (F,\omega)$ be four functions satisfying the following:*

$$\int_0^{\gamma_1\{\omega_1(Rx,Sy)\}} \lambda(t)dt \le \int_0^{\upsilon\{\Omega(x,y)\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x,y)\}} \lambda(t)dt; \tag{3}$$

with

$$\int_0^{\gamma_1(t)} \lambda(z)dz - \int_0^{\upsilon(t)} \lambda(z)dz + \int_0^{\phi(t)} \lambda(z)dz > 0;$$

for all

$$r > 0,\ \lim_{t\to r}\int_0^{\gamma_1(t)} \lambda(z)dz - \lim_{t\to r}\int_0^{\upsilon(t)} \lambda(z)dz + \liminf_{t\to r}\int_0^{\phi(t)} \lambda(z)dz > 0;$$

where $\gamma_1 \in \Gamma$, $\upsilon \in \Upsilon$, $\phi \in \Phi$ and $\lambda \in \Lambda$, and

$\Omega(x,y) = \max\Big\{\omega_1(Px,Rx), \omega_1(Qy,Sy), \omega_1(Qy,Rx), \omega_2(Px,Sy), \omega_2(Px,Qy)\Big\}$.

Also,

(a) *$R \subseteq Q$ and $S \subseteq P$,*

(b) *$\{P,R\}$ and $\{Q,S\}$ is weakly compatible; either P or R is continuous.*

Then P, Q, R and S have a unique common fixed point in F.

Proof. Let x_0 be any arbitrary element in X. From condition (a), there exist two elements x_1 and x_2 in X such that $Rx_0 = Qx_1 = y_0$ and $Sx_1 = Px_2 = y_1$. Proceeding inductively we can construct a sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Rx_{2n} = Qx_{2n+1} \ \ and \ \ y_{2n+1} = Sx_{2n+1} = Px_{2n+2}$$

for all $n \in \mathbb{N}$.

Case-1: For some $n \in \mathbb{N}$, $y_n = y_{n+1} \Rightarrow y_{n+1} = y_{n+2}$.

If n is even, i.e., $n = 2k$, $k \in \mathbb{N}$, we have

$$y_{2k} = y_{2k+1}. \tag{4}$$

If $y_{2k+1} \neq y_{2k+2}$, then $\omega_1(y_{2k+1}, y_{2k+2}) > 0$.

Now,

$$\begin{aligned}
\Omega(y_{2k+2}, y_{2k+1}) &= \max\Big\{\omega_1(Px_{2k+2}, Rx_{2k+2}), \omega_1(Qx_{2k+1}, Sx_{2k+1}), \omega_1(Qx_{2k+1}, Rx_{2k+2}),\\
&\quad \omega_2(Px_{2k+2}, Sx_{2k+1}), \omega_2(Px_{2k+2}, Qx_{2k+1})\Big\}\\
&= \max\Big\{\omega_1(y_{2k+1}, y_{2k+2}), \omega_1(y_{2k}, y_{2k+1}), \omega_1(y_{2k}, y_{2k+2}),\\
&\quad \omega_2(y_{2k+1}, y_{2k+1}), \omega_2(y_{2k+1}, y_{2k})\Big\}\\
&= \max\Big\{\omega_1(y_{2k+1}, y_{2k+2}), \omega_1(y_{2k}, y_{2k+2})\Big\}\\
&= \max\Big\{\omega_1(y_{2k+1}, y_{2k+2}), \omega_1(y_{2k+1}, y_{2k+2})\Big\}\\
&= \omega_1(y_{2k+1}, y_{2k+2}).
\end{aligned}$$

Hence, $\Omega(y_{2k+2}, y_{2k+1}) = \omega_1(y_{2k+1}, y_{2k+2})$.

Now,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k+2}, y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+2}, Sx_{2k+1})\}} \lambda(t)dt\\
&\leq \int_0^{\upsilon\{\Omega(x_{2k+2}, x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k+2}, x_{2k+1})\}} \lambda(t)dt\\
&= \int_0^{\upsilon\{\Omega(y_{2k+1}, y_{2k+2})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(y_{2k+1}, y_{2k+2})\}} \lambda(t)dt\\
&< \int_0^{\gamma\{\Omega(y_{2k+2}, y_{2k+1})\}} \lambda(t)dt
\end{aligned}$$

which is a contradiction.

Hence,

$$y_{2k} = y_{2k+1} \Rightarrow y_{2k+1} = y_{2k+2}. \tag{5}$$

If, n is odd, i.e., for $n = 2k+1$, $k \in \mathbb{N} \cup \{0\}$, we have

$$y_{2k+1} = y_{2k+2} \tag{6}$$

and

$$\Omega(x_{2k+2}, x_{2k+3}) = \omega_1(x_{2k+2}, x_{2k+3}). \tag{7}$$

Now,

$$\begin{aligned}
\Omega(x_{2k+2},x_{2k+3}) &= \max\Big\{\omega_1(Px_{2k+2},Rx_{2k+2}),\omega_1(Qx_{2k+3},Sx_{2k+3}),\omega_1(Qx_{2k+3},Rx_{2k+2}),\\
&\quad \omega_2(Px_{2k+2},Sx_{2k+3}),\omega_2(Px_{2k+2},Qx_{2k+3})\Big\}\\
&= \max\Big\{\omega_1(y_{2k+1},y_{2k+2}),\omega_1(y_{2k+2},y_{2k+3}),\omega_1(y_{2k+2},y_{2k+2}),\omega_2(y_{2k+1},y_{2k+3}),\\
&\quad \omega_2(y_{2k+1},y_{2k+2})\Big\}\\
&\le \max\Big\{\omega_1(y_{2k+2},y_{2k+3}),\omega_2(y_{2k+1},y_{2k+3})\Big\}\\
&\le \max\Big\{\omega_1(y_{2k+2},y_{2k+3}),\frac{\omega_1(y_{2k+1},y_{2k+2})+\omega_1(y_{2k+2},y_{2k+3})}{2}\Big\}\\
&= \omega_1(y_{2k+2},y_{2k+3}).
\end{aligned}$$

Hence, $\Omega(x_{2k+2},x_{2k+3})=\omega_1(y_{2k+2},y_{2k+3})$.

Now,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k+2},y_{2k+3})\}}\lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+2},Sx_{2k+3})\}}\lambda(t)dt\\
&\le \int_0^{\upsilon\{\Omega(x_{2k+2},x_{2k+3})\}}\lambda(t)dt-\int_0^{\phi\{\Omega(x_{2k+2},x_{2k+3})\}}\lambda(t)dt\\
&= \int_0^{\upsilon\{\Omega(y_{2k+2},y_{2k+3})\}}\lambda(t)dt-\int_0^{\phi\{\Omega(y_{2k+2},y_{2k+3})\}}\lambda(t)dt\\
&< \int_0^{\gamma\{\Omega(y_{2k+2},y_{2k+3})\}}\lambda(t)dt
\end{aligned}$$

which is a contradiction.

Hence,

$$y_{2k+1}=y_{2k+2}\Rightarrow y_{2k+2}=y_{2k+3}.$$

If we continue thus process, then we obtain $y_n=y_{n+1}\Rightarrow y_n=y_{n+k}$ for $k=1,2,\cdots$. Therefore $\{y_n\}$ is a constant sequence and hence ω-Cauchy sequence in F.

Case-2: Let $y_n\neq y_{n+1}$ for all $n\in\mathbb{N}$. If n is even, i.e., $n=2k$, $k\in\mathbb{N}$, we have

$$\begin{aligned}
\Omega(x_{2k},x_{2k+1}) &= \max\Big\{\omega_1(Px_{2k},Rx_{2k}),\omega_1(Qx_{2k+1},Sx_{2k+1}),\omega_1(Qx_{2k+1},Rx_{2k}),\omega_2(Px_{2k},Sx_{2k+1}),\\
&\quad \omega_2(Px_{2k},Qx_{2k+1})\Big\}\\
&= \max\Big\{\omega_1(y_{2k-1},y_{2k}),\omega_1(y_{2k},y_{2k+1}),\omega_1(y_{2k},y_{2k}),\omega_2(y_{2k-1},y_{2k+1}),\omega_2(y_{2k-1},y_{2k})\Big\}\\
&\le \max\Big\{\omega_1(y_{2k-1},y_{2k}),\omega_1(y_{2k},y_{2k+1}),\frac{\omega_1(y_{2k-1},y_{2k})+\omega_1(y_{2k},y_{2k+1})}{2},\omega_2(y_{2k-1},y_{2k})\Big\}
\end{aligned}$$

If $\Omega(x_{2k}, x_{2k+1}) = \omega(y_{2k}, y_{2k+1})$, then

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k},Sx_{2k+1})\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt \\
&= \int_0^{\upsilon\{\Omega(y_{2k},y_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(y_{2k},y_{2k+1})\}} \lambda(t)dt \\
&< \int_0^{\gamma\{\Omega(y_{2k},y_{2k+1})\}} \lambda(t)dt
\end{aligned}$$

which is a contradiction. Hence,

$$\Omega(x_{2k}, x_{2k+1}) = \omega_1(y_{2k-1}, y_{2k}). \tag{8}$$

Now from (1)

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k},Sx_{2k+1})\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt \\
&= \int_0^{\upsilon\{\omega_1(y_{2k-1},y_{2k})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(y_{2k-1},y_{2k})\}} \lambda(t)dt \\
&< \int_0^{\gamma_1\{\Omega(y_{2k-1},y_{2k})\}} \lambda(t)dt
\end{aligned}$$

Since γ_1 is strictly increasing, so we have

$$\omega_1(y_{2k}, y_{2k+1}) < \omega_1(y_{2k-1}, y_{2k}). \tag{9}$$

If n is odd, i.e., $n = 2k+1$, $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\Omega(x_{2k+1}, x_{2k+2}) &= \max\Big\{\omega_1(Px_{2k+1}, Rx_{2k+1}), \omega_1(Qx_{2k+2}, Sx_{2k+2}), \omega_1(Qx_{2k+2}, Rx_{2k+1}), \\
&\quad \omega_2(Px_{2k+1}, Sx_{2k+2}), \omega_2(Px_{2k+1}, Qx_{2k+2})\Big\} \\
&= \max\Big\{\omega_1(y_{2k}, y_{2k+1}), \omega_1(y_{2k+1}, y_{2k+2}), \omega_1(y_{2k+1}, y_{2k+1}), \\
&\quad \omega_2(y_{2k}, y_{2k+2}), \omega_2(y_{2k}, y_{2k+1})\Big\} \\
&\leq \max\Big\{\omega_1(y_{2k}, y_{2k+1}), \omega_1(y_{2k+1}, y_{2k+2}), \\
&\quad \frac{\omega_1(y_{2k}, y_{2k+1}) + \omega_1(y_{2k+1}, y_{2k+2})}{2}, \omega_2(y_{2k}, y_{2k+1})\Big\}.
\end{aligned}$$

If $\Omega(x_{2k+1}, x_{2k+2}) = \omega_1(y_{2k+1}, y_{2k+2})$, then

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k+1},y_{2k+2})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+1},Sx_{2k+2})\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(x_{2k+1},x_{2k+2})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k+1},x_{2k+2})\}} \lambda(t)dt \\
&= \int_0^{\upsilon\{\omega_1(y_{2k+1},y_{2k+2})\}} \lambda(t)dt - \int_0^{\phi\{\omega_1(y_{2k+1},y_{2k+2})\}} \lambda(t)dt \\
&< \int_0^{\gamma\{\omega_1(y_{2k+1},y_{2k+2})\}} \lambda(t)dt
\end{aligned}$$

which is a contradiction.

Hence,

$$\Omega(x_{2k+1}, x_{2k+2}) = \omega_1(x_{2k}, x_{2k+1}) \tag{10}$$

Now, from (1)

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k},Sx_{2k+1})\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt \\
&= \int_0^{\upsilon\{\omega_1(y_{2k-1},y_{2k})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(y_{2k-1},y_{2k})\}} \lambda(t)dt \\
&< \int_0^{\gamma_1\{\omega_1(y_{2k-1},y_{2k})\}} \lambda(t)dt.
\end{aligned}$$

Since γ_1 is strictly increasing, so we have

$$\omega_1(y_{2k}, y_{2k+1}) < \omega_1(y_{2k-1}, y_{2k}). \tag{11}$$

From (9) and (11) we conclude that

$$\omega_1(y_n, y_{n+1}) < \omega_1(y_{n-1}, y_n) foralln = 1, 2, 3, \cdots. \tag{12}$$

Therefore $\{\omega_1(y_n, y_{n+1})\}$ is monotone decreasing and bounded below, so convergent.

Let

$$\lim_{n\to\infty} \omega_1(y_n, y_{n+1}) = l \quad and \quad \lim_{n\to\infty} \int_0^{\gamma_1\{\omega_1(y_n,y_{n+1})\}} \lambda(t)dt = l^*,$$

where l and $l^* \geq 0$.

Claim: $l = 0$.

If not, then $l > 0$. Then $\lim_{k\to\infty} \omega_1(y_{2k}, y_{2k+1}) = l$ and $\lim_{n\to\infty} \int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}})\lambda(t)dt = l^*$.

Now, from (1) we have

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}}\lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k},Sx_{2k+1})\}}\lambda(t)dt\\ &\leq \int_0^{\upsilon\{\Omega(x_{2k},x_{2k+1})\}}\lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k},x_{2k+1})\}}\lambda(t)dt\\ &= \int_0^{\upsilon\{\omega_1(y_{2k-1},y_{2k})\}}\lambda(t)dt - \int_0^{\phi\{\omega_1(y_{2k-1},y_{2k})\}}\lambda(t)dt\end{aligned}$$

Taking limit as $k \to \infty$ in the above inequalities, we have

$$\lim_{k\to\infty}\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}}\lambda(t)dt \leq \lim_{k\to\infty}\int_0^{\upsilon\{\omega_1(y_{2k-1},y_{2k})\}}\lambda(t)dt - \lim_{k\to\infty}\inf\int_0^{\phi\{\omega(y_{2k-1},y_{2k})\}}\lambda(t)dt$$

This implies

$$\lim_{k\to\infty}\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}}\lambda(t)dt \leq \lim_{k\to\infty}\int_0^{\gamma_1\{\omega_1(y_{2k-1},y_{2k})\}}\lambda(t)dt,$$

i.e., $l^* \leq l^*$ which is a contradiction. Hence, $l = 0$, i.e.,

$$\lim_{n\to\infty}\omega_1(y_n, y_{n+1}) = 0. \tag{13}$$

Since (X,ω) be a convex modular space and F is a ω-complete subset of X satisfying Δ_2-type condition and $\lim_{n\to\infty}\omega_1(y_n, y_{n+1}) = 0$, so by Lemma 2.14 $\{y_n\}$ is ω-Cauchy in F. Since F is ω-complete, there exists $z \in F$ such that $\lim_{n\to\infty}\omega(y_n, z) = 0$.

Therefore

$$\lim_{n\to\infty}\omega_1(y_{2n}, z) = \lim_{n\to\infty}\omega_1(Rx_{2n}, z) = \lim_{n\to\infty}\omega_1(Qx_{2n+1}, z) = 0$$

and

$$\lim_{n\to\infty}\omega_1(y_{2n+1}, z) = \lim_{n\to\infty}\omega_1(Sx_{2n+1}, z) = \lim_{n\to\infty}\omega_1(Px_{2n+2}, z) = 0.$$

Since the mappings P and R are compatible, so $\lim_{k\to\infty}RPx_{2k} = \lim_{k\to\infty}PRx_{2k}$.

We assume that $Pz = z$. If not, then we will arrives a contradiction.

$$\begin{aligned}
\Omega(Px_{2k}, x_{2k+1}) &= \max\Big\{\omega_1(P^2x_{2k}, RPx_{2k}), \omega_1(Qx_{2k+1}, Sx_{2k+1}), \omega_1(Qx_{2k+1}, RPx_{2k}),\\
&\quad \omega_2(P^2x_{2k}, Sx_{2k+1}), \omega_2(P^2x_{2k}, Qx_{2k+1})\Big\}\\
&= \max\Big\{\omega_1(P^2x_{2k}, PRx_{2k}), \omega_1(Qx_{2k+1}, Sx_{2k+1}), \omega_1(Qx_{2k+1}, PRx_{2k}),\\
&\quad \omega_2(P^2x_{2k}, Sx_{2k+1}), \omega_2(P^2x_{2k}, Qx_{2k+1})\Big\}\\
&= \max\Big\{\omega_1(Py_{2k-1}, Py_{2k}), \omega_1(y_{2k}, y_{2k+1}), \omega_1(y_{2k}, Py_{2k}),\\
&\quad \omega_2(Py_{2k-1}, y_{2k+1}), \omega_2(Py_{2k-1}, y_{2k})\Big\}.
\end{aligned}$$

Taking limit as $k \to \infty$,

$$\begin{aligned}
\lim_{k\to\infty} \Omega(Px_{2k}, x_{2k+1}) &= \max\Big\{\omega_1(Pz, Pz), \omega_1(z, z), \omega_1(z, Pz), \omega_2(Pz, z), \omega_2(Pz, z)\Big\}\\
&= \omega_1(Pz, z)
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(Py_{2k}, y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(PRx_{2k}), Sx_{2k+1}\}} \lambda(t)dt\\
&= \int_0^{\gamma_1\{\omega_1(R(Px_{2k}), Sx_{2k+1})\}} \lambda(t)dt\\
&\leq \int_0^{\upsilon\{\Omega(Px_{2k}, x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(Px_{2k}, x_{2k+1})\}} \lambda(t)dt\\
\Rightarrow \int_0^{\gamma_1\{\omega_1(Pz, z)\}} \lambda(t)dt &< \int_0^{\gamma_1\{\omega_1(Pz, z)\}} \lambda(t)dt
\end{aligned}$$

which is a contradiction. Hence, $Pz = z$.

We assume that $Rz = z$. If not, then we will arrive a contradiction.

$$\begin{aligned}
\Omega(z, x_{2k}) &= \max\Big\{\omega_1(Pz, Rz), \omega_1(Qx_{2k}, Sx_{2k}), \omega_1(Qx_{2k}, Rz),\\
&\quad \omega_2(Pz, z), \omega_2(Pz, Qx_{2k})\Big\}.
\end{aligned}$$

Taking limit as $k \to \infty$,

$$\begin{aligned}
\lim_{k\to\infty} \Omega(z, x_{2k}) &= \max\Big\{\omega_1(z, Rz), \omega_1(z, z), \omega_1(z, Rz),\\
&\quad \omega_2(Pz, z), \omega_2(Pz, z)\Big\}\\
&= \omega_1(z, Rz).
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(Rz,y_{2k})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rz,Sx_{2k}\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(z,x_{2k})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(z,x_{2k})\}} \lambda(t)dt \\
\Rightarrow \int_0^{\gamma_1\{\omega_1(Pz,z)\}} \lambda(t)dt &< \int_0^{\gamma_1\{\omega_1(Pz,z)\}} \lambda(t)dt
\end{aligned}$$

which is a contradiction. Hence, $Rz = z$.

We assume that $Qz = z$. If not, then we will arrive a contradiction. Since the mappings S and Q are compatible, so $SQx_{2k+2} = QSx_{2k+2}$.

$$\begin{aligned}
\Omega(x_{2k+1}, Qx_{2k+2}) &= \max\Big\{\omega_1(Px_{2k+1}, Rx_{2k+1}), \omega_1(Q^2x_{2k+2}, SQx_{2k+2}), \omega_1(Q^2x_{2k+2}, Rx_{2k+1}), \\
&\quad \omega_2(Px_{2k+1}, SQx_{2k+2}), \omega_2(Px_{2k+1}, Q^2x_{2k+2})\Big\} \\
&= \max\Big\{\omega_1(y_{2k}, y_{2k+1}), \omega_1(Q^2x_{2k+2}, QSx_{2k+2}), \omega_1(Q^2x_{2k+2}, Rx_{2k+1}), \\
&\quad \omega_2(Px_{2k+1}, QSx_{2k+2}), \omega_2(Px_{2k+1}, Q^2x_{2k+2})\Big\}.
\end{aligned}$$

Taking limit as $k \to \infty$, we have

$$\begin{aligned}
\lim_{k\to\infty} \Omega(x_{2k+1}, Qx_{2k+2}) &= \max\Big\{\omega_1(z,z), \omega_1(Qz,Qz), \omega_1(Qz,z), \omega_2(z,Qz), \omega_2(z,Qz)\Big\} \\
&= \omega_1(z,Qz).
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k+1},Qy_{2k+2})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+1},Q(Sx_{2k+2}))\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(x_{2k+1},Qx_{2k+2})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k+1},Qx_{2k+2})\}} \lambda(t)dt \\
&= \int_0^{\upsilon\{\Omega(x_{2k+1},Qx_{2k+2})\}} \lambda(t)dt.
\end{aligned}$$

In both side taking limit as $k \to \infty$, we have

$$\int_0^{\gamma_1\{\omega_1(z,Qz)\}} \lambda(t)dt < \int_0^{\gamma_1\{\omega_1(z,Qz)\}} \lambda(t)dt$$

which is a contradiction. Hence $Qz = z$.

We assume that $Sz = z$. If not, then we will arrive a contradiction.

$$\begin{aligned}\Omega(x_{2k+1}, z) &= \max\Big\{\omega_1(Px_{2k+1}, Rx_{2k+1}), \omega_1(Qz, Sz), \omega_1(Qz, Rx_{2k+1}),\\ &\quad \omega_2(Px_{2k+1}, Sz), \omega_2(Px_{2k+1}, Qz)\Big\}\\ &= \max\Big\{\omega_1(Px_{2k+1}, Rx_{2k+1}), \omega_1(z, Sz), \omega_1(z, Rx_{2k+1}),\\ &\quad \omega_2(Px_{2k+1}, Sz), \omega_2(Px_{2k+1}, z)\Big\}\end{aligned}$$

Taking limit as $k \to \infty$

$$\begin{aligned}\lim_{k\to\infty} \Omega(x_{2k+1}, z) &= \max\Big\{\omega_1(z, z), \omega_1(z, Sz), \omega_1(z, z), \omega_2(z, Sz), \omega_2(z, z)\Big\}\\ &= \omega_1(z, Sz).\end{aligned}$$

Now,

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(y_{2k+1}, Sz)\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+1}, Sz\}} \lambda(t)dt\\ &\leq \int_0^{\upsilon\{\Omega(x_{2k+1}, z)\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k+1}, z)\}} \lambda(t)dt.\end{aligned}$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \int_0^{\gamma_1\{\omega_1(y_{2k+1}, Sz)\}} \lambda(t)dt \leq \lim_{k\to\infty} \int_0^{\upsilon\{\Omega(x_{2k+1}, z)\}} \lambda(t)dt - \liminf_{k\to\infty} \int_0^{\phi\{\Omega(x_{2k+1}, z)\}} \lambda(t)dt$$

$$\Rightarrow \int_0^{\gamma_1\{\omega_1(z, Sz)\}} \lambda(t)dt < \int_0^{\gamma_1\{\omega_1(z, Sz)\}} \lambda(t)dt$$

which is a contradiction. Hence, $Sz = z$.

Therefore z is a common fixed point of P, Q, R and S.

To prove the uniqueness, we assume that $w(\neq z)$ is also a fixed point of P, Q, R and S. Then $Pz = Qz = Rz = Sz = z$ and $Pw = Qw = Rw = Sw = w$.

Now,

$$\begin{aligned}\Omega(z, w) &= \max\Big\{\omega_1(Pz, Rz), \omega_1(Qw, Sw), \omega_1(Qw, Rz), \omega_2(Pz, Sw), \omega_2(Pz, Qw)\Big\}\\ &= \max\Big\{\omega_1(z, z), \omega_1(w, w), \omega_1(w, z), \omega_2(z, w), \omega_2(z, w)\Big\}\\ &= \omega_1(w, z).\end{aligned}$$

From (1) we have,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(z,w)\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rz,Sw\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(z,w)\}} \lambda(t)dt - \int_0^{\phi\{\Omega(z,w)\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\omega(z,w)\}} \lambda(t)dt - \int_0^{\phi\{\omega(z,w)\}} \lambda(t)dt \\
&< \int_0^{\gamma\{\omega(z,w)\}} \lambda(t)dt
\end{aligned}$$

which contradicts our hypothesis. Hence, $z = w$. This completes the proof.

4. Conclusion

This study contributes to the ongoing development of fixed point theory by establishing new results for four mappings that satisfy integral-type contraction conditions within the framework of convex modular metric spaces. These findings not only extend classical results like the Banach Contraction Principle but also build upon recent advancements in modular metric and function spaces. Given the rich structure and flexibility of modular frameworks, there remains substantial potential for further exploration in this direction. Future work may focus on broader classes of contractive conditions, additional structural generalizations, and and examine practical applications across diverse mathematical and applied contexts within the modular framework.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Jayanta Das]: Thought and designed the research/problem; Contributed to research method and evaluation of data; Collected the data; Wrote the manuscript. (60%).

Author [Ashoke Das]: Collected the data; Contributed to completing the research and solving the problem. (40%).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

$\xi(\mathrm{Ric})$-Vector Fields on Sequential Warped Product Manifolds

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Abstract: In this paper, we study $\xi(\mathrm{Ric})$-vector fields on sequential warped products. Assuming that a vector field is a $\xi(\mathrm{Ric})$-vector field on the sequential product, we investigate the necessary conditions under which its components are also $\xi(\mathrm{Ric})$-vector fields on the factor manifolds. We also examine the conditions in which a vector field on a sequential warped product could be a $\xi(\mathrm{Ric})$-vector field. Furthermore, we examine $\xi(\mathrm{Ric})$-vector fields on a sequential warped space-times.

Keywords: Sequential warped product manifold, $\xi(\mathrm{Ric})$- vector field, generalized Robertson-Walker space-times, standard static space-times.

1. Introduction

In the literature, various special types of smooth vector fields have been studied, such as Killing, concircular, and conformal vector fields. Each of these types carries important geometric properties; for instance, concircular vector fields preserve concircularity of geodesics, while torse-forming vector fields satisfy a specific covariant derivative structure involving the metric tensor. These special vector fields are deeply connected to the underlying geometry of the manifold (see [2, 3, 6, 13, 14]). The concept of $\xi(\mathrm{Ric})$-vector fields, first introduced by Hinterleitner and Kiosak in [9], Tgeneralizes the notion of torse-forming and concircular vector fields by incorporating the Ricci tensor into the defining condition. These vector fields have recently attracted significant attention due to their rich geometric behavior and potential relevance in physical models (see [5, 7, 8, 10–12]).

Warped product manifolds, originally introduced by O'Neill and Bishop in [1], were developed as a method for constructing manifolds with prescribed curvature properties-particularly those of negative curvature. Beyond pure geometry, warped products and their generalizations (such as doubly and multiply warped products) have found substantial applications in general relativity, where they are used to model spacetime geometries under various physical assumptions

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In this paper, we focus on $\xi(\mathrm{Ric})$-vector fields on sequential warped product manifolds which are a recently introduced generalization of the warped product construction. These manifolds allow a richer and more flexible geometric framework, suitable for both theoretical and applied investigations. Our main goal is to obtain necessary and sufficient conditions for the existence of $\xi(\mathrm{Ric})$-vector fields on sequential warped product manifolds and their factor spaces. Furthermore, we explore the structure of these vector fields in the context of sequential warped product spacetimes, highlighting their potential geometric and physical significance.

2. Preliminaries

Consider the Riemannian manifolds (M_i, g_i) for $1 \leq i \leq 3$ along with the smooth functions $f : M_1 \longrightarrow \mathbb{R}^+$ and $h : M_1 \times M_2 \longrightarrow \mathbb{R}^+$. The sequential warped product manifold M is defined as the manifold $M = (M_1 \times_f M_2) \times_h M_3$ endowed with the metric tensor $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ [4].

From now on, (M, g) will be regarded as sequential warped product where $M^n = (M_1^{n_1} \times_f M_2^{n_2}) \times_h M_3^{n_3}$ with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$.

The following lemmas are needed to prove our results.

Lemma 2.1 *[4] Assume that (M, g) be a sequential warped product and $Y_i, Z_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 3$. Then*

i. $\nabla_{Y_1} Z_1 = \nabla^1_{Y_1} Z_1$,

ii. $\nabla_{Y_1} Y_2 = \nabla_{Y_2} Y_1 = Y_1(lnf) Y_2$,

iii. $\nabla_{Y_2} Z_2 = \nabla^2_{Y_2} Z_2 - f g_2(Y_2, Z_2) \nabla^1 f$,

iv. $\nabla_{Y_3} Y_1 = \nabla_{Y_1} Y_3 = Y_1(\ln h) Y_3$,

v. $\nabla_{Y_2} Y_3 = \nabla_{Y_3} Y_2 = Y_2(\ln h) Y_3$,

vi. $\nabla_{Y_3} Z_3 = \nabla^3_{Y_3} Z_3 - h g_3(Y_3, Z_3)\, \nabla h$.

Lemma 2.2 *[4] Assume that (M, g) be a sequential warped product and $Y_i, Z_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 3$. The following conditions are satisfied:*

i. $\mathrm{Ric}(Y_1, Z_1) = \mathrm{Ric}^1(Y_1, Z_1) - \frac{n_2}{f} \mathrm{Hess}^1 f(Y_1, Z_1) - \frac{n_3}{h} \overline{\mathrm{Hess}} h(Y_1, Z_1)$,

ii. $\mathrm{Ric}(Y_2, Z_2) = \mathrm{Ric}^2(Y_2, Z_2) - f^\sharp g_2\,(Y_2, Z_2) - \frac{n_3}{h} \overline{\mathrm{Hess}} h(Y_2, Z_2)$,

iii. $\mathrm{Ric}(Y_3, Z_3) = \mathrm{Ric}^3(Y_3, Z_3) - h^\sharp g_3\,(Y_3, Z_3)$, *iv.* $\mathrm{Ric}(Y_i, Z_j) = 0$ *when* $i \neq j$, *where* $f^\sharp = \left(f\Delta^1 f + (n_2 - 1)\left\|\nabla^1 f\right\|^2\right)$ *and* $h^\sharp = \left(h\Delta h + (n_3 - 1)\left\|\nabla h\right\|^2\right)$.

3. Main Results

In this section, we investigate the properties of $\xi(\mathrm{Ric})$-vector fields on sequential warped product manifolds.

Firstly, we state the following tensor:

$$D(X,Y) = g(\nabla_X \xi, Y) - \mu \mathrm{Ric}(X,Y). \tag{1}$$

When $D \equiv 0$, the vector field ξ is said to be a $\xi(\mathrm{Ric})$-vector field with scalar μ.

Theorem 3.1 *Assume that* (M,g) *be sequential warped product. If* $\xi = \xi_1 + \xi_2 + \xi_3$ *is a* $\xi(\mathrm{Ric})$*-vector field on* M*, in this case one of the following cases is true:*

(i) f *and* h *are constants and hence* ξ_i, $\xi_i(\mathrm{Ric})$*-vector field on* M_i, $i = 1,2,3$.

(ii) $\xi_2 = 0$ *and* $\xi_3 = 0$ *and therefore* M_2 *and* M_3 *are Einstein manifolds if* $\overline{\mathrm{Hess}}h = \varphi g$ *and* ξ_1, $\xi_1(\mathrm{Ric})$*-vector field on* M_1 *if* $\frac{n_2\mu}{f}\mathrm{Hess}f(X_1,Y_1) + \frac{n_3\mu}{h}\overline{\mathrm{Hess}}h(X_1,Y_1) = 0$.

Proof Assume that (M,g) is a sequential warped product manifold and $\xi = \xi_1 + \xi_2 + \xi_3$ is a vector field on M. The vector field ξ is a $\xi(\mathrm{Ric})$ vector field with scalar μ if and only if $D = 0$. Hence from the (1), we get

$$g(\nabla_X \xi, Y) - \mu \mathrm{Ric}(X,Y) = 0$$

for all $X, Y \in \chi(M)$.

Let $X = X_1$ and $Y = Y_1$. Using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
0 = D(X_1,Y_1) &= g(\nabla_{X_1}\xi, Y_1) - \mu \mathrm{Ric}(X_1,Y_1) \\
&= g(\nabla_{X_1}\xi_1 + \nabla_{X_1}\xi_2 + \nabla_{X_1}\xi_3, Y_1) \\
&\quad -\mu(\mathrm{Ric}^1(X_1,Y_1) - \frac{n_2}{f}\mathrm{Hess}^1 f(Y_1,Z_1) - \frac{n_3}{h}\overline{\mathrm{Hess}}h(Y_1,Z_1)) \\
&= g_1(\nabla^1_{X_1}\xi_1, Y_1) - \mu\mathrm{Ric}^1(X_1,Y_1) + \frac{\mu n_2}{f}\mathrm{Hess}^1 f(X_1,Y_1) + \frac{\mu n_3}{h}\overline{\mathrm{Hess}}h(X_1,Y_1)).
\end{aligned}$$

Here, ξ_1 is a $\xi_1(\mathrm{Ric})$-vector field on M_1 if one of the following conditions holds:

(a) f and h are constants, or

(b) $\frac{\mu n_2}{f}\mathrm{Hess}^1 f(Y_1,Z_1) + \frac{\mu n_3}{h}\overline{\mathrm{Hess}}h(Y_1,Z_1) = 0$

Now, let $X = X_2$ and $Y = Y_2$. Then

$$\begin{aligned} 0 = D(X_2, Y_2) &= g(\nabla_{X_2}\xi, Y_2) - \mu\mathrm{Ric}(X_2, Y_2) \\ &= f\xi_1(f)g_2(X_2, Y_2) + f^2 g_2(\nabla^2_{X_2}\xi_2, Y_2) \\ &\quad -\mu(\mathrm{Ric}^2(X_2, Y_2) + f^\sharp g_2(X_2, Y_2) - \frac{n_3}{h}\overline{\mathrm{Hess}}h(X_2, Y_2)). \end{aligned}$$

Here, if f and h are constants, then ξ_2 is a $\xi_2(\mathrm{Ric})$-vector field on M_2 or $\xi_2 = 0$ which implies that M_2 is an Einstein manifold.

For $X = X_3$ and $Y = Y_3$, by using the same pattern, we have h is constant and then ξ_3 is a $\xi_3(\mathrm{Ric})$-vector field on M_3 or $\xi_3 = 0$ and M_3 is an Einstein manifold.

For $1 \leq i, j \leq 3$ and $i \neq j$, when $X = X_i$ and $Y = Y_j$, f and h are constants or $\xi_2 = \xi_3 = 0$.

Hence the proof is completed.

□

The following theorem provides the required criterion for the vector field ξ to be a $\xi(\mathrm{Ric})$ vector field.

Theorem 3.2 *Assume that (M, g) be a sequential warped product. If one of the conditions below is satisfied:*

(i) f and h are constants and ξ_i, $\xi_i(\mathrm{Ric})$-vector field on M_i, $i = 1, 2, 3$ with scalars μ, $\frac{\mu}{f^2}$ and $\frac{\mu}{h^2}$ respectively, or

(ii) $\xi_2 = 0$, $\xi_3 = 0$ and

$$\mu\mathrm{Ric}^1(X_1, Y_1) = g_1(\nabla^1_{X_1}\xi_1, Y_1) + \frac{\mu n_2}{f}\mathrm{Hess}f(X_1, Y_1) + \frac{\mu n_3}{h}\overline{\mathrm{Hess}}h(X_1, Y_1),$$

$$\mu\mathrm{Ric}^2(X_2, Y_2) = [f\xi_1(f) + \mu f^\sharp]g_2(X_2, Y_2) + \frac{\mu n_3}{h}\overline{\mathrm{Hess}}h(X_2, Y_2),$$

$$\mu\mathrm{Ric}^3(X_3, Y_3) = [h\xi_1(h) + \mu h^\sharp]g_3(X_3, Y_3),$$

then $\xi = \xi_1 + \xi_2 + \xi_3$ is a $\xi(\mathrm{Ric})$-vector field on M.

Proof Since the tensor D is linear, one might easily conclude the following equalities:

- $D(X_1, Y_1) = g_1(\nabla^1_{X_1}\xi_1, Y_1) - \mu\mathrm{Ric}^1(X_1, Y_1) + \frac{\mu n_2}{f}\mathrm{Hess}^1 f(X_1, Y_1) + \frac{\mu n_3}{h}\overline{\mathrm{Hess}}h(X_1, Y_1)$,
- $D(X_2, Y_2) = f^2 g_2(\nabla^2_{X_2}\xi_2, Y_2) + f\xi_1(f)g_2(X_2, Y_2) - \mu\mathrm{Ric}^2(X_2, Y_2)$
  $+ \mu f^\sharp g_2(X_2, Y_2) - \frac{\mu n_3}{h}\overline{\mathrm{Hess}}h(X_2, Y_2)$,

- $D(X_3,Y_3) = h^2 g_3(\nabla^3_{X_3}\xi_3, Y_3) + h(\xi_1+\xi_2)(h) g_3(X_3,Y_3) - \mu \mathrm{Ric}^3(X_3,Y_3)$
  $+ \mu h^\sharp g_3(X_3,Y_3)$,
- $D(X_1,Y_2) = f X_1(f) g_2(\xi_2,Y_2)$,
- $D(X_1,Y_3) = h X_1(h) g_3(\xi_3,Y_3)$,
- $D(X_2,Y_1) = -f g_2(X_2,\xi_2) g_1(\nabla f, Y_1)$,
- $D(X_2,Y_3) = h X_2(h) g_3(\xi_3,Y_3)$,
- $D(X_3,Y_1) = -h g_3(X_3,\xi_3) g_1((\nabla h)^T, Y_1)$,
- $D(X_3,Y_2) = -h g_3(X_3,\xi_3) f^2 g_2((\nabla h)^\perp, Y_2)$.

If the above equalities vanish, then the tensor D vanishes. Assume that the functions f and h are constants. Then we get

$D(X_1,Y_1) = g_1(\nabla^1_{X_1}\xi_1, Y_1) - \mu \mathrm{Ric}^1(X_1,Y_1)$,

$D(X_2,Y_2) = f^2 g_2(\nabla^2_{X_2}\xi_2, Y_2) - \mu \mathrm{Ric}^2(X_2,Y_2)$,

$D(X_3,Y_3) = h^2 g_3(\nabla^3_{X_3}\xi_3, Y_3) - \mu \mathrm{Ric}^3(X_3,Y_3)$,

$D(X_i,Y_j) = 0,\ 1 \le i,j \le 3,\ i \ne j.$

Hence if ξ_i is $\xi_i(\mathrm{Ric})$-vector fields on M_i for any $i = 1,2,3$ with scalars μ, $\frac{\mu}{f^2}$ and $\frac{\mu}{h^2}$ respectively, then ξ is $\xi(\mathrm{Ric})$-vector field on M since all components of D would be zero.

Now, assume that $\xi_2 = \xi_3 = 0$. Then we have

$$\begin{aligned} D(X_1,Y_1) &= g_1(\nabla^1_{X_1}\xi_1, Y_1) - \mu \mathrm{Ric}^1(X_1,Y_1) \\ &\quad + \frac{\mu n_2}{f}\mathrm{Hess}^1 f(X_1,Y_1) + \frac{\mu n_3}{h}\overline{\mathrm{Hess}} h(X_1,Y_1), \end{aligned}$$

$$\begin{aligned} D(X_2,Y_2) &= f\xi_1(f) g_2(X_2,Y_2) - \mu \mathrm{Ric}^2(X_2,Y_2) \\ &\quad + \mu f^\sharp g_2(X_2,Y_2) - \frac{\mu n_3}{h}\overline{\mathrm{Hess}} h(X_2,Y_2), \end{aligned}$$

$$\begin{aligned} D(X_3,Y_3) &= h\xi_1(h) g_3(X_3,Y_3) - \mu \mathrm{Ric}^3(X_3,Y_3) \\ &\quad + \mu h^\sharp g_3(X_3,Y_3), \end{aligned}$$

$$D(X_i,Y_j) = 0, 1 \le i,j \le 3, i \ne j.$$

Here, if we use the conditions in the hypothesis, then we have $D = 0$ which implies that ξ is a $\xi(\mathrm{Ric})$-vector field on M. □

4. Applications

In this section, we will characterize the $\xi(\mathrm{Ric})$-vector fields on sequential standard static space-times and sequential generalized Robertson-Walker space-times.

Consider (M_i, g_i) as Riemannian manifolds for $1 \leq i \leq 2$, along with $f : M_1 \longrightarrow \mathbb{R}^+$, $h : M_1 \times M_2 \longrightarrow \mathbb{R}^+$. The (n_1+n_2+1)-dimensional sequential standard static space-time (sequential SSS-T) $\overline{M}$ is defined as the product manifold $\overline{M} = (M_1 \times_f M_2) \times_h I$ equipped with the metric tensor $\overline{g} = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$ [4]. Here I represents a connected, open subinterval of $\mathbb{R}$ and dt^2 denotes the standard Euclidean metric tensor on I.

Proposition 4.1 *[4] Assume that $(\overline{M}, \overline{g})$ be a sequential SSS-T and $Y_i, Z_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 2$. In this case, the following conditions hold:*

i. $\overline{\nabla}_{Y_1} Z_1 = \nabla^1_{Y_1} Z_1$,

ii. $\overline{\nabla}_{Y_1} Y_2 = \overline{\nabla}_{Y_2} Y_1 = Y_1(\ln f) Y_2$,

iii. $\overline{\nabla}_{Y_2} Z_2 = \nabla^2_{Y_2} Z_2 - f g_2(Y_2, Z_2) \nabla^1 f$,

iv. $\overline{\nabla}_{Y_i} \partial_t = \overline{\nabla}_{\partial_t} Y_i = Y_i(\ln h) \partial_t$, $i = 1, 2$,

v. $\overline{\nabla}_{\partial_t} \partial_t = h \nabla h$.

Proposition 4.2 *[4] Assume that $(\overline{M}, \overline{g})$ be a sequential SSS-T and $Y_i, Z_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 2$. In this case, the following conditions are satisfied:*

i. $\overline{\mathrm{Ric}}(Y_1, Z_1) = \mathrm{Ric}^1(Y_1, Z_1) - \frac{n_2}{f} \mathrm{Hess}^1 f(Y_1, Z_1) - \frac{1}{h} \overline{\mathrm{Hess}} h(Y_1, Z_1)$,

ii. $\overline{\mathrm{Ric}}(Y_2, Z_2) = \mathrm{Ric}^2(Y_2, Z_2) - f^{\sharp} g_2(Y_2, Z_2) - \frac{1}{h} \overline{\mathrm{Hess}} h(Y_2, Z_2)$,

iii. $\overline{\mathrm{Ric}}(\partial_t, \partial_t) = h \Delta h$,

iv. $\overline{\mathrm{Ric}}(Y_i, Z_j) = 0$ *when* $i \neq j$, *where* $f^{\sharp} = \left(f \Delta^1 f + (n_2 - 1) \left\| \nabla^1 f \right\|^2 \right)$.

Remark 4.3 *A vector field of the form $\omega \partial_t$ on $(I, -dt^2)$ is $\xi_1(\mathrm{Ric})$-vector field if and only if $\dot{\omega} = 0$ on I.*

Hence we could state the following corollary:

Corollary 4.4 *Assume that $(\overline{M}, \overline{g})$ be a sequential SSS-T and $X_i, Y_i \in \chi(M_i)$ for $1 \leq i \leq 2$. In this case, the following conditions hold:*

i. $D(X_1,Y_1)=g_1(\nabla^1_{X_1}\xi_1,Y_1)-\mu\mathrm{Ric}^1(X_1,Y_1)+\frac{\mu n_2}{f}\mathrm{Hess}^1 f(X_1,Y_1)$
$+\frac{\mu}{h}\overline{\mathrm{Hess}}h(X_1,Y_1)$,

ii. $D(X_2,Y_2)=f^2g_2(\nabla^2_{X_2}\xi_2,Y_2)-\mu\mathrm{Ric}^2(X_2,Y_2)+f\xi_1(f)g_2(X_2,Y_2)$
$+\mu f^{\sharp}g_2(X_2,Y_2)-\frac{\mu}{h}\overline{\mathrm{Hess}}h(X_2,Y_2)$,

iii. $D(\partial_t,\partial_t)=-h(\xi_1+\xi_2)(h)-\dot{\omega}h^2-\mu h\Delta h$,

iv. $D(X_1,Y_2)=fX_1(f)g_2(\xi_2,Y_2)$,

v. $D(X_1,\partial_t)=-\omega hX_1(h)$,

vi. $D(X_2,Y_1)=-fY_1(f)g_2(X_2,\xi_2)$,

vii. $D(X_2,\partial_t)=-\omega hX_2(h)$,

viii. $D(\partial_t,Y_1)=\omega hY_1(h)$,

ix. $D(\partial_t,Y_2)=\omega f^2hY_2(h)$.

Now, based on Theorem 3.1, we can obtain the following result:

Theorem 4.5 *Assume that* $(\overline{M},\overline{g})$ *be a sequential SSS-T. If* $\overline{\xi}=\xi_1+\xi_2+\omega\partial_t$ *is a* $\overline{\xi}(\overline{\mathrm{Ric}})$*-vector field on* $\overline{M}$*, then one of the conditions below holds:*

(i) $w=c$ *for some* $c\in\mathbb{R}$*, where* f *and* h *are constant and* ξ_i*,* $\xi_i(\mathrm{Ric})$*-vector field on* M_i*,* $i=1,2$ *with factor* μ *and* $\frac{\mu}{f^2}$ *respectively.*

(ii) $\omega=0$ *and* $\xi_2=0$ *where*

$\xi_1(h)=-\mu\Delta h$,

$\mu\mathrm{Ric}^1(X_1,Y_1)=g_1(\nabla^1_{X_1}\xi_1,Y_1)+\frac{\mu n_2}{f}\mathrm{Hess}f(X_1,Y_1)+\frac{\mu}{h}\overline{\mathrm{Hess}}h(X_1,Y_1)$,

$\mu\mathrm{Ric}^2(X_2,Y_2)=[f\xi_1(f)+\mu f^{\sharp}]g_2(X_2,Y_2)+\frac{\mu}{h}\overline{\mathrm{Hess}}h(X_2,Y_2)$.

Proof Assume that $(\overline{M},\overline{g})$ be a sequential warped product manifold and $\overline{\xi}=\xi_1+\xi_2+\omega\partial_t$ a $\overline{\xi}(\overline{\mathrm{Ric}})$- vector field on $\overline{M}$. Then $D(\overline{X},\overline{Y})=0$ for all $\overline{X},\overline{Y}\in\chi(\overline{M})$. Since the tensor D is linear in each component, by Corollary 4.4, the proof is clear. □

We now delve into the structure of sequential generalized Robertson-Walker (GRW) space-times, beginning with a brief review of their definition.

Consider (M_i,g_i) to be Riemannian manifolds for $2\le i\le 3$ and let $f:I\longrightarrow\mathbb{R}^+$, $h:I\times M_2\longrightarrow\mathbb{R}^+$ be smooth functions. The sequential GRW space-time $\overline{M}$ of dimension (n_2+n_3+1)-

is constructed as the triple product manifold $\overline{M} = I \times_f M_2 \times_h M_3$ equipped with the metric tensor $\overline{g} = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3$. Here I represents a connected, open subinterval of $\mathbb{R}$ and dt^2 denotes the standard Euclidean metric tensor on I [4].

Proposition 4.6 *[4] Assume that* $(\overline{M}, \overline{g})$ *be a sequential GRW space-time and* $Y_i, Z_i \in \mathfrak{X}(M_i)$ *for* $2 \leq i \leq 3$. *In this case, the following conditions are satisfied:*

i. $\overline{\nabla}_{\partial_t} \partial_t = 0$,

ii. $\overline{\nabla}_{\partial_t} Y_i = \nabla_{Y_i} \partial_t = \frac{\dot{f}}{f} Y_i$, $i = 2, 3$,

iii. $\overline{\nabla}_{Y_2} Z_2 = \nabla^2_{Y_2} Z_2 - f \dot{f} g_2(Y_2, Z_2) \partial_t$,

iv. $\overline{\nabla}_{Y_2} Y_3 = \overline{\nabla}_{Y_3} Y_2 = Y_2(\ln h) Y_3$,

v. $\overline{\nabla}_{Y_3} Z_3 = \overline{\nabla}^3_{Y_3} Z_3 - h g_3(Y_3, Z_3) \nabla h$.

Proposition 4.7 *[4] Assume that* $(\overline{M}, \overline{g})$ *be a sequential GRW space-time and* $Y_i, Z_i \in \mathfrak{X}(M_i)$ *for* $2 \leq i \leq 3$. *Then*

i. $\overline{\mathrm{Ric}}(\partial_t, \partial_t) = \frac{n_2}{f} \ddot{f} + \frac{n_3}{h} \frac{\partial^2 h}{\partial t^2}$,

ii. $\overline{\mathrm{Ric}}(Y_2, Z_2) = \mathrm{Ric}^2(Y_2, Z_2) - f^{\diamond} g_2(Y_2, Z_2) - \frac{n_3}{h} \overline{\mathrm{Hess}} h(Y_2, Z_2)$,

iii. $\overline{\mathrm{Ric}}(Y_3, Z_3) = \mathrm{Ric}^3(Y_3, Z_3) - h^{\sharp} g_3(Y_3, Z_3)$,

iv. $\overline{\mathrm{Ric}}(Y_i, Z_j) = 0$ *when* $i \neq j$, *where* $f^{\diamond} = -f \ddot{f} + (n_2 - 1) \dot{f}^2$ *and* $h^{\sharp} = h \Delta h + (n_3 - 1) \left\| \nabla h \right\|^2$.

Corollary 4.8 *Let* $(\overline{M}, \overline{g})$ *be a sequential GRW space-time and* $X_i, Y_i \in \chi(M_i)$ *for* $1 \leq i \leq 2$. *Then the following conditions hold:*

i. $D(\partial_t, \partial_t) = \dot{\omega} - \dfrac{\mu n_2}{f} \ddot{f} + \dfrac{\mu n_3}{h} \dfrac{\partial^2 h}{\partial t^2}$,

ii. $D(X_2, Y_2) = f^2 g_2(\nabla^2_{X_2} \xi_2, Y_2) - \mu \mathrm{Ric}^2(X_2, Y_2) + \omega f \dot{f} g_2(X_2, Y_2)$
$+ \mu f^{\diamond} g_2(X_2, Y_2) - \frac{\mu n_3}{h} \overline{\mathrm{Hess}} h(X_2, Y_2)$,

iii. $D(X_3, Y_3) = h^2 g_3(\nabla^3_{X_3} \xi_3, Y_3) - \mu \mathrm{Ric}^3(X_3, Y_3) + h \xi_2(h) g_3(X_3, Y_3)$
$+ \omega h^2 \frac{\dot{f}}{f} g_3(X_3, Y_{33}) + \mu f^{\sharp} g_3(X_3, Y_3)$,

iv. $D(\partial_t, Y_2) = f \dot{f} g_2(\xi_2, Y_2)$, *v.* $D(\partial_t, Y_3) = h^2 \frac{\dot{f}}{f} g_3(\xi_3, Y_3)$,

vi. $D(X_2, \partial_t) = -X_2(\omega) + f\dot{f} g_2(X_2, \xi_2)$,

vii. $D(X_2, Y_3) = hX_2(h) g_3(\xi_3, Y_3)$,

viii. $D(X_3, \partial_t) = -X_3(\omega) + h\frac{\partial h}{\partial t} g_3(X_3, \xi_3)$,

ix. $D(X_3, Y_2) = -hf^2 Y_2(h) g_3(\xi_3, Y_3)$.

As an application of Theorem 3.1, we give the following theorem.

Theorem 4.9 *Let* $(\overline{M}, \overline{g})$ *be a sequential GRW space-time. If* $\overline{\xi} = \omega\partial_t + \xi_2 + \xi_3$ *is a* $\overline{\xi}(\overline{\text{Ric}})$*-vector field on* $\overline{M}$*, then one of the following conditions holds:*

(i) f *and* h *are constants and hence* ω *is constant and*

$$\mu \text{Ric}^2(X_2, Y_2) = f^2 g_2(\nabla^2_{X_2}\xi_2, Y_2) + \mu f^{\diamond} g_2(X_2, Y_2) + \frac{\mu n_3}{h}\overline{\text{Hess}}h(X_2, Y_2),$$

$$\mu \text{Ric}^3(X_3, Y_3) = h^2 g_3(\nabla^2_{X_3}\xi_3, Y_3) + \mu h^{\sharp} g_3(X_3, Y_3) + h\xi_2(h) g_3(X_3, Y_3),$$ *or*

(ii) $\xi_2 = \xi_3 = 0$ *where*

$$\dot{\omega} = \frac{\mu n_2}{f}\ddot{f} + \frac{\mu n_3}{h}\frac{\partial^2 h}{\partial t^2},$$

M_2 *is Einstein if* $\overline{Hess}h = \psi\overline{g}$,

M_3 *is Einstein.*

Proof Let $(\overline{M}, \overline{g})$ be a sequential warped product manifold and $\overline{\xi} = \omega\partial_t + \xi_2 + \xi_3$ a $\overline{\xi}(\overline{\text{Ric}})$-vector field on $\overline{M}$. Then $D(\overline{X}, \overline{Y}) = 0$ for all $\overline{X}, \overline{Y} \in \chi(\overline{M})$. Since the tensor D is linear in each component, by Corollary 4.8, the proof is clear.

□

5. Conclusion

In this study, we investigated $\xi(\text{Ric})$-vector fields on sequential warped product manifolds and their associated space-times. We derived necessary and sufficient conditions for the existence of such vector fields on the total manifold and analyzed when their components satisfy the $\xi(\text{Ric})$-condition on the factor spaces. These findings clarify how the geometry of sequential warping interacts with curvature-based vector fields, thereby extending prior results on torse-forming and concircular vector fields in a Ricci-related setting.

To the best of our knowledge, these directions have not yet been extensively studied in the context of sequential warped products and may provide a fruitful avenue for further research. In particular, exploring the behavior of $\xi(\mathrm{Ric})$-vector fields under broader curvature constraints or in physically motivated space-time models could deepen the understanding of their geometric and physical significance.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Research Article

Some Results on Eigenvalue Intervals for Positive Solutions of the 3^{rd}-order Impulsive Boundary Value Problem's Iterative System with *p*-Laplacian Operator

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Abstract: The objective of this paper is to determine the eigenvalue intervals for which positive solutions are guaranteed for the iterative system of the 3^{rd}-order impulsive boundary value problem. The existence of solutions is established by applying the well-known Guo-Krasnosel'skii fixed point theorem. An illustrative example is provided to demonstrate the applicability of the theoretical results.

Keywords: Fixed point, impulsive BVP, iterative systems, eigenvalue interval.

1. Introduction

Boundary value problems (BVPs) serve as fundamental tools for modeling complex phenomena in physics, biology, and engineering. A specific subclass, impulsive boundary value problems (IBVPs), offers a robust framework for analyzing systems subject to sudden, discontinuous changes. Foundational contributions by Lakshmikantham [12], Bainov [5] and Simeonov [4] established the core theory of impulsive differential equations, extending to higher-order systems. Subsequently, advanced mathematical techniques-such as fixed point theorems and variational methods-have been employed to address more complex formulations [3, 12]. In addition to their theoretical strength, IBVPs are widely applicable across various scientific and engineering domains. For instance, in mechanical engineering, they are used to analyze structural vibrations under sudden loads, such as during seismic events affecting bridges [17]. In biomedical modeling, they support the optimization of oscillatory behavior in drug delivery systems [30]. In control theory, they aid in investigating the controllability of impulsive dynamic systems [1]. Furthermore, they find meaningful applications in population dynamics [26] and financial market modeling [27]. Moreover, the study by Zhang et al. [33] in the references demonstrates significant potential for applications in autonomous robot

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swarms, drone fleets, distributed sensor networks, and intelligent transportation systems.

Determining eigenvalue intervals is crucial for analyzing the existence and uniqueness of solutions to boundary value problems, as eigenvalues characterize the spectral properties of differential operators and thus dictate the system's stability and behavior [16]. These intervals reveal the parameter values (typically denoted by λ) at which nontrivial solutions arise, which is vital for various applications-including vibration analysis in mechanical systems to determine resonance frequencies in beams [34], estimation of energy levels in quantum mechanics via Schrödinger equations [8]. Recent studies have also emphasized their significance in fractional and nonlinear BVPs, thereby enhancing system design and optimization across disciplines [9].

Although extensive research has been conducted on third-order impulsive boundary value problems (BVPs) [6, 10, 11] and iterative systems [13, 18, 19, 22, 24], the literature still lacks focused investigations on eigenvalue intervals for third-order impulsive systems with iterative structures. Addressing this gap, the present study is, to the best of our knowledge, the first to explore eigenvalue intervals in this specific context. By employing the Guo–Krasnosel'skii fixed point theorem [12], we establish the existence of positive solutions and identify the corresponding eigenvalue intervals.

Compared to previous studies, this work significantly advances the field by unifying third-order dynamics, impulsive effects, iterative structures, and eigenvalue intervals into a single framework. In contrast to earlier studies-for instance, Zhang and Yao [31], who investigated solution multiplicity for second-order p-Laplacian impulsive equations using variational methods, or Oz and Karaca [19], who examined eigenvalue intervals for second-order m-point impulsive BVPs via fixed-point theory-our study focuses on third-order systems. Likewise, although Zhang and Ao [32] studied some third-order BVPs with eigenparameter-dependent boundary conditions on specific time scales, they did not consider iterative systems. Other works, such as those by Bi and Liu [6], Feliz and Rui [10], primarily addressed the existence of solutions, without investigating the role of eigenvalue intervals. In 2022, Bouabdallah et al. [7] studied eigenvalue boundary value problem with impulsive conditions, but the problem they considered is neither of third order nor does it involve an iterative system. Therefore, our study not only fills a significant gap in the existing literature but also provides a novel and comprehensive perspective for future research on complex impulsive systems. Based on the above-mentioned results and the importance of theoretical solutions to contribute to the application areas, in this work, we handle the following nonlinear

3^{rd}-order with p-Laplacian impulsive boundary value problem (IBVP)'s iterative system:

$$\begin{cases}(\phi_p(\kappa_i''(t)))' + \mu_i q_i(t)h_i(\kappa_{i+1}(t)) = 0, \ t \in I = [0,1], \ t \neq t_m, i \in \{1,2,3,\cdots,n\} \\ \kappa_{n+1}(t) = \kappa_1(t) \\ \triangle\kappa_i|_{t=t_m} = \mu_i I_{im}(\kappa_{i+1}(t_m)), \ m \in \{1,2,\cdots...,k\} \\ \triangle\kappa_i'|_{t=t_m} = -\mu_i J_{im}(\kappa_{i+1}(t_m)), \ m \in \{1,2,\cdots...,k\} \\ a_1\kappa_i(0) - a_2\kappa_i'(0) = 0 \\ a_3\kappa_i(1) + a_4\kappa_i'(1) = 0 \\ \kappa_i''(0) = 0, \end{cases} \tag{1}$$

where $t \neq t_m$, $m \in \{1,2,3,\cdots\ldots,k\}$ such that $0 < t_1 < t_2 < \cdots... < t_k < 1$. Furthermore, for $i \in \{1,2,3,\cdots...,n\}$, the functions $\triangle\kappa_i$ and $\triangle\kappa_i'$ at the point $t = t_m$ stand for the jump of $\kappa_i(t)$ and $\kappa_i'(t)$ at the point $t = t_m$, i.e.,

$$\triangle\kappa_i|_{t=t_m} = \kappa_i(t_m^+) - \kappa_i(t_m^-), \quad \triangle\kappa_i'|_{t=t_m} = \kappa_i'(t_m^+) - \kappa_i'(t_m^-),$$

where the values $\kappa_i(t_m^+)$, $\kappa_i'(t_m^+)$ state the right-hand limit of $\kappa_i(t)$ and $\kappa_i'(t)$ at the point $t = t_m$, $m \in \{1,2,3,\cdots,k\}$, and similarly $\kappa_i(t_m^-)$, $\kappa_i'(t_m^-)$ state left-hand limit of $\kappa_i(t)$ and $\kappa_i'(t)$ at the point $t = t_m$, $m \in \{1,2,3,\cdots,k\}$. In addition, the function $\phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$.

In this paper, we assume that the following conditions are given:

$(C1)$ a_1, a_2, a_3, a_4 are positive real constants.

$(C2)$ For $i = 1,\cdots,n$, h_i is a continuous function from the set $\mathbb{R}^+$ to $\mathbb{R}^+$.

$(C3)$ For $i \in \{1,2,3,\cdots,n\}$, $q_i \in C(I,\mathbb{R}^+)$ and on any closed subinterval of I, q_i does not vanish identically.

$(C4)$ For $i \in \{1,2,3,\cdots,n\}$, $I_{im} \in C(\mathbb{R},\mathbb{R}^+)$ and $J_{im} \in C(\mathbb{R},\mathbb{R}^+)$ are bounded functions and the inequality $[a_4 + a_3(1-t_m)]J_{im}(\eta) > a_3 I_{im}(\eta)$, $t < t_m$, $m \in \{1,2,3,\cdots,k\}$ is satisfied, where η be any nonnegative number.

$(C5)$ Each of the following expressions is a positive real number:

$$h_i^0 = \lim_{\kappa\to 0^+} \frac{h_i(\kappa)}{\kappa^{p-1}}, \quad I_{im}^0 = \lim_{\kappa\to 0^+} \frac{I_{im}(\kappa)}{\kappa}.$$

$$J_{im}^0 = \lim_{\kappa\to 0^+} \frac{J_{im}(\kappa)}{\kappa}, \text{ and } h_i^\infty = \lim_{\kappa\to\infty} \frac{h_i(\kappa)}{\kappa^{p-1}}, \ i \in \{1,2,3,...,n\},$$

where positive solutions of the nonlinear 3^{rd}-order IBVP (1)'s iterative system with p-Laplacian exist for μ_i, $i \in \{1, 2, 3, \cdots, n\}$.

The primary structure of this manuscript unfolds as follows. Section 2 introduces several definitions and fundamental lemmas, which serve as key tools for establishing our main result. Section 3 determines the eigenvalue intervals that ensure the existence of positive solutions in the 3^{rd}-order IBVP (1)'s iterative system with the p-Laplacian operator. Section 4 provides an illustrative example to demonstrate the applicability of the main results.

2. Preliminaries

In this section, we introduce fundamental definitions in Banach spaces and supply several supplementary lemmas that will be utilized later.

Define $I' = I \backslash \{t_1, t_2, \cdots, t_k\}$. The space $C(I)$ denotes the Banach space of all continuous mappings $\kappa : I \to \mathbb{R}$ equipped with the norm $\|\kappa\| = \sup_{t \in I} |\kappa(t)|$. The space $PC(I)$ consists of functions $\kappa : I \to \mathbb{R}$ such that $\kappa \in C(I')$, $\kappa(t_m^+)$ and $\kappa(t_m^-)$ exist and $\kappa(t_m^-) = \kappa(t_m)$ for $m \in \{1, 2, \cdots, k\}$. $PC(I)$ is also a Banach space with the norm $\|\kappa\|_{PC} = \sup_{t \in I} |\kappa(t)|$. Additionally, The space $C^2(I')$ consists of all twice continuously differentiable functions defined on an interval I' to $\mathbb{R}$.

Let $\mathbb{B} = PC(I) \cap C^2(I')$. A function $(\kappa_1, \cdots, \kappa_n) \in \mathbb{B}^n$ is considered a solution of the 3^{rd}-order IBVP (1)'s iterative system if it satisfies the conditions of the 3^{rd}-order IBVP (1)'s iterative system.

We first consider the case $i = 1$ in the 3^{rd}-order IBVP (1). Accordingly, the solution κ_1 of the 3^{rd}-order IBVP (2) is obtained. Once κ_1 is determined, we proceed to compute κ_n. Continuing in this manner, we successively determine $\kappa_{n-1}, \kappa_{n-2}, \ldots,$ until we reach κ_2. In this way, the complete solution $(\kappa_1, \ldots, \kappa_n)$ of the iterative system associated with the 3^{rd}-order IBVP (1) is constructed.

Assume that $x(t) \in C(I)$, then we deal with the following 3^{rd}-order IBVP:

$$\begin{cases} (\phi_p(\kappa_1''(t)))' + x(t) = 0, \quad t \in I = [0,1], \ t \neq t_m \\ \triangle \kappa_1|_{t=t_m} = \mu_1 I_{1m}(\kappa_2(t_m)), \ m \in \{1, 2, \cdots, k\} \\ \triangle \kappa_1'|_{t=t_m} = -\mu_1 J_{1m}(\kappa_2(t_m)), \ m \in \{1, 2, \cdots, k\} \\ a_1\kappa_1(0) - a_2\kappa_1'(0) = 0 \\ a_3\kappa_1(1) + a_4\kappa_1'(1) = 0 \\ \kappa_1''(0) = 0. \end{cases} \tag{2}$$

The following homogeneous equation's solutions are specified via τ and η.

$$\phi_p(\kappa_i''(t))' = 0,\ t \in I \tag{3}$$

under the initial conditions

$$\begin{cases} \tau(0) = a_2, \quad \tau'(0) = a_1 \\ \eta(1) = a_4, \quad \eta'(1) = -a_3. \end{cases} \tag{4}$$

Using the initial conditions (4), we can deduce from (3) for τ and η the following equations:

$$\tau(t) = a_2 + a_1 t,\ \textit{and}\ \eta(t) = a_4 + a_3(1-t). \tag{5}$$

Set

$$\delta := a_1 a_4 + a_1 a_3 + a_2 a_3. \tag{6}$$

Lemma 2.1 *Assume that the conditions (C1)-(C5) are satisfied. If κ_1, which is belonging to set $\mathbb{B}$, is a solution of the following equation*

$$\kappa_1(t) = \int_0^1 \mathcal{G}(t,s)\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds + \sum_{m=1}^{k} H_{1m}(t,t_m), \tag{7}$$

where

$$\mathcal{G}(t,s) = \frac{1}{\delta}\begin{cases} (a_2 + a_1 s)[a_4 + a_3(1-t)],\ s \le t \\ (a_2 + a_1 t)[a_4 + a_3(1-s)],\ t \le s \end{cases} \tag{8}$$

and

$$H_{1m}(t,t_m) = \frac{1}{\delta}\begin{cases} (a_2 + a_1 t)[-a_3\mu_1 I_{1m}(\kappa_2(t_m)) + (a_4 + a_3(1-t_m))\mu_1 J_{1m}(\kappa_2(t_m))], \quad t < t_m \\ (a_4 + a_3(1-t))[a_1\mu_1 I_{1m}(\kappa_2(t_m)) + (a_2 + a_1 t_m)\mu_1 J_{1m}(\kappa_2(t_m))], \quad t_m \le t, \end{cases} \tag{9}$$

then κ_1 is a solution of the 3^{rd}-order IBVP (2).

Proof Let κ_1 satisfy (7), then we will show that y is a solution of the IBVP (2). Because κ_1 satisfies (7), then we obtain

$$\kappa_1(t) = \int_0^1 \mathcal{G}(t,s)\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds + \sum_{m=1}^{k} H_{1m}(t,t_m),$$

i.e.,

$$\begin{aligned}\kappa_1(t) &= \frac{1}{\delta}\int_0^t (a_2+a_1 s)[a_4+a_3(1-t)]\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds\\ &+\frac{1}{\delta}\int_t^1 (a_2+a_1 t)[a_4+a_3(1-s)]\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds\\ &+\frac{1}{\delta}\sum_{0<t_m<t}(a_4+a_3(1-t))[a_1\mu_1 I_{1m}(\kappa_2(t_m))+(a_2+a_1t_m)\mu_1 J_{1m}(\kappa_2(t_m))]\\ &+\frac{1}{\delta}\sum_{t<t_m<1}(a_2+a_1 t)[-a_3\mu_1 I_{1m}(\kappa_2(t_m))+(a_4+a_3(1-t_m))\mu_1 J_{1m}(\kappa_2(t_m))],\end{aligned}$$

$$\begin{aligned}\kappa_1'(t) &= \frac{1}{\delta}\int_0^t (-a_3)(a_2+a_1 s)\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds\\ &+\frac{1}{\delta}\int_t^1 (a_1)[a_4+a_3(1-s)]\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds\\ &+\frac{1}{\delta}\sum_{0<t_m<t}(-a_3)[a_1\mu_1 I_{1m}(\kappa_2(t_m))+(a_2+a_1t_m)\mu_1 J_{1m}(\kappa_2(t_m))]\\ &+\frac{1}{\delta}\sum_{t<t_m<1}(a_1)[-a_3\mu_1 I_{1m}(\kappa_2(t_m))+(a_4+a_3(1-t_m))\mu_1 J_{1m}(\kappa_2(t_m))].\end{aligned}$$

Thus,

$$\begin{aligned}\kappa_1''(t) &= \frac{1}{\delta}\left[-a_3(a_2+a_1t)-a_1(a_4+a_3(1-t))\right]\phi_p^{-1}\left(\int_0^t x(\omega)d\omega\right)\\ &= -\phi_p^{-1}\left(\int_0^t x(\omega)d\omega\right)\end{aligned}$$

and

$$\kappa_1^{''}(0)=0.$$

So that

$$(\phi_p(\kappa_1''(t)))' = -x(t),$$

i.e.,

$$(\phi_p(\kappa_1''(t)))' + x(t) = 0.$$

Since

$$\kappa_1(0) = \frac{1}{\delta}\int_0^1 a_2[a_4 + a_3(1-s)]\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds$$

$$+\frac{1}{\delta}\sum_{m=1}^{k} a_2[-a_3\mu_1 I_{1m}(\kappa_2(t_m)) + (a_4 + a_3(1-t_m))\mu_1 J_{1m}(\kappa_2(t_m))]$$

and

$$\kappa_1'(0) = \frac{1}{\delta}\int_0^1 (a_1)[a_4 + a_3(1-s)]\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds$$

$$+\frac{1}{\delta}\sum_{m=1}^{k} a_1[-a_3\mu_1 I_{1m}(\kappa_2(t_m)) + (a_4 + a_3(1-t_m))\mu_1 J_{1m}(\kappa_2(t_m))],$$

we get

$$a_1\kappa_1(0) - a_2\kappa_1'(0) = 0.$$

Since

$$\kappa_1(1) = \frac{1}{\delta}\int_0^1 (a_2 + a_1 s)(a_3 + a_4)\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds$$

$$+\frac{1}{\delta}\sum_{m=1}^{k} (a_3 + a_4)[a_1\mu_1 I_{1m}(\kappa_2(t_m)) + (a_2 + a_1 t_m)\mu_1 J_{1m}(\kappa_2(t_m))]$$

and

$$\kappa_1'(1) = \frac{1}{\delta}\int_0^1 (-a_3)(a_2 + a_1 s)\phi_p^{-1}\left(\int_0^s x(\omega)d\omega\right)ds$$

$$+\frac{1}{\delta}\sum_{m=1}^{k} (-a_3)[a_1\mu_1 I_{1m}(\kappa_2(t_m)) + (a_2 + a_1 t_m)\mu_1 J_{1m}(\kappa_2(t_m))],$$

we have

$$a_3\kappa_1(1) + a_4\kappa'(1) = 0.$$

□

Lemma 2.2 *Let (C1)-(C5) hold. For* $\kappa_1 \in \mathbb{B}$ *with* $x(t) \geq 0$ *for* $t \in I$*, the solution* κ_1 *of the* 3^{rd}*-order IBVP (2) satisfies, for* $t \in I$, $\kappa_1(t) \geq 0$.

Proof Initially, for $t, s \in I \times I$, it is apparent from the description that Green's function $\mathcal{G}$ is positive. In addition, since the functions I_{1m} and J_{1m} are positive, we have the positivity of H_{1m}. Consequently, for $t \in I$, $\kappa_1(t)$ is positive. □

Lemma 2.3 *[13] Assume that (C1)-(C5) are satisfied. For $t \in I$, the 3^{rd}-order IBVP (2)'s solution, i.e., $\kappa_1 \in \mathbb{B}$ satisfy the inequality $\kappa_1'(t) \geq 0$.*

Lemma 2.4 *Suppose that the conditions (C1)-(C5) are satisfied. Therefore, for any $t, s \in I$, we get the following inequality*

$$\mathcal{G}(s,s) \geq \mathcal{G}(t,s) \geq 0, \tag{10}$$

where the function $\mathcal{G}(t,s)$ defined as in (8).

Proof The claimed inequality can be easily obtained from (8). □

Lemma 2.5 *[13] Assume that the conditions (C1)-(C5) are fulfilled. Let $\sigma \in (0, \frac{1}{2})$. Therefore, for any $t, s \in I$, we get*

$$\mathcal{G}(s,s) \leq \frac{1}{\gamma}\mathcal{G}(t,s), \tag{11}$$

where $\gamma := \min\left\{\frac{a_2 + a_1\sigma}{a_2 + a_1}, \frac{a_4 + a_3\sigma}{a_4 + a_3}\right\}$.

The set $\mathcal{P}$ defined as $\mathcal{P} = \{\kappa_1 \in PC(I) : \kappa_1(t)$ is nonnegative, nondecreasing and concave on $I\}$ is a cone of the set $PC(I)$.

Lemma 2.6 *Assume that the conditions (C1)-(C5) are satisfied and $\kappa_1(t) \in \mathcal{P}$. Then, the following inequality is satisfied,*

$$\min_{t \in [\sigma, 1-\sigma]} \kappa_1(t) \geq \sigma \|\kappa_1\|_{PC}, \tag{12}$$

where $\sigma \in (0, \frac{1}{2})$ *and* $\|\kappa_1\|_{PC} = \sup_{t \in I} |\kappa_1(t)|$.

Proof Since κ_1 is an element of $\mathcal{P}$, we can say that $\kappa_1(t)$ is concave on I. As a consequence of this, $\|\kappa_1\|_{PC} = \sup_{t \in I} |\kappa_1(t)| = \kappa_1(1)$ and $\min_{t \in [\sigma, 1-\sigma]} \kappa_1(t) = \kappa_1(\sigma)$. As κ_1's graph is concave downward on the interval I, we achieve

$$\frac{\kappa_1(1) - \kappa_1(0)}{1 - 0} \leq \frac{\kappa_1(\sigma) - \kappa_1(0)}{\sigma - 0},$$

i.e., $\kappa_1(\sigma) \geq \sigma\kappa_1(1) + (1-\sigma)\kappa_1(0)$. Thus, $\kappa_1(\sigma) \geq \sigma\kappa_1(1)$. □

If and only if

$$\kappa_1(t) = \int_0^1 \mathcal{G}(t,s_1)\,\phi_p^{-1}\bigg(\mu_1 \int_0^{s_1} q_1(\omega_1)\,h_1\bigg(\cdots h_{n-1}\bigg(\int_0^{s_n} \mathcal{G}(\omega_{n-1},s_n)\,\phi_p^{-1}\bigg(\mu_n \int_0^{s_n} q_n(\omega_n)\,h_n\big(\kappa_1(\omega_n)\big)\,d\omega_n\bigg)ds_n$$

$$+ \sum_{m=1}^{k} H_{nm}(\omega_{n-1},t_m)\bigg)\cdots\bigg)d\omega_1\bigg)ds_1 + \sum_{m=1}^{k} H_{1m}(t,t_m)$$

where for $i = 1,2,\cdots,n$

$$\kappa_i(t) = \int_0^1 \mathcal{G}(t,s)\phi_p^{-1}\bigg(\mu_i \int_0^s q_i(\omega)h_i(\kappa_{i+1}(\omega))d\omega\bigg)ds + \sum_{m=1}^{k} H_{im}(t,t_m),\ t \in I,$$

$$\kappa_{n+1}(t) = \kappa_1(t),$$

$$H_{im}(t,t_m) = \frac{1}{\delta}\begin{cases}(a_2+a_1t)[-a_3\mu_i I_{im}(\kappa_{i+1}(t_m)) + (a_4+a_3(1-t_m))\mu_i J_{im}(\kappa_{i+1}(t_m))], & t < t_m\\ (a_4+a_3(1-t))[a_1\mu_i I_{im}(\kappa_{i+1}(t_m)) + (a_2+a_1t_m)\mu_1 J_{im}(\kappa_{i+1}(t_m))], & t_m \leq t.\end{cases}$$

We state that an n-tuple $(\kappa_1(t),\kappa_2(t),\cdots,\kappa_n(t))$ is a solution of the 3^{rd}-order IBVP (1)'s iterative system. We will employ a fixed point theorem called Guo-Krasnosel'skii [12] to determine the eigenvalue intervals wherein the 3^{rd}-order IBVP (1)'s iterative system possesses at least one positive solution within a cone.

Theorem 2.7 *[12] Let X denote a Banach space and $P \subset X$ be a cone within X. Suppose Ω_1 and Ω_2 are two bounded open subsets of X such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Consider $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ as a completely continuous operator, satisfying either of the following conditions:*

i. For all $x \in P \cap \partial\Omega_1$, $\|Ax\| \leq \|x\|$, and for all $x \in P \cap \partial\Omega_2$, $\|Ax\| \geq \|x\|$,

ii. For all $x \in P \cap \partial\Omega_1$, $\|Ax\| \geq \|x\|$, and for all $x \in P \cap \partial\Omega_2$, $\|Ax\| \leq \|x\|$.

Under these conditions, the operator A possesses at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3. Main Result

In this section, we establish the conditions necessary to identify the eigenvalues for which the iterative system associated with the third-order impulsive boundary value problem (2) has at least one positive solution in a cone. Then, we define an integral operator $T : \mathcal{P} \to \mathbb{B}$ for $\kappa_1 \in \mathcal{P}$, where

$$T\kappa_1(t) = \int_0^1 \mathcal{G}(t,s_1)\,\phi_p^{-1}\Big(\mu_1 \int_0^{s_1} q_1(\omega_1)\,h_1\Big(\cdots h_{n-1}\Big(\int_0^{s_n} \mathcal{G}(\omega_{n-1},s_n)\,\phi_p^{-1}\Big(\mu_n \int_0^{s_n} q_n(\omega_n)\,h_n(\kappa_1(\omega_n))\,d\omega_n\Big)ds_n$$

$$+ \sum_{m=1}^{k} H_{nm}(\omega_{n-1},t_m)\Big)\cdots\Big)d\omega_1\Big)ds_1 + \sum_{m=1}^{k} H_{1m}(t,t_m), \tag{13}$$

thereby setting the foundation for analyzing the behavior of the solutions within this framework.

From conditions (C1)–(C5), Lemmas 2.2 and 2.3, and the definition of T, it follows that for $\kappa_1 \in \mathcal{P}$, the following hold: $T\kappa_1(t) \geq 0$, $(T\kappa_1)'(t) \geq 0$, and $(T\kappa_1)'(t)$ is concave on I. Therefore, $T(\mathcal{P}) \subset \mathcal{P}$. Moreover, one can show that the operator T is completely continuous by applying the Arzelà–Ascoli Theorem.

We now explore the relevant fixed points of T within the cone $\mathcal{P}$. For convenience, we introduce the following notation. Let

$$N_1 := \max_{1\leq i\leq n}\left\{\left[\phi_p\left(\gamma\sigma \int_\sigma^{1-\sigma} \mathcal{G}(s,s)\left(\int_0^s q_i(\omega)d\omega\right)ds\right)h_i^\infty\right]^{-1}\right\}$$

and

$$N_2 = \min_{1\leq i\leq n}\left\{\left[\mu_i^{\frac{2-p}{p-1}}\left(\int_0^1 \mathcal{G}(s,s)\left(\int_0^s q_i(\omega)d\omega\right)ds + \frac{k}{\delta}(2a_1+a_2)(a_3+a_4)\right)\cdot\left(\max\{\phi_p^{-1}(h_i^0), I_{im}^0, J_{im}^0\}\right)\right]^{-1}\right\}.$$

Theorem 3.1 *Suppose that the conditions (C1)-(C5) are met. Therefore, for each $\mu_1, \mu_2, \cdots, \mu_n$ satisfying*

$$N_1 < \mu_i < N_2, \;\; i = 1,2,\cdots,n \tag{14}$$

an n-tuple $(\kappa_1, \kappa_2, \cdots, \kappa_n)$ exists, satisfying (1), with each $\kappa_i(t) > 0$ for $i \in \{1,2,3,\ldots,n\}$ on I.

Proof Assume μ_r, for $1 \leq r \leq n$, be as defined in (14). Choose $\varepsilon > 0$ such that

$$\max_{1\leq i\leq n}\left\{\left[\phi_p\left(\gamma\sigma \int_\sigma^{1-\sigma} \mathcal{G}(s,s)\phi_p^{-1}\left(\int_0^s q_i(\omega)d\omega\right)ds\right)(h_i^\infty - \varepsilon)\right]^{-1}\right\} \leq \min_{1\leq r\leq n} \mu_r$$

and

$$\max_{1\leq r\leq n}\mu_r \leq \min_{1\leq i\leq n}\left\{\left[\left(\mu_i^{\frac{2-p}{p-1}} \int_0^1 \mathcal{G}(s,s)\phi_p^{-1}\left(\int_0^s q_i(\omega)d\omega\right)ds + \frac{k}{\delta}(2a_1+a_2)(a_3+a_4)\right)\right.\right.$$

$$\left.\left.\cdot\left(\max\{\phi_p^{-1}(h_i^0+\varepsilon), I_{im}^0+\varepsilon, J_{im}^0+\varepsilon\}\right)\right]^{-1}\right\}.$$

We investigate the fixed points of the completely continuous operator $T:\mathcal{P}\to\mathcal{P}$, as defined in (13). Utilizing the definitions of $h_i^0, I_{im}^0, J_{im}^0$, there exists a constant $K_1>0$ such that, for each $i\in\{1,2,3,\cdots,n\}$ and $1\le m\le k$,

$$h_i(\kappa)\le(h_i^0+\varepsilon)\kappa^{p-1},\quad I_{im}(\kappa)\le(I_{im}^0+\varepsilon)\kappa,\quad J_{im}(\kappa)\le(J_{im}^0+\varepsilon)\kappa,\quad 0<\kappa<K_1.$$

Suppose that $\kappa_1\in\mathcal{P}$ with $\|\kappa_1\|=K_1$. We begin by verifying that $\kappa_n\le K_1$ holds in the case when $i=n$. For $0\le s_{n-1}\le 1$, by applying Lemma 2.4 and the choice of ε, we obtain

$$\begin{aligned}
&\int_0^1 \mathcal{G}(s_{n-1},s_n)\phi_p^{-1}\left(\mu_n\int_0^{s_n} q_n(\omega_n)h_n(\kappa_1(\omega_n))d\omega_n\right)ds_n+\sum_{m=1}^{k}H_{nm}(s_{n-1},t_m)\\
&\le\mu_n\left[\left(\mu_n^{\frac{2-p}{p-1}}\int_0^1\mathcal{G}(s_n,s_n)\phi_p^{-1}\left(\int_0^{s_n}q_n(\omega_n)d\omega_n\right)ds_n+\frac{k}{\delta}(2a+b)(c+d)\right)\right.\\
&\qquad\left.\cdot\left(\max\{\phi_p^{-1}(h_n^0+\varepsilon),I_{nm}^0+\varepsilon,J_{nm}^0+\varepsilon\}\right)\right]\|\kappa_1\|\\
&\le K_1.
\end{aligned}$$

Proceeding with the case $i=n-1$, we now demonstrate that κ_{n-1} is also less than K_1. This pattern persists with Lemma 2.4, where, for $0\le s_{n-2}\le 1$, it holds that

$$\begin{aligned}
&\int_0^1\mathcal{G}(s_{n-2},s_{n-1})\,\phi_p^{-1}\Bigg(\mu_{n-1}\int_0^{s_{n-1}}q_{n-1}(\omega_{n-1})\,h_{n-1}\Bigg(\int_0^1\mathcal{G}(\omega_{n-1},s_n)\,\phi_p^{-1}\left(\mu_n\int_0^{s_n}q_n(\omega_n)\,h_n(\kappa_1(\omega_n))\,d\omega_n\right)ds_n\\
&\quad+\sum_{m=1}^{k}H_{nm}(\omega_{n-1},t_m)\Bigg)d\omega_{n-1}\Bigg)ds_{n-1}+\sum_{m=1}^{k}H_{n-1,m}(s_{n-2},t_m)\\
&\le\mu_{n-1}\left[\left(\mu_{n-1}^{\frac{2-p}{p-1}}\int_0^1\mathcal{G}(s_{n-1},s_{n-1})\,\phi_p^{-1}\left(\int_0^{s_{n-1}}q_{n-1}(\omega_{n-1})\,d\omega_{n-1}\right)ds_{n-1}+\frac{k}{\delta}(2a_1+a_2)(a_3+a_4)\right)\right.\\
&\qquad\left.\cdot\left(\max\left\{\phi_p^{-1}(h_{n-1}^0+\varepsilon),\,I_{n-1,m}^0+\varepsilon,\,J_{n-1,m}^0+\varepsilon\right\}\right)\right]\|\kappa_1\|\\
&\le\|\kappa_1\|=K_1.
\end{aligned}$$

Proceeding with this argument, we obtain

$$\begin{aligned}
&\int_0^1\mathcal{G}(t,s_1)\phi_p^{-1}\left(\mu_1\int_0^{s_1}q_1(\omega_1)h_1(\mu_2\cdots)d\omega_1\right)ds_1+\sum_{m=1}^{k}H_{1m}(t,t_m)\\
&\le\mu_1\left[\left(\mu_1^{\frac{2-p}{p-1}}\int_0^1\mathcal{G}(s_1,s_1)\phi_p^{-1}\left(\int_0^{s_1}q_1(\omega_1)d\omega_1\right)ds_1+\frac{k}{\delta}(2a_1+a_2)(a_3+a_4)\right)\right.\\
&\qquad\left.\cdot\left(\max\{\phi_p^{-1}(h_1^0+\varepsilon),I_{1m}^0+\varepsilon,J_{1m}^0+\varepsilon\}\right)\right]K_1\\
&\le K_1=\|\kappa_1\|.
\end{aligned}$$

Thereby, $\|T\kappa_1\| \le K_1 = \|\kappa_1\|$. If we define $\Omega_1 = \{\kappa \in \mathbb{B} : \|\kappa\| < K_1\}$, then the inequality

$$\|T\kappa_1\| \le \|\kappa_1\| \text{ holds for } \kappa_1 \in \mathcal{P} \cap \partial\Omega_1. \tag{15}$$

From the definitions of h_i^∞, $i = 1, 2, \cdots, n$, there is a $\bar{K}_2 > 0$ such that, for each $1 \le i \le n$, $h_i(\kappa) \ge (h_i^\infty - \varepsilon)\kappa^{p-1}$, $\kappa \ge \bar{K}_2$. Let $K_2 = \max\{2K_1, \frac{\bar{K}_2}{\sigma}\}$. Let $\kappa_1 \in \mathcal{P}$ and $\|\kappa_1\| = K_2$. Therefore, $\min_{t\in[\sigma,1-\sigma]} \kappa_1(t) \ge \sigma\|\kappa_1\| \ge \bar{K}_2$ is gained with the help of the Lemmas 2.5 and 2.6. We begin by verifying that $\kappa_n \ge K_2$ holds in the case when $i = n$.

Consequently, utilizing Lemmas 2.5 and 2.6, and given the selection of ε, we obtain

$$\begin{aligned}
&\int_0^1 \mathcal{G}(s_{n-1}, s_n)\phi_p^{-1}\left(\mu_n \int_0^{s_n} q_n(\omega_n)h_n(\kappa_1(\omega_n))d\omega_n\right)ds_n + \sum_{m=1}^{k} H_{nm}(s_{n-1}, t_m)\\
&\ge \gamma \int_\sigma^{1-\sigma} \mathcal{G}(s_n, s_n)\phi_p^{-1}\left(\mu_n \int_0^{s_n} q_n(\omega_n)h_n(\kappa_1(\omega_n))d\omega_n\right)ds_n\\
&\ge \phi_p^{-1}(\mu_n)\phi_p^{-1}(h_n^\infty - \varepsilon)\gamma \int_\sigma^{1-\sigma} \mathcal{G}(s_n, s_n)\phi_p^{-1}\left(\int_0^{s_n} q_n(\omega_n)d\omega_n\right)\kappa_1(s_n)ds_n\\
&\ge \phi_p^{-1}(\mu_n)\phi_p^{-1}(h_n^\infty - \varepsilon)\gamma\sigma \int_\sigma^{1-\sigma} \mathcal{G}(s_n, s_n)\phi_p^{-1}\left(\int_0^{s_n} q_n(\omega_n)d\omega_n\right)ds_n\|\kappa_1\|\\
&\ge \|\kappa_1\| = K_2 \ for \ 0 \le s_{n-1} \le 1.
\end{aligned}$$

We now consider the case $i = n-1$ and show that $\kappa_{n-1} > K_2$. Following the approach used in Lemmas 2.5 and 2.6, and using the selected ε, we obtain

$$\begin{aligned}
&\int_0^1 \mathcal{G}(s_{n-2}, s_{n-1})\phi_p^{-1}\Bigg(\mu_{n-1} \int_0^{s_{n-1}} q_{n-1}(\omega_{n-1})h_{n-1}\Bigg(\int_0^1 \mathcal{G}(\omega_{n-1}, s_n)\phi_p^{-1}\left(\mu_n \int_0^{s_n} q_n(\omega_n)h_n(\kappa_1(\omega_n))d\omega_n\right)ds_n\\
&\quad + \sum_{m=1}^{k} H_{nm}(\omega_{n-1}, t_m)\Bigg)d\omega_{n-1}\Bigg)ds_{n-1} + \sum_{m=1}^{k} H_{n-1,m}(s_{n-2}, t_m)\\
&\ge \phi_p^{-1}(h_{n-1}^\infty - \varepsilon)\gamma \int_\sigma^{1-\sigma} \mathcal{G}(s_{n-2}, s_{n-1})\phi_p^{-1}\left(\mu_{n-1} \int_0^{s_{n-1}} q_{n-1}(\omega_{n-1})d\omega_{n-1}\right)ds_{n-1}K_2\\
&\ge \phi_p^{-1}(\mu_{n-1})\phi_p^{-1}(h_{n-1}^\infty - \varepsilon)\gamma\sigma \int_\sigma^{1-\sigma} \mathcal{G}(s_{n-1}, s_{n-1})\phi_p^{-1}\left(\int_0^{s_{n-1}} q_{n-1}(\omega_{n-1})d\omega_{n-1}\right)ds_{n-1}K_2\\
&\ge K_2 \quad for \ 0 \le s_{n-2} \le 1.
\end{aligned}$$

Once more, employing a bootstrapping argument leads us to conclude that

$$\int_0^1 \mathcal{G}(t, s_1)\phi_p^{-1}\left(\mu_1 \int_0^{s_1} q_1(\omega_1)h_1\left(\int_0^1 \cdots\right)d\omega_1\right)ds_1 + \sum_{m=1}^{k} H_{1m}(t, t_m) \ge K_2.$$

Thus, $T\kappa_1(t) \ge K_2 = \|\kappa_1\|$.

Therefore, $\|T\kappa_1\| \geq \|\kappa_1\|$. Putting $\Omega_2 = \{\kappa \in \mathbb{B} : \|\kappa\| < K_2\}$, then

$$\|T\kappa_1\| \geq \|\kappa_1\|, \qquad \kappa_1 \in \mathcal{P} \cap \partial\Omega_2. \tag{16}$$

Applying Lemma 2.1 to (15) and (16), we can conclude that T has a fixed point $\kappa_1 \in \mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$. In conclusion, setting $\kappa_{n+1} = \kappa_1$ yields a positive solution $(\kappa_1, \kappa_2, \cdots, \kappa_n)$ for the 3^{rd}-order IBVP (1)'s iterative system, where iteratively,

$$\kappa_r(t) = \int_0^1 \mathcal{G}(t,s)\phi_p^{-1}\left(\mu_r \int_0^s q_r(\omega)h_r(\kappa_{r+1}(\omega))d\omega\right)ds + \sum_{m=1}^{k} H_{rm}(t,t_m), \quad r \in \{n, n-1, \cdots, 1\}.$$

□

Example 3.2 *Assume that* $k = 4$, $n = 4$ *and* $p = 2$, $q_i(t) = 1$ *for* $1 \leq i \leq 4$, $a_1 = a_3 = 4, a_2 = a_4 = 2, \sigma = \dfrac{1}{4}$ *in the IBVP* (1)*'s iterative system, i.e.,*

$$\begin{cases} (\phi_1(\kappa_i''(t)))' + \mu_i h_i(\kappa_{i+1}(t)) = 0, \ t \neq t_m, \ t \in I = [0,1], \ t \neq t_m, \ i \in \{1,2,3,4\} \\ \kappa_{n+1}(t) = \kappa_1(t) \\ \triangle\kappa_i|_{t=t_m} = \mu_i I_{im}(\kappa_{i+1}(t_m)), \ m = 1,2 \\ \triangle\kappa_i'|_{t=t_m} = -\mu_i J_{im}(\kappa_{i+1}(t_m)), \ m = 1,2 \\ 3\kappa_i(0) - 2\kappa_i'(0) = 0 \\ 3\kappa_i(1) + 2\kappa_i'(1) = 0 \\ \kappa_i''(0) = 0, \end{cases} \tag{17}$$

where

$$h_1(\kappa_2) = \kappa_2\left(3 \cdot 10^4 - \frac{29999}{\ln(e+\kappa_2)}\right), \ h_2(\kappa_3) = 2\kappa_3(10^4 - 9999e^{-5\kappa_3}),$$

$$h_3(\kappa_4) = \kappa_4\left(4 \cdot 10^4 - 39999\frac{e^{-4\kappa_4}}{\ln(e+\kappa_4)}\right), \ h_4(\kappa_1) = \frac{\kappa_1}{5}\kappa_1(10^5 - (99995)e^{-\kappa_1}),$$

$$I_{1m}(\kappa_2) = \frac{6\kappa_2^2 + 4\kappa_2}{3+\kappa_2}, \ I_{2m}(\kappa_3) = \frac{2\kappa_3^3 + 4\kappa_3}{8+\kappa_3^2}, \ I_{3m}(\kappa_4) = \frac{8\kappa_4^3 + 4\kappa_4}{7+4\kappa_4^2}, \ I_{4m}(\kappa_1) = \frac{10\kappa_1^2 + 2\kappa_1}{11+\kappa_1},$$

$$J_{1m}(\kappa_2) = \frac{9\kappa_2^2 + 6\kappa_2}{2+\kappa_2}, \ J_{2m}(\kappa_3) = \frac{3\kappa_3^3 + 6\kappa_3}{5+\kappa_3^2}, \ J_{3m}(\kappa_4) = \frac{12\kappa_4^3 + 6\kappa_4}{5+4\kappa_4^2}, \ J_{4m}(\kappa_1) = \frac{15\kappa_1^2 + 3\kappa_1}{8+\kappa_1}.$$

Using the definitions of the functions h_i, I_{im} *and* J_{im} *for* $i \in \{1,2,3,4\}$, *we achieve the following numbers:*

$h_1^0 = 1$, $h_2^0 = 2$, $h_3^0 = 1$ *and* $h_4^0 = 1$, $h_1^\infty = 3 \cdot 10^4$, $h_2^\infty = 2 \cdot 10^4$, $h_3^\infty = 4 \cdot 10^4$ *and* $h_4^\infty = 2 \cdot 10^4$,

$I_{1m}^0 = \dfrac{4}{3}$, $I_{2m}^0 = \dfrac{1}{2}$, $I_{3m}^0 = \dfrac{4}{7}$ *and* $I_{4m}^0 = \dfrac{2}{11}$, $J_{1m}^0 = 3$, $J_{2m}^0 = \dfrac{6}{5}$, $J_{3m}^0 = \dfrac{6}{5}$ *and* $J_{4m}^0 = \dfrac{3}{8}$.

It is easy to see that conditions (C1)-(C5) are satisfied. With the help of some basic *computations, for* $1 \le i \le 4$*, we obtain* $\rho = 21, \gamma = \dfrac{11}{20}$ *and*

$$\mathcal{G}(t,s) = \frac{1}{21}\begin{cases}(2+3s)(5-3t), & s \le t\\ (2+3t)(5-3s), & t \le s.\end{cases}$$

Additionally, if we use descriptions, we get $N_1 = 0,025322$ *and* $N_2 = 0,081632$*. With the help of the Theorem 3.1, we determine that the optimal eigenvalue interval is*

$$0,025322 < \mu_i < 0,081632 \; for \; i = 1,2,3,4$$

ensuring a positive solution of the 3^{rd}*-order IBVP (17)'s iterative system.*

4. Conclusion

This study explores eigenvalue intervals for third-order impulsive boundary value problems (IBVPs) with p-Laplacian and iterative structures, addressing a previously underexplored area. By applying the Guo–Krasnosel'skii fixed point theorem [12], we establish the existence of positive solutions for the iterative system (1) and determine the eigenvalue intervals of parameters $\mu_1, \mu_2, \cdots, \mu_n$. Beginning with the initial solution $\kappa_1(t)$ of the third-order IBVP (2), the iterative construction of the solution set $(\kappa_1(t), \cdots, \kappa_n(t))$ provides a robust analytical framework for understanding such systems' dynamics.

Beyond theoretical contributions, this work has significant practical implications. In mechanical engineering, it aids in analyzing vibrational modes of structures subjected to impulsive forces (e.g., seismic events or explosions), contributing to safer designs [17, 32]. In biological modeling, these intervals reveal oscillatory patterns in drug delivery systems, optimizing dosing strategies [30]. For control theory, they enhance stability algorithms in robotics and signal processing where abrupt changes occur [1, 15].

Our study advances the field by unifying third-order impulsive systems with iterative structures-a gap in existing literature. Unlike prior work on second-order impulsive BVPs [19, 31] or non-iterative third-order systems [6, 32], we incorporate eigenvalue intervals and higher-order dynamics, offering novel perspectives for complex impulsive systems.

Future research could extend this framework to higher-order systems or complex boundary conditions – for instance, the boundary parameters $a_i's$ could be generalized from positive constants to functions. Furthermore, combining numerical solution methods may enhance computational efficiency, stability analyses under parameter variations [15, 16, 23] will provide critical insights for engineering applications.

In summary, this study comprehensively advances the theory and applications of third-order iterative IBVPs. By elucidating eigenvalue intervals and their cross-disciplinary relevance, we pave the way for mathematical and practical breakthroughs.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Research Article

Semiprime Ideal of Rings with Symmetric Bi-Derivations

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Abstract: Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\Im$ an ideal of Ω, $\Delta : \Omega \times \Omega \to \Omega$ a symmetric bi-derivation and δ be the trace of Δ. In the present paper, we shall prove that δ is a $\wp$-commuting map on $\Im$ if any one of the following holds: i. $\delta(\sigma) \circ \kappa \in \wp$, ii. $\delta([\sigma,\kappa]) \pm [\delta(\sigma),\kappa] \in \wp$, iii. $\delta(\sigma \circ \kappa) \pm (\delta(\sigma) \circ \kappa) \in \wp$, iv. $\delta([\sigma,\kappa]) \pm \delta(\sigma) \circ \kappa \in \wp$, v. $\delta(\sigma \circ \kappa) \pm [\delta(\sigma),\kappa] \in \wp$, vi. $\delta(\sigma) \circ \kappa \pm [\delta(\kappa),\sigma] \in \wp$, vii. $\delta([\sigma,\kappa]) \pm \delta(\sigma) \circ \kappa - [\delta(\kappa),\sigma] \in \wp$, viii. $\delta([\sigma,\kappa]) \pm [\delta(\sigma),\kappa] + [\delta(\kappa),\sigma] \in \wp$, ix. $\Delta(\sigma,\kappa\kappa_3) \pm \Delta(\sigma,\kappa)\kappa_3 \in \wp$, x. $\Delta(\delta(\sigma),\sigma) \in \wp$, xi. $\delta(\delta(\sigma)) = g(\sigma)$, xii. $\delta(\sigma)\kappa \pm \sigma g(\kappa) \in \wp$, xiii. $[\delta(\sigma),\kappa] \pm [g(\kappa),\sigma] \in \wp$, xiv. $\delta(\sigma) \circ \kappa \pm (\sigma \circ g(\kappa)) \in \wp$, xv. $[\delta(\sigma),\kappa] \pm (\sigma \circ g(\kappa)) \in \wp$, xvi. $\delta(\sigma) \circ \kappa \pm [g(\kappa),\sigma] \in \wp$ for all $\sigma,\kappa \in \Im$ where $G : \aleph \times \aleph \to \aleph$ is a symmetric bi-derivation such that g is the trace of G.

Keywords: Rings, ideals, semiprime ideals, derivations, symmetric bi-derivations.

1. Introduction

Let Ω be an associative ring with center Z. A proper ideal $\wp$ of Ω is termed prime if for any elements $\sigma,\kappa \in \Omega$, the inclusion $\sigma\Omega\kappa \subseteq \wp$ implies that either $\sigma \in \wp$ or $\kappa \in \wp$. Equivalently, the ring Ω is said to be prime if (0), the zero ideal, is a prime ideal. In addition to prime ideals, the concept of semiprime ideals is also fundamental in ring theory. A proper ideal $\wp$ is semiprime if for any $\sigma \in \Omega$, the condition $\sigma\Omega\sigma \subseteq \wp$ implies $\sigma \in \wp$. The ring Ω is semiprime if (0) is a semiprime ideal. While every prime ideal is semiprime, the converse is not generally true. For any $\sigma,\kappa \in \Omega$, the symbol $[\sigma,\kappa]$ stands for the commutator $\sigma\kappa - \kappa\sigma$ and the symbol $\sigma \circ \kappa$ stands for the commutator $\sigma\kappa + \kappa\sigma$. An additive mapping $\delta : \Omega \to \Omega$ is called a derivation if $\delta(\sigma\kappa) = \delta(\sigma)\kappa + \sigma\delta(\kappa)$ holds for all $\sigma,\kappa \in \Omega$. A mapping $\Delta(.,.) : \Omega \times \Omega \to \Omega$ is said to be symmetric if $\Delta(\sigma,\kappa) = \Delta(\kappa,\sigma)$ for all $\sigma,\kappa \in \Omega$. A mapping $\delta : \Omega \to \Omega$ is called the trace of $\Delta(.,.)$ if $\delta(\sigma) = \Delta(\sigma,\sigma)$ for all $\sigma \in \Omega$.It is obvious that if $\Delta(.,.)$ is bi-additive (i.e., additive in both arguments), then the trace δ of $\Delta(.,.)$

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2020 *AMS Mathematics Subject Classification*: 16U80, 16W10, 16W25 satisfies the identity $\delta(\sigma+\kappa)=\delta(\sigma)+\delta(\kappa)+2\Delta(\sigma,\kappa)$ for all $\sigma,\kappa\in\Omega$. If $\Delta(.,.)$ is bi-additive and satisfies the identities

$$\Delta(\sigma\kappa,\varsigma)=\Delta(\sigma,\varsigma)\kappa+\sigma\Delta(\kappa,\varsigma)$$

and

$$\Delta(\sigma,\kappa\varsigma)=\Delta(\sigma,\kappa)\varsigma+\kappa\Delta(\sigma,\varsigma)$$

for all $\sigma,\kappa,\varsigma\in\Omega$, then $\Delta(.,.)$ is called a symmetric bi-derivation.

Example 1.1 *Suppose the ring*

$$\Omega=\left\{\left(\begin{array}{cc} a & b\\ 0 & 0\end{array}\right)\middle|\, a,b\in\mathbb{R}\right\}.$$

Define maps $\Delta:\Omega\times\Omega\to\Omega$ *as follows:*

$$\Delta\left(\left(\begin{array}{cc} a & b\\ 0 & 0\end{array}\right),\left(\begin{array}{cc} c & \delta\\ 0 & 0\end{array}\right)\right)=\left(\left(\begin{array}{cc} 0 & ac\\ 0 & 0\end{array}\right)\right).$$

Then it is easy to verify that Δ *is a symmetric bi-derivation on* Ω. *Also, the trace of* Δ *is*

$$\delta\left(\begin{array}{cc} a & b\\ 0 & 0\end{array}\right)=\left(\begin{array}{cc} 0 & a^2\\ 0 & 0\end{array}\right).$$

Let S be a nonempty subset of Ω. A mapping T from Ω to Ω is called commuting on S if $[T(\sigma),\sigma]=0$ for all $\sigma\in S$. This definition has been generalized such as: A map $T:\Omega\to\Omega$ is called a U-commuting map on S if $[T(\sigma),\sigma]\in U$ for all $\sigma\in S$ and some $U\subseteq\Omega$. In particular, if $U=0$, then T is called a commuting map on S if $[T(\sigma),\sigma]=0$. Note that every commuting map is a $U-$commuting map (put $0=U$). But the converse is not true in general. Take U some a set of Ω has no zero such that $[T(\sigma),\sigma]\in U$, then T is a $U-$commuting map but it is not a commuting map. The notion of additive commuting mapping is closely connected with the notion of bi-derivation. Every additive commuting mapping $T:\Omega\to\Omega$ gives rise to a bi-derivation on Ω. Namely, linearizing $[T(\sigma),\sigma]\in\wp$, we get $[T(\sigma),\kappa]=[\sigma,T(\kappa)]$ and we note that the map $(\sigma,\kappa)\longmapsto[T(\sigma),\kappa]$ is a bi-derivation. The concept of bi-derivation was introduced by Maksa in [8]. It is shown in [9] that symmetric bi-derivations are related to general solution of some functional expressions.

The property of interchangeability of prime or semiprime rings with derivation was first discussed by Posner [10]. Later, many authors studied the commutativity conditions in prime and semiprime rings. In recent years, the effects of these conditions on the derivation of prime and semiprime ideals have begun to be examined.

In 1992, Daif and Bell showed that if the derivativon δ on a semiprime ring Ω satisfies the condition $\sigma\kappa \pm \delta(\sigma\kappa) = \kappa\sigma \pm \delta(\kappa\sigma)$ for each $\sigma, \kappa \in \Omega$, then the ring Ω is commutative [5]. In 1999, Ashraf considered the same condition for the symmetric bi-derivation on a prime ring [1]. In 2015, Reddy, Rao, and Reddy generalized this theorem for semiprime rings [11]. In 2001, Ashraf and Rehman showed that if the derivation δ on an ideal $\Im$ of a prime ring Ω satisfies one of the conditions $\delta(\sigma\kappa) - \sigma\kappa \in \varsigma$ or $\delta(\sigma\kappa) - \kappa\sigma \in \varsigma$ for each $\sigma, \kappa \in \Omega$, then $\Im$ is commutative [2]. These conditions were investigated by Koç Sögütcü and Gölbaşı in 2021 for inverse bi-derivation Lie ideals [6].

On the other hand, Vukman proved in 1990 that if Ω is a semiprime ring, Δ is the symmetric bi-derivation on the ring Ω and δ, Δ is the trace of the symmetric bi-derivation, then $\Delta = 0$ if $\Delta(\delta(\sigma), \sigma) = 0$ and $\delta(\delta(\sigma)) = g(\sigma)$ for all $\sigma \in \Omega$ [12]. In 2017, Reddy and Naik considered the above conditions for the symmetric reverse bi-derivation. It was investigated by Koç Sögütcü and Gölbaşı in 2021 for reverse bi-derivation Lie ideals on the semiprime ring [7].

Ashraf et al. in 2005 studied the commutativity of a prime ring Ω, which allows a generalized derivation T and the associated derivation δ, satisfying any of these properties: $\delta(\sigma) \circ T(\kappa) = 0$ or $[\delta(\sigma), T(\kappa)] = 0$ for all $\sigma, \kappa \in \Omega$ [3]. In 2024, Çelik and Koç Sögütcü considered these conditions for multiplicative derivation with semiprime ideal [4].

In this paper, we investigate the algebraic identities mentioned above for symmetric bi-derivation acting on a semiprime ideal without making any assumptions on the ideal of the ring. We will make some extensive use of the basic commutator identities: 1) $[\sigma, \kappa\varsigma] = \kappa[\sigma, \varsigma] + [\sigma, \kappa]\varsigma$,
2) $[\sigma\kappa, \varsigma] = [\sigma, \varsigma]\kappa + \sigma[\kappa, \varsigma]$,
3) $\sigma\kappa \circ \varsigma = (\sigma \circ \varsigma)\kappa + \sigma[\kappa, \varsigma] = \sigma(\kappa \circ \varsigma) - [\sigma, \varsigma]\kappa$,
4) $\sigma \circ \kappa\varsigma = \kappa(\sigma \circ \varsigma) + [\sigma, \kappa]\varsigma = (\sigma \circ \kappa)\varsigma + \kappa[\varsigma, \sigma]$.

2. Main Results

Lemma 2.1 *Let Ω be a ring with $\wp$ a semiprime ideal of R, $\Im$ an ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega \times \Omega \to \Omega$ a symmetric bi-derivation and δ be the trace of Δ. If $\delta(\sigma) \circ \kappa \in \wp$ for all $\sigma, \kappa \in \Im$, then δ is $\wp$-commuting map on $\Im$.*

Proof By the hypothesis, we get

$$\delta(\sigma) \circ \kappa \in \wp, \text{ for all } \sigma, \kappa \in \Im.$$

Taking κ by $\kappa\varsigma$ in the last expression, we obtain that

$$\kappa(\delta(\sigma) \circ \varsigma) + [\delta(\sigma), \kappa]\varsigma \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \Im.$$

Using the hypothesis, we have

$$[\delta(\sigma),\kappa]\varsigma \in \wp \text{ for all } \sigma,\kappa,\varsigma \in \Im.$$

Taking ς by $t[\delta(\sigma),\kappa], t \in \Omega$ in the above expression, we get

$$[\delta(\sigma),\kappa]t[\delta(\sigma),\kappa] \in \wp \text{ for all } \sigma,\kappa \in \Im, t \in \Omega.$$

Since $\wp$ is a semiprime ideal of $\Omega,$ we conclude that

$$[\delta(\sigma),\kappa] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

Replacing κ by σ in the last expression, we get

$$[\delta(\sigma),\sigma] \in \wp \text{ for all } \sigma \in \Im.$$

Hence we conclude that δ is $\wp-$commuting on $\Im$. □

Theorem 2.2 *Let Ω be a ring with $\wp$ a semiprime ideal of $\Omega,$ $\Im$ an ideal of $\Omega,$ $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega \times \Omega \to \Omega$ a symmetric bi-derivation and δ be the trace of Δ. If any of the following conditions is satisfied for all $\sigma,\kappa \in \Omega$, then δ is $\wp-$commuting map on $\Im$.*

i) $\delta(\sigma \circ \kappa) \pm (\delta(\sigma) \circ \kappa) \in \wp,$

ii) $\delta([\sigma,\kappa]) \pm (\delta(\sigma) \circ \kappa) \in \wp,$

iii) $\delta(\sigma \circ \kappa) \pm [\delta(\sigma),\kappa] \in \wp,$

iv) $\delta(\sigma) \circ \kappa \pm [\delta(\kappa),\sigma] \in \wp.$

Proof i) By the hypothesis, we get

$$\delta(\sigma \circ \kappa) \pm (\delta(\sigma) \circ \kappa) \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \Im,$ we have

$$\delta(\sigma \circ \kappa) + \delta(\sigma \circ \varsigma) + 2\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \pm \delta(\sigma) \circ \kappa \pm \delta(\sigma) \circ \varsigma \in \wp.$$

Using the hypothesis, we arrive at

$$2\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \in \wp.$$

Since $char(\Omega/\wp) \neq 2,$ we have

$$\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \in \wp.$$

Replacing ς by κ in this expression, we find that

$$\Delta(\sigma \circ \kappa, \sigma \circ \kappa) \in \wp \text{ for all } \sigma,\kappa \in \Im$$

and so

$$\delta(\sigma \circ \kappa) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Again using the hypothesis, we obtain that

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

We see that δ is $\wp-$commuting map on $\mathfrak{I}$ by Lemma 2.1.

ii) By the hypothesis, we have

$$\delta([\sigma, \kappa]) \pm (\delta(\sigma) \circ \kappa) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \mathfrak{I}$ in the hypothesis, we get

$$\delta([\sigma, \kappa]) + \delta([\sigma, \varsigma]) + 2\Delta([\sigma, \kappa], [\sigma, \varsigma]) \pm \delta(\sigma) \circ \kappa \pm \delta(\sigma) \circ \varsigma \in \wp.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2$, we find that

$$\Delta([\sigma, \kappa], [\sigma, \varsigma]) \in \wp.$$

Replacing ς by κ in the last expression, we see that

$$\Delta([\sigma, \kappa], [\sigma, \kappa]) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

That is

$$\delta([\sigma, \kappa]) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Using this expression in our hypothesis, we get

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

By Lemma 2.1, we obtain that δ is $\wp-$commuting on $\mathfrak{I}$.

iii) By the hypothesis, we have

$$\delta(\sigma \circ \kappa) \pm [\delta(\sigma), \kappa] \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \mathfrak{I}$, we get

$$\delta(\sigma \circ \kappa) + \delta(\sigma \circ \varsigma) + 2\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \pm [\delta(\sigma), \kappa] \pm [\delta(\sigma), \varsigma] \in \wp.$$

Using the hypothesis, we see that

$$\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \in \wp.$$

Replacing ς by κ in the last expression, we get

$$\Delta(\sigma \circ \kappa, \sigma \circ \kappa) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}$$

and so

$$\delta(\sigma \circ \kappa) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Using the hypothesis, we have

$$[\delta(\sigma), \kappa] \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Replacing κ by σ in the last expression, we obtain that δ is $\wp-$commuting on $\mathfrak{I}$.

iv) We have

$$\delta(\sigma) \circ \kappa \pm [\delta(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Replacing κ by $\kappa + \varsigma, \varsigma \in \mathfrak{I}$, we get

$$\delta(\sigma) \circ \kappa + \delta(\sigma) \circ \varsigma \pm 2[\Delta(\kappa, \varsigma), \sigma] \pm [\delta(\kappa), \sigma] \pm [\delta(\varsigma), \sigma] \in \wp.$$

By the hypothesis, we have

$$2[\Delta(\kappa, \varsigma), \sigma] \in \wp.$$

Using $char(\Omega/\wp) \neq 2$, we find that

$$[\Delta(\kappa, \varsigma), \sigma] \in \wp.$$

Replacing ς by κ in this expression, we get

$$[\Delta(\kappa, \kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}$$

and so

$$[\delta(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Using the hypothesis, we have

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

By Lemma 2.1, we conclude that δ is $\wp-$commuting on $\mathfrak{I}$. □

Theorem 2.3 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\mathfrak{I}$ an ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega \times \Omega \to \Omega$ a symmetric bi-derivation and δ be the trace of Δ. If any of the following conditions is satisfied for all $\sigma, \kappa \in \Omega$, then δ is $\wp-$commuting map on $\mathfrak{I}$.*

i) $\delta([\sigma, \kappa]) \pm (\delta(\sigma) \circ \kappa) + [\delta(\kappa), \sigma] \in \wp$,

ii) $\delta([\sigma, \kappa]) \pm [\delta(\sigma), \kappa] + [\delta(\kappa), \sigma] \in \wp$.

Proof i) By the hypothesis, we obtain that

$$\delta([\sigma, \kappa]) \pm (\delta(\sigma) \circ \kappa) + [\delta(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Replacing κ by $\kappa + \varsigma, \varsigma \in \mathfrak{I}$ in the last expression, we get

$$\delta([\sigma, \kappa]) + \delta([\sigma, \varsigma]) + 2\Delta([\sigma, \kappa], [\sigma, \varsigma]) \pm \delta(\sigma) \circ \kappa \pm \delta(\sigma) \circ \varsigma + [\delta(\kappa), \sigma] + [\delta(\varsigma), \sigma] + 2[\Delta(\kappa, \varsigma), \sigma] \in \wp.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2$, we have

$$\Delta([\sigma,\kappa],[\sigma,\varsigma]) + [\Delta(\kappa,\varsigma),\sigma] \in \wp.$$

Replacing κ by ς in the above expression, we see that

$$\Delta([\sigma,\kappa],[\sigma,\kappa]) + [\Delta(\kappa,\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

That is

$$\delta([\sigma,\kappa]) + [\delta(\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

We can write this expression such as

$$\delta([\sigma,\kappa]) + [\delta(\kappa),\sigma] \pm \delta(\kappa)\circ\sigma \mp \delta(\kappa)\circ\sigma \in \wp.$$

Using the hypothesis, we arrive at

$$\delta(\kappa)\circ\sigma \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

By Lemma 2.1, we conclude that δ is $\wp-$commuting on $\Im$.

ii) By the hypothesis, we get

$$\delta([\sigma,\kappa]) \pm [\delta(\sigma),\kappa] + [\delta(\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

Writing κ by $\kappa+\varsigma$, $\varsigma \in \Im$ in this expression, we obtain that

$$\delta([\sigma,\kappa]) + \delta([\sigma,\varsigma]) + 2\Delta([\sigma,\kappa],[\sigma,\varsigma]) \pm [\delta(\sigma),\kappa] \pm [\delta(\sigma),\varsigma] + [\delta(\kappa),\sigma] + [\delta(\varsigma),\sigma] + 2[\Delta(\kappa,\varsigma),\sigma] \in \wp.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2$, we see that

$$\Delta([\sigma,\kappa],[\sigma,\varsigma]) + [\Delta(\kappa,\varsigma),\sigma] \in \wp \text{ for all } \sigma,\kappa,\varsigma \in \Im.$$

Replacing ς by κ in the last expression, we have

$$\delta([\sigma,\kappa]) + [\delta(\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

The rest of the proof is the same as above. This completes proof. □

Theorem 2.4 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega\times\Omega \to \Omega$ a symmetric bi-derivation, δ be the trace of Δ. If $\Delta(\sigma,\kappa\varsigma) - \Delta(\sigma,\kappa)\varsigma \in \wp$ for all $\sigma,\kappa,\varsigma \in \Omega$, then δ is $\wp-$commuting on Ω.*

Proof By the hypothesis, we get

$$\Delta(\sigma,\kappa\varsigma) - \Delta(\sigma,\kappa)\varsigma \in \wp \text{ for all } \sigma,\kappa,\varsigma \in \Omega.$$

Since Δ is a bi-derivation, we can write

$$\Delta(\sigma,\kappa\varsigma)=\Delta(\sigma,\kappa)\varsigma+\kappa\Delta(\sigma,\varsigma).$$

Using the hypothesis, we obtain that

$$\kappa\Delta(\sigma,\varsigma)\in\wp \text{ for all } \sigma,\kappa,\varsigma\in\Omega.$$

Replacing κ by $\Delta(\sigma,\varsigma)\kappa$ in the last expression, we get

$$\Delta(\sigma,\varsigma)\kappa\Delta(\sigma,\varsigma)\in\wp \text{ for all } \sigma,\kappa,\varsigma\in\Omega.$$

Since $\wp$ is semiprime ideal of Ω, we obtain that $\Delta(\sigma,\varsigma)\in\wp$ for all $\sigma,\varsigma\in\Omega$. Replacing ς by σ, we get $\delta(\sigma)\in\wp$, and so $[\delta(\sigma),\sigma]\in\wp$ for all $\sigma\in\Im$. Hence we conclude that δ is $\wp-$commuting map.

□

Theorem 2.5 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $char(\Omega/\wp)\neq 2$ and $\Delta:\Omega\times\Omega\to\Omega$ a symmetric bi-derivation, δ be the trace of Δ. If $\Delta(\delta(\sigma),\sigma)\in\wp$, for all $\sigma\in\Omega$, then δ is $\wp-$commuting on Ω.*

Proof By the hypothesis, we have

$$\Delta(\delta(\sigma),\sigma)\in\wp \text{ for all } \sigma\in\Omega.$$

Replacing σ by $\sigma+\kappa$, $\kappa\in\Omega$ in the last expression, we get

$$\Delta(\delta(\sigma),\sigma)+\Delta(\delta(\sigma),\kappa)+\Delta(\delta(\kappa),\sigma)+\Delta(\delta(\kappa),\kappa)+2\Delta(\Delta(\sigma,\kappa),\sigma)+2\Delta(\Delta(\sigma,\kappa),\kappa)\in\wp.$$

Using our hypothesis, we see that

$$\Delta(\delta(\sigma),\kappa)+\Delta(\delta(\kappa),\sigma)+2\Delta(\Delta(\sigma,\kappa),\sigma)+2\Delta(\Delta(\sigma,\kappa),\kappa)\in\wp.$$

Replacing σ by $-\sigma$ in the above expression, we obtain that

$$\Delta(\delta(\sigma),\kappa)-\Delta(\delta(\kappa),\sigma)+2\Delta(\Delta(\sigma,\kappa),\sigma)-2\Delta(\Delta(\sigma,\kappa),\kappa)\in\wp.$$

We obtained from the last two expressions

$$\Delta(\delta(\sigma),\kappa)+2\Delta(\Delta(\sigma,\kappa),\sigma)\in\wp. \quad (1)$$

Taking κ by $\sigma\kappa$ in (1) and using the hypothesis, we get

$$\sigma\Delta(\delta(\sigma),\kappa)+2\delta(\sigma)\Delta(\sigma,\kappa)+2\sigma\Delta(\Delta(\sigma,\kappa),\sigma)+2\delta(\sigma)\Delta(\kappa,\sigma)\in\wp. \quad (2)$$

Multipliying in (1) by σ on left hand side, we see that

$$\sigma\Delta(\delta(\sigma),\kappa)+2\sigma\Delta(\Delta(\sigma,\kappa),\sigma)\in\wp. \quad (3)$$

Subtracting (2) from (3), we arrive at

$$4\delta(\sigma)\Delta(\sigma,\kappa)\in\wp.$$

Using $char(\Omega/\wp)\neq 2,$ we get

$$\delta(\sigma)\Delta(\sigma,\kappa)\in\wp.$$

Replacing κ by $\kappa\sigma$ in the last expression, we have

$$\delta(\sigma)\kappa\delta(\sigma)\in\wp \text{ for all } \sigma,\kappa\in\Omega.$$

Since $\wp$ is semiprime ideal of R, we get $\delta(\sigma)\in\wp,$ and so $[\delta(\sigma),\sigma]\in\wp$ for all $\sigma\in\Im.$ Hence we conclude that δ is $\wp-$commuting map. □

Theorem 2.6 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω $char(\Omega/\wp)\neq 2,$, $char\Omega/\wp\neq 3$ and $\Delta:\Omega\times\Omega\to\Omega,$ $G:\Omega\times\Omega\to\Omega$ two symmetric reverse bi-derivations where δ is the trace of Δ and g is the trace of G. If $\delta(\delta(\sigma))\pm g(\sigma)\in\wp$ for all $\sigma\in\Omega,$ then g is $\wp-$commuting on Ω.*

Proof By our hypothesis, we have

$$\delta(\delta(\sigma))\pm g(\sigma)\in\wp \text{ for all } \sigma\in\Omega.$$

Replacing σ by $\sigma+\kappa,$ $\kappa\in\Omega,$ we get

$$\begin{gathered}\delta(\delta(\sigma))+\delta(\delta(\kappa))+2\Delta(\delta(\sigma),\delta(\kappa))+4\delta(\Delta(\sigma,\kappa))+4\Delta(\delta(\sigma),\Delta(\sigma,\kappa))+4\Delta(\delta(\kappa),\Delta(\sigma,\kappa))\\ \pm g(\sigma)\pm g(\kappa)\pm 2G(\sigma,\kappa)\in\wp.\end{gathered}$$

Using the hypothesis and $char(\Omega/\wp)\neq 2,$ we obtain that

$$\Delta(\delta(\sigma),\delta(\kappa))+2\delta(\Delta(\sigma,\kappa))+2\Delta(\delta(\sigma),\Delta(\sigma,\kappa))+2\Delta(\delta(\kappa),\Delta(\sigma,\kappa))\pm G(\sigma,\kappa)\in\wp. \tag{4}$$

Replacing σ by $-\sigma$ in the above expression, we see that

$$\Delta(\delta(\sigma),\delta(\kappa))+2\delta(\Delta(\sigma,\kappa))-2\Delta(\delta(\sigma),\Delta(\sigma,\kappa))-2\Delta(\delta(\kappa),\Delta(\sigma,\kappa))\mp G(\sigma,\kappa)\in\wp. \tag{5}$$

Subtracting (4) from (5), we arrive at

$$4\Delta(\delta(\sigma),\Delta(\sigma,\kappa))+4\Delta(\delta(\kappa),\Delta(\sigma,\kappa))\pm 2G(\sigma,\kappa)\in\wp.$$

Since $char(\Omega/\wp)\neq 2,$ we get

$$2\Delta(\delta(\sigma),\Delta(\sigma,\kappa))+2\Delta(\delta(\kappa),\Delta(\sigma,\kappa))\pm G(\sigma,\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Omega. \tag{6}$$

Replacing σ by 2σ in the last expression, we see that

$$16\Delta(\delta(\sigma),\Delta(\sigma,\kappa))+4\Delta(\delta(\kappa),\Delta(\sigma,\kappa))\pm 2G(\sigma,\kappa)\in\wp. \tag{7}$$

Using (6), we have

$$4\Delta(\delta(\sigma),\Delta(\sigma,\kappa))+4\Delta(\delta(\kappa),\Delta(\sigma,\kappa))\pm 2G(\sigma,\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Omega. \tag{8}$$

Subtracting (7) from (8), we arrive at

$$12\Delta(\delta(\sigma),\Delta(\sigma,\kappa))\in\wp.$$

Using $char(\Omega/\wp)\neq 2$ and $char\Omega/\wp\neq 3$, we get

$$\Delta(\delta(\sigma),\Delta(\sigma,\kappa))\in\wp \text{ for all } \sigma,\kappa\in\Omega.$$

Replacing κ by σ in the above expression, we see that

$$\Delta(\delta(\sigma),\Delta(\sigma,\sigma))\in\wp.$$

That is

$$\delta(\delta(\sigma))\in\wp \text{ for all } \sigma\in\Omega.$$

Returning our hypothesis and using this, we get $g(\sigma)\in\wp$, and so $[g(\sigma),\sigma]\in\wp$ for all $\sigma\in\Im$. Hence we conclude that g is $\wp-$commuting map. This completes proof. □

Theorem 2.7 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\Im$ an ideal of Ω, $char(\Omega/\wp)\neq 2$ and $\Delta:\Omega\times\Omega\to\Omega$, $G:\Omega\times\Omega\to\Omega$ two symmetric bi-derivations where δ is the trace of Δ and g is the trace of G. If $\delta(\sigma)\kappa\pm\sigma g(\kappa)\in\wp$ for all $\sigma,\kappa\in\Im$, then δ is $\wp-$commuting map.*

Proof Let assume that

$$\delta(\sigma)\kappa\pm\sigma g(\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Im.$$

Writing κ by $\kappa+\varsigma,\varsigma\in\Im$, we have

$$\delta(\sigma)\kappa+\delta(\sigma)\varsigma\pm\sigma g(\kappa)\pm\sigma g(\varsigma)\pm 2\sigma G(\kappa,\varsigma)\in\wp.$$

Using the hypothesis, we get

$$2\sigma G(\kappa,\varsigma)\in\wp.$$

Since $char(\Omega/\wp)\neq 2$ and replacing ς by κ, we see that

$$\sigma G(\kappa,\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Im.$$

That is

$$\sigma g(\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Im.$$

By the hypothesis, we get

$$\delta(\sigma)\kappa\in\wp \text{ for all } \sigma,\kappa\in\Im.$$

Thus we can write $\delta(\sigma)\sigma\in\wp$ for all $\sigma\in\Im$, and so $[\delta(\sigma),\sigma]\in\wp$ for all $\sigma\in\Im$. Hence we conclude that δ is $\wp-$commuting map. □

Theorem 2.8 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\Im$ an ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega \times \Omega \to \Omega$, $G : \Omega \times \Omega \to \Omega$ two symmetric bi-derivations where δ is the trace of Δ and g is the trace of G. If any of the following conditions is satisfied for all $\sigma, \kappa \in \Omega$, then δ is $\wp-$ commuting map on $\Im$.*

i) $[\delta(\sigma), \kappa] \pm [g(\kappa), \sigma] \in \wp$,

ii) $\delta(\sigma) \circ \kappa \pm (\sigma \circ g(\kappa)) \in \wp$,

iii) $[\delta(\sigma), \kappa] \pm (\sigma \circ g(\kappa)) \in \wp$,

iv) $\delta(\sigma) \circ \kappa \pm [g(\kappa), \sigma] \in \wp$.

Proof i) By the hypothesis, we have

$$[\delta(\sigma), \kappa] \pm [g(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \Im$, we get

$$[\delta(\sigma), \kappa] + [\delta(\sigma), \varsigma] \pm [g(\kappa), \sigma] \pm [g(\varsigma), \sigma] \pm 2[G(\kappa, \varsigma), \sigma] \in \wp.$$

Using the hypothesis, we have

$$2[G(\kappa, \varsigma), \sigma] \in \wp.$$

Since $\Omega/\wp$ is characteristic not two ring, we obtain that

$$[G(\kappa, \varsigma), \sigma] \in \wp.$$

Replacing ς by κ in the last expression, we have

$$[G(\kappa, \kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

That is,

$$[g(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Using this in our hypothesis, we obtain that $[\delta(\sigma), \kappa] \in \wp$, and so $[\delta(\sigma), \sigma] \in \wp$ for all $\sigma \in \Im$. Hence δ is $\wp-$commuting on $\Im$.

ii) By the hypothesis, we get

$$\delta(\sigma) \circ \kappa \pm (\sigma \circ g(\kappa)) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Replacing κ by $\kappa + \varsigma$, $\varsigma \in \Im$, we obtain that

$$\delta(\sigma) \circ \kappa + \delta(\sigma) \circ \varsigma \pm (\sigma \circ g(\kappa)) \pm (\sigma \circ g(\varsigma)) \pm 2(\sigma \circ G(\kappa, \varsigma)) \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \Im.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2$, we see that

$$(\sigma \circ G(\kappa, \varsigma)) \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \Im.$$

Taking ς by κ in this expression, we have

$$\sigma \circ g(\kappa) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Hence we get from our hypothesis,

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Replacing κ by $\kappa\varsigma$, $\varsigma \in \mathfrak{I}$, we find that

$$\kappa[\varsigma, \delta(\sigma)] \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \mathfrak{I}.$$

Replacing κ by $[\varsigma, \delta(\sigma)]\Omega$ in the last expression, we have

$$[\varsigma, \delta(\sigma)]t[\varsigma, \delta(\sigma)] \in \wp \text{ for all } \sigma, \varsigma \in \mathfrak{I},\ t \in \Omega.$$

Since $\wp$ is semiprime ideal of Ω, we get

$$[\varsigma, \delta(\sigma)] \in \wp \text{ for all } \sigma, \varsigma \in \mathfrak{I}.$$

In particular we have $[\sigma, \delta(\sigma)] \in \wp$ for all $\sigma \in \mathfrak{I}$, and so δ is $\wp-$commuting on $\mathfrak{I}$.

iii) By the hypothesis, we have

$$[\delta(\sigma), \kappa] \pm \sigma \circ g(\kappa) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \mathfrak{I}$, we get

$$[\delta(\sigma), \kappa] + [\delta(\sigma), \varsigma] \pm (\sigma \circ g(\varsigma)) \pm 2(\sigma \circ G(\kappa, \varsigma)) \in \wp.$$

Using $char(\Omega/\wp) \neq 2$, we see that

$$(\sigma \circ G(\kappa, \varsigma)) \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \mathfrak{I}.$$

Taking ς by κ in this expression, we have

$$\sigma \circ g(\kappa) \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Hence we get from our hypothesis,

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

By Lemma 2.1, we conclude that δ is $\wp-$commuting on $\mathfrak{I}$.

iv) By the hypothesis, we get

$$\delta(\sigma) \circ \kappa \pm [g(\kappa), \sigma] \in \wp, \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Taking κ by $\kappa+\varsigma, \varsigma \in \mathfrak{I}$, we get

$$\delta(\sigma) \circ \kappa + \delta(\sigma) \circ \varsigma \pm [g(\kappa), \sigma] \pm [g(\varsigma), \sigma] \pm 2[G(\kappa, \varsigma), \sigma] \in \wp.$$

Using the hypothesis, we have

$$2[G(\kappa, \varsigma), \sigma] \in \wp.$$

Since $\Omega/\wp$ is characteristic not two ring, we obtain that

$$[G(\kappa, \varsigma), \sigma] \in \wp.$$

Replacing ς by κ in the last expression, we have

$$[G(\kappa, \kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

That is,

$$[g(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \mathfrak{I}.$$

Using this in our hypothesis, we obtain that $[\delta(\sigma), \kappa] \in \wp$, and so $[\delta(\sigma), \sigma] \in \wp$ for all $\sigma \in \mathfrak{I}$. Hence δ is $\wp-$commuting on $\mathfrak{I}$. □

3. Conclusion

In this study, the subject of symmetric bi-derivations of a ring under the influence of a semiprime ideal is considered without imposing a condition on the ring. Here, the conditions used in the literature to prove that a ring is commutative are examined for symmetric bi-derivations and semiprime ideals. The results obtained provide a new perspective on the structural properties of rings with derivations. Based on this research, further research can be conducted by considering different algebraic structures such as generalized derivations instead of symmetric bi-derivations, homoderivations and alternating rings instead of rings, near rings, operator algebras, Banach algebras and other areas where ring theory plays a fundamental role.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Emine Koç Sögütcü]: Thought and designed the research/problem, contributed to research method or evaluation of data, wrote the manuscript (%60).

Author [Öznur Gölbaşı]: Collected the data, contributed to research method or evaluation of data (%40).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

On non-Newtonian Helices in Multiplicative Euclidean Space

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Abstract: In this article, spherical indicatrices of a curve and helices are re-examined using both the algebraic structure and the geometric structure of non-Newtonian (multiplicative) Euclidean space. Indicatrices of a multiplicative curve on the multiplicative sphere in multiplicative space are obtained. In addition, multiplicative general helix, multiplicative slant helix and multiplicative clad and multiplicative g-clad helix characterizations are provided. Finally, examples and drawings are given.

Keywords: Non-Newtonian calculus, spherical indicatrices, helices, multiplicative differential geometry.

1. Introduction

Classical analysis, which is a widely used mathematical theory today, was defined by Gottfried Leibniz and Isaac Newton in the second half of the 17th century based on the concepts of derivatives and integrals. Constructed upon algebra, trigonometry, and analytic geometry, the classical analysis consists of concepts such as limits, derivatives, integrals, and series. These concepts are regarded as simple versions of addition and subtraction, leading to the designation of this analysis as summational analysis. Classical analysis finds applications in various fields, including natural sciences, computer science, statistics, engineering, economics, business, and medicine, where mathematical modeling is required, and optimal solution methods are sought. However, there are situations in some mathematical models where classical analysis falls short. Therefore, alternative analyses have been defined based on different arithmetic operations while building upon classical analysis. For instance, in 1887, Volterra developed an approach known as Volterra-type analysis or multiplicative analysis since it is founded on the multiplication operation [30]. In multiplicative analysis, the roles of addition and subtraction operations in classical analysis are assumed by the multiplication and division operations, respectively. Following the definition of Volterra analysis, Grossman and Katz conducted some new studies between 1972 and 1983. This led to the development of the non-Newtonian analysis, which also involves fundamental definitions

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and concepts [14, 15]. These analyses have been referred to as geometric analysis, bigeometric analysis, and anageometric analysis. Multiplicative analysis has emerged as an alternative approach to classical analysis and has become a significant area of research and development in the field of mathematics. These new analyses may allow for a more effective resolution of various problems by examining different mathematical structures. Furthermore, these studies contribute to the expansion of the boundaries of mathematical analysis and find applications in various disciplines.

Arithmetic is an integer field which is a subset of the real numbers. An arithmetic system is the structure obtained by algebraic operations defined in this field. In fact, this field can be considered as a different interpretation of the real number field such that a countable number of infinitely ordered objects can be formed and these structures are equivalent or isomorphic to each other. The generator function, which is used to create arithmetic systems, is a one-to-one and bijective transformation whose domain is real numbers and whose value set is a subset of positive real numbers. The unit function I and the function e^x are examples of generator functions. Just as each generator produces a single arithmetic, each arithmetic can be produced with the help of a single generator. Multiplicative analysis has its own multiplicative space. In this special space, the classical number system has turned into a multiplicative number system consisting of positive real numbers, denoted by $\mathbb{R}_*$. Likewise, the basic mathematical operations in classical analysis have also turned into their purely multiplicative versions. This is clearly shown in the table below.

Table 1. Basic multiplicative operations

$a +_* b$	$e^{\log a + \log b}$	ab
$a -_* b$	$e^{\log a - \log b}$	$\frac{a}{b}$
$a \cdot_* b$	$e^{\log a \log b}$	$a^{\log b}$
$a /_* b$	$e^{\log a / \log b}$	$a^{\frac{1}{\log b}}, \ b \neq 1$

Multiplicative analysis, contrast to not a completely new topic, has recently started to be explored and discovered more in today's context. The main reason behind this lies in the successful modeling of problems that cannot be addressed using classical analysis, achieved through the application of multiplicative analysis. This characteristic has led many mathematicians to prefer multiplicative analysis for solving challenging problems that are otherwise difficult to model within their respective fields. Stanley took the lead in this regard and re-announced geometric analysis as multiplicative analysis [27]. On this subject, fractal growths of fatigue defects in materials are studied by Rybaczuk and Stoppel [24] and the physical and fractional dimension concepts are studied by Rybaczuk and Zielinski [25]. In addition, there are many studies on multiplicative analysis in the field of pure mathematics. For example, the non-Newtonian efforts in complex analysis are [6, 29], in numerical analysis [1, 7, 34], in differential equations [5, 31, 33].

Also, Bashirov et al. reconsider multiplicative analysis with some basic definitions, theorems, propositions, properties and examples [4]. The multiplicative Dirac system and multiplicative time scale are studied by Emrah et al. [13, 16].

Georgiev brought a completely different perspective to multiplicative analysis with the books titled *Multiplicative Differential Calculus*, *Multiplicative Differential Geometry* and *Multiplicative Analytic Geometry* published in 2022 [10–12]. Unlike previous studies, Georgiev used operations as purely multiplicative operations and almost reconstructed the multiplicative space. These books have been recorded as the initial studies in particular for multiplicative geometry. Georgiev's book [10] serves as a guide for researchers in this field by encompassing numerous fundamental definitions and theorems pertaining to curves, surfaces, and manifolds. The book elucidates how to associate basic geometric objects such as curves, surfaces, and manifolds with multiplicative analysis, shedding light on their properties in multiplicative spaces. Additionally, it emphasizes the connections between multiplicative geometry and other mathematical domains, making it a valuable resource for researchers working in various branches of mathematics. Afterward, Nurkan et al. tried to construct geometry with geometric calculus. In addition, Gram-Schmidt vectors are obtained [23]. On the other hand, Aydın et al. studied rectifying curves in multiplicative Euclidean space. The multiplicative rectifying curves are fully classified and visualized through multiplicative spherical curves and they studied multiplicative submanifolds and of multiplicative Euclidean space [2, 3]. Has and Yılmaz constructed multiplicative conics using multiplicative arguments [17] and in another study they investigated multiplicative magnetic curves [18]. Has , Yılmaz and Yıldırım have worked on the multiplicative Lorentz-Minkowski space [19]. Ceyhan et al. performed optical fiber examined with multiplicative quaternions [8].

A helix curve is the curve that a point follows as it rotates around a fixed axis in a three-dimensional space. The helix curve is formed as a result of this rotational movement, and the rotation time around the axis determines the stability of the curve. While the helix curve is important in terms of geometry, it is also increasing in different branches of science. For example, helix is a term used for the connections of DNA. The double helix structure of DNA is called an image helix [32]. In computer graphics and 3D applications, helix curves are used in sections of complex surfaces and their results [9]. The helix is used in blades and aerospace engineering for the design and performance analysis of propellers and rotor blades [26]. In addition, helices have been traditionally studied by many researchers with their different properties [20–22, 28, 35].

In this study, spherical indicatrices and helix curves, which are important for differential geometry, are examined in multiplicative space. Spherical indicatrices, general helix, slant helix, clad helix and g-clad helix are rearranged with reference to multiplicative operations. Moreover, in the multiplicative Euclidean space, basic concepts such as orthogonal vectors, orthogonal system, curves, Frenet frame, etc. are mentioned. In addition, it is aimed to make these basic concepts more memorable by visualizing them.

2. Multiplicative Calculus and Multiplicative Space

The definitions and theorems that will be presented in this section are taken from the works of Georgiev [10–12].

Since the multiplicative space has an exponential structure, the sets of multiplicative real numbers are we have

$$\mathbb{R}_* = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+, \quad \mathbb{R}_*^+ = \{e^x : x \in \mathbb{R}^+\} = (1,\infty) \text{ and } \mathbb{R}_*^- = \{e^x : x \in \mathbb{R}\} = (0,1). \quad (1)$$

The basic multiplicative operations for all $m,n \in \mathbb{R}_*$, are

$$m +_* n = e^{\log m + \log n} = mn, \qquad m -_* n = e^{\log m - \log n} = m/n,$$

$$m \cdot_* n = e^{\log m \log n} = m^{\log n}, \qquad m/_* n = e^{\log m / \log n} = m^{\frac{1}{\log n}}, \ n \neq 1.$$

According to the multiplicative addition operation, the multiplicative neutral and unit element are $0_* = 1$ and $1_* = e$, respectively.

The inverse elements of multiplicative addition and multiplicative multiplication operations for all $m \in \mathbb{R}_*$ are as follows, respectively:

$$-_* m = 1/m, \quad m^{-1_*} = e^{\frac{1}{\log m}}.$$

Absolute value function in multiplicative space, we have

$$|m|_* = \begin{cases} m, & m \geq 0_* \\ -_* m, & m < 0_*. \end{cases}$$

With the help of multiplicative arguments, the multiplicative power function can be given as for all $m \in \mathbb{R}_*$ and $k \in \mathbb{N}$

$$m^{k_*} = e^{(\log m)^k}, \quad m^{\frac{1}{2}_*} = \sqrt[*]{m} = e^{\sqrt{\log m}}.$$

A vector whose components are elements of the space $\mathbb{R}_*$ is called a multiplicative vector and satisfies the following properties $\vec{\mathbf{r}} = (r_1, r_2, \cdots, r_n), \vec{\mathbf{s}} = (s_1, s_2, \cdots, s_n) \in \mathbb{R}_*^n$ multiplicative vectors and $\lambda \in \mathbb{R}_*$, as follows

$$\vec{\mathbf{r}} +_* \vec{\mathbf{s}} = (r_1 +_* s_1, \cdots, r_n +_* s_n) = (r_1 s_1, \cdots, r_n s_n),$$

$$\lambda \cdot_* \vec{\mathbf{r}} = (\lambda \cdot_* r_1, \cdots, \lambda \cdot_* r_n) = (r_1^{\log \lambda}, \cdots, r_n^{\log \lambda}) = e^{\log \vec{\mathbf{r}} \log \lambda},$$

where $\log \vec{\mathbf{r}} = (\log r_1, \log r_2, \cdots, \log r_n)$. Let $\vec{\mathbf{r}} = (r_1, r_2, \cdots, r_n)$ and $\vec{\mathbf{s}} = (s_1, s_2, \cdots, s_n) \in \mathbb{R}_*^n$ be two multiplicative vectors in the multiplicative vector space $\mathbb{R}_*^n$. Thus the multiplicative inner product of two multiplicative vectors is follow

$$\langle \vec{\mathbf{r}}, \vec{\mathbf{s}} \rangle_* = r_1 \cdot_* s_1 +_* \cdots +_* r_n \cdot_* s_n = e^{\langle \log \vec{\mathbf{r}}, \log \vec{\mathbf{s}} \rangle}.$$

If the multiplicative vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ are multiplicative orthogonal to each other, they are denoted by $\vec{\mathbf{r}} \perp_* \vec{\mathbf{s}}$ and this relation is as follows

$$\langle \vec{\mathbf{r}}, \vec{\mathbf{s}} \rangle_* = 0_*.$$

In Figure 1, we present the graph of the multiplicative orthogonal vectors.

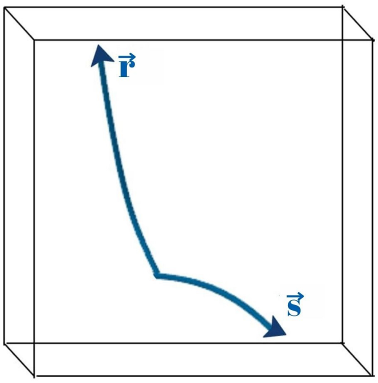

Figure 1: Multiplicative orthogonal vectors $\vec{\mathbf{r}} = (e^{\frac{1}{2}}, e^{-\frac{3}{4}}, e^{\frac{3}{2}})$ and $\vec{\mathbf{s}} = (e^{\frac{3}{4}}, e, e^{\frac{1}{4}})$

The multiplicative norm of the multiplicative vector $\vec{\mathbf{r}} \in \mathbb{R}_*^n$ is given by the multiplicative inner product is defined as follows:

$$\| \vec{\mathbf{r}} \|_* = e^{\langle \log \vec{\mathbf{r}}, \log \vec{\mathbf{r}} \rangle^{\frac{1}{2}}}.$$

Let $\vec{\mathbf{r}} = (r_1, r_2, r_3)$ and $\vec{\mathbf{s}} = (s_1, s_2, s_3)$ be 3D multiplicative vectors, and the multiplicative cross products of $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$, we have

$$\vec{\mathbf{r}} \times_* \vec{\mathbf{s}} = (e^{\log r_2 \log s_3 - \log r_3 \log s_2}, e^{\log r_3 \log s_1 - \log r_1 \log s_3}, e^{\log r_1 \log s_2 - \log r_2 \log s_1}).$$

Multiplicative cross product preserves the properties of traditional cross product with its arguments. For example, cross products of multiplicative vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ are multiplicative orthogonal to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$. We give this visually in Figure 2 The multiplicative angle between the multiplicative unit direction vectors $\vec{\mathbf{r}}, \vec{\mathbf{s}} \in \mathbb{R}_*^n$ is given by

$$\phi = \arccos_* (e^{\langle \log \vec{\mathbf{r}}, \log \vec{\mathbf{s}} \rangle}).$$

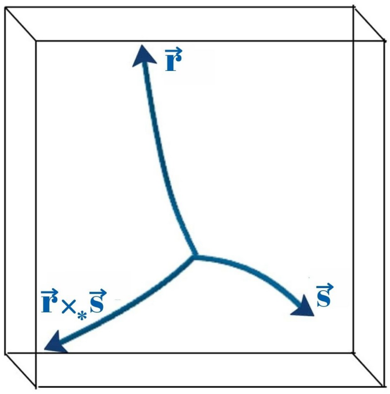

Figure 2: Multiplicative orthogonal system $\overrightarrow{\mathbf{r}} = (e^{\frac{1}{2}}, e^{-\frac{3}{4}}, e^{\frac{3}{2}})$, $\overrightarrow{\mathbf{s}} = (e^{\frac{3}{4}}, e, e^{\frac{1}{4}})$ and $\overrightarrow{\mathbf{r}} \times_* \overrightarrow{\mathbf{s}} = (e^{-\frac{27}{16}}, e, e^{\frac{17}{16}})$

Multiplicative trigonometric functions with the help of multiplicative angles

$$\sin_* \phi = e^{\sin \log \phi}, \quad \cos_* \phi = e^{\cos \log \phi},$$

$$\tan_* \phi = e^{\tan \log \phi}, \quad \cot_* \phi = e^{\cot \log \phi}.$$

Multiplicative trigonometric functions provide the same algebraic properties as traditional trigonometric functions, but with their own arguments. For example, there is the equality $\sin_*^{2*} \theta +_* \cos_*^{2*} \theta = 1_*$. For other relations, see [11].

The multiplicative derivative of the multiplicative function $f(t) \subset \mathbb{R}_*$ for $t \in I \subset \mathbb{R}_*$ is as follows

$$\begin{aligned} f^*(t) &= \lim_{h \to 0_*} \left(\left(f(t +_* h) -_* f(t) \right) /_* h \right) \\ &= \lim_{h \to 1} \exp \left[\frac{\log f(th) - \log f(t)}{\log(h)} \right] \\ &= \lim_{h \to 1} \exp \left[\frac{th f'(th)}{f(th)} \right] \\ &= e^{t \frac{f'(t)}{f(t)}}. \end{aligned}$$

Multiplicative differentiation realizes many properties provided in classical differentiation, such as linearity, Leibniz rule, chain rules, etc., based on multiplicative arguments. For examples $(f(x) \cdot_* g(x))^* = f^*(x) \cdot_* g(x) +_* g^*(x) \cdot_* f(x)$. It can also be stated as $f^*(x) = d_* f /_* d_* x$. For other relations, see [11].

The multiplicative integral of the multiplicative function $f(t) \subset \mathbb{R}_*$ is as follows for $t \in I \subset \mathbb{R}_*$

$$\int_* f(x) \cdot_* d_* x = e^{\int \frac{1}{x} \log f(x) dx}, \quad x \in \mathbb{R}_*.$$

The geometric location of points with equal multiplicative distances from a point in multiplicative space is called a multiplicative sphere. The equation of the sphere with centered at $C(a,b,c)$ and radius r is

$$\|P -_* C\|_* = r,$$

where $P = (x,y,z)$ is the representation point of the multiplicative sphere, so

$$e^{(\log x-\log a)^2+(\log y-\log b)^2+(\log z-\log c)^2} = e^{(\log r)^2}.$$

In Figure 3 we show the multiplicative sphere with centered at multiplicative origin $O(0_*,0_*,0_*)$ and radius 1_*

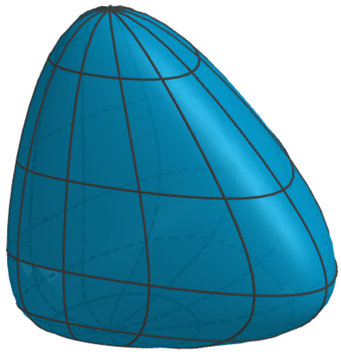

Figure 3: A multiplicative sphere with centered at multiplicative origin $O(0_*,0_*,0_*)$ and radius 1_*

3. Differential Geometry of Curves in Multiplicative Space

A multiplicative parametrization of class C_*^k $(k \geq 1_*)$ for a curve $\mathbf{x}$ in $\mathbb{R}_*^3$ (i.e., the component-functions of $\mathbf{x}$ are k-times continuously multiplicative differentiable), is a multiplicative vector valued function $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$, where s is mapped to $\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s))$. In particular, a parametric multiplicative curve $\mathbf{x}$ is regular if and only if $\|\mathbf{x}^*(s)\|_* \neq 0_*$ for any $s \in I$. Looking at it dynamically, the multiplicative vector $\mathbf{x}^*(s)$ represents the multiplicative velocity of the multiplicative curve at time s. For a multiplicative curve $\mathbf{x}$ to have multiplicative naturally parameters, the necessary and sufficient condition is that the curve is from the class C_*^k and $\|\mathbf{x}^*(s)\|_* = 1_*$ for each $s \in I$.

Given $s_0 \in I$, the multiplicative arc length of a multiplicative regular parameterized curve $\mathbf{x}(s)$ from the point s_0, is by definition

$$h(s) = \int_{*s_0}^{s} \|\mathbf{x}^*(t)\|_* \cdot_* d_* t. \tag{2}$$

As an example, the multiplicative circle curve in multiplicative plane with center $(0_*, 0_*, 0_*)$ and radius $r = e^{-2}$ is given by the equation $\mathbf{x}(s) = e^{-2} \cdot_* (e^{\frac{1}{2}} \cos_* 2s, e^{\frac{1}{2}} \cdot_* \sin_* 2s, e^{\sqrt{3}})$ in $\mathbb{R}^3_*$. It can be plotted as in Figure 4.

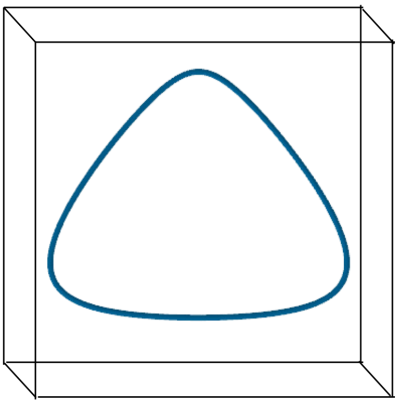

Figure 4: A multiplicative circle in the plane $z = e^{\frac{\sqrt{3}}{2}}$ with centered at $(0_*, 0_*, 0_*)$, radius $r = 1/e^2$ and $0_* < s < e^{2\pi}$

The multiplicative Frenet trihedron of a naturally parameterized multiplicative curve $\mathbf{x}(s)$ are

$$\mathbf{t}(s) = \mathbf{x}^*(s), \qquad \mathbf{n}(s) = \mathbf{x}^{**}(s) /_* \|\mathbf{x}^{**}(s)\|_*, \qquad \mathbf{b}(s) = \mathbf{t}(s) \times_* \mathbf{n}(s).$$

The vector field $\mathbf{t}(s)$ (resp. $\mathbf{n}(s)$ and $\mathbf{b}(s)$) along $\mathbf{x}(s)$ is said to be multiplicative tangent (resp. multiplicative principal normal and multiplicative binormal). It is direct to prove that $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is mutually multiplicative orthogonal and $\mathbf{n}(s) \times_* \mathbf{b}(s) = \mathbf{t}(s)$ and $\mathbf{b}(s) \times_* \mathbf{t}(s) = \mathbf{n}(s)$. We also point out that the arc length parameter and multiplicative Frenet frame are independent from the choice of multiplicative parametrization [10].

To give an example, the multiplicative Frenet vectors of the multiplicative curve

$$\mathbf{x}(s) = \left((e^3 /_* e^5) \cdot_* \cos_* s, (e^3 /_* e^5) \cdot_* \sin_* s, e^4 /_* e^5 \cdot_* e^s\right)$$

are

$$\begin{aligned}
\mathbf{t}(s) &= (-_* (e^3 /_* e^5) \cdot_* \sin_* s, (e^3 /_* e^5) \cdot_* \cos_* s, e^4 /_* e^5), \\
\mathbf{n}(s) &= (-_* \cos_* s, -_* \sin_* s, 0_*), \\
\mathbf{b}(s) &= ((e^4 /_* e^5) \cdot_* \sin_* s, -_* (e^4 /_* e^5) \cdot_* \cos_* s, e^3 /_* e^5).
\end{aligned}$$

In Figure 5, we present the graph of the multiplicative Frenet frame on the multiplicative curve $\mathbf{x}(s)$.

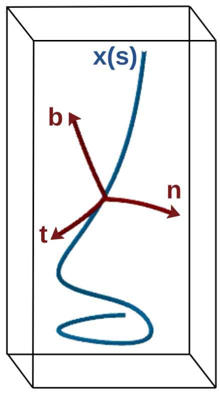

Figure 5: Multiplicative curve and its multiplicative Frenet frame

The multiplicative Frenet formulae of $\mathbf{x}$ are given by

$$\begin{aligned}\mathbf{t}^* &= \kappa \cdot_* \mathbf{n},\\ \mathbf{n}^* &= -_*\kappa \cdot_* \mathbf{t} +_* \tau \cdot_* \mathbf{b},\\ \mathbf{b}^* &= -_*\tau \cdot_* \mathbf{n},\end{aligned}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion functions of $\mathbf{x}$, calculated by

$$\kappa(s) = \|\mathbf{x}^{**}(s)\|_* = e^{\langle \log \mathbf{x}^{**}, \log \mathbf{x}^{**}\rangle^{\frac{1}{2}}}, \tag{3}$$

$$\tau(s) = \langle \mathbf{n}^*(s), \mathbf{b}(s)\rangle_* = e^{\langle \log \mathbf{n}^*(s), \log \mathbf{b}(s)\rangle}. \tag{4}$$

4. Main Results

4.1. Multiplicative Spherical Indicatries

Consider a multiplicative curve $\mathbf{x}(s) \in \mathbb{R}^3_*$. The multiplicative Frenet vectors of $\mathbf{x}$ also evolve along the curve as a multiplicative vector field. The thing to note here is that since the multiplicative Frenet vectors of the multiplicative curve $\mathbf{x}$ are multiplicative unit vectors, they form a curve on the multiplicative sphere. In this section, such curves will be examined.

The multiplicative curve $\mathbf{x}(s)$ is associated with multiplicative Frenet vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. Now, let us consider the unit multiplicative tangent vectors along $\mathbf{x}(s)$. These vectors collectively form another curve, denoted by $\mathbf{x}_t = \mathbf{t}$. This new curve resides on the surface of a multiplicative sphere with a radius of 1_* and centered at the multiplicative origin $O = (0_*, 0_*, 0_*)$. The multiplicative curve $\mathbf{x}_t$ is often referred to as the multiplicative spherical indicatrix associated with the unit multiplicative tangent vector $\mathbf{t}$. We will call this curve multiplicative tangent indicatrix of the original multiplicative curve $\mathbf{x}$, in line with the more conventional notation. With similar thought, we will call multiplicative curves $\mathbf{x}_n = \mathbf{n}$ and $\mathbf{x}_b = \mathbf{b}$ as multiplicative normal indicatrix and multiplicative binormal indicatrix of $\mathbf{x}$, respectively.

Proposition 4.1 *Let $\mathbf{x}_t$ be the multiplicative tangent indicatrix of a multiplicative naturally parameterized curve $\mathbf{x}$. The multiplicative naturally parameter s_t of $\mathbf{x}_t$ is given by*

$$s_t = e^{\int^s \frac{1}{u}\kappa(u)du},$$

where s is multiplicative naturally parameter of $\mathbf{x}$ and $\kappa(s)$ is multiplicative curvature of $\mathbf{x}$.

Proof Let $\mathbf{x}(s)$ be the multiplicative naturally parameterized curve. Also, let $\mathbf{x}_t(s) = \mathbf{t}(s)$ be the multiplicative tangent indicatrix of $\mathbf{x}(s)$. Considering (2), we get the multiplicative naturally parameter of $\mathbf{x}_t$ as follows

$$s_t = \int_*^s \|\mathbf{t}^*(u)\|_* \cdot_* d_* u.$$

Then from the definition of multiplicative curvature, we get

$$s_t = \int_*^s \kappa(u) \cdot_* d_* u$$

or equivalently

$$s_t = e^{\int^s \frac{1}{u}\kappa(u)du}.$$

□

Theorem 4.2 *Let $\mathbf{x}_t$ be the multiplicative tangent indicatrix of a multiplicative naturally parameterized curve $\mathbf{x}$ with $\kappa \neq 0_*$ on I. The multiplicative Frenet vectors $\{T_t, N_t, B_t\}$ of $\mathbf{x}_t$ satisfy*

$$\begin{aligned} T_t &= \mathbf{n}, \\ N_t &= (-_*\mathbf{t} +_* f \cdot_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \\ B_t &= (f \cdot_* \mathbf{t} +_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \end{aligned}$$

where $f = f(s)$ and $f = \tau/_*\kappa$.

Proof Since $\mathbf{x}_t$ is the multiplicative tangent indicatrix of $\mathbf{x}$, we have

$$\mathbf{x}_t = \mathbf{t}.$$

Taking the multiplicative derivative of both sides of the above equation with respect to s,

$$(d_*\mathbf{x}_t/_* d_* s_t) \cdot_* (d_* s_t/_* d_* s) = \kappa \cdot_* \mathbf{n}.$$

Using Proposition 4.1 and putting $T_t = d_* \mathbf{x}_t /_* d_* s_t$, we get

$$T_t = \mathbf{n}. \tag{5}$$

If we take the multiplicative derivative of (5) with respect to s and apply multiplicative Frenet formulas, we have

$$\kappa \cdot_* (d_* T_t /_* d_* s_t) = -_* \kappa \cdot_* \mathbf{t} +_* \tau \cdot_* \mathbf{b}$$

and

$$d_* T_t /_* d_* s_t = -_* \mathbf{t} +_* (\tau /_* \kappa) \cdot_* \mathbf{b}.$$

Considering the multiplicative norm, the following equation is obtained:

$$\begin{aligned} \| d_* T_t /_* d_* s_t \|_* &= e^{(\langle -\log \mathbf{t}, -\log \mathbf{t} \rangle + (\frac{\log \tau}{\log \kappa})^2 \langle \log \mathbf{b}, \log \mathbf{b} \rangle)^{\frac{1}{2}}} \\ &= e^{\sqrt{(1+(\frac{\log \tau}{\log \kappa})^2)}}. \end{aligned} \tag{6}$$

In that case, we can see that

$$N_t = (d_* T_t /_* d_* s_t) /_* \| d_* T_t /_* d_* s_t \|_* = (-_* \mathbf{t} +_* (\tau /_* \kappa) \cdot_* \mathbf{b}) /_* e^{(1+(\frac{\log \tau}{\log \kappa})^2)^{\frac{1}{2}}}.$$

Setting $\tau /_* \kappa = f$,

$$N_t = (-_* \mathbf{t} +_* f \cdot_* \mathbf{b}) /_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}. \tag{7}$$

On the other hand, if we take into account (5) and (7) along with the multiplicative Frenet formulas, we obtain the final Frenet vector as

$$B_t = [\mathbf{n} \times_* (-_* \mathbf{t} +_* f \cdot_* \mathbf{b})] /_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}.$$

When we organize the multiplicative operations, we obtain

$$B_t = (f \cdot_* \mathbf{t} +_* \mathbf{b}) /_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}. \tag{8}$$

□

Proposition 4.3 *Let $\mathbf{x}_t$ be the multiplicative tangent indicatrix of a multiplicative naturally parameterized curve $\mathbf{x}$ with $\kappa \neq 0_*$ on I. The multiplicative curvatures of $\mathbf{x}_t$ are*

$$\kappa_t = e^{(1+(\log f(s))^2)^{\frac{1}{2}}} \quad \text{and} \quad \tau_t = \sigma \cdot_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}},$$

where $\sigma = f^* /_* (e^{\log \kappa (1+(\log f(s))^2)^{\frac{3}{2}}})$.

Proof The first equality follows by (6),

$$\kappa_t = e^{\langle \log \mathbf{x}_t^{**}, \log \mathbf{x}_t^{**} \rangle^{\frac{1}{2}}} = e^{(1+(\log f(s))^2)^{\frac{1}{2}}}.$$

Next considering calculate the multiplicative derivative of N_t with respect to s_t, considering (7). Also, is chosen $e^{(1+(\log f(s))^2)^{\frac{1}{2}}} = \lambda$ in (7), so

$$(d_* N_t /_* d_* s_t) \cdot_* (d_* s_t /_* d_* s) = [(-_* \kappa \cdot_* \mathbf{n} +_* f^* \cdot_* \mathbf{b} -_* f \cdot_* \tau \cdot_* \mathbf{n}) \cdot_* \lambda -_* \lambda \cdot_* (-_* \mathbf{t} +_* f \cdot \mathbf{b})] /_* \lambda^{2*}.$$

After this we can write

$$N_t^* \cdot_* \kappa = [\lambda^* \cdot_* \mathbf{t} -_* \lambda \cdot_* (\kappa +_* f \cdot_* \tau) \cdot_* \mathbf{n} +_* (\lambda \cdot_* f^* -_* \lambda^* \cdot_* f) \cdot_* \mathbf{b}] /_* \lambda^{2*}$$

and so

$$N_t^* = [\lambda^* \cdot_* \mathbf{t} -_* \lambda \cdot_* (\kappa +_* f \cdot_* \tau) \cdot_* \mathbf{n} +_* (\lambda \cdot_* f^* -_* \lambda^* \cdot_* f) \cdot_* \mathbf{b}] /_* \lambda^{2*} \cdot_* \kappa. \tag{9}$$

Then from (8) and (9), we obtain

$$\begin{aligned} \tau_t = \langle \log N_t^*, \log B_t \rangle_* &= (f \cdot_* \lambda^*) /_* \lambda^{3*} \cdot_* \kappa +_* (\lambda \cdot_* f^* -_* \lambda^* \cdot_* f) /_* \lambda^{3*} \cdot_* \kappa \\ &= (f^* \cdot_* \lambda) /_* \lambda^{3*} \cdot_* \kappa. \end{aligned}$$

Here again let's consider the choice $e^{(1+(\log f(s))^2)^{\frac{1}{2}}} = \lambda$, so we get

$$[f^* /_* (e^{(1+(\log f(s))^2)^{\frac{3}{2}}} \cdot_* \kappa)] \cdot_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}.$$

Finally, if a choice is made in the form $\sigma = f^* /_* (e^{\log \kappa (1+(\log f(s))^2)^{\frac{3}{2}}} \cdot_* \kappa)$, the above-mentioned equation becomes

$$\tau_t = \sigma \cdot_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}.$$

□

Using similar arguments, we may have the following results.

Proposition 4.4 *Let $\mathbf{x}_n$ be the multiplicative normal indicatrix of a multiplicative naturally parameterized curve $\mathbf{x}$. The multiplicative arc parameter s_n of the multiplicative curve $\mathbf{x}_n$ provides*

$$s_n = \int_* \kappa(s) \cdot_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}} \cdot_* d_* s, \tag{10}$$

where $f = f(s)$ and $f = \tau /_* \kappa$.

Theorem 4.5 *Let* $\mathbf{x}_n$ *be the multiplicative normal indicatrix of a multiplicative naturally parameterized curve* $\mathbf{x}$. *The multiplicative Frenet vectors* $\{T_n, N_n, B_n\}$ *of* $\mathbf{x}_n$ *as follows*

$$\begin{aligned} T_n &= (-_*\mathbf{t} +_* f \cdot_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \\ N_n &= (\sigma/_* e^{(1+(\log \sigma(s))^2)^{\frac{1}{2}}}) \cdot_* [((f \cdot_* \mathbf{t} +_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}) -_* \mathbf{n}/_* \sigma], \\ B_n &= (e/_* e^{(1+(\log \sigma(s))^2)^{\frac{1}{2}}}) \cdot_* [((f \cdot_* \mathbf{t} +_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}} -_* \mathbf{n} \cdot_* \sigma], \end{aligned}$$

where $\sigma = f^* /_* (e^{\log \kappa (1+(\log f(s))^2)^{\frac{3}{2}}})$.

Proposition 4.6 *Let* $\mathbf{x}_n$ *be the multiplicative normal indicatrix of the multiplicative curve* $\mathbf{x}$. *The multiplicative curvatures of the normal indicatrix* $\mathbf{x}_n$ *are described as follows*

$$\kappa_n = e^{(1+(\log \sigma(s))^2)} \quad and \quad \tau_n = \Gamma \cdot_* e^{(1+(\log f(s))^2)}, \tag{11}$$

where $\Gamma = \sigma^* /_* (e^{\log \kappa (1+(\log f(s))^2)(1+(\log \sigma(s))^2)^{\frac{3}{2}}}$.

Proposition 4.7 *Let* $\mathbf{x}_b$ *be the multiplicative binormal indicatrix of a multiplicative naturally parameterized curve* $\mathbf{x}$. *The multiplicative arc parameter* s_b *of the multiplicative curve* $\mathbf{x}_b$ *provides*

$$s_b = \int_* \tau(s) d_* s.$$

Theorem 4.8 *Let* $\mathbf{x}_b$ *be the multiplicative binormal indicatrix of a multiplicative naturally parameterized curve* $\mathbf{x}$. *The multiplicative Frenet vectors* $\{T_b, N_b, B_b\}$ *of* $\mathbf{x}_b$ *satisfy*

$$\begin{aligned} T_b &= -_*\mathbf{n}, \\ N_b &= (\mathbf{t} -_* f \cdot_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \\ B_b &= (f \cdot_* \mathbf{t} +_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \end{aligned}$$

where $f = f(s)$ *and* $f = \tau/_*\kappa$.

Proposition 4.9 *Let the multiplicative curve, denoted as* $\mathbf{x}_b$, *be the binormal indicatrix of the multiplicative curve* $\mathbf{x}$. *Then, the multiplicative curvatures of the* $\mathbf{x}_b$ *are described as follows*

$$\kappa_b = e^{(1+(\log f(s))^2)^{\frac{1}{2}}} /_* f \quad and \quad \tau_b = (-_*\sigma \cdot_* e^{(1+(\log \sigma(s))^2)^{\frac{1}{2}}})/_* f,$$

where $\sigma = f^* /_* (\kappa \cdot_* (e^{1+(\log f(s))^2})^{\frac{3}{2}*})$.

Example 4.10 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$ *be multiplicative naturally parametrized curve in* $\mathbb{R}_*^3$ *parameterized by*

$$\mathbf{x}(s) = \left(e^s, e^{\frac{e^2}{2}}, e^{\frac{e^3}{6}}\right).$$

In Figure 6, we present the graph of the multiplicative spherical indicatrices of $\mathbf{x}$.

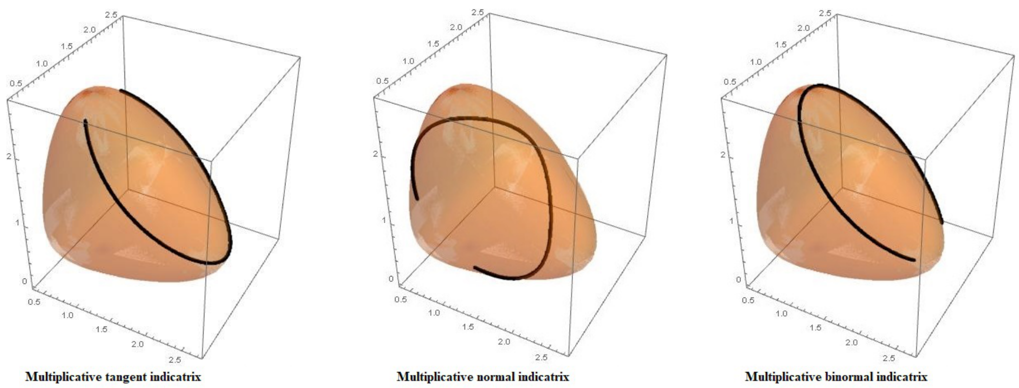

Figure 6: Multiplicative spherical indicatrices

4.2. Multiplicative Helices

Definition 4.11 *Consider the multiplicative curve* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$ *with* $\kappa \neq 0_*$. *If the multiplicative tangent vector field of the curve* $\mathbf{x}$ *makes a constant multiplicative angle with a constant multiplicative vector, then the curve* $\mathbf{x}$ *is referred to as a multiplicative general helix [10].*

Theorem 4.12 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$ *be a multiplicative curve with* $\kappa \neq 0_*$. *The multiplicative space curve* $\mathbf{x}$ *is a multiplicative general helix if and only if the multiplicative ratio of multiplicative torsion and multiplicative curvature is constant. In other words, it is*

$$\kappa /_* \tau = c, \quad c \in \mathbb{R}_*.$$

Proof The proof of the theorem is explained by Georgiev (see [10]). □

Example 4.13 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$ *be multiplicative naturally parametrized general helix curve in* $\mathbb{R}_*^3$ *parameterized by*

$$\mathbf{x}(s) = \left(e^3 /_* e^5 \cdot_* \cos_* s, e^3 /_* e^5 \cdot_* \sin_* s, e^4 /_* e^5 \cdot_* e^s\right).$$

With the help of multiplicative curvature formulas from (4), we give

$$\kappa(s) = e^3 /_* e^5 \quad \textit{and} \quad \tau = e^4 /_* e^5.$$

Since

$$\tau/_*\kappa = e^{(\log e^4/\log e^5)/(\log e^3/\log e^5)} = e^4/_* e^3$$

is a multiplicative constant, $\mathbf{x}$ *is a multiplicative helix. In Figure 7, we present the graph of the multiplicative general helix.*

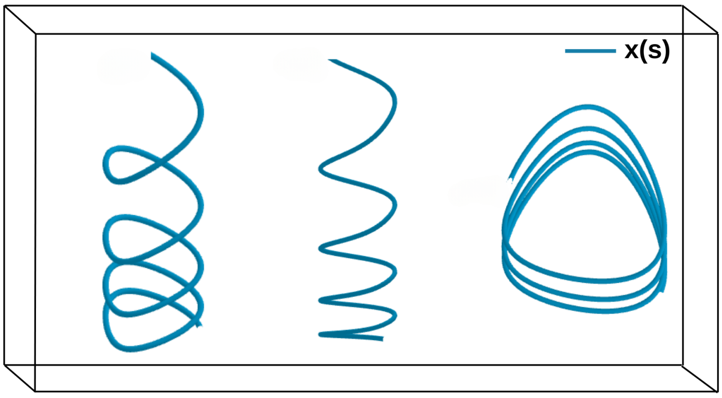

Figure 7: Multiplicative general helix

Definition 4.14 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}^3_*$ *be the multiplicative curve with* $\kappa \neq 0_*$. *If the multiplicative normal vector field of the curve* $\mathbf{x}$ *makes a constant multiplicative angle with a constant multiplicative vector, then the curve* $\mathbf{x}$ *is referred to as a multiplicative slant helix.*

Theorem 4.15 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}^3_*$ *be multiplicative curve with* $\kappa \neq 0_*$. *The multiplicative curve* $\mathbf{x}$ *is a multiplicative slant helix if and only if the following equality is a multiplicative constant function*

$$\sigma(s) = [\kappa^{2*}(s)/_*(\kappa^{2*}(s) +_* \tau^{2*}(s))^{\frac{3}{2}*}] \cdot_* (\tau(s)/_*\kappa(s))^*. \tag{12}$$

Proof Suppose that multiplicative naturally parametrized curve $s \mapsto \mathbf{x}(s)$ is a multiplicative slant helix. Since the multiplicative normal vector field of the multiplicative curve $\mathbf{x}$ makes a constant multiplicative angle with $\mathbf{v}$, which is a constant multiplicative vector, we have

$$\langle \mathbf{n}, \mathbf{v} \rangle_* = \cos_* \theta, \tag{13}$$

where θ constant multiplicative angle. Taking a multiplicative derivative of (13), we get

$$\langle \mathbf{n}^*, \mathbf{v} \rangle_* = 0_* \tag{14}$$

and

$$-_*\kappa \cdot_* \langle \mathbf{t}, \mathbf{v} \rangle_* +_* \tau \cdot_* \langle \mathbf{b}, \mathbf{v} \rangle_* = 0_*.$$

As can be seen from the elements of multiplicative Frenet frame and (13), there is a constant angle between $\mathbf{n}$ and fixed direction $\mathbf{v}$ and there is also a constant angle between $\mathbf{b}$ and fixed direction $\mathbf{v}$. Then the following equations are provided,

$$\langle \mathbf{t}, \mathbf{v} \rangle_* = (c \cdot_* \tau)/_* \kappa, \tag{15}$$

$$\langle \mathbf{b}, \mathbf{v} \rangle_* = c, \ c \in \mathbb{R}_*. \tag{16}$$

In terms of the multiplicative Frenet frame, we can write the decomposition for $\mathbf{v}$ as

$$\begin{aligned} \mathbf{v} &= e^{\langle \log \mathbf{t}, \log \mathbf{v} \rangle \log \mathbf{t} + \langle \log \mathbf{n}, \log \mathbf{v} \rangle \log \mathbf{n} + \langle \log \mathbf{b}, \log \mathbf{v} \rangle \log \mathbf{b}} \\ &= \langle \mathbf{t}, \mathbf{v} \rangle_* \cdot_* \mathbf{t} +_* \langle \mathbf{n}, \mathbf{v} \rangle_* \cdot_* \mathbf{n} +_* \langle \mathbf{b}, \mathbf{v} \rangle_* \cdot_* \mathbf{b}. \end{aligned}$$

The constant direction $\mathbf{v}$ from (13), (15) and (16) is obtained as follows

$$\mathbf{v} = (c \cdot_* \tau)/_* \kappa \cdot_* \mathbf{t} +_* \cos_* \theta \cdot_* \mathbf{n} +_* c \cdot_* \mathbf{b}. \tag{17}$$

Since $\mathbf{v}$ is the multiplicative unit vector, taking the multiplicative norm of both sides of the above equation, we get

$$\begin{aligned} e^{\langle \log \mathbf{v}, \log \mathbf{v} \rangle^{\frac{1}{2}}} &= ((c \cdot_* \tau)/_* \kappa)^{2*} \cdot_* e^{\langle \log \mathbf{t}, \log \mathbf{t} \rangle^{\frac{1}{2}}} +_* \cos_*^{2*} \theta \cdot_* e^{\langle \log \mathbf{n}, \log \mathbf{n} \rangle^{\frac{1}{2}}} \\ +_* & \ c^{2*} \cdot_* e^{\langle \log \mathbf{b}, \log \mathbf{b} \rangle^{\frac{1}{2}}} \end{aligned}$$

or

$$c^{2*} \cdot_* (\tau^{2*}/_* \kappa^{2*} +_* e) = \sin_*^{2*} \theta.$$

If the necessary algebraic operations are performed here, we obtain

$$c = (\kappa/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* \sin_* \theta.$$

Therefore, we can easily write $\mathbf{v}$ as

$$\mathbf{v} = \tau/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* sin_* \theta \cdot_* \mathbf{t} +_* \cos_* \theta \cdot_* \mathbf{n} +_* \kappa/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* sin_* \theta \cdot_* \mathbf{b}. \tag{18}$$

Take the multiplicative derivative of (14), we get

$$\langle \mathbf{n}^{**}, \mathbf{v} \rangle_* = 0_*. \tag{19}$$

From multiplicative Frenet frame and (18) and (19), we have

$$\begin{aligned} &\langle -_* \kappa^* \cdot_* \mathbf{t} -_* (\kappa^{2*} +_* \tau^{2*}) \cdot_* \mathbf{n} +_* \tau^* \mathbf{b}, \tau/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* sin_* \theta \cdot_* \mathbf{t} \\ &+_* \cos_* \theta \cdot_* \mathbf{n} +_* \kappa/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* sin_* \theta \cdot_* \mathbf{b} \rangle_* = 0_*. \end{aligned}$$

Here the following equation exists

$$(\kappa \cdot_* \tau^* - \tau \cdot_* \kappa^*)/_*(\kappa^{2*} +_* \tau^{2*})^{\frac{3}{2}*} \cdot_* \tan_* \theta + e = 0_*$$

and finally, we get

$$\tan_* \theta = (\kappa \cdot_* \tau^* - \tau \cdot_* \kappa^*)/_*(\kappa^{2*} +_* \tau^{2*})^{\frac{3}{2}*}.$$

Since the multiplicative angle θ is constant, after the necessary adjustments, we obtain that

$$\kappa^{2*}/_*(\kappa^{2*} +_* \tau^{2*})^{\frac{3}{2}*} \cdot_* (\tau/_*\kappa)^* = c, \quad c \in \mathbb{R}_*.$$

□

Example 4.16 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$ *be multiplicative naturally parametrized slant helix curve in* $\mathbb{R}_*^3$ *as*

$$\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s)),$$

where

$$\begin{aligned} x_1(s) &= e^9/_*e^{400} \cdot_* e^{\sin\log 25s} +_* e^{25}/_*e^{144} \cdot_* e^{\sin\log 9s}, \\ x_2(s) &= -_*e^9/_*e^{400} \cdot_* e^{\cos\log 25s} +_* e^{25}/_*e^{144} \cdot_* e^{\cos\log 9s}, \\ x_3(s) &= e^{15}/_*e^{136} \cdot_* e^{\sin\log 17s}. \end{aligned}$$

In Figure 8, we present the graph of the multiplicative slant helix.

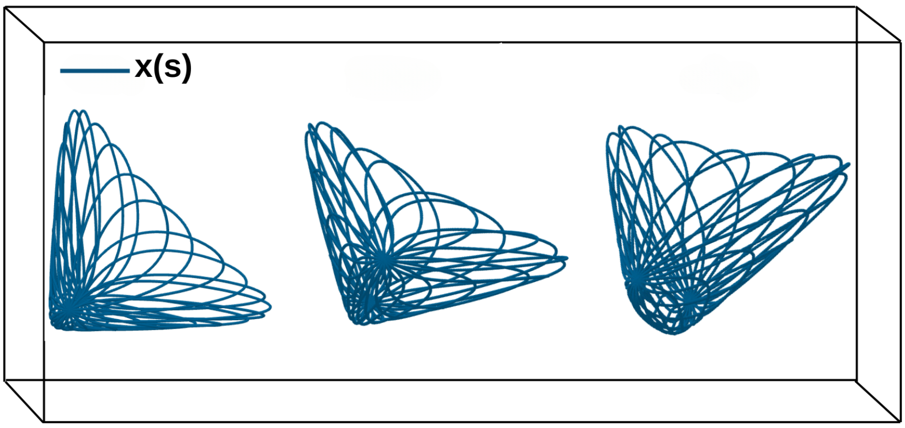

Figure 8: Multiplicative slant helix

Definition 4.17 *Let a regular multiplicative curve* $\mathbf{x}$ *be given in the multiplicative space with* $\kappa \neq 0_*$. *The multiplicative curve* $\mathbf{x}$ *is called the multiplicative clad helix if the multiplicative spherical image of the multiplicative principal normal vector* $\mathbf{n} : I \to \mathbb{S}^2_*$ *(*$\mathbb{S}^2_*$ *denotes the multiplicative sphere) of the curve* $\mathbf{x}$ *is part of the multiplicative cylindrical helix in* $\mathbb{S}^2_*$.

Therefore we remark that a multiplicative slant helix is a multiplicative clad helix. We have the following characterization of clad helices.

Theorem 4.18 *Let* $\mathbf{x}$ *be a multiplicative naturally parametrized curve with* $\kappa \neq 0_*$. *Then* $\mathbf{x}$ *is a multiplicative clad helix if and only if*

$$\Gamma = \sigma^* /_* [\kappa \cdot_* (e +_* f^{2*}) \cdot_* (e +_* \sigma^{2*})^{\frac{3}{2}*}]$$

is a constant function. Here, $f = \tau /_* \kappa$ *and* $\sigma = f^* /_* (\kappa \cdot_* (e +_* f^{2*})^{\frac{3}{2}*})$.

Proof With the multiplicative normal indicatrix of the multiplicative curve $\mathbf{x}$ being $\mathbf{x}_n$, we know from (11) that the multiplicative curvatures of $\mathbf{x}_n$ are as follows

$$\kappa_n = (e +_* \sigma^{2*})^{\frac{1}{2}*},$$

$$\tau_n = \Gamma \cdot_* (e +_* \sigma^{2*})^{\frac{1}{2}*}.$$

It follows that $\Gamma = \tau_n /_* \kappa_n$. For a part of $\mathbf{x}_n$ to be a multiplicative cylindrical helix, $\tau_n /_* \kappa_n$ must be a multiplicative constant. This means that Γ is a multiplicative constant. □

Definition 4.19 *Let a regular multiplicative curve* $\mathbf{x}$ *be given in the multiplicative space with* $\kappa \neq 0_*$. *The multiplicative curve* $\mathbf{x}$ *is called the multiplicative g-clad helix if the multiplicative spherical image of the multiplicative principal normal vector* $\mathbf{n} : I \to \mathbb{S}^2_*$ *of the curve* $\mathbf{x}$ *is part of the multiplicative slant helix in* $\mathbb{S}^2_*$.

We have the following characterization of g-clad helices.

Theorem 4.20 *Let* $\mathbf{x}$ *be a multiplicative naturally parametrized curve with* $\kappa \neq 0_*$. *Then* $\mathbf{x}$ *is a multiplicative g-clad helix if and only if*

$$\psi(s) = \Gamma^*(s) /_* \left((\kappa^{2*}(s) +_* \tau^{2*}(s))^{\frac{1}{2}*} \cdot_* (e +_* \sigma^{2*}(s))^{\frac{1}{2}*} \cdot_* (e +_* \Gamma^{2*}(s))^{\frac{3}{2}*} \right)$$

is a constant function.

Proof With the multiplicative normal indicatrix of the multiplicative curve $\mathbf{x}$ being $\mathbf{x}_n$, from (11) the multiplicative curvatures of $\mathbf{x}_n$ are as follows

$$\begin{aligned}\kappa_n &= (e +_* \sigma^{2*})^{\frac{1}{2}*},\\ \tau_n &= \Gamma \cdot_* (e +_* \sigma^{2*})^{\frac{1}{2}*}.\end{aligned}$$

If the necessary algebraic operations are performed here, we get

$$\kappa_n^{2*} +_* \tau_n^{2*} = (e +_* \sigma^{2*}) \cdot_* (e +_* \Gamma^{2*}).$$

From (12), we know that

$$\left(\kappa_n^{2*} /_* (\kappa_n^{2*} +_* \tau_n^{2*})^{\frac{3}{2}*}\right) \cdot_* (\tau_n /_* \kappa_n)^* = c, c \in \mathbb{R}_*.$$

So, we can easily see that

$$\Gamma^* /_* \left((\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* (e +_* \sigma^{2*})^{\frac{1}{2}*} \cdot_* (e +_* \Gamma^{2*})^{\frac{3}{2}*}\right). \tag{20}$$

If the normal indicatrix of the multiplicative curve $\mathbf{x}$ is a slant helix, (20) is a constant function. This completes the proof. □

Proposition 4.21 *Considering Theorems 4.18 and 4.20 that a multiplicative slant helix is a multiplicative helix with the condition $\sigma \equiv 0_*$, a multiplicative clad helix is a multiplicative slant helix with the condition $\Gamma \equiv 0_*$ and a multiplicative g-clad helix is a multiplicative clad helix with the condition $\psi \equiv 0_*$. Hence, we have the following relation*

$$\left\{\begin{array}{c}\textit{the family of}\\ \textit{multiplicative}\\ \textit{helices}\end{array}\right\} \subset \left\{\begin{array}{c}\textit{the family of}\\ \textit{multiplicative}\\ \textit{slant helices}\end{array}\right\} \subset \left\{\begin{array}{c}\textit{the family of}\\ \textit{multiplicative}\\ \textit{clad helices}\end{array}\right\} \subset \left\{\begin{array}{c}\textit{the family of}\\ \textit{multiplicative}\\ \textit{g-clad helices}\end{array}\right\}.$$

Example 4.22 *Let $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$ be multiplicative naturally parametrized clad helix curve in $\mathbb{R}_*^3$ parameterized by*

$$\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s)),$$

where

$$\begin{aligned}x_1(s) &= e^{18} \cdot_* \cos_* 3s \cdot_* \cos_*(e^6 \cdot_* \cos_* 3s),\\ x_2(s) &= e^{-18} \cdot_* \cos_* 3s \cdot_* \sin_*(e^6 \cdot_* \cos_* 3s),\\ x_3(s) &= \sin_* 2s.\end{aligned}$$

In Figure 9, we present the graph of the multiplicative clad helix

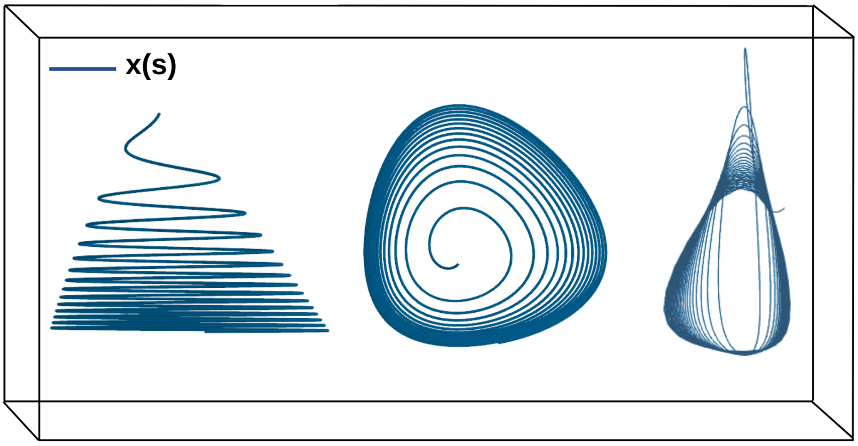

Figure 9: Multiplicative clad helix.

5. Conclusion

In this article, helices were examined using multiplicative arguments. The key point of this study is that the concept of metric, which is very important for geometry, is different from the traditional Euclidean metric. The metric here is a metric of multiplicative space based on proportional difference. Thanks to the multiplicative metric, helices, which are an important field of study in differential geometry, have been re-characterized and this change has been supported with examples. In this way, some applications of multiplicative space in differential geometry have been introduced and will be an example for other researchers.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Aykut Has]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).

Author [Beyhan Yılmaz]: Thought and designed the research/problem, contributed to research method or evaluation of data, wrote the manuscript (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

Exploring the Novel Wave Structures of the Kairat-X Equation via Two Analytical Methods

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Abstract: This paper aims to investigate the Kairat-X equation in the context of the ferromagnetic materials, optical fibers, differential geometry of curves, and equivalence aspects. Two efficient techniques are used to obtain new solutions: the modified extended tanh expansion method and the $\left(\frac{G'}{G^2}\right)$-expansion function method. By applying these methods, the nonlinear ordinary differential form of the analyzed equation is obtained using the appropriate wave transform. The effective application of the proposed approaches has yielded a substantial number of analytical solutions for the model, including hyperbolic, bright-dark soliton, W-shaped soliton, and mixed-type trigonometric, rational, and trigonometric solutions. These methods are advantageous in deriving a wide variety of exact solutions; however, they can also present limitations in terms of computational complexity and the scope of applicable equations. Various graphical representations are given to enhance the understanding of the obtained solutions. To the best of our knowledge, all derived solutions are novel. Furthermore, the correctness of each solution has been verified using Maple software.

Keywords: Kairat-X equation, the modified extended tanh expansion method, the $\left(\frac{G'}{G^2}\right)$-expansion function method.

1. Introduction

Nonlinear partial differential equations (NLPDEs) are used to model complex physical phenomena in physics, mechanics, biology, chemistry, and engineering [8, 10, 26, 30]. The study of nonlinear wave phenomena has attracted significant attention in recent years, including breathing waves, rogue waves, and solitons. The derivation of soliton solutions for NLPDEs has become an extremely fascinating and active field of research for many scientists working in engineering and applied sciences. Solitons, which are widely used in science and engineering, play a crucial technological role in enabling the transmission of digital information through optical fibers. In electromagnetics,

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solitons are also studied, such as transverse electromagnetic waves between two superconducting metal strips. Optical solitons are an important area of study in nonlinear optics, covering a wide range of topics such as metasurfaces, crystals, birefringence, magneto-optics, optical fibers, and optical couplers [27]. Optical solitons, also known as soliton wave packets, are characterized by their stability over long propagation distances. High-speed data transmission over optical fibers and the operation of technologies such as all-optical switches depend on this property. For modern telecommunications to be reliable and efficient, optical solitons and their stability are crucial [31]. Researchers are increasingly recognizing the significant contributions that mathematical approaches and computational technologies make to science, especially in areas where technological advancements and real-world applications are involved [28].

In recent years, finding exact solutions to NLPDEs has become crucial. Many direct and effective approaches have been developed to help engineers and physicists better understand the mechanisms governing these physical models, as well as the associated challenges and potential applications. Several efficient methods have been proposed for determining the implicit soliton solutions of nonlinear equations, including the tanh-function method [19], the modified simplest equation approach [18], the sine-cosine method [5, 29], the complete discriminant system method [3], the Jacobi elliptic function method [12, 14, 15], the first integral method [6], the modified sub-equation method [11, 24], the modified expansion rational function method [7] and the sub-equation method [1, 22].

This paper focuses on the Kairat-X equation (K-XE), a NLPDE that emerges in contexts such as nonlinear optics, ferromagnetic media, and optical fiber systems. The K-XE is given by [13, 23]:

$$\rho_{tt} + \rho_{xxxt} - 3\left(\rho_x \rho_t\right)_x = 0, \tag{1}$$

where $\rho = \rho(x,t)$ denotes the real wave function, with the nonlinear interaction and dispersion effects represented by the terms $(\rho_x\rho_t)_x$ and ρ_{xxxt}, respectively. The K-XE was initially formulated by Myrzakulova, who studied its Lax pair representation in order to demonstrate its integrable properties [23]. Faridi et al. employed the new auxiliary equation method to derive new soliton solutions for the same equation [13]. In their study, numerous soliton solutions with diverse characteristics such as complex waves, plane waves, shock waves, and exponential wave forms were obtained. Iqbal et al. investigated the fractional form of this equation and applied the extended simple equation method to obtain various solutions in trigonometric, exponential, and rational forms [17]. Ahmad et al. utilized the unified method to derive several exact analytical solutions for the same model [2]. Samina et al. used the generalized auxiliary equation method to obtain soliton solutions of the (1) and also performed a detailed analysis of its bifurcation structure, chaotic dynamics, and sensitivity characteristics [25].

The aim of this work is to improve wave behavior through study and enhance its practical applications, particularly in the field of telecommunications [9]. It takes an interdisciplinary approach by combining physics, computer science, and mathematics, emphasizing the role of active scientific research in solving real-world problems and advancing technology.

The existence of a Lax pair implies that the model possesses infinitely many conservation laws and can admit soliton-type solutions. This feature justifies the use of powerful analytical techniques such as the modified extended tanh expansion method (METEM) and the $\left(\frac{G'}{G^2}\right)$-expansion function method, as applied in this study. By utilizing different solution prototypes for the considered model, new approaches are presented to improve data transmission rates, optimize optical systems, and advance nonlinear optics toward more reliable and efficient communication technologies.

2. Methodology

Suppose that the presence of a NLPDE of the form:

$$N(\rho, \rho_x, \rho_t, \rho_{xx}, \rho_{xt}, \rho_{tt}, \cdots) = 0, \tag{2}$$

in which $\rho = \rho(x,t)$ is an arbitrary function of x and t with its partial derivatives.

Applying the next wave transformation

$$\rho(x,t) = V(\xi), \quad \xi = (\kappa x - \eta t), \tag{3}$$

then (2) reduces to the following form:

$$O(V, V', V'', V''', \cdots) = 0. \tag{4}$$

Here, η and κ are real constants different from zero.

2.1. Basic Steps of the METEM

This section presents the fundamental steps of the METEM approach [20].

Step 1: Consider the general solution of (4) in the form:

$$V(\xi) = M_0 + \sum_{s=1}^{R} \left(M_s \Phi^s(\xi) + L_s \Phi^{-s}(\xi)\right) \quad (M_R \neq 0 \text{ or } L_R \neq 0), \tag{5}$$

where $\Phi(\xi)$ defined as follows:

$$\frac{d\Phi(\xi)}{d\xi} = \Theta + \left(\Phi(\xi)\right)^2, \tag{6}$$

in which Θ is arbitrary constant. The following expressions represent the general solutions of (6):

Case 1: When $\Theta < 0$, the corresponding **hyperbolic solutions** can be written as follows:

$$\Phi_1(\xi) = -\sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\xi+\xi_0)\right), \tag{7}$$

$$\Phi_2(\xi) = -\sqrt{-\Theta}\coth\left(\sqrt{-\Theta}\,(\xi+\xi_0)\right), \tag{8}$$

$$\Phi_3(\xi) = -\sqrt{-\Theta}\left(\tanh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right) + i\varepsilon\,\mathrm{sech}\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)\right), \tag{9}$$

$$\Phi_4(\xi) = \frac{-\sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\xi+\xi_0)\right)+\Theta}{\sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\xi+\xi_0)\right)+1}, \tag{10}$$

$$\Phi_5(\xi) = \frac{\sqrt{-\Theta}\left(-4\cosh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)+5\right)}{4\sinh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)+3}, \tag{11}$$

$$\Phi_6(\xi) = \frac{\varepsilon\sqrt{-\Theta\,(c^2+d^2)} - c\sqrt{-\Theta}\cosh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)}{c\sinh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)+d}, \tag{12}$$

$$\Phi_7(\xi) = \varepsilon\sqrt{-\Theta}\left[1 - \frac{2c}{c+\cosh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right) - \varepsilon\sinh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)}\right]. \tag{13}$$

Case 2: If $\Theta > 0$, the desired **trigonometric solutions** can be expressed as follows:

$$\Phi_8(\xi) = \sqrt{\Theta}\tan\left(\sqrt{\Theta}\,(\xi+\xi_0)\right), \tag{14}$$

$$\Phi_9(\xi) = -\sqrt{\Theta}\cot\left(\sqrt{\Theta}\,(\xi+\xi_0)\right), \tag{15}$$

$$\Phi_{10}(\xi) = \sqrt{\Theta}\left(\tan\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right) + \varepsilon\sec\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)\right), \tag{16}$$

$$\Phi_{11}(\xi) = -\frac{\sqrt{\Theta}\left(1-\tan\left(\sqrt{\Theta}\,(\xi+\xi_0)\right)\right)}{1+\tan\left(\sqrt{\Theta}\,(\xi+\xi_0)\right)}, \tag{17}$$

$$\Phi_{12}(\xi) = \frac{\sqrt{\Theta}\left(-5\cos\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)+4\right)}{5\sin\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)+3}, \tag{18}$$

$$\Phi_{13}(\xi) = \frac{\varepsilon\sqrt{\Theta\,(c^2+d^2)} - c\sqrt{\Theta}\cos\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)}{c\sin\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)+d}, \tag{19}$$

$$\Phi_{14}(\xi) = i\varepsilon\sqrt{\Theta}\left[1 - \frac{2c}{c+\cos\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right) - i\varepsilon\sin\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)}\right]. \tag{20}$$

Case 3: For $\Theta = 0$, the relevant **rational solution** can be derived as follows:

$$\Phi_{15}(\xi) = -\frac{1}{\xi + \xi_0}. \tag{21}$$

Here, $\varepsilon = \pm 1,\ c \neq 0,\ d,\ \Theta,\ \xi_0$ are real arbitrary parameters.

Step 2: By taking the homogeneous balance between the highest order derivative and the most considerable nonlinear term in (4), the value of R is obtained.

Step 3: Inserting (5) and its derivatives into (4), with respect to (6), we obtain a polynomial in terms of $V(\xi)$. By setting the coefficients of each power of $V(\xi)$ to zero, we obtain a system of equations involving the unknown parameters $\Theta,\ M_s,\ L_s\ (s = 1, 2, \ldots, R)$. By solving this system, we derive the analytical solutions of (4).

Step 4: Lastly, the application of the transformation in (3) to the solutions of (4) enables the construction of several analytical solutions for (2). Under three distinct cases, the corresponding solutions to (6) have been obtained.

2.2. Description of the $\left(\frac{G'}{G^2}\right)$ Expansion Function Method

The principal steps of the $\left(\frac{G'}{G^2}\right)$-expansion function method is specified in this subsection [21]. To solve (1), we assume a solution of the form:

$$H(\xi) = m_0 + \sum_{i=1}^{K} \left(m_i \left(\frac{G'}{G^2}\right)^{i} + n_i \left(\frac{G'}{G^2}\right)^{-i} \right) \quad (m_i \neq 0 \text{ or } n_i \neq 0), \tag{22}$$

where $G = G(\xi)$ defined as follows:

$$\left(\frac{G'}{G^2}\right)' = \tau + \varphi \left(\frac{G'}{G^2}\right)^2. \tag{23}$$

Here, the constants $\varphi \neq 0$ and $\tau \neq 1$ are assumed, and the unknown constants $m_0,\ m_i,\ n_i$ $(i = 1, 2, 3, \cdots, K)$ will be defined later. The corresponding three families of solutions to (22) are as follows:

When $\tau\varphi > 0$, we have the following **trigonometric solution**:

$$\left(\frac{G'}{G^2}\right) = \sqrt{\frac{\tau}{\varphi}} \left(\frac{A_1 \cos\left(\sqrt{\tau\varphi}\xi\right) + A_2 \sin\left(\sqrt{\tau\varphi}\xi\right)}{A_2 \cos\left(\sqrt{\tau\varphi}\xi\right) - A_1 \sin\left(\sqrt{\tau\varphi}\xi\right)} \right). \tag{24}$$

When $\tau\varphi < 0$, we derive the subsequent **hyperbolic solution**:

$$\left(\frac{G'}{G^2}\right) = -\sqrt{\frac{\tau\varphi}{\varphi}} \left(\frac{A_1 \sinh\left(2\sqrt{\tau\varphi}\xi\right) + A_1 \cosh\left(2\sqrt{\tau\varphi}\xi\right) + A_2}{A_1 \sinh\left(2\sqrt{\tau\varphi}\xi\right) + A_1 \cosh\left(2\sqrt{\tau\varphi}\xi\right) - A_2} \right). \tag{25}$$

When $\varphi \neq 0, \tau = 0$, we have the next **rational solution**:

$$\left(\frac{G'}{G^2}\right) = \left(-\frac{A_1}{\varphi\left(A_1\xi + A_2\right)}\right). \tag{26}$$

Here, A_1, A_2 are constants. Substituting (22) and (23) into (4), and equating the coefficients of like powers of $\left(\frac{G'}{G^2}\right)$ to zero, yields a system of algebraic equations solved via Maple software program.

3. Application of the Offered Methods

Consider the wave transformation given by:

$$\rho(x,t) = V(\xi), \quad \xi = \kappa x - \eta t. \tag{27}$$

Substituting (3) into (1), then we reach

$$\eta V''(\xi) - \kappa^3 V(\xi)'''' + 3\kappa^2 \left(V'^2(\xi)\right)' = 0. \tag{28}$$

When integrating (28) with respect to ξ, we obtain

$$\eta V'(\xi) - \kappa^3 V(\xi)''' + 3\kappa^2 \left(V'^2(\xi)\right) + c_0 = 0. \tag{29}$$

where suppose that the integration constant c_0 is zero. Assuming $V'(\xi) = S$, where $S(\xi)$ is real-valued, (1) reduces to the following ODE:

$$\eta S(\xi) - \kappa^3 S(\xi)'' + 3\kappa^2 S^2(\xi) = 0. \tag{30}$$

3.1. The Solutions to the Proposed Model Using the METEM

Applying the equilibrium principle to (30) yields $n = 2$. Therefore, the (5) turns into

$$S(\xi) = M_0 + M_1\Phi(\xi) + M_2\Phi^2(\xi) + \frac{L_1}{\Phi(\xi)} + \frac{L_2}{\Phi^2(\xi)}. \tag{31}$$

In this case, M_0, M_1, M_2, L_1 and L_2 are parameters. Adhering to the suggested method, then we reach the subsequent equation system:

$$\begin{cases} \eta M_0 - \kappa^3\left(2M_2\Theta^2 + 2L_2\right) + 3\kappa^2\left(2L_2M_2 + 2L_1M_1 + M_0^2\right) = 0, \\ \eta L_2 - 8\kappa^3 L_2\Theta + 3\kappa^2\left(L_1^2 + 2L_2M_0\right) = 0, \\ \eta L_1 - 2\kappa^3 L_1\Theta + 3\kappa^2\left(2L_1M_0 + 2L_2M_1\right) = 0, \\ 6\kappa^2 M_1M_2 - 2\kappa^3 M_1 = 0, \\ 3\kappa^2 M_2^2 - 6\kappa^3 M_2 = 0, \\ \eta M_2 - 8\kappa^3 M_2\Theta + 3\kappa^2\left(2M_0M_2 + M_1^2\right) = 0, \\ \eta M_1 - 2\kappa^3 M_1\Theta + 3\kappa^2\left(2L_1M_2 + 2M_1M_0\right) = 0, \\ -6\kappa^3 L_2\Theta^2 + 3\kappa^2 L_2^2 = 0, \\ -2\kappa^3 L_1\Theta^2 + 6\kappa^2 L_2L_1 = 0. \end{cases}$$

Solving the above system of algebraic equation, we obtain the following sets:

Set 1:

$$M_0 = -\frac{4\kappa\Theta}{3}, \quad M_1 = 0, \quad M_2 = 2\kappa, \quad L_1 = 0, \quad L_2 = 2\kappa\Theta^2, \quad \eta = 16\kappa^3\Theta. \tag{32}$$

Set 2:

$$M_0 = \frac{2\kappa\Theta}{3}, \quad M_1 = 0, \quad M_2 = 2\kappa, \quad L_1 = 0, \quad L_2 = 0, \quad \eta = 4\kappa^3\Theta. \tag{33}$$

By using **Set 1**, we get the following solutions:

Case 1: If $\Theta < 0$, then the kink type solution obtained as

$$\rho_{1,1}(x,t) = -\frac{4\kappa\Theta}{3} - 2\kappa\Theta\tanh\left(\sqrt{-\Theta}\left(\kappa x - \eta t\right)\right)^2 - \frac{2\kappa\Theta}{\tanh\left(\sqrt{-\Theta}\left(\kappa x - \eta t\right)\right)^2}. \tag{34}$$

The solitary wave solution reached as

$$\rho_{1,2}(x,t) = -\frac{4\kappa\Theta}{3} - 2\kappa\Theta\coth\left(\sqrt{-\Theta}\left(\kappa x - \eta t\right)\right)^2 - \frac{2\kappa\Theta}{\coth\left(\sqrt{-\Theta}\left(\kappa x - \eta t\right)\right)^2}. \tag{35}$$

The mixed complex bright-dark soliton solution attained as

$$\rho_{1,3}(x,t) = -\frac{4\kappa\Theta}{3} - 2\kappa\Theta\Big(\tanh\left(2\sqrt{-\Theta}\left(\kappa x - \eta t\right)\right) + i\,\mathrm{sech}\left(2\sqrt{-\Theta}\left(\kappa x - \eta t\right)\right)\Big)^2$$
$$-\frac{2\kappa\Theta}{\left(\tanh\left(2\sqrt{-\Theta}\left(\kappa x - \eta t\right)\right) + i\,\mathrm{sech}\left(2\sqrt{-\Theta}\left(\kappa x - \eta t\right)\right)\right)^2}. \tag{36}$$

The kink type solution reached as

$$\rho_{1,4}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\left(-\Theta + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(1 + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}$$

$$+ \frac{2\kappa\Theta^2\left(1 + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(-\Theta + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{37}$$

The solitary wave solutions obtained as

$$\rho_{1,5}(x,t) = -\frac{4\kappa\Theta}{3} - \frac{2\kappa\Theta\left(5 - 4\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(3 + 4\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}$$

$$- \frac{2\kappa\Theta\left(3 + 4\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(5 - 4\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}, \tag{38}$$

$$\rho_{1,6}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\left(\sqrt{-\Theta\,(c^2+d^2)} - c\sqrt{-\Theta}\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(c\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) + d\right)^2}$$

$$+ \frac{2\kappa\Theta^2\left(c\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) + d\right)^2}{\left(\sqrt{-\Theta\,(c^2+d^2)} - c\sqrt{-\Theta}\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}, \tag{39}$$

$$\rho_{1,7}(x,t) = -\frac{4\kappa\Theta}{3} - \frac{2\kappa\Theta\left(-c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}$$

$$- \frac{2\kappa\Theta\left(c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(-c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{40}$$

Case 2: If $\Theta > 0$, then we reached the following singular periodic wave solutions:

$$\rho_{1,8}(x,t) = -\frac{4\kappa\Theta}{3} + 2\kappa\Theta\tan\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2 + \frac{2\kappa\Theta}{\tan\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2}, \tag{41}$$

$$\rho_{1,9}(x,t) = -\frac{4\kappa\Theta}{3} + 2\kappa\Theta\cot\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2 + \frac{2\kappa\Theta}{\cot\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2}. \tag{42}$$

The mixed type trigonometric soliton solutions attained as

$$\rho_{1,10}(x,t) = -\frac{4\kappa\Theta}{3} + 2\kappa\Theta\left(\tan\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right) + \sec\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right)\right)^2$$

$$+ \frac{2\kappa\Theta}{\left(\tan\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right) + \sec\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{43}$$

The explicit periodic type solution reached as

$$\rho_{1,11}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\Theta\left(1-\tan\left(\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2}{\left(1+\tan\left(\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2} + \frac{2\kappa\Theta\left(1+\tan\left(\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2}{\left(1-\tan\left(\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2}, \tag{44}$$

$$\rho_{1,12}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\Theta\left(4-5\cos\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2}{\left(3+5\sin\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2} + \frac{2\kappa\Theta\left(3+5\sin\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2}{\left(4-5\cos\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2}, \tag{45}$$

$$\rho_{1,13}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\left(\sqrt{\Theta\,(c^2-d^2)} - c\sqrt{\Theta}\cos\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2}{\left(c\sin\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)+d\right)^2} + \frac{2\kappa\Theta^2\left(c\sin\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)+d\right)^2}{\left(\sqrt{\Theta\,(c^2-d^2)} - c\sqrt{\Theta}\cos\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^2}, \tag{46}$$

$$\rho_{1,14}(x,t) = -\frac{4\kappa\Theta}{3} - 2\kappa\Theta\left(1-\frac{2c}{c+\cos\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)-i\sin\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)}\right)^2 - \frac{2\kappa\Theta}{\left(1-\frac{2c}{c+\cos\left(2\sqrt{\Theta}(\kappa x-\eta t)\right)-i\sin\left(2\sqrt{\Theta}(\kappa x-\eta t)\right)}\right)^2}. \tag{47}$$

Case 3: If $\Theta = 0$, then we get the rational solution as below:

$$\rho_{1,15}(x,t) = \frac{2\kappa}{(\kappa x-\eta t)^2}. \tag{48}$$

For **Set 2**, we reach the next solutions:

Case 1: If $\Theta < 0$, then the kink type solution obtained as

$$\rho_{2,1}(x,t) = -\frac{2\kappa\Theta}{3}\left(3\tanh\left(\sqrt{-\Theta}\,(\kappa x-\eta t)\right)^2 - 1\right). \tag{49}$$

The solitary wave solution obtained as

$$\rho_{2,2}(x,t) = -\frac{2\kappa\Theta}{3}\left(3\coth\left(\sqrt{-\Theta}\,(\kappa x-\eta t)\right)^2 - 1\right). \tag{50}$$

The mixed complex bright-dark soliton solution attained as

$$\rho_{2,3}(x,t)=\frac{2\kappa\Theta}{3}-2\kappa\Theta\Big(\tanh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)+i\,\mathrm{sech}\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)\Big)^{2}. \tag{51}$$

The kink type solution reached as

$$\rho_{2,4}(x,t)=\frac{2\kappa\Theta}{3}+\frac{2\kappa\left(-\Theta+\sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}{\left(1+\sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}. \tag{52}$$

The mixed type hyperbolic solutions obtained as

$$\rho_{2,5}(x,t)=\frac{2\kappa\Theta}{3}-\frac{2\kappa\Theta\left(5-4\cosh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}{\left(3+4\sinh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}, \tag{53}$$

$$\rho_{2,6}(x,t)=\frac{2\kappa\Theta}{3}+\frac{2\kappa\Big(\sqrt{-\Theta\,(c^{2}+d^{2})}-c\sqrt{-\Theta}\cosh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)\Big)^{2}}{\left(c\sinh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)+d\right)^{2}}, \tag{54}$$

$$\rho_{2,7}(x,t)=\frac{2\kappa\Theta}{3}-\frac{2\kappa\Theta\left(-c+\cosh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)-\sinh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}{\left(c+\cosh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)-\sinh\left(2\sqrt{-\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}. \tag{55}$$

Case 2: If $\Theta>0$, then we get as follows:

The singular periodic wave solutions obtained as

$$\rho_{2,8}(x,t)=\frac{2\kappa\Theta}{3}\left(3\tan\left(\sqrt{\Theta}\,(\kappa x-\eta t)\right)^{2}+1\right), \tag{56}$$

$$\rho_{2,9}(x,t)=\frac{2\kappa\Theta}{3}\left(3\cot\left(\sqrt{\Theta}\,(\kappa x-\eta t)\right)^{2}+1\right). \tag{57}$$

The combo trigonometric soliton solution attained as

$$\rho_{2,10}(x,t)=\frac{2\kappa\Theta}{3}+2\kappa\Theta\Big(\tan\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)+\sec\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\Big)^{2}. \tag{58}$$

The explicit periodic type solution reached as

$$\rho_{2,11}(x,t)=\frac{2\kappa\Theta}{3}+\frac{2\kappa\Theta\left(-1+\tan\left(\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}{\left(1+\tan\left(\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}. \tag{59}$$

The combo trigonometric soliton solution obtained as

$$\rho_{2,12}(x,t)=\frac{2\kappa\Theta}{3}+\frac{2\kappa\Theta\left(4-5\cos\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}{\left(3+5\sin\left(2\sqrt{\Theta}\,(\kappa x-\eta t)\right)\right)^{2}}, \tag{60}$$

$$\rho_{2,13}(x,t)=\frac{2\kappa\Theta}{3}+\frac{2\kappa\left(\sqrt{\Theta\left(c^2-d^2\right)}-c\sqrt{\Theta}\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}{\left(c\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)+d\right)^2}. \tag{61}$$

The complex trigonometric wave solutions attained as

$$\rho_{2,14}(x,t)=\frac{2\kappa\Theta}{3}-2\kappa\Theta\left(1-\frac{2c}{c+\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)-i\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)}\right)^2. \tag{62}$$

Case 3: If $\Theta=0$, then we have the following rational solution:

$$\rho_{2,15}(x,t)=\frac{2\kappa}{\left(\kappa x-\eta t\right)^2}. \tag{63}$$

3.2. Utilizing the $\left(\frac{G'}{G^2}\right)$ Expansion Function Method

Using the homogenous balance principle, (22) is as follows:

$$H(\xi)=m_0+m_1\left(\frac{G'}{G^2}\right)+m_2\left(\frac{G'}{G^2}\right)^2+n_1\left(\frac{G'}{G^2}\right)^{-1}+n_2\left(\frac{G'}{G^2}\right)^{-2}. \tag{64}$$

When (64) is inserted into (30) with all coefficients set to zero, we obtain

$$\begin{cases}
-2\kappa^3n_2\varphi^2-2\kappa^3m_2\tau^2+3\kappa^2m_0^2+6\kappa^2m_1n_1+6\kappa^2m_2n_2+\eta m_0=0,\\
-8\kappa^3\varphi\tau m_2+6\kappa^2m_0m_2+3\kappa^2m_1^2+\eta m_2=0,\\
-2\kappa^3\varphi\tau m_1+6\kappa^2m_0m_1+6\kappa^2m_2n_1+\eta m_1=0,\\
-2\kappa^3\tau\varphi n_1+6\kappa^2m_0n_1+6\kappa^2m_1n_2+\eta n_1=0,\\
-8\kappa^3\varphi\tau n_2+6\kappa^2m_0n_2+3\kappa^2n_1^2+\eta n_2=0,\\
-2\kappa^3\varphi^2m_1+6\kappa^2m_1m_2=0,\\
-2\kappa^3\tau^2n_1+6\kappa^2n_1n_2=0,\\
-6\kappa^3\varphi^2m_2+3\kappa^2m_2^2=0,\\
-6\kappa^3\tau^2n_2+3\kappa^2n_2^2=0.
\end{cases}$$

By solving above the algebraic system, we get the following solution sets:

Set 1:

$$\eta=4\kappa^3\varphi\tau,\quad m_0=\frac{2\kappa\varphi\tau}{3},\quad m_1=0,\quad m_2=2\kappa\varphi^2,\quad n_1=0,\quad n_2=0.$$

Set 2:

$$\eta=16\kappa^3\varphi\tau,\quad m_0=-\frac{4\kappa\varphi\tau}{3},\quad m_1=0,\quad m_2=2\kappa\varphi^2,\quad n_1=0,\quad n_2=2\kappa\tau^2.$$

By using **Set 1**, we have the following soliton solutions:

If $\tau\varphi > 0$, then the trigonometric solution is given by the following form:

$$\rho_1(x,t) = \frac{2\kappa\tau\varphi}{3} + \frac{2\kappa\tau\varphi\left(A_1\cos\left(\sqrt{\tau\varphi}\xi\right) + A_2\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}{\left(A_2\cos\left(\sqrt{\tau\varphi}\xi\right) - A_1\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}. \tag{65}$$

If $\tau\varphi < 0$, then the hyperbolic solution is found as follow:

$$\rho_2(x,t) = \frac{2\kappa\tau\varphi}{3} - \frac{2\kappa\tau\varphi\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(-\sqrt{\tau\varphi}\xi\right) + A_2\right)^2}{\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(2\sqrt{-\tau\varphi}\xi\right) - A_2\right)^2}. \tag{66}$$

If $\tau = 0,\ \varphi \neq 0$, then the rational solution is given by the following form:

$$\rho_3(x,t) = \frac{2\kappa A_1^2}{\left(\xi A_1 + A_2\right)^2}. \tag{67}$$

For **Set 2**, we reach the following solutions:

If $\tau\varphi > 0$, then the trigonometric solution is given by the following form:

$$\begin{aligned}\rho_{1,0}(x,t) = &-\frac{4\kappa\tau\varphi}{3} + \frac{2\kappa\tau\varphi\left(A_1\cos\left(\sqrt{\tau\varphi}\xi\right) + A_2\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}{\left(A_2\cos\left(\sqrt{\tau\varphi}\xi\right) - A_1\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}\\ &+ \frac{2\kappa\tau\varphi\left(A_2\cos\left(\sqrt{\tau\varphi}\xi\right) - A_1\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}{\left(A_1\cos\left(\sqrt{\tau\varphi}\xi\right) + A_2\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}.\end{aligned} \tag{68}$$

If $\tau\varphi < 0$, then the hyperbolic solution is found as below:

$$\begin{aligned}\rho_{2,0}(x,t) = &-\frac{2\kappa\tau\varphi}{3} - \frac{2\kappa\tau\varphi\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(\sqrt{-\tau\varphi}\xi\right) + A_2\right)^2}{\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(2\sqrt{-\tau\varphi}\xi\right) - A_2\right)^2}\\ &- \frac{2\kappa\tau\varphi\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(\sqrt{-\tau\varphi}\xi\right) - A_2\right)^2}{\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(2\sqrt{-\tau\varphi}\xi\right) + A_2\right)^2}.\end{aligned} \tag{69}$$

If $\tau = 0,\ \varphi \neq 0$, then the rational solution is given by the following form:

$$\rho_{3,0}(x,t) = \frac{2\kappa A_1^2}{\left(\xi A_1 + A_2\right)^2}. \tag{70}$$

4. Graphical Explanation

This section demonstrates the structural properties of the obtained soliton solutions using various graphical representations. Specifically, 3D, contour, and 2D graphs corresponding to analytical solutions derived in previous sections are presented. These visualizations help demonstrate the localization, amplitude, and wave propagation properties of the solutions more intuitively and comprehensively.

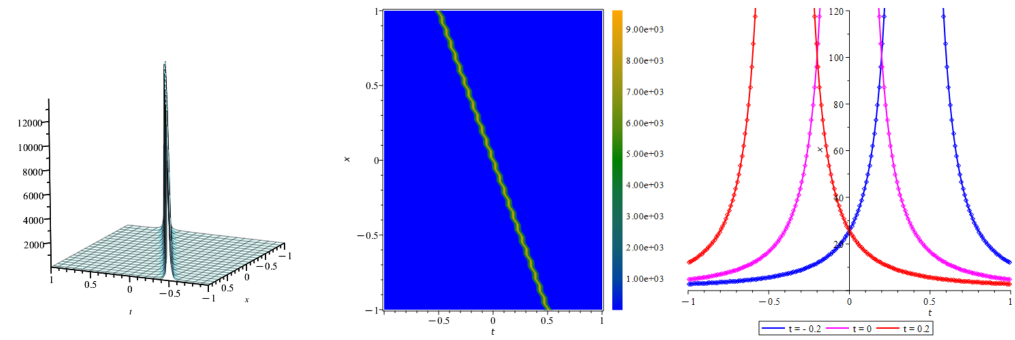
Figure 1. 3D, 2D, and contour plots of the bright soliton solution for $\rho_{1,1}(x,t)$ are presented when $\Theta = -0.5$, $\kappa = 0.5$.

Figure 2. The W-shaped soliton solution of $|\rho_{1,3}(x,t)|$ is illustrated using 3D, 2D, and contour plots for $\Theta = -0.5$, $\kappa = 0.5$.

Figure 3. The singular solitary wave structure of $\rho_{1,8}(x,t)$ is depicted through 3D, 2D, and contour plots for $\Theta = 1$, $\kappa = 1$.

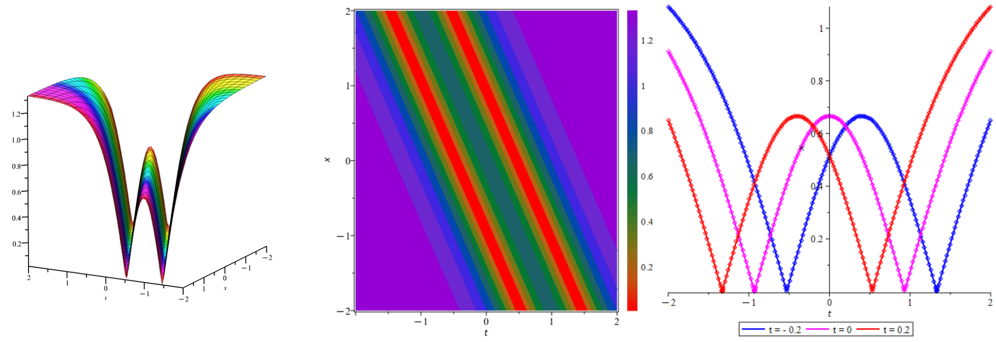
Figure 4. The dark soliton solution of $\rho_{2,1}(x,t)$ is illustrated using 3D, 2D, and contour plots for $\Theta = -0.5$, $\kappa = 1$.

The solutions in this study have characteristic graphical properties of nonlinear wave systems. These solutions are bright, dark, W-shaped soliton and singular solitary wave solutions. A bright soliton is a smooth, bell-shaped peak that does not change as it moves. In contrast, a W-shaped soliton has two peaks, meaning the wave amplitude differs between the peak and trough, forming a profile that resembles the letter “W”. A singular solitary wave solution differs in that it has infinite or undefined amplitude at certain points, resulting in sharp discontinuities or singularities in the wave profile. Conversely, a dark soliton appears as a localized notch (or trough) on a smooth background wave with a phase shift and stable motion. Dark and bright solitons have smooth, localized forms at the graphical level, while W-shaped soliton and singular solitary wave introduce more complex dynamics into the wave profile. This impacts applications in fluid dynamics, optics, and Bose-Einstein condensates.



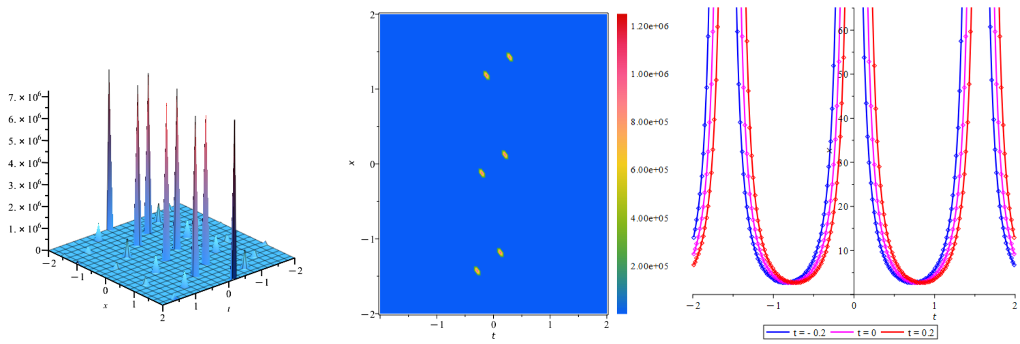
(a) 3D (b) Contour (c) 2D

Figure 1: Graphs of $\rho_{1,1}(x,t)$



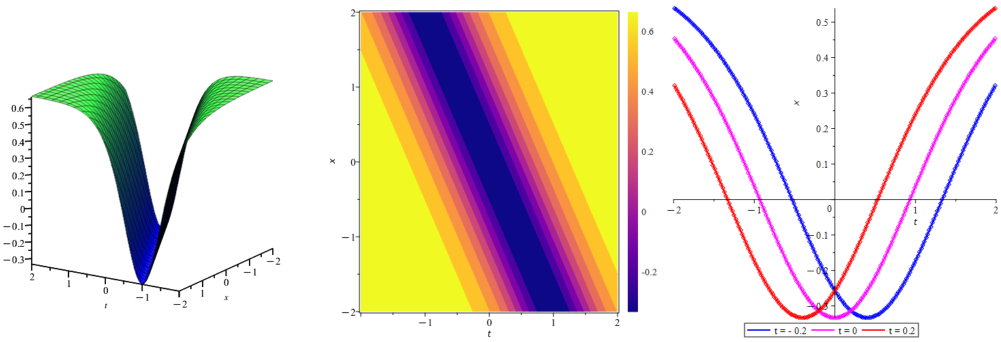
(a) 3D (b) Contour (c) 2D

Figure 2: Graphical representations of $|\rho_{1,3}(x,t)|$



(a) 3D (b) Contour (c) 2D

Figure 3: Graphical representations of $\rho_{1,8}(x,t)$



(a) 3D (b) Contour (c) 2D

Figure 4: Graphical representations of $\rho_{2,1}(x,t)$.

5. Conclusion

In this study, METEM and the $\left(\frac{G'}{G^2}\right)$-expansion function method were used to derive new analytical solutions of nonlinear K-XE. The results include a wide range of exact solutions, such as W-shaped solitons, singular solitary waves, bright and dark solitons, as well as rational, hyperbolic, and trigonometric forms. Several of these solutions were visualized using 3D, contour, 2D plots generated via Maple software. These graphical representations effectively capture the physical behavior of the solutions and validate their consistency. According to the obtained results, these two approaches provide highly accurate analytical solutions for K-XE. Another advantage of these methods is their proven ability to efficiently generate solutions. These solutions are crucial for understanding the wave dynamics of the model. All solutions have been verified using software programs. In the future, the study will be expanded to include the fractional and variable-coefficient forms of the K-XE. These efforts are expected to further enhance the model's physical interpretability and applicability in nonlinear science.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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